A Study of the Mean Value of the Error Term in the Mean Square Formula of the Riemann Zeta-function in the Critical Strip $3/4 \le \sigma < 1$

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Abstract

Let $E_\sigma(T)$ be the error term in the mean square formula of the Riemann Zeta-function in the critical strip $1/2 < \sigma < 1$. It is an analogue of the error term $E(T)$ for the case $\sigma = 1/2$. The research of $E(T)$ has a long history but the investigation of $E_\sigma(T)$ is quite new. In particular there is only a few information known about $E_\sigma(T)$ for $3/4 < \sigma < 1$. As one way of exploration, we approach this problem by looking at its mean value $\int_1^T E_\sigma(u)\,du$. In this paper, we give it an Atkinson-type series expansion and explore its properties as a function of T. From this, we can obtain Ω-results of $E_\sigma(T)$ and its mean square for $3/4 < \sigma < 1$.

Key words: Mean value, Mean square formula, Riemann Zeta-function

1. INTRODUCTION

Let $\zeta(s)$ be the Riemann Zeta-function, and let

$$E(T)=\int_0^T|\zeta(1/2+it)|^2\,dt-T(\log\frac{T}{2\pi}+2\gamma-1)$$

denote the error term in the mean-square formula for $\zeta(s)$ (on the critical line). The behaviour of $E(T)$ is interesting and many papers are devoted to study this function. However, the conjecture about its order of magnitude, $E(T)\ll T^{1/4+\epsilon}$, remains open. Analogously, it is defined for $1/2<\sigma<1$,

$$E_\sigma(T)=\int_0^T|\zeta(\sigma+it)|^2\,dt-\left(\zeta(2\sigma)T+(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma}\right).$$

The behaviour of $E_\sigma(T)$ is very interesting too, and in fact, more delicate analysis is required to explore its properties such as the Atkinson-type series expansion and mean square formula, see ([14]-[17]). Excellent surveys are given in [10] and [15].

In the critical strip $1/2<\sigma<1$, our knowledge of $E_\sigma(T)$ is not 'uniform', for example, the mean square formula and $\Omega_\pm$-results are available for $1/2<\sigma\le 3/4$ but not for the other part. In fact, not much is known for the case $3/4<\sigma<1$, except perhaps some upper bound estimates and

$$\int_1^T E_\sigma(t)^2\,dt\ll T\qquad(3/4<\sigma<1).\tag{1.1}$$

To furnish this part, we look at the mean value $\int_1^T E_\sigma(u)\,du$. The mean values of $E(T)$ and $E_\sigma(T)$ $(1/2<\sigma<3/4)$ are respectively studied in [2] and [6], each of which gives an asymptotic expansion. Correspondingly, we can represent it in this case $3/4\le\sigma<1$ by an asymptotic formula with a good error term. The proof of the asymptotic formula relies on the argument of [2] and uses the tools available in [2] and [16]. But there is a difficulty which we need to get around. In [2], Hafner and Ivić used a result of Jutila [8] on transformation of Dirichlet Polynomials, which depends on the formula

$$\textstyle\sum'_{a\le n\le b}d(n)f(n)=\displaystyle\int_a^b(\log x+2\gamma)f(x)\,dx+\sum_{n=1}^{\infty}d(n)\int_a^b f(x)\alpha(nx)\,dx$$

where α is a combination of Bessel functions. It is not available in our case but this can be avoided by using the idea in [16].

In addition, we shall regard the mean value as a function of T and study its behaviour; more precisely, we consider

$$G_\sigma(T)=\int_1^T E_\sigma(t)\,dt+2\pi\zeta(2\sigma-1)T.$$

Unlike the case $1/2 \le \sigma < 3/4$, $G_\sigma(T)$ is now more fluctuating. Nevertheless we can still explore many interesting properties, including some power moments, $\Omega_\pm$-results, gaps between sign-changes and limiting distribution functions. The second and third power moments are obtained by the methods used in [16] and [20]. The underlying principle in proving the $\Omega_\pm$-results are the same as other authors but the technical treatment is different (for example, compare our Section 6 with [2, Section 5]). To discuss the width of gaps between sign-changes, we need the method in [4], though the case $1/2 \le \sigma < 3/4$ can be handled by the argument in [11]. We can determine the exact order of magnitude of the gaps (see Theorems 5 and 6). This is because the series representation of $G_\sigma(t)$ has a fast rate of convergence. We apply the results in [1] and [3] to deal with the problem about limiting distribution function. Such an investigation is actually not done in the case $1/2 \le \sigma < 3/4$, perhaps because it is less interesting in the sense that the exact order of magnitude of $G_\sigma(t)$ $(1/2 \le \sigma < 3/4)$ is known; therefore, the limiting distribution is 'compactly supported'. Here, we say that a limiting distribution is compactly supported if it equals 0 or 1 outside a compact set. (Note that a distribution function is non-decreasing.) However, in our case it never vanishes (i.e. never equal to 0 or 1), and we shall investigate the rate of decay. Our approach here can give a more precise result than that of [1].

Finally, we want to remark that our work leads to

$$\int_1^T E_\sigma(t)\,dt = -2\pi\zeta(2\sigma-1)T + O(\sqrt{T}\log T) \qquad (3/4 < \sigma < 1).$$

This suggests that we should further split $E_\sigma(t)$ into two parts: $E_\sigma(t) = -2\pi\zeta(2\sigma-1) + E^*_\sigma(t)$.

2. STATEMENT OF RESULTS

Throughout the paper, we assume $3/4 \le \sigma < 1$ and use c, c' and c'' to denote some constants which may differ at each occurrence. The implied constants in $\ll$- or O-symbols and the unspecified positive constants c_i $(i = 1, 2, \ldots)$ may depend on σ.

Let $\sigma_a(n) = \sum_{d|n} d^a$ and $\operatorname{arsinh} x = \log(x + \sqrt{x^2+1})$. We define

$$\begin{aligned}
\Sigma_1(t, X) &= \sqrt{2}\left(\frac{t}{2\pi}\right)^{5/4-\sigma} \sum_{n \le X} (-1)^n \frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} e_2(t,n) \sin f(t,n) \\
\Sigma_2(t, X) &= 2\left(\frac{t}{2\pi}\right)^{1/2-\sigma} \sum_{n \le B(t,\sqrt{X})} \frac{\sigma_{1-2\sigma}(n)}{n^{1-\sigma}} \left(\log\frac{t}{2\pi n}\right)^{-2} \sin g(t,n),
\end{aligned}$$

where

$$\begin{aligned}
e_2(t,n) &= \left(1+\frac{\pi n}{2t}\right)^{-1/4}\left(\sqrt{\frac{2t}{\pi n}}\ \mathrm{arsinh}\sqrt{\frac{\pi n}{2t}}\right)^{-2},\\
f(t,n) &= 2t\ \mathrm{arsinh}\sqrt{\frac{\pi n}{2t}}+(2\pi nt+\pi^2n^2)^{1/2}-\frac{\pi}{4},\\
g(t,n) &= t\log\frac{t}{2\pi n}-t+\frac{\pi}{4},\\
B(t,\sqrt{X}) &= \frac{t}{2\pi}+\frac{X}{2}-\sqrt{X}\left(\frac{t}{2\pi}+\frac{X}{4}\right)^{1/2}=\left(\left(\frac{t}{2\pi}+\frac{X}{4}\right)^{1/2}-\frac{\sqrt{X}}{2}\right)^2.
\end{aligned}$$

Theorem 1 *Let $T\ge 1$ and $N\asymp T$. We have*

$$\int_1^T E_\sigma(t)\,dt=-2\pi\zeta(2\sigma-1)T+\Sigma_1(T,N)-\Sigma_2(T,N)+O(\log^2 T).$$

Remark: Bounding $\Sigma_1(T,N)$ and $\Sigma_2(T,N)$ trivially, we infer with (1.1) a result for $E_\sigma(t)$.

Corollary We have $\int_1^T E_\sigma(t)\,dt=-2\pi\zeta(2\sigma-1)T+O(\sqrt{T}\min((\sigma-3/4)^{-1},\log T))$.

Define $G_\sigma(t)=\int_1^t E_\sigma(u)\,du+2\pi\zeta(2\sigma-1)t$. Then $G_\sigma(t)\ll t^{1/2}$, and it is not hard to get

$$\int_T^{2T} G_\sigma(t)\,dt=o(T^{1+(5/4-\sigma)}). \tag{2.1}$$

Results of higher power moments are proved in the next theorem.

Theorem 2 *Let $T\ge 1$. We have*

(1) $\int_T^{2T} G_\sigma(t)^2\,dt=B(\sigma)\int_T^{2T}(t/(2\pi))^{5/2-2\sigma}\,dt+O(T^{3-2\sigma})$,

(2) $\int_T^{2T} G_\sigma(t)^3\,dt=-C(\sigma)\int_T^{2T}(t/(2\pi))^{15/4-3\sigma}\,dt+O(T^{(13-8\sigma)/3})$,

(3) for any real $k\in[0,A_0)$ and any odd integer $0\le l<A_0$ where $A_0=(\sigma-3/4)^{-1}$,

$$\int_1^T |G_\sigma(t)|^k\,dt\sim\alpha_k(\sigma)T^{1+k(5/4-\sigma)}\quad\text{and}\quad\int_1^T G_\sigma(t)^l\,dt\sim\beta_l(\sigma)T^{1+l(5/4-\sigma)}.$$

where $\alpha_k(\sigma)\ge 0$ and $\beta_l(\sigma)$ are some constants, $B(\sigma)$ and $C(\sigma)$ are defined by

$$\begin{aligned}
B(\sigma) &= \sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)^2n^{2\sigma-7/2}=\zeta(7/2-2\sigma)\zeta(3/2+2\sigma)\zeta(5/2)^2\zeta(5)^{-1},\\
C(\sigma) &= \frac{3}{2}\sum_{s=1}^{\infty}\frac{\mu(s)^2}{s^{21/4-3\sigma}}\sum_{a,b=1}^{\infty}\frac{\sigma_{1-2\sigma}(sa^2)}{a^{7/2-2\sigma}}\frac{\sigma_{1-2\sigma}(sb^2)}{b^{7/2-2\sigma}}\frac{\sigma_{1-2\sigma}(s(a+b)^2)}{(a+b)^{7/2-2\sigma}}.
\end{aligned}$$

It is expected that $G_\sigma(t)$ is oscillatory in nature and its order of magnitude of $G_\sigma(t)$ is about $t^{5/4-\sigma}$. Although (2.1) tells us that there is plenty of cancellation between the positive and negative parts, Theorem 2 (2) shows the negative part should be more dominating. This phenomenon also appears in the case $1/2\le\sigma<3/4$. Now, we look at its distribution of values from the probabilistic viewpoint.

Theorem 3 *The limiting distribution* $D_\sigma(u)$ *of* $t^{\sigma-5/4}G_\sigma(t)$ *exists, and is equal to the distribution of the random series* $\eta = \sum_{n=1}^{\infty} a_n(t_n)$ *where*

$$a_n(t) = \sqrt{2}\frac{\mu(n)^2}{n^{7/4-\sigma}}\sum_{r=1}^{\infty}(-1)^{nr}\frac{\sigma_{1-2\sigma}(nr^2)}{r^{7/2-2\sigma}}\sin(2\pi rt - \pi/4)$$

and t_n *are independent random variables uniformly distributed on* $[0,1]$. *Define tail(*$D_\sigma(u)$*)* $= 1 - D_\sigma(u)$ *for* $u \geq 0$ *and* $D_\sigma(u)$ *for* $u < 0$. *Then*

$$\left.\begin{matrix}\exp(-c_1\exp(|u|)) \\ \\ \exp(-c_2|u|^{4/(4\sigma-3)})\end{matrix}\right\} \ll \textit{tail}(D_\sigma(u)) \ll \begin{cases}\exp(-c_3\exp(|u|)) & \textit{if } \sigma = 3/4, \\ \\ \exp(-c_4|u|^{4/(4\sigma-3)}) & \textit{if } 3/4 < \sigma < 1.\end{cases} \tag{2.2}$$

Remark: $D_\sigma(u)$ is non-symmetric and should skew towards the negative side because of Theorem 2 (2). Again it is also true for $1/2 \leq \sigma < 3/4$. But in the case $1/2 \leq \sigma < 3/4$, the set $\{u \in \mathbf{R} : 0 < D_\sigma(u) < 1\}$ is compact and it differs from our case.

To investigate the oscillatory nature, we consider the extreme values of $G_\sigma(t)$ and the frequency of occurrence of large values. These are revealed in the following three results.

Theorem 4 *As* $T \to \infty$, *we have*

$$G_{3/4}(T) = \Omega_+(\sqrt{T}\log\log\log T) \textit{ and } G_{3/4}(T) = \Omega_-(\sqrt{T}\log\log T);$$

if $3/4 < \sigma < 1$, *then*

$$G_\sigma(T) = \Omega_+(T^{5/4-\sigma}\exp(c_5\frac{(\log\log T)^{\sigma-3/4}}{(\log\log\log T)^{7/4-\sigma}})) \textit{ and } G_\sigma(T) = \Omega_-(T^{5/4-\sigma}(\log T)^{\sigma-3/4}).$$

Theorem 5 *For every sufficiently large* T, *there exist* t_1, $t_2 \in [T, T + c_6\sqrt{T}]$ *such that* $G_\sigma(t_1) \geq c_7 t_1^{5/4-\sigma}$ *and* $G_\sigma(t_2) \leq -c_7 t_2^{5/4-\sigma}$. *In particular,* $G_\sigma(t)$ *has (at least) one sign change in every interval* $[T, T + c_8\sqrt{T}]$.

Theorem 6 *Let* $\delta > 0$ *be a fixed small number. Then for all sufficiently large* $T \geq T_0(\delta)$, *there are two sets* S^+ *and* S^- *of disjoint intervals in* $[T, 2T]$ *such that*

1. *any interval in* $S^\pm$ *is of length* $c_9\delta\sqrt{T}$,
2. *cardinality of* $S^\pm \geq c_{10}\delta^{4(1-\sigma)}\sqrt{T}$,
3. *if* $I \in S^\pm$, *then* $\pm G_\sigma(t) \geq (c_{11} - \delta^{5/2-2\sigma})t^{5/4-\sigma}$ *for all* $t \in I$ *(so no sign-changes on* I*)*.

3. SERIES REPRESENTATION

This section is to prove Theorem 1 and we need two lemmas.

Lemma 3.1 *Let α, β, γ, a, b, k, T be real numbers such that α, β, γ are positive and bounded, $\alpha \neq 1$, $0 < a < 1/2$, $a < T/(8\pi k)$, $b \geq T$, $k \geq 1$, $T \geq 1$,*

$$U(t) = \left(\frac{t}{2\pi k} + \frac{1}{4}\right)^{1/2}, V(t) = 2arsinh\sqrt{\frac{\pi k}{2t}},$$

$$\begin{aligned} L_k(t) &= (2ki\sqrt{\pi})^{-1} t^{1/2} V(t)^{-\gamma-1} U(t)^{-1/2} \left(U(t) - \frac{1}{2}\right)^{-\alpha} \left(U(t) + \frac{1}{2}\right)^{-\beta} \\ &\quad \times \exp\left(itV(t) + 2\pi i k U(t) - \pi i k + \frac{\pi i}{4}\right), \end{aligned}$$

and

$$J(T) = \int_T^{2T} \int_a^b y^{-\alpha}(1+y)^{-\beta} \left(\log\frac{1+y}{y}\right)^{-\gamma} \exp(it\log(1+1/y) + 2\pi i k y)\, dy dt.$$

Then uniformly for $|\alpha - 1| \geq \epsilon$, $1 \leq k \leq T+1$, we have

$$J(T) = L_k(2T) - L_k(T) + O(a^{1-\alpha}) + O(Tk^{-1}b^{\gamma-\alpha-\beta}) + O((T/k)^{(\gamma+1-\alpha-\beta)/2} T^{-1/4} k^{-5/4}).$$

In the case $-k$ in place of k, the result holds without $L_k(2T) - L_k(T)$ for the corresponding integral.

This is [2, Lemma 3].

Lemma 3.2 *Let*

$$\Delta_{1-2\sigma}(t) = {\sum}'_{n\leq t}\sigma_{1-2\sigma}(n) - \left(\zeta(2\sigma)t + \frac{\zeta(2-2\sigma)}{2-2\sigma}t^{2-2\sigma} - \frac{1}{2}\zeta(2\sigma-1)\right)$$

where the sum ${\sum}'_{n\leq t}$ counts half of the last term only when t is an integer. Define $\tilde{\Delta}_{1-2\sigma}(\xi) = \int_0^\xi \Delta_{1-2\sigma}(t)\,dt - \zeta(2\sigma-2)/12$. Assuming $3/4 \leq \sigma < 1$, we have for $\xi \geq 1$,

$$\begin{aligned} \tilde{\Delta}_{1-2\sigma}(\xi) &= C_1\xi^{5/4-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-7/4}\cos(4\pi\sqrt{n\xi} + \pi/4) \\ &\quad + C_2\xi^{3/4-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-9/4}\cos(4\pi\sqrt{n\xi} - \pi/4) + O(\xi^{1/4-\sigma}) \end{aligned}$$

where the two infinite series on the right-hand side are uniformly convergent on any finite closed subinterval in $(0,\infty)$, and the values of the constants are $C_1 = -1/(2\sqrt{2}\pi^2)$, $C_2 = (5-4\sigma)(7-4\sigma)/(64\sqrt{2}\pi^3)$. In addition, we have for $3/4 \leq \sigma < 1$,

$$\Delta_{1-2\sigma}(v) \ll v^{1-\sigma}, \quad \int_1^x \Delta_{1-2\sigma}(v)^2\,dv \ll x\log x,$$

$$\tilde{\Delta}_{1-2\sigma}(\xi) \ll \xi^r\log\xi, \quad \int_1^x \tilde{\Delta}_{1-2\sigma}(v)^2\,dv \ll x^{7/2-2\sigma}$$

where $0 < r = -(4\sigma^2 - 7\sigma + 2)/(4\sigma - 1) \leq 1/2$.

This comes from [16, Lemma 1] and the result in [19].

Proof of Theorem 1. From [14, (3.4)] and [17, (3.1)], we have

$$\begin{aligned}
&\int_{-t}^{t} |\zeta(\sigma+iu)|^2\,du \\
= \;& 2\zeta(2\sigma)t + 2\zeta(2\sigma-1)\Gamma(2\sigma-1)\frac{\sin(\pi\sigma)}{1-\sigma}t^{2-2\sigma} - 2i\int_{\sigma-it}^{\sigma+it} g(u, 2\sigma-u)\,du + O(\min(1,|t|^{-2\sigma})).
\end{aligned}$$

(Note that the value of c_3 in [17, (3.1)] is zero.) Hence, we have

$$E_\sigma(t) = -i\int_{\sigma-it}^{\sigma+it} g(u,2\sigma-u)\,du + O(\min(1,t^{-2\sigma})).$$

Define

$$h(u,\xi) = 2\int_0^\infty y^{-u}(1+y)^{u-2\sigma}\cos(2\pi\xi y)\,dy. \tag{3.1}$$

Assume $AT \le X \le T$ and X is not an integer where $0 < A < 1$ is a constant. Then, following [16, p.364-365], we define

$$\begin{aligned}
\mathrm{G}_1(t) &= \sum_{n\le X}\sigma_{1-2\sigma}(n)\int_{\sigma-it}^{\sigma+it} h(u,n)\,du, \\
\mathrm{G}_2(t) &= \Delta_{1-2\sigma}(X)\int_{\sigma-it}^{\sigma+it} h(u,X)\,du, \\
\mathrm{G}_3(t) &= \int_{\sigma-it}^{\sigma+it}\int_X^\infty (\zeta(2\sigma)+\zeta(2-2\sigma)\xi^{1-2\sigma})h(u,\xi)\,d\xi\,du, \\
\mathrm{G}_4^*(t) &= \tilde{\Delta}_{1-2\sigma}(X)\int_{\sigma-it}^{\sigma+it}\frac{\partial h}{\partial \xi}(u,X)\,du, \\
\mathrm{G}_4^{**}(t) &= \int_X^\infty \tilde{\Delta}_{1-2\sigma}(\xi)\int_{\sigma-it}^{\sigma+it}\frac{\partial^2 h}{\partial \xi^2}(u,\xi)\,du\,d\xi.
\end{aligned} \tag{3.2}$$

Then, we have

$$\begin{aligned}
\int_T^{2T} E_\sigma(t)\,dt &= -i\int_T^{2T}\mathrm{G}_1(t)\,dt + i\int_T^{2T}\mathrm{G}_2(t)\,dt - i\int_T^{2T}\mathrm{G}_3(t)\,dt \\
&\quad -i\int_T^{2T}\mathrm{G}_4^*(t)\,dt - i\int_T^{2T}\mathrm{G}_4^{**}(t)\,dt + O(1)
\end{aligned} \tag{3.3}$$

1) Evaluation of $\int_T^{2T}\mathrm{G}_1(t)\,dt$. We take $\gamma=1$, $\alpha=\beta=\sigma$ in Lemma 3.1, we have from (3.1),

$$\begin{aligned}
&\int_T^{2T}\int_{\sigma-it}^{\sigma+it} h(u,n)\,du\,dt \\
= \;& 4i\int_T^{2T}\int_0^\infty (y(1+y))^{-\sigma}(\log(1+1/y))^{-1}\sin(t\log((1+y)/y))\cos(2\pi ny)\,dy\,dt \\
= \;& 2i\ \mathrm{Im}\int_T^{2T}\int_0^\infty (y(1+y))^{-\sigma}(\log(1+1/y))^{-1}\{\exp(i(t\log((1+y)/y)+2\pi ny)) \\
&\quad + \exp(i(t\log((1+y)/y)-2\pi ny))\}\,dy\,dt \\
= \;& 2i\ \mathrm{Im}\ (L_n(2T)-L_n(T)) + O(T^{3/4-\sigma}n^{\sigma-9/4})
\end{aligned}$$

Noting that $L_n(t) = (i\sqrt{2})^{-1}(t/(2\pi))^{5/4-\sigma}(-1)^n n^{\sigma-7/4} e_2(t,n)\exp(i(f(t,n)+\pi/2))$, we get, with (3.2),

$$\begin{aligned}
&\int_T^{2T} \mathrm{G}_1(t)\,dt \\
&= \sqrt{2}i\left(\frac{t}{2\pi}\right)^{5/4-\sigma}\sum_{n\le X}(-1)^n\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}e_2(t,n)\sin f(t,n)\Bigg|_T^{2T} + O(T^{3/4-\sigma}) \qquad (3.4)
\end{aligned}$$

2) Evaluation of $\int_T^{2T}\mathrm{G}_2(t)\,dt$. The treatment is similar to G_1. From (3.2) and Lemma 3.1, $\int_T^{2T}\int_{\sigma-it}^{\sigma+it} h(u,X)\,du\,dt = 2i\,\mathrm{Im}(L_X(2T) - L_X(T)) + O(T^{3/4-\sigma}X^{\sigma-9/4})$. Since $L_X(t) \ll t^{5/4-\sigma}X^{\sigma-7/4} \ll T^{-1/2}$ for $t = T$ or $2T$, we have

$$\int_T^{2T}\mathrm{G}_2(t)\,dt \ll \Delta_{1-2\sigma}(X)T^{-1/2} \ll T^{1/2-\sigma}. \qquad (3.5)$$

3) Evaluation of $\int_T^{2T}\mathrm{G}_3(t)\,dt$. Using [14, (4.6)], we have

$$\begin{aligned}
&\mathrm{G}_3(t) \\
&= -2i\pi^{-1}(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma})\int_0^\infty y^{-\sigma-1}(1+y)^{-\sigma}(\log(1+1/y))^{-1} \\
&\quad\times\sin(2\pi Xy)\sin(t\log(1+1/y))\,dy \\
&\quad+(1-2\sigma)\pi^{-1}\zeta(2-2\sigma)X^{1-2\sigma}\int_0^\infty y^{-1}(1+y)^{1-2\sigma}\sin(2\pi Xy) \\
&\quad\times\int_{\sigma-it}^{\sigma+it}(u+1-2\sigma)^{-1}\left(\frac{1+y}{y}\right)^u du\,dy.
\end{aligned}$$

Direct computation shows that for $y > 0$,

$$\begin{aligned}
&\int_{\sigma-it}^{\sigma+it}(u+1-2\sigma)^{-1}(1+1/y)^u\,du \\
&= 2\pi i\left(\frac{1+y}{y}\right)^{2\sigma-1} + \left(\int_{-\infty+it}^{\sigma+it}+\int_{\sigma-it}^{-\infty-it}\right)(1+1/y)^u(u+1-2\sigma)^{-1}\,du.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\int_T^{2T}\mathrm{G}_3(t)\,dt \\
&= 2i(1-2\sigma)\zeta(2-2\sigma)TX^{1-2\sigma}I_1 - 2i\pi^{-1}(\zeta(2\sigma)+\zeta(2-2\sigma)X^{1-2\sigma})I_2 \\
&\quad+\pi^{-1}(1-2\sigma)\zeta(2-2\sigma)X^{1-2\sigma}I_3 \qquad (3.6)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^\infty y^{-2\sigma}\sin(2\pi Xy)\,dy \\
I_2 &= \int_T^{2T}\int_0^\infty y^{-1-\sigma}(1+y)^{-\sigma}(\log(1+1/y))^{-1}\sin(2\pi Xy)\sin(t\log(1+1/y))\,dy\,dt, \\
I_3 &= \int_T^{2T}\int_0^\infty y^{-1}(1+y)^{1-2\sigma}\sin(2\pi Xy) \\
&\quad\times\left(\int_{-\infty+it}^{\sigma+it}+\int_{\sigma-it}^{-\infty-it}\right)(1+1/y)^u(u+1-2\sigma)^{-1}\,du\,dy\,dt.
\end{aligned}$$

Then, $I_1 = 2^{2\sigma-2}\pi^{2\sigma}X^{2\sigma-1}/(\Gamma(2\sigma)\sin(\pi\sigma))$ which is the main contribution. Interchanging the integrals, we have

$$I_2 = -\int_0^\infty y^{-1-\sigma}(1+y)^{-\sigma}\,(\log(1+1/y))^{-2}\sin(2\pi Xy)\cos(t\log(1+1/y))|_{t=T}^{t=2T}\,dy.$$

We split the integral into two parts $\int_0^c + \int_c^\infty$ for some large constant $c > 0$. Expressing the product $\sin(\cdots)\cos(\cdots)$ as a combination of $\exp(i(t\log(1+1/y) \pm 2\pi Xy))$, since $(d/dy)(t\log(1+1/y) \pm 2\pi Xy) = \pm 2\pi X - t/(y(1+y)) \gg X$ for $y \geq c$ (recall $t = T$ or $2T$), the integral $\int_c^\infty$ is $\ll X^{-1}$ by the first derivative test. Applying the mean value theorem for integrals, we have

$$\int_0^c \ll \left|\int_{c'}^{c''} y^{-1-\sigma}(1+y)^{-1}\sin(2\pi Xy)\cos(t\log(1+1/y))\,dy\right|.$$

Integration by parts yields that the last integral $\int_{c'}^{c''}$ equals

$$\begin{aligned} & t^{-1}\left(y^{-\sigma}\sin(2\pi Xy)\sin(t\log(1+1/y))|_{c'}^{c''} - \int_{c'}^{c''} O(y^{-\sigma-1}|\sin(2\pi Xy)| + y^{-\sigma}X)\,dy\right) \\ \ll\ & 1. \end{aligned} \tag{3.7}$$

Hence $I_2 \ll 1$. For I_3, the extra integration over t is in fact not necessary to yield our bound. Thus, we write $I_3 = \int_T^{2T}(I_{31} + I_{32})\,dt$, separated according to the integrals over u. I_{31} and I_{32} are treated in the same way, so we work out I_{31} only. Using integration by parts over u,

$$\begin{aligned} I_{31} = & \int_0^\infty y^{-1}(1+y)^{1-2\sigma}(\log(1+1/y))^{-1}\sin(2\pi Xy)\exp(it\log(1+1/y)) \\ & \times\left\{\frac{(1+1/y)^\alpha}{\alpha+1-2\sigma+it}\Big|_{\alpha=-\infty}^{\alpha=\sigma} + \int_{-\infty}^{\sigma}(1+1/y)^\alpha\,\frac{d\alpha}{(\alpha+1-2\sigma+it)^2}\right\}dy. \end{aligned}$$

Then we consider $\int_0^\infty y^{-1}(1+y)^{1-2\sigma}(\log(1+1/y))^{-1}\sin(2\pi Xy)\exp(it\log(1+1/y))(1+1/y)^\alpha\,dy$. Again, we split it into $\int_0^c + \int_c^\infty$. Then $\int_c^\infty \ll X^{-1}$. If $\alpha \leq -2$, then $\int_0^c \ll 1$ trivially; otherwise, we have (see (3.7))

$$\int_0^c \ll \left|\int_{c'}^{c''} y^{-1-\alpha}(1+y)^{-1}\sin(2\pi Xy)\exp(it\log(1+1/y))\,dy\right| \ll 1.$$

Therefore, $I_{31} \ll T^{-1}$ and so $I_3 \ll 1$. Putting these estimates into (3.6), we get

$$\begin{aligned} \int_T^{2T} \mathrm{G}_3(t)\,dt &= i2^{2\sigma-1}\pi^{2\sigma}\frac{(1-2\sigma)\zeta(2-2\sigma)}{\Gamma(2\sigma)\sin(\pi\sigma)}T + O(1) \\ &= -2\pi i\zeta(2\sigma-1)T + O(1). \end{aligned} \tag{3.8}$$

4) Evaluation of $\int_T^{2T}\mathrm{G}_4^*(t)\,dt$. From [16, Section 4], we obtain

$$\int_T^{2T}\mathrm{G}_4^*(t)\,dt = 4i\tilde{\Delta}_{1-2\sigma}(X)((2\sigma-1)I_1 + I_2 - \sigma I_3 - I_4)$$

where by Lemma 3.1, (recall $L_X(t) \ll T^{-1/2} \ll X^{-1/2}$ for $t = T$ or $2T$)

$$\begin{aligned}
I_1 &= X^{2\sigma-2} \int_T^{2T} \int_0^\infty \frac{\cos(2\pi y)\sin(t\log(1+X/y))}{y^\sigma (X+y)^\sigma \log(1+X/y)}\,dy\,dt \\
&= X^{-1} \int_T^{2T} \int_0^\infty \frac{\cos(2\pi X y)\sin(t\log(1+1/y))}{y^\sigma (1+y)^\sigma \log(1+1/y)}\,dy\,dt \ll X^{-3/2} \\
I_2 &= X^{2\sigma-1} \int_T^{2T} t \int_0^\infty \frac{\cos(2\pi y)\cos(t\log(1+X/y))}{y^\sigma (X+y)^{\sigma+1} \log(1+X/y)}\,dy\,dt \\
&\ll X^{-1} T \sup_{T \le T_1 \le T_2 \le 2T} \left| \int_{T_1}^{T_2} \int_0^\infty \frac{\cos(2\pi X y)\cos(t\log(1+1/y))}{y^\sigma (1+y)^{\sigma+1} \log(1+1/y)}\,dy\,dt \right| \ll X^{-1/2}
\end{aligned}$$

and similarly I_3, $I_4 \ll X^{-3/2}$. With Lemma 3.2,

$$\int_T^{2T} \mathrm{G}_4^*(t)\,dt \ll T^{r-1/2}\log T \ll \log T. \tag{3.9}$$

5) Evaluation of $\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt$. [16, (3.6) and Section 5] gives

$$\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt = -4iI_1 + 4iI_2 + 4iI_3. \tag{3.10}$$

I_1, I_2 and I_3 are defined as follows: write

$$w(\xi, y) = \tilde{\Delta}_{1-2\sigma}(\xi)\xi^{-2} y^{-\sigma}(1+y)^{-\sigma-2}(\log(1+1/y))^{-1}\cos(2\pi\xi y), \tag{3.11}$$

then

$$\begin{aligned}
I_1 &= \int_X^\infty \int_T^{2T} t^2 \int_0^\infty w(\xi,y)\sin(t\log(1+1/y))\,dy\,dt\,d\xi \\
I_2 &= \int_X^\infty \int_T^{2T} t \int_0^\infty w(\xi,y) H_1(y)\cos(t\log(1+1/y))\,dy\,dt\,d\xi \\
I_3 &= \int_X^\infty \int_T^{2T} \int_0^\infty w(\xi,y) H_0(y)\sin(t\log(1+1/y))\,dy\,dt\,d\xi
\end{aligned}$$

where $H_0(y)$ and $H_1(y)$ are linear combinations of $y^\mu(\log(1+1/y))^{-\nu}$ with $\mu+\nu \le 2$ and $\mu+\nu \le 1$ respectively. (Remark: It is stated in [16] $\mu+\nu \le 2$ only for both $H_0(y)$ and $H_1(y)$.)

When $\xi \ge X \asymp T \asymp t$ and $\mu+\nu \le 2$, we have

$$\int_T^{2T} \int_0^\infty \frac{\exp(it\log(1+1/y))\cos(2\pi\xi y)}{y^{\sigma-\mu}(1+y)^{\sigma+2}(\log(1+1/y))^{\nu+1}}\,dy\,dt \ll 1, \tag{3.12}$$

$$\int_0^\infty \frac{\exp(it\log(1+1/y))\cos(2\pi\xi y)}{y^{\sigma-\mu}(1+y)^{\sigma+2}(\log(1+1/y))^{\nu+1}}\,dy \ll T^{-1/2}. \tag{3.13}$$

(3.13) can be seen from [16, p.368]. To see (3.12), we split the inner integral into $\int_0^c + \int_c^\infty$. First derivative test gives $\int_c^\infty \ll \xi^{-1}$. For the first part $\int_0^c$, we integrate over t first and it is plain that $\int_0^c \int_T^{2T} \ll 1$.

Using (3.12) and Lemma 3.2, we have $I_3 \ll \int_X^\infty \tilde{\Delta}_{1-2\sigma}(\xi)\xi^{-2}\,d\xi \ll T^{1/4-\sigma}$. Applying integration by parts to the t-integral, we find that $I_2 \ll T^{3/4-\sigma}$ with (3.12) and (3.13). (Here we have used $\mu+\nu \le 1$ for $H_1(y)$.) Since

$$\begin{aligned}
&\int_T^{2T} t^2 \sin(t\log(1+1/y))\,dt \\
= \;& -t^2(\log(1+1/y))^{-1}\cos(t\log(1+1/y))\Big|_T^{2T} \\
&+ 2t(\log(1+1/y))^{-2}\sin(t\log(1+1/y))\Big|_T^{2T} - 2(\log(1+1/y))^{-2}\int_T^{2T}\sin(t\log(1+1/y))\,dt,
\end{aligned}$$

the last two terms contribute $T^{3/4-\sigma}$ and $T^{1/4-\sigma}$ in I_1 respectively by using (3.13) and (3.12). Substituting into (3.10), we get with [14, Lemma 3] (or [5, Lemma 15.1]) and (3.11)

$$\begin{aligned}
&\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt \\
= \;& 4it^2\int_X^\infty\int_0^\infty w(\xi,y)(\log(1+1/y))^{-1}\cos(t\log(1+1/y))\,dy\,d\xi\Big|_{t=T}^{t=2T} + O(T^{3/4-\sigma}) \\
= \;& i\pi^{-1/2}t^{5/2}\int_X^\infty \frac{\tilde{\Delta}_{1-2\sigma}(\xi)\cos(tV+2\pi\xi U-\pi\xi+\pi/4)}{\xi^3V^2U^{1/2}(U-1/2)^\sigma(U+1/2)^{\sigma+2}}\,d\xi\Big|_T^{2T} + O(T^{3/4-\sigma}) \qquad (3.14)
\end{aligned}$$

where U and V are defined as in Lemma 3.1 with k replaced by ξ. Applying the argument in [16, Section 6] to (3.14), we get

$$\begin{aligned}
&\int_T^{2T} \mathrm{G}_4^{**}(t)\,dt \\
= \;& -2i\left(\frac{t}{2\pi}\right)^{1/2-\sigma}\sum_{n\le B(t,\sqrt{X})}\frac{\sigma_{1-2\sigma}(n)}{n^{1-\sigma}}\left(\log\frac{t}{2\pi n}\right)^{-2}\sin g(t,n) + O(\log T). \qquad (3.15)
\end{aligned}$$

(Remark: The σ in [16, Lemma 4] should be omitted, as mentioned in [15].)

Inserting (3.4), (3.5), (3.8), (3.9), (3.15) into (3.3), we obtain

$$\int_T^{2T} E_\sigma(t)\,dt = -2\pi\zeta(2\sigma-1)T + \Sigma_1(t,X)|_T^{2T} - \Sigma_2(t,X)|_T^{2T} + O(\log T). \qquad (3.16)$$

6) Transformation of Dirichlet Polynomial. Let X_1, $X_2 \asymp T$ (both are not integers) and denote $B_1 = B(T,\sqrt{X_1})$ and $B_2 = B(T,\sqrt{X_2})$. Assume $X_1 < X_2$. Write

$$F(x) = x^{\sigma-1}\left(\log\frac{T}{2\pi x}\right)^{-2}\exp(i(T\log\frac{T}{2\pi x} + 2\pi x - T + \frac{\pi}{4})),$$

then we have

$$\begin{aligned}
&\sum_{B(T,\sqrt{X_2})<n\le B(T,\sqrt{X_1})} \sigma_{1-2\sigma}(n)n^{\sigma-1}(\log(T/(2\pi n)))^{-2}\sin g(T,n) \\
= \;& \mathrm{Im}\sum_{B_2<n\le B_1}\sigma_{1-2\sigma}(n)F(n). \qquad (3.17)
\end{aligned}$$

Stieltjes integration gives

$$\begin{aligned}
&\sum_{B_2<n\le B_1}\sigma_{1-2\sigma}(n)F(n)\\
=\ &\int_{B_2}^{B_1}F(t)(\zeta(2\sigma)+\zeta(2-2\sigma)t^{1-2\sigma})\,dt+\Delta_{1-2\sigma}(t)F(t)|_{B_2}^{B_1}-\int_{B_2}^{B_1}\Delta_{1-2\sigma}(t)F'(t)\,dt\\
=\ &I_1+I_2-I_3,\ \text{say.}
\end{aligned}\tag{3.18}$$

Now, since $(d/dt)(g(T,t)+2\pi t)=2\pi-T/t<-c$ when $B_2<t<B_1$, we have

$$\begin{aligned}
I_1 &= \int_{B_2}^{B_1}(\zeta(2\sigma)+\zeta(2-2\sigma)t^{1-2\sigma})t^{\sigma-1}(\log(T/(2\pi t)))^{-2}\exp(i(g(T,t)+2\pi t))\,dt\\
&\ll T^{\sigma-1}.
\end{aligned}$$

By Lemma 3.2, $I_2\ll 1$. Direct computation gives

$$F'(t)=i(2\pi-\frac{T}{t})t^{\sigma-1}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(T\log\frac{T}{2\pi t}+2\pi t-T+\frac{\pi}{4}))+O(t^{\sigma-2})$$

where $B_2\le t\le B_1$. As $\int_{B_2}^{B_1}|\Delta_{1-2\sigma}(t)|t^{\sigma-2}\,dt\ll T^{\sigma-1}\sqrt{\log T}$, we have by (3.18),

$$\begin{aligned}
&\sum_{B_2<n\le B_1}\sigma_{1-2\sigma}(n)F(n)\\
=\ &-i\exp(i(T\log\frac{T}{2\pi}-T+\frac{\pi}{4}))\int_{B_2}^{B_1}\Delta_{1-2\sigma}(t)(2\pi-\frac{T}{t})t^{\sigma-1}\\
&\times\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))\,dt+O(1).
\end{aligned}\tag{3.19}$$

The integral $\int_{B_1}^{B_2}$ in (3.19) is, after by parts,

$$\begin{aligned}
&\tilde{\Delta}_{1-2\sigma}(t)(2\pi-\frac{T}{t})t^{\sigma-1}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))\Bigg|_{B_1}^{B_2}\\
&-\int_{B_2}^{B_1}\tilde{\Delta}_{1-2\sigma}(t)\frac{d}{dt}\left\{(2\pi-\frac{T}{t})t^{\sigma-1}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))\right\}dt
\end{aligned}$$

The first term is $\ll T^{\sigma-1/2}\log T$ by Lemma 3.2. Besides, computing directly shows, for $B_2\le t\le B_1$,

$$\frac{d}{dt}\{\cdots\}=i(2\pi t-T)^2t^{\sigma-3}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))+O(t^{\sigma-2}).$$

Treating the O-term with Lemma 3.2, (3.19) becomes

$$\begin{aligned}
&\sum_{B_2<n\le B_1}\sigma_{1-2\sigma}(n)F(n)\\
=\ &-\exp(i(T\log\frac{T}{2\pi}-T+\frac{\pi}{4}))\int_{B_2}^{B_1}\tilde{\Delta}_{1-2\sigma}(t)(2\pi t-T)^2t^{\sigma-3}\\
&\times\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))\,dt+O(T^{\sigma-1/2}\log T).
\end{aligned}\tag{3.20}$$

Inserting the Voronoi-type series of $\tilde{\Delta}_{1-2\sigma}(t)$ (see Lemma 3.2) into (3.20), we get

$$\begin{aligned}
&\sum_{B_2<n\le B_1}\sigma_{1-2\sigma}(n)F(n)\\
=\;& -\exp(i(T\log\frac{T}{2\pi}-T+\frac{\pi}{4}))\left\{C_1\sum_{n=1}^{\infty}\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}J_1(n)+C_2\sum_{n=1}^{\infty}\frac{\sigma_{1-2\sigma}(n)}{n^{9/4-\sigma}}J_2(n)\right\}\\
&+O(T^{\sigma-1/2}\log T) \qquad (3.21)
\end{aligned}$$

where

$$\begin{aligned}
J_1(n) &= \int_{B_2}^{B_1}(2\pi t-T)^2t^{-7/4}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))\cos(4\pi\sqrt{nt}+\frac{\pi}{4})\,dt\\
J_2(n) &= \int_{B_2}^{B_1}(2\pi t-T)^2t^{-9/4}\left(\log\frac{T}{2\pi t}\right)^{-2}\exp(i(2\pi t-T\log t))\cos(4\pi\sqrt{nt}-\frac{\pi}{4})\,dt.
\end{aligned}$$

Applying the first derivative test or bounding trivially, we have $J_2(n)\ll T^{-1/4}$ for $n\le cT$, $J_2(n)\ll T^{3/4}$ for $cT<n<c'T$ and $\ll T^{1/4}n^{-1/2}$ for $n\ge c'T$. Thus, the second sum in (3.21) is

$$\begin{aligned}
&\ll \left(T^{-1/4}\sum_{n\le cT}+T^{3/4}\sum_{cT<n<c'T}\right)\sigma_{1-2\sigma}(n)n^{\sigma-9/4}+T^{1/4}\sum_{n\ge c'T}\sigma_{1-2\sigma}(n)n^{\sigma-11/4}\\
&\ll T^{\sigma-1/2}. \qquad (3.22)
\end{aligned}$$

After a change of variable $t=x^2$,

$$\begin{aligned}
J_1(n) &= \int_{\sqrt{B_2}}^{\sqrt{B_1}}(2\pi x^2-T)^2x^{-5/2}\left(\log\frac{T}{2\pi x^2}\right)^{-2}\{\exp(i(2\pi x^2-2T\log x+4\pi\sqrt{n}x+\frac{\pi}{4}))\\
&\quad+\exp(i(2\pi x^2-2T\log x-4\pi\sqrt{n}x-\frac{\pi}{4}))\}\,dx.
\end{aligned}$$

Then we use [5, Theorem 2.2], with $f(x)=x^2-\pi^{-1}T\log x$, $\Phi(x)=x^{3/2}$, $F(x)=T$, $\mu(x)=x/2$ and $k=\pm2\sqrt{n}$. Thus,

$$\begin{aligned}
J_1(n) &= \delta_n 2\pi^2\left(\frac{T}{2\pi}\right)^{3/4}e_2(T,n)\exp(i(f(T,n)-T\log\frac{T}{2\pi}+T-\pi n+\frac{3\pi}{4}))+O(\delta_nT^{-1/4})\\
&\quad+O(T^{3/4}\exp(-c\sqrt{nT}-cT))+O(T^{3/4}\min(1,|\sqrt{X_1}\pm\sqrt{n}|^{-1}))\\
&\quad+O(T^{3/4}\min(1,|\sqrt{X_2}\pm\sqrt{n}|^{-1}))
\end{aligned}$$

where $\delta_n=1$ if $B_2<x_0<B_1$ and $k>0$, or $\delta_n=0$ otherwise. ($x_0=\sqrt{T/(2\pi)+n/4}-\sqrt{n}/2$ is the saddle point.) Note that $B_2<x_0<B_1$ is equivalent to $X_1<n<X_2$. Thus, for the first term in (3.21), we have

$$\begin{aligned}
&-C_1\exp(i(T\log\frac{T}{2\pi}-T+\frac{\pi}{4}))\sum_{n=1}^{\infty}\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}J_1(n)\\
=\;&\frac{1}{\sqrt{2}}\left(\frac{T}{2\pi}\right)^{3/4}\sum_{X_1<n<X_2}\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}e_2(T,n)\exp(i(f(T,n)-\pi n+\pi))+O(T^{\sigma-1/2}\log T)
\end{aligned}$$

Together with (3.22), (3.21) and (3.17), we obtain

$$\begin{aligned}
&\sum_{B(T,\sqrt{X_2})<n\le B(T,\sqrt{X_1})} \sigma_{1-2\sigma}(n)n^{\sigma-1}(\log T/(2\pi n))^{-2}\sin g(T,n) \\
= \ & -\frac{1}{\sqrt{2}}\left(\frac{T}{2\pi}\right)^{3/4}\sum_{X_1<n<X_2}(-1)^n\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}e_2(T,n)\sin f(T,n) \\
& + O(T^{\sigma-1/2}\log T).
\end{aligned} \tag{3.23}$$

We can complete our proof now. Taking $X=[T]-1/2$ in (3.16), we have $\Sigma_i(t,X)-\Sigma_i(t,T)\ll\log T$ for $i=1,2$ and $t=T,2T$; hence

$$\begin{aligned}
\int_T^{2T} E_\sigma(t)\,dt \ &= \ -2\pi\zeta(2\sigma-1)T+\Sigma_1(t,t)|_T^{2T}-\Sigma_2(t,t)|_T^{2T} \\
&\quad -((\Sigma_1(2T,2T)-\Sigma(2T,T))+(\Sigma_2(2T,T)-\Sigma_2(2T,2T)))+O(\log T).
\end{aligned}$$

Choosing X_1 and X_2 in (3.23) to be half-integers closest to T and $2T$ respectively, then $(\Sigma_1(2T,2T)-\Sigma(2T,T))+(\Sigma_2(2T,T)-\Sigma_2(2T,2T))\ll\log T$. Hence,

$$\int_1^T E_\sigma(t)\,dt \ = \ -2\pi\zeta(2\sigma-1)T+\Sigma_1(T,T)-\Sigma_2(T,T)+O(\log^2 T).$$

The extra $\log T$ in the O-term comes from the number of dyadic intervals. Suppose $N\asymp T$. We apply (3.23) again with $X_1=[N]+1/2$ and $X_2=[T]+1/2$ to yield our theorem.

4. THE SECOND AND THIRD POWER MOMENTS

Using the arguments in [16, p.374-375] and the argument in [18, p.341], we can obtain the lemma below.

Lemma 4.1 *We have for $N\asymp T$, $\int_T^{2T}\Sigma_2(t,N)^2\,dt\ll T$, $\int_T^{2T}\Sigma_1(t,N)\Sigma_2(t,N)\,dt\ll T\log T$ and*

$$\int_T^{2T}\Sigma_1(t,N)^2\,dt=B(\sigma)\int_T^{2T}\left(\frac{t}{2\pi}\right)^{5/2-2\sigma}dt+O(T^{3-2\sigma}).$$

Moreover, if we define for $1\le M\le M'\ll T$,

$$\Sigma_{M,M'}(t)=\sqrt{2}\left(\frac{t}{2\pi}\right)^{5/4-\sigma}\sum_{M\le n<M'}(-1)^n\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}e_2(t,n)\sin f(t,n)$$

then, $\int_T^{2T}\Sigma_{M,M'}(t)^2\,dt\ll T^{7/2-2\sigma}M^{2\sigma-5/2}$.

Remark: Part (1) of Theorem 2 follows immediately.

Lemma 4.2 *Let $0\le A<(\sigma-3/4)^{-1}$. Then, we have*

$$\int_T^{2T}|G_\sigma(t)|^A\,dt\ll T^{1+A(5/4-\sigma)}.$$

Proof. The case $0 \le A \le 2$ is proved by Hölder's inequality and part (1) of Theorem 2. Consider the situation $2 < A < (\sigma - 3/4)^{-1}$. Then, for $T \le t \le 2T$ and $N \asymp T$, we have $\Sigma_2(t,N) \ll T^{1/2}$ and hence $\int_T^{2T} |\Sigma_2(t,N)|^A\,dt \ll T^{A/2}$ by Lemma 4.1. We take $N = 2^R - 1 \asymp T$ and write $M = 2^r$ in the proof of this lemma. Then $\Sigma_1(t,N) \le \sum_{r\le R} |\Sigma_{M,2M}(t)|$. By Hölder's inequality, we have

$$|\Sigma_1(t,N)|^A \ll \left(\sum_{r\le R} \alpha_r^A |\Sigma_{M,2M}(t)|^A\right)\left(\sum_{r\le R} \alpha_r^{-A/(A-1)}\right)^{A-1}.$$

By taking $\alpha_r = M^{(1-A(\sigma-3/4))/(2A)}$ and using the trivial bound $\Sigma_{M,2M}(t) \ll T^{5/4-\sigma}M^{\sigma-3/4}$, we have

$$\begin{aligned}\int_T^{2T} |\Sigma_1(t,N)|^A\,dt &\ll T^{(5/4-\sigma)(A-2)}\left(\sum_{r\le R} 2^{-r(1-A(\sigma-3/4))/(2(A-1))}\right)^{A-1} \\ &\quad \times \sum_{r\le R} \alpha_r^A M^{(\sigma-3/4)(A-2)} \int_T^{2T} \Sigma_{M,2M}(t)^2\,dt \\ &\ll_A T^{1+A(5/4-\sigma)} \sum_{r\le R} \alpha_r^A M^{A(\sigma-3/4)-1} \ll_A T^{1+A(5/4-\sigma)} \qquad (4.1)\end{aligned}$$

by Lemma 4.1.

Now we can prove part (2) of Theorem 2 but part (3) will be proved in next section.

Proof of Theorem 2 (2). We have, with $M = [\delta T^{1/3}]$ for some small constant $\delta > 0$,

$$\int_T^{2T} G_\sigma(t)^3\,dt = \int_T^{2T} \Sigma_{1,M}(t)^3\,dt + O\Big(\int_T^{2T} |G_\sigma(t) - \Sigma_{1,M}(t)|(G_\sigma(t)^2 + \Sigma_{1,M}^2(t))\,dt\Big).$$

Lemma 4.2 and (4.1) give $\int_T^{2T} G_\sigma(t)^4\,dt$ and $\int_T^{2T} \Sigma_{1,M}(t)^4\,dt \ll T^{1+4(5/4-\sigma)}$. Since

$$\begin{aligned}\int_T^{2T} (G_\sigma(t) - \Sigma_{1,M}(t))^2\,dt &\ll \int_T^{2T} \Sigma_{M,T}(t)^2\,dt + T\log^4 T \\ &\ll T^{4(2-\sigma)/3},\end{aligned}$$

we get

$$\int_T^{2T} G_\sigma(t)^3\,dt = \int_T^{2T} \Sigma_{1,M}(t)^3\,dt + O(T^{(13-8\sigma)/3}). \qquad (4.2)$$

After multiplying out, rearranging and renaming the terms, we obtain

$$\begin{aligned}&\int_T^{2T} \Sigma_{1,M}(t)^3\,dt \\ &= \frac{3}{\sqrt{2}} \sum_{m,n,k<M} (-1)^{m+n+k} r(m,n,k) \int_T^{2T} \left(\frac{t}{2\pi}\right)^{15/4-3\sigma} \sin(f(t,m)+f(t,n)-f(t,k))\,dt \\ &\quad + O(T^{17/4-3\sigma} \sum_{m,n,k<M} r(m,n,k)(\sqrt{m}+\sqrt{n}+\sqrt{k})^{-1}) \qquad (4.3)\end{aligned}$$

where $r(m,n,k) = \sigma_{1-2\sigma}(m)\sigma_{1-2\sigma}(n)\sigma_{1-2\sigma}(k)(mnk)^{\sigma-7/4}$. Here, we have used the first derivative test and the fact that $(d/dt)(f(t,m)+f(t,n)+f(t,k)) \gg (\sqrt{m}+\sqrt{n}+\sqrt{k})t^{-1/2}$. The O-term in (4.3) is $\ll T^{17/4-3\sigma+\epsilon}\sum_{m\le n\le k\le M}(mn)^{\sigma-7/4}k^{\sigma-9/4}$, so absorbed by the O-term in (4.2).

To estimate the main term in (4.3), we note that

$$\begin{aligned}\frac{d}{dt}(f(t,m)+f(t,n)-f(t,k)) &= 2(\operatorname{arsinh}\sqrt{\frac{\pi m}{2t}}+\operatorname{arsinh}\sqrt{\frac{\pi n}{2t}}-\operatorname{arsinh}\sqrt{\frac{\pi k}{2t}}) \\ &= \sqrt{\frac{2\pi}{t}}(\sqrt{m}+\sqrt{n}-\sqrt{k})+O\left(\left(\frac{\max(m,n,k)}{t}\right)^{3/2}\right).\end{aligned}$$

Write $\Delta = \sqrt{m}+\sqrt{n}-\sqrt{k}$, then provided the choice of δ in M is small enough, the last line is $\gg |\Delta|t^{-1/2}$ as $|\Delta| \ge (\max(m,n,k))^{-3/2}$ when $\Delta \ne 0$, from [20, Lemma 2]. If J denotes the sum over non-diagonal terms $\sum_{\Delta\ne 0}$ in (4.3), then

$$J \ll T^{17/4-3\sigma}\sum_{\substack{\Delta\ne 0\\ m,n,k<M}} r(m,n,k)|\Delta|^{-1}.$$

We separate the sum $\sum_{\Delta\ne0}$ into three parts $\sum_{0<|\Delta|\ll m^{-1/2}}+\sum_{m^{-1/2}\ll|\Delta|\ll m^{1/2}}+\sum_{|\Delta|\gg m^{1/2}}$. Suppose $n\le m$ are fixed. When $|\Delta|\le m^{-1/2}$, it is apparent that only one k can satisfy this condition and $k\asymp m$. As $|\Delta|\ge m^{-3/2}$, we have $\sum_{0<|\Delta|\ll m^{-1/2}} \ll T^{\epsilon}\sum_{n\le m<M}(nm^2)^{\sigma-7/4}m^{3/2}$ $\ll T^{\sigma-7/12+\epsilon}$. If $m^{-1/2}\ll|\Delta|\ll\sqrt{m}$, we divide this range into dyadic intervals $I_r=(u_r,2u_r]$, say. When $|\Delta|\in I_r$, the number of such k is $\ll 1+u_r\sqrt{m}$. Thus, $\sum_{m^{-1/2}\ll|\Delta|\ll m^{1/2}}$ $\ll T^{\epsilon}\sum_{n\le m<M}(nm^2)^{\sigma-7/4}\sqrt{m} \ll T^{\sigma-11/12+\epsilon}$. The last case $\sum_{|\Delta|\gg m^{1/2}}$ is $\ll T^{\sigma/3-1/4+\epsilon}$. Therefore, $J\ll T^{11/3-2\sigma+\epsilon}\ll T^{(13-8\sigma)/3}$.

The diagonal term $\sum_{\Delta=0}$ in (4.3) contributes

$$\frac{3}{\sqrt{2}}\sum_{\substack{m,n,k<M\\ \sqrt{m}+\sqrt{n}=\sqrt{k}}}(-1)^{m+n+k}r(m,n,k)\int_T^{2T}\left(\frac{t}{2\pi}\right)^{15/4-3\sigma}(\sin(-\frac{\pi}{4})+O(\frac{k^{3/2}}{\sqrt{t}}))\,dt. \tag{4.4}$$

Crude estimation gives

$$\begin{aligned}\sum_{\substack{\Delta=0\\ m,n,k<M}}(mn)^{\sigma-7/4}k^{\sigma-1/4} &\ll \sum_{n\le m<M} m^{2\sigma-2}n^{\sigma-7/4}\ll M^{3\sigma-7/4} \text{ and}\\ \sum_{\substack{\Delta=0\\ k\ge M}}(mnk)^{\sigma-7/4} &\ll \sum_{n\le m,\, m\ge M} n^{\sigma-7/4}m^{2\sigma-7/2}\ll M^{3\sigma-13/4}.\end{aligned}$$

Hence, (4.4) becomes

$$-\frac{3}{2}\sum_{\Delta=0}(-1)^{m+n+k}r(m,n,k)\int_T^{2T}(t/2\pi)^{15/4-3\sigma}\,dt+O(T^{11/3-2\sigma+\epsilon}).$$

This completes the proof, with the fact that $\sqrt{m}+\sqrt{n}=\sqrt{k}$ holds only if $m=sa^2$, $n=sb^2$ and $k=s(a+b)^2$ for squarefree positive integer s and some positive integers a, b. (Note that $m+n+k$ is then even.)

5. LIMITING DISTRIBUTION FUNCTION

We first quote some results from [1, Theorem 4.1] and [3, Theorem 6]

Let F be a real-valued function defined on $[1,\infty)$, and let $a_1(t)$, $a_2(t),\ldots$ be continuous real-valued periodic functions of period 1 such that $\int_0^1 a_n(t)\,dt=0$ and $\sum_{n=1}^{\infty}\int_0^1 a_n(t)^2\,dt<\infty$. Suppose that there are positive constants γ_1, $\gamma_2,\ldots$ which are linearly independent over $\mathbf{Q}$, such that

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \min(1,|F(t)-\sum_{n\le N}a_n(\gamma_n t)|)\,dt=0.$$

Fact I. For every continuous bounded function g on $\mathbf{R}$, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g(F(t))\,dt=\int_{-\infty}^{\infty} g(x)\nu(dx),$$

where $\nu(dx)$ is the distribution of the random series $\eta=\sum_{n=1}^{\infty}a_n(t_n)$ and t_n are independent random variables uniformly distributed on $[0,1]$. Equivalently, the distribution function of F defined by $P_T(u)=T^{-1}\mu\{t\in[1,T]:F(t)\le u\}$ converges weakly to a function $P(u)$, called the limiting distribution, as $T\to\infty$.

Fact II. If $\int_1^T |F(t)|^A\,dt\ll T$, then for any real $k\in[0,A)$ and any odd integer $l\in[0,A)$, the following limits exist:

$$\lim_{T\to\infty}T^{-1}\int_1^T |F(t)|^k\,dt \text{ and } \lim_{T\to\infty}T^{-1}\int_1^T F(t)^l\,dt.$$

Now, let us take $F(t)=t^{2\sigma-5/2}G_\sigma(2\pi t^2)$, $\gamma_n=2\sqrt{n}$ and

$$a_n(t)=\sqrt{2}\frac{\mu(n)^2}{n^{7/4-\sigma}}\sum_{r=1}^{\infty}(-1)^{nr}\frac{\sigma_{1-2\sigma}(nr^2)}{r^{7/2-2\sigma}}\sin(2\pi rt-\pi/4). \tag{5.1}$$

By Lemma 4.1, we see that for $N\le\sqrt{T}$,

$$\int_T^{2T}(t^{2\sigma-5/2}G_\sigma(2\pi t^2)-t^{2\sigma-5/2}\Sigma_{1,N}(2\pi t^2))^2\,dt\ll TN^{2\sigma-5/2}.$$

Using the fact that $e_2(2\pi t^2,n)=1+O(n/t^2)$ and $f(2\pi t^2,n)=4\pi\sqrt{n}t-\pi/4+O(n^{3/2}/t)$ for $n\le\sqrt{t}$ and following the computation in [3, p.402], we can show

$$\int_T^{2T}(t^{2\sigma-5/2}G_\sigma(2\pi t^2)-\sum_{n\le N}a_n(2\sqrt{n}t))^2\,dt\ll TN^{2\sigma-5/2}\qquad(N\le\sqrt{T}).$$

This completes the proofs of Theorem 2 (c) and the first part of Theorem 3, by Facts I and II with Lemma 4.2.

We proceed to prove the lower bounds in (2.2) with the idea in [1, Section V]. Because t_n's are independent random variables, we firstly consider the size of the set over which $a_n(t)$ is quite large.

Lemma 5.1 *Let n be squarefree. Define $A_n = \{t \in [0,1] : a_n(t) > B^{-1}\sigma_{1-2\sigma}(n)n^{\sigma-7/4}\}$ where $B = 4A(\sum_{r=1}^{\infty} r^{4\sigma-7})^{-1}$ and $A = \sqrt{2}\sum_{r=1}^{\infty} \sigma_{1-2\sigma}(r^2)r^{2\sigma-7/2}$. Then, we have $\mu(A_n) \geq 1/(AB)$ where μ is the Lebesgue measure.*

Proof. Since $\int_0^1 a_n(t)\,dt = 0$ from (5.1), we have $\int_0^1 a_n^+(t)\,dt = \int_0^1 a_n^-(t)\,dt$ where $a_n^{\pm}(t) = \max(0, \pm a_n(t))$. Observing that

$$\sup_{0\leq t\leq 1} |a_n(t)| \leq A\sigma_{1-2\sigma}(n)n^{\sigma-7/4}, \tag{5.2}$$

$$\int_0^1 a_n(t)^2\,dt = \frac{1}{n^{7/2-2\sigma}}\sum_{r=1}^{\infty}\frac{\sigma_{1-2\sigma}(nr^2)^2}{r^{7-4\sigma}}, \tag{5.3}$$

we have

$$\int_0^1 a_n^+(t)^2\,dt + \int_0^1 a_n^-(t)^2\,dt = \frac{1}{n^{7/2-2\sigma}}\sum_{r=1}^{\infty}\frac{\sigma_{1-2\sigma}(nr^2)^2}{r^{7-4\sigma}}$$

and $\int_0^1 a_n^{\pm}(t)^2\,dt \leq A\sigma_{1-2\sigma}(n)n^{\sigma-7/4}\int_0^1 a_n^+(t)\,dt$. Hence $\int_0^1 a_n^+(t)\,dt \geq 2B^{-1}\sigma_{1-2\sigma}(n)n^{\sigma-7/4}$. Since

$$\sup_{0\leq t\leq 1} |a_n(t)|\mu(A_n) + \frac{1}{B}\frac{\sigma_{1-2\sigma}(n)}{n^{\sigma-7/4}}\mu(A_n^c) \geq \int_0^1 a_n^+(t)\,dt,$$

we get $\mu(A_n) \geq 1/(AB)$ from (5.2) and $\mu(A_n^c) \leq 1$.

Proof of lower bounds in Theorem 3. Let n be a large positive integer. By Markov's inequality, we have

$$\Pr(|\sum_{m=n+1}^{\infty} a_m(t_m)| \leq 2\sqrt{K}) \geq 1 - \frac{1}{4K}\sum_{m=1}^{\infty}\int_0^1 a_m(t)^2\,dt \geq \frac{3}{4}$$

where $\Pr(\#)$ denotes the probability of the event $\#$ and $K = \sum_{m=1}^{\infty}\int_0^1 a_m(t)^2\,dt < +\infty$. For $1 \leq m \leq n$, define A_m as in Lemma 5.1 if m is squarefree and $A_m = [0,1]$ otherwise. Consider the set

$$E_n = \{(t_1, t_2, \ldots) : t_m \in A_m \text{ for } 1 \leq m \leq n \text{ and } |\sum_{m=n+1}^{\infty} a_m(t_m)| \leq 2\sqrt{K}\}.$$

Then,

$$\Pr(E_n) \;=\; \prod_{m=1}^{n} \Pr(A_m)\Pr(|\sum_{m=n+1}^{\infty} a_m(t_m)| \le 2\sqrt{K}) \ge \frac{3}{4(AB)^n}$$

due to $\Pr(A_m) = \mu(A_m)$ and Lemma 5.1. When $(t_1, t_2, \ldots) \in E_n$, we have

$$\begin{aligned}\sum_{m=1}^{\infty} a_m(t_m) &\ge \frac{1}{B} \sum_{\substack{m\le n \\ m \text{ squarefree}}} \frac{\sigma_{1-2\sigma}(m)}{m^{7/4-\sigma}} - 2\sqrt{K} \\ &\ge B^{-1} \sum_{m \le n} \mu(m)^2 m^{\sigma-7/4} - 2\sqrt{K} \\ &\gg \begin{cases} \log n & \text{if } \sigma = 3/4, \\ n^{\sigma-3/4} & \text{if } 3/4 < \sigma < 1. \end{cases}\end{aligned}$$

Our result for $1 - D_\sigma(u)$ follows after we replace n by $[e^u]$ if $\sigma = 3/4$ and by $[u^{4/(4\sigma-3)}]$ if $3/4 < \sigma < 1$. The case of $D_\sigma(-u)$ can be proved in the same way.

To derive the upper bounds, we need a result related to the Laplace transform of limiting distribution functions.

Lemma 5.2 *Let X be a real random variable with the probability distribution $D(x)$. Suppose $D(x) > 0$ for any $x > 0$. For the two cases: (i) $\psi(x) = x \log x$ and $\phi(x) = \log x$, or (ii) $\psi(x) = x^{4/(7-4\sigma)}$ and $\phi(x) = x^{(4\sigma-3)/4}$, there exist two positive numbers L and L' such that*

- *(a) if $\limsup_{\lambda\to\infty} \psi(\lambda)^{-1} \log E(\exp(\lambda X)) \le L$, then $\limsup_{x\to\infty} x^{-1} \log(1 - D(\phi(x))) \le -L'$,*

- *(b) if $\limsup_{\lambda\to\infty} \psi(\lambda)^{-1} \log E(\exp(-\lambda X)) \le L$, then $\limsup_{x\to\infty} x^{-1} \log D(-\phi(x)) \le -L'$.*

This can be seen from [12, Lemma 3.1].

Proof of upper bounds in Theorem 3. We take $N = \lambda$ if $\sigma = 3/4$, and $N = \lambda^{4/(7-4\sigma)}$ if $3/4 < \sigma < 1$. When $n \le N$, we use

$$\int_0^1 \exp(\pm\lambda a_n(t))\,dt \le \exp(\lambda A \frac{\sigma_{1-2\sigma}(n)\mu(n)^2}{n^{7/4-\sigma}}).$$

Recall that $A = \sqrt{2}\sum_{r=1}^{\infty} \sigma_{1-2\sigma}(r^2) r^{7/2-2\sigma}$. If $n > N$ and $\lambda A \sigma_{1-2\sigma}(n) < n^{7/4-\sigma}$, then by the inequality $e^x \le 1 + x + x^2$ for $x \le 1$ and (5.3),

$$\begin{aligned}\int_0^1 \exp(\pm\lambda a_n(t))\,dt &\le \int_0^1 (1 \pm \lambda a_n(t) + \lambda^2 a_n(t)^2)\,dt \\ &\le 1 + (\lambda A)^2 \frac{\sigma_{1-2\sigma}(n)^2 \mu(n)^2}{n^{7/2-2\sigma}} \\ &\le \exp((\lambda A)^2 \frac{\sigma_{1-2\sigma}(n)^2 \mu(n)^2}{n^{7/2-2\sigma}})\end{aligned}$$

as $\int_0^1 a_n(t)\,dt = 0$. Finally, for $n > N$ and $\lambda A\sigma_{1-2\sigma}(n) \ge n^{7/4-\sigma}$, we can get

$$\int_0^1 \exp(\pm\lambda a_n(t))\,dt \le \exp(\lambda A\frac{\sigma_{1-2\sigma}(n)\mu(n)^2}{n^{7/4-\sigma}}) \le \exp((\lambda A)^2\frac{\sigma_{1-2\sigma}(n)^2\mu(n)^2}{n^{7/2-2\sigma}}).$$

Therefore,

$$\begin{aligned}\log E(\exp(\pm\lambda X)) &\le \lambda A\sum_{n\le N}\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} + (\lambda A)^2\sum_{n>N}\frac{\sigma_{1-2\sigma}(n)^2}{n^{7/2-2\sigma}} \\ &\ll \begin{cases}\lambda\log\lambda & \text{if } \sigma = 3/4, \\ \lambda^{4/(7-4\sigma)} & \text{if } 3/4 < \sigma < 1.\end{cases}\end{aligned}$$

The proof is complete with Lemma 5.2.

6. $\Omega_\pm$-RESULT

We firstly prepare a result.

Lemma 6.1 *We have*

$$\begin{aligned}&\sum_{n\le B^2}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} \\ =& \begin{cases}2^{-1-2\sigma}(\sigma-3/4)^{-1}(\sigma-1/4)^{-1}\zeta(2\sigma)B^{2\sigma-3/2}+O(1) & \textit{if } \sigma > 3/4, \\ 2^{-1/2}\zeta(3/2)\log B + O(1) & \textit{if } \sigma = 3/4,\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\sum_{n\le B^2}(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} \\ =& \begin{cases}2^{-1}(\sigma-3/4)^{-1}(\sigma-1/4)^{-1}\zeta(2\sigma)B^{2\sigma-3/2}+O(1) & \textit{if } \sigma > 3/4, \\ 2\zeta(3/2)\log B + O(1) & \textit{if } \sigma = 3/4,\end{cases}\end{aligned}$$

Proof. To prove the first equality, we consider

$$\Delta_{1-2\sigma}(t,1/2) = {\sum}'_{n\le t}(-1)^n\sigma_{1-2\sigma}(n) - \left(2^{-2\sigma}\zeta(2\sigma)t + 2^{2\sigma-2}\frac{\zeta(2-2\sigma)}{2-2\sigma}t^{2-2\sigma} - E_{1-2\sigma}(0,1/2)\right),$$

where $E_{1-2\sigma}(0,1/2)$ is a constant. Then, we have $\Delta_{1-2\sigma}(t,1/2) \ll_\epsilon t^{1/(1+4\sigma)+\epsilon}$ from [9, Remark 1]. Then,

$$\begin{aligned}&\sum_{n\le B^2}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} \\ =& \int_1^{B^2}(1-\frac{\sqrt{t}}{B})t^{\sigma-7/4}(2^{-2\sigma}\zeta(2\sigma)+2^{2\sigma-2}\zeta(2-2\sigma)t^{1-2\sigma})\,dt \\ &+\int_1^{B^2}(1-\frac{\sqrt{t}}{B})t^{\sigma-7/4}\,d\Delta_{1-2\sigma}(t,1/2).\end{aligned}$$

The second integral is $\ll 1$ after by parts. The first integral is

$$2^{-2\sigma}\zeta(2\sigma)\int_1^{B^2}(t^{\sigma-7/4}-B^{-1}t^{\sigma-5/4})\,dt+O(1).$$

Separating into two cases $\sigma=3/4$ or not, our assertion follows. The second one is proved similarly with Lemma 3.2.

Proof of Theorem 4. We convolve $G_\sigma(t)$ with the kernel

$$K(u)=2B\left(\frac{\sin 2\pi Bu}{2\pi Bu}\right)^2.$$

Let $t\asymp\sqrt{T}$ and $1\ll B\ll L^{1/4}\ll T^{1/16}$. Similar to [13, (4.7)], we have $\int_{-L}^{L}\Sigma_2(2\pi(t+u)^2,T)K(u)\,du\ll 1$, and

$$\begin{aligned}&\int_{-L}^{L}\Sigma_1(2\pi(t+u)^2,T)K(u)\,du\\ =\;&\sqrt{2}\sum_{n\le T}(-1)^n\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}\int_{-L}^{L}(t+u)^{5/2-2\sigma}e_2(2\pi(t+u)^2,n)\sin f(2\pi(t+u)^2,n)K(u)\,du.\end{aligned}$$

The integral can be treated in the same way as [13, (4.8)] when $B^4\ll n\le T$. Hence, $\sum_{B^4\ll n\le T}\ll\sum_{B^4\ll n\le T}\sigma_{1-2\sigma}(n)n^{\sigma-7/4}\,(B^2n^{-1}t^{5/2-2\sigma}+BLn^{-1/2}t^{3/2-2\sigma})\ll t^{5/2-2\sigma}B^{4\sigma-5}$. For $n\ll B^4$, we note that

$$\begin{aligned}&\operatorname{Im}\int_{-L}^{L}(t+u)^{5/2-2\sigma}e_2(2\pi(t+u)^2,n)K(u)e^{if(2\pi(t+u)^2,n)}\,du\\ =\;&t^{5/2-2\sigma}\int_{-L}^{L}\sin(4\pi\sqrt{n}(t+u)-\pi/4)K(u)\,du+O(t^{3/2-2\sigma}(n^{3/2}+L))\\ =\;&t^{5/2-2\sigma}\max(0,1-\sqrt{n}/B)\sin(4\pi\sqrt{n}t-\pi/4)+O(t^{3/2-2\sigma}(n^{3/2}+L)).\end{aligned}$$

Hence denoting

$$S_B(t)=\sqrt{2}\sum_{n\le B^2}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}\sin(4\pi\sqrt{n}t-\frac{\pi}{4}),\tag{6.1}$$

we have, for $1\ll B\ll L^{1/4}\ll T^{1/16}$,

$$t^{2\sigma-5/2}\int_{-L}^{L}G_\sigma(2\pi(t+u)^2)K(u)\,du=S_B(t)+O(B^{4\sigma-5}).\tag{6.2}$$

With (6.2), it suffices to consider the $\Omega_\pm$-results of $S_B(t)$. Let $\delta>0$ be a sufficiently small constant. Take $B=[\delta\sqrt{\log T}/10]$ and $L=B^4$, then Dirichlet's Theorem gives that there exists l such that $\|l\sqrt{n}\|<\delta$ and $T^{1/10}\le l\le(1+\delta^{-B^2})T^{1/10}$. Then, we have

$$S_B(l)=\sqrt{2}\sum_{n\le B^2}(-1)^n(1-\frac{\sqrt{n}}{B})\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}\sin(-\frac{\pi}{4})+E\tag{6.3}$$

where

$$|E| \le 2\sqrt{2} \sum_{n \le B^2} (1 - \frac{\sqrt{n}}{B}) \frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} |\sin(2\pi\sqrt{n}l)|.$$

Hence, by Lemma 4.1, the first sum in (6.3) is $\le -c_{12} B^{2\sigma-3/2} \min((\sigma - 3/4)^{-1}, \log B)$. In addition, $|E| \le c_{13} \delta B^{2\sigma-3/2} \min((\sigma - 3/4)^{-1}, \log B)$ (note that c_{13} is independent of δ). We thus infer (as $B \asymp \sqrt{\log t}$)

$$S_B(t) = \begin{cases} \Omega_-(\log\log t) & \text{if } \sigma = 3/4, \\ \Omega_-((\log t)^{\sigma-3/4}) & \text{if } 3/4 < \sigma < 1. \end{cases}$$

To prove the Ω_+-result, we take $x = \delta \log\log T \log\log\log T$ and $B = T^{1/100}$ ($L = B^4$). Again, $\delta > 0$ is a small number. Define $\mathbf{P}_x = \{p \le x : \ p \text{ is an odd prime}\}$ and $\mathbf{Q}_x$ to be the set of positive squarefree integers whose prime factors lie in $\mathbf{P}_x$. Consider the kernel

$$T_x(u) = \prod_{q \in \mathbf{Q}_x} (1 + \cos(4\pi\sqrt{q}u)) = \prod_{q \in \mathbf{Q}_x} \left(1 + \frac{e^{4\pi i \sqrt{q} u} + e^{-4\pi i \sqrt{q} u}}{2}\right)$$

and

$$\tilde{\eta}(x) = \inf\{|\sqrt{m} + \mu| : \ m \text{ is a natural number and } \mu \in S_x\}$$

where $S_x = \{\mu = \sum_{q \in \mathbf{Q}_x} r_q \sqrt{q} : r_q = 0, \pm 1 \text{ and } \sum r_q^2 \ge 2\}$. It is known from [2, Lemma 4] that

$$x \ll -\log \tilde{\eta}(x) \ll \exp(c\frac{x}{\log x}). \tag{6.4}$$

Write $\epsilon = 2\pi\tilde{\eta}(x)$, then $\epsilon \gg \exp(-\sqrt{\log T})$. Note that

$$\epsilon \int_{-\infty}^{\infty} \left(\frac{\sin \epsilon\pi u}{\epsilon\pi u}\right)^2 e^{2\pi i u v} \, du = \max(1 - \left|\frac{v}{\epsilon}\right|, 0).$$

Hence,

$$\begin{aligned} & \epsilon \int_{-\infty}^{\infty} T_x(u) \sin(4\pi\sqrt{n}(t+u) - \frac{\pi}{4}) \left(\frac{\sin \epsilon\pi u}{\epsilon\pi u}\right)^2 du \\ = \ & \epsilon \ \mathrm{Im} \ e^{i(4\pi\sqrt{n}t - \pi/4)} \int_{-\infty}^{\infty} \left\{1 + \frac{1}{2}\sum_{q \in \mathbf{Q}_x} e^{4\pi i \sqrt{q} u} + \frac{1}{2}\sum_{q \in \mathbf{Q}_x} e^{-4\pi i \sqrt{q} u} + \sum_{\mu \in S_x} h_\mu e^{4\pi i \mu u}\right\} \\ & \times e^{4\pi i \sqrt{n} u} \left(\frac{\sin \epsilon\pi u}{\epsilon\pi u}\right)^2 du \\ = \ & \begin{cases} \sin(4\pi\sqrt{n}t - \pi/4), & n \in \mathbf{Q}_x \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{6.5}$$

where h_μ ($\mu \in S_x$) are some constants. Hence, we have

$$\begin{aligned} & \epsilon \int_{-\infty}^{\infty} T_x(u) S_B(t+u) \left(\frac{\sin \epsilon\pi u}{\epsilon\pi u}\right)^2 du \\ = \ & \sqrt{2} \sum_{\substack{n \le B^2 \\ n \in \mathbf{Q}_x}} (-1)^n (1 - \frac{\sqrt{n}}{B}) \frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} \sin(4\pi\sqrt{n}t - \frac{\pi}{4}). \end{aligned}$$

Note that $\max_{q\in\mathbf{Q}_x}|q| \le e^{2x} \le B/2$, $|\mathbf{Q}_x| \ll \exp(cx/(\log x)) \ll \sqrt{\log T}$ and n is odd when $n \in \mathbf{Q}_x$. Choose l such that $\|l\sqrt{n}\| < \delta'$ and $T^{1/10} \le l \le (1+\delta^{-|\mathbf{Q}_x|})T^{1/10} \ll T^{1/4}$ where $\delta' > 0$ is a sufficiently small constant, then (recall $n \in \mathbf{Q}_x$ is odd)

$$\begin{aligned}\epsilon\int_{-\infty}^{\infty} T_x(u)S_B(l+u)\left(\frac{\sin\epsilon\pi u}{\epsilon\pi u}\right)^2 du &\gg \sum_{n\in\mathbf{Q}_x}\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}}\\ &\gg \begin{cases}\log x & \text{if } \sigma = 3/4,\\ \exp(cx^{\sigma-3/4}/(\log x)) & \text{if } \sigma > 3/4.\end{cases}\end{aligned}$$

Plainly, we have

$$\epsilon\int_{|u|\ge T^{1/20}} T_x(u)S_B(l+u)\left(\frac{\sin\epsilon\pi u}{\epsilon\pi u}\right)^2 du \ll \epsilon^{-1}T^{-1/20}2^{|\mathbf{Q}_x|}\sum_{n\le B^2}\frac{\sigma_{1-2\sigma}(n)}{n^{7/4-\sigma}} \ll 1,$$

and $\epsilon\int_{-T^{1/20}}^{T^{1/20}} T_x(u)(\sin(\epsilon\pi u)/(\epsilon\pi u))^2\,du \ll 1$ by (6.5) with $n=0$. Thus, we obtain

$$\sup_{T^{1/10}\ll u\ll T^{1/4}} S_B(u) \gg \begin{cases}\log\log\log T & \text{if } \sigma = 3/4,\\ \exp(c(\log\log T)^{\sigma-3/4}(\log\log\log T)^{\sigma-7/4}) & \text{if } \sigma > 3/4.\end{cases}$$

7. OCCURRENCE OF LARGE VALUES

Proof of Theorem 5. Define $K_\tau(u) = (1-|u|)(1+\tau\sin(4\pi\alpha u))$ where $\tau = -1$ or $+1$ and α is a large constant. Then, we have

$$\int_{-1}^{1} K_\tau(u)\sin(4\pi\sqrt{n}(t+\alpha u) - \frac{\pi}{4})\,du = -\frac{\tau}{2}\delta_{1,n}\cos(4\pi t - \pi/4) + O(\alpha^{-2}n^{-1})$$

where $\delta_{1,n} = 1$ if $n=1$ and 0 otherwise. This gives with (6.2)

$$\begin{aligned}&\int_{-1}^{1}(t+u)^{2\sigma-5/2}\int_{-L}^{L} G_\sigma(2\pi(t+u+v)^2)K(v)\,dvK_\tau(u)\,du\\ =\ &\frac{\tau}{2}(1-B^{-1})\cos(4\pi t-\pi/4) + O(\alpha^{-2}) + O(B^{4\sigma-5}).\end{aligned}$$

Our assertion follows once we take sufficiently large constants B and α ($L = B^4$), and choose $\|4t\| \le 1/8$ with $t \in [\sqrt{T}, \sqrt{T}+1]$. (Note that τ can be $+1$ or -1 at our disposal.)

The next lemma is used to prove Theorem 6.

Lemma 7.1 *For $T^{5/12} \le H \le T^{1/2}$,*

$$\int_T^{2T}\max_{0\le h\le H}(G_\sigma(t+h) - G_\sigma(t))^2\,dt \ll TH^{5-4\sigma}$$

where the implied constant depends on σ.

Proof. Following the arguments in [7], we have

$$\int_T^{2T} (G_\sigma(t+h) - G_\sigma(t))^2 \, dt \ll Th^{5-4\sigma} \min((\sigma - 3/4)^{-1}, \log(T/h^2)) \tag{7.1}$$

where $\log^2 T \leq h \leq \sqrt{T}$. Let $b = T^{1/24}$ and $H = 2^\lambda b$. Suppose $(j-1)b \leq h < jb$. Then, using the bound $E_\sigma(t) \ll t^{2(1-\sigma)/3+\epsilon}$, we have

$$\begin{aligned} |G_\sigma(t+h) - G_\sigma(t)| &\leq |G_\sigma(t+jb) - G_\sigma(t)| + |G_\sigma(t+jb) - G_\sigma(h)| \\ &= |G_\sigma(t+jb) - G_\sigma(t)| + O(T^{2(1-\sigma)/3+\epsilon} b) \end{aligned}$$

and hence for any fixed t,

$$\max_{0 \leq h \leq H} |G_\sigma(t+h) - G_\sigma(t)| \leq \max_{1 \leq j \leq 2^\lambda} |G_\sigma(t+jb) - G_\sigma(t)| + O(T^{2(1-\sigma)/3+\epsilon} b). \tag{7.2}$$

Let us take $1 \leq j_0 = j_0(t) \leq 2^\lambda$ such that

$$|G_\sigma(t+j_0 b) - G_\sigma(t)| = \max_{1 \leq j \leq 2^\lambda} |G_\sigma(t+jb) - G_\sigma(t)|.$$

Then we can express $j_0 = 2^\lambda \sum_{\mu \in S_t} 2^{-\mu}$ where S_t is a certain set of non-negative integers. Hence,

$$G_\sigma(t+j_0 b) - G_\sigma(t) = \sum_{\mu \in S_t} G_\sigma(t + (\nu+1)2^{\lambda-\mu} b) - G_\sigma(t + \nu 2^{\lambda-\mu} b)$$

where $0 \leq \nu = \nu_{t,\mu} < 2^\mu$ is an integer. By Cauchy-Schwarz inequality and inserting all ν's other than $\nu_{t,\mu}$, we get

$$\begin{aligned} &(G_\sigma(t+j_0 b) - G_\sigma(t))^2 \\ \leq &\left(\sum_{\mu \in S_t} 2^{-(1-\sigma)\mu} \right) \sum_{\mu \in S_t} 2^{(1-\sigma)\mu} (G_\sigma(t + (\nu+1)2^{\lambda-\mu} b) - G_\sigma(t + \nu 2^{\lambda-\mu} b))^2 \\ \ll &\sum_{\mu \in S_t} \sum_{0 \leq \nu < 2^\mu} 2^{(1-\sigma)\mu} (G_\sigma(t + (\nu+1)2^{\lambda-\mu} b) - G_\sigma(t + \nu 2^{\lambda-\mu} b))^2 \end{aligned}$$

as $\sum_{\mu \in S_t} 2^{-(1-\sigma)\mu} \ll 1$. Integrating over $[T, 2T]$ and using (7.1), we see that

$$\begin{aligned} &\int_T^{2T} \max_{0 \leq h \leq H} (G_\sigma(t+h) - G_\sigma(t))^2 \, dt \\ \ll &\sum_{\mu \in S_t} \sum_{0 \leq \nu < 2^\mu} 2^{(1-\sigma)\mu} \int_T^{2T} (G_\sigma(t + (\nu+1)2^{\lambda-\mu} b) - G_\sigma(t + \nu 2^{\lambda-\mu} b))^2 \, dt + T^{1+4(1-\sigma)/3+\epsilon} b^2 \\ \ll &\sum_{\mu \in S_t} \sum_{0 \leq \nu < 2^\mu} 2^{(1-\sigma)\mu} \int_{T+\nu 2^{\lambda-\mu} b}^{2T+\nu 2^{\lambda-\mu} b} (G_\sigma(t + 2^{\lambda-\mu} b) - G_\sigma(t))^2 \, dt + T^{17/12+\epsilon} \\ \ll &\; TH^{5-4\sigma} \sum_{\mu \in S_t} \sum_{0 \leq \nu < 2^\mu} 2^{-(4-3\sigma)\mu} \\ \ll &\; TH^{5-4\sigma}. \end{aligned}$$

Proof of Theorem 6. Define $G_\sigma^{\pm}(t) = \max(0, \pm G_\sigma(t))$. By using Theorem 2 (c), we have $\int_T^{2T} |G_\sigma(t)|^3 \, dt \ll T^{1+3(5/4-\sigma)}$. Hence, by Cauchy-Schwarz inequality,

$$\left(\int_T^{2T} G_\sigma(t)^2 \, dt \right)^2 \leq \int_T^{2T} |G_\sigma(t)| \, dt \int_T^{2T} |G_\sigma(t)|^3 \, dt.$$

we have $\int_T^{2T} |G_\sigma(t)| \, dt \gg T^{1+(5/4-\sigma)}$. Together with (2.1), $\int_T^{2T} G_\sigma^{\pm}(t) \, dt \geq c_{14} \int_T^{2T} t^{5/4-\sigma} \, dt$.

Consider $K^{\pm}(t) = G_\sigma^{\pm}(t) - (c_{14} - \epsilon) t^{5/4-\sigma}$ where $\epsilon = \delta^{5/2-2\sigma}$, we have

$$\int_T^{2T} K^{\pm}(t) \, dt \geq \epsilon \int_T^{2T} t^{5/4-\sigma} \, dt$$

and $K^{\pm}(t+h) - K^{\pm}(t) = G_\sigma^{\pm}(t+h) - G_\sigma^{\pm}(t) + O(T^{1/4-\sigma} h)$. This gives

$$\int_T^{2T} \max_{h \leq H} |K^{\pm}(t+h) - K^{\pm}(t)| \, dt \ll \int_T^{2T} \max_{h \leq H} |G_\sigma(t+h) - G_\sigma(t)| \, dt + T^{5/4-\sigma} H$$

since $|G_\sigma^{\pm}(t+h) - G_\sigma^{\pm}(t)| \leq |G_\sigma(t+h) - G_\sigma(t)|$. Lemma 7.1 yields that

$$\int_T^{2T} \max_{h \leq H} |K^{\pm}(t+h) - K^{\pm}(t)| \, dt \ll T H^{5/2-2\sigma} + T^{5/4-\sigma} H.$$

Define $\omega^{\pm}(t) = K^{\pm}(t) - \max_{h \leq H} |K^{\pm}(t+h) - K^{\pm}(t)|$. Taking $H = c' \epsilon^{2/(5-4\sigma)} \sqrt{T}$ $(= c' \delta \sqrt{T})$ for some sufficiently small constant $c' > 0$, we have

$$\int_T^{2T} \omega^{\pm}(t) \, dt \geq \epsilon \int_T^{2T} t^{5/4-\sigma} \, dt - \int_T^{2T} \max_{h \leq H} |K^{\pm}(t+h) - K^{\pm}(t)| \, dt \gg \epsilon T^{1+(5/4-\sigma)}.$$

Let $\mathcal{I}^{\pm} = \{t \in [T, 2T] : \omega^{\pm}(t) > 0\}$. Then

$$\begin{aligned} \int_T^{2T} \omega^{\pm}(t) \, dt &\leq \int_{\mathcal{I}^{\pm}} \omega^{\pm}(t) \, dt \leq \int_{\mathcal{I}^{\pm}} K^{\pm}(t) \, dt \\ &\leq \left(\int_{\mathcal{I}^{\pm}} dt \right)^{1/2} \left(\int_T^{2T} K^{\pm}(t)^2 \, dt \right)^{1/2}. \end{aligned}$$

We infer $|\mathcal{I}^{\pm}| \gg \epsilon^2 T$ as $\int_T^{2T} K^{\pm}(t)^2 \, dt \ll \int_T^{2T} G_\sigma(t)^2 \, dt + T^{7/2-2\sigma}$. When $t \in \mathcal{I}^{\pm}$, we have $K^{\pm}(t) \geq \max_{h \leq H} |K^{\pm}(t+h) - K^{\pm}(t)| \geq 0$. Hence, $K^{\pm}(u) \geq 0$ for all $u \in [t, t+H]$, i.e. $G_\sigma^{\pm}(t) \geq (c_{14} - \epsilon) t^{5/4-\sigma}$. The number of such intervals is not less than $|\mathcal{I}^{\pm}|/H \gg c_{15} \delta^{4(1-\sigma)} \sqrt{T}$.

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