Proof of Chvátal's Conjecture on Maximal Stable Sets and Maximal Cliques in Graphs

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*The work described in this paper was partially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CityU1056/01E).

†Supported by the fund from NSFC under grant numbers 10271065 and 60373025.

‡Supported in part by the Research Grants Council of Hong Kong (Project No. HKU7109/01P).

Abstract

Grillet established conditions on a partially ordered set under which each maximal antichain meets each maximal chain. Chvátal made a conjecture in terms of graphs that strengthens Grillet's theorem. The purpose of this paper is to prove this conjecture.

1 Introduction

Grillet [6] proved that in every partially ordered set containing no quadruple (a, b, c, d) such that

$$a < b, c < d, b \text{ covers } c,$$
$$\text{and the remaining three pairs of elements are incomparable,}$$

each maximal antichain meets each maximal chain. (As usual, we say that b *covers* c if $c < b$ and $c < x \leq b$ implies $x = b$. Throughout this paper, the adjective *maximal* is always meant with respect to set-inclusion rather than size.) Berge [1] pointed out that Grillet's theorem can be stated in terms of graphs rather than partially ordered sets: if a comparability graph has the property that every (induced) P_4 is contained in an (induced) A (see Fig.1), then each maximal stable set meets each maximal clique. (The vertices of a *comparability graph* are the elements of a partially ordered set, with two vertices adjacent if and only if they are comparable: see [4], [5].) Then he went on to make a conjecture and suggest a problem that strengthen this statement; both the conjecture and the problem were solved in the negative [7]. Chvátal [2, 7] proposed the following conjecture as a variation on Berge's problem.

Conjecture *Let G be a graph with no induced subgraph isomorphic to F or $\bar{F}$ (the complement graph of F). Then each maximal stable set meets each maximal clique in G if and only if each P_4 extends into an A in G.*

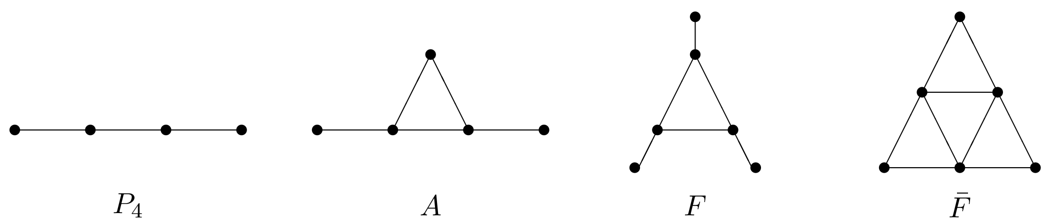

Figure 1. P_4, A, F, and $\bar{F}$

Clearly, if a graph G enjoys the property that each maximal stable set meets each maximal clique, then every P_4 extends into an A in G. However, the converse need not hold in general: both F and $\bar{F}$ are counterexamples. Chvátal's conjecture asserts that actually F and $\bar{F}$ are the only obstructions to the above property.

Gallai [3] characterized comparability graphs in terms of 19 forbidden induced subgraphs. Since both F and $\bar{F}$ are included in this list, Chvátal's conjecture generalizes Grillet's theorem.

Two theorems in the spirit of Berge's problem that are weaker than Chvátal's conjecture but stronger than Grillet's theorem were proved in [7]. The purpose of this paper is to prove Chvátal's conjecture.

Theorem *Let G be a graph with no induced subgraph isomorphic to F or $\bar{F}$. Then each maximal stable set meets each maximal clique in G if and only if each P_4 extends into an A in G.*

Outline of the proof. The "only if" part is trivial. To prove the "if" part, we assume that G is a counterexample with the smallest number of vertices. In section 2, we prove that if a maximal stable set S is disjoint from a maximal clique C, then the configuration between S and C can be fully described. Based on this observation, we can further show that G contains a subgraph Σ as depicted in Figure 2; this intermediate structure plays an important role in our proof. In section 3, we show that there exist some vertices outside Σ which have certain nice adjacency properties. In section 4, we exhibit an F or $\bar{F}$ in G by using the vertices obtained in section 3, and thus we reach a contradiction.

2 Structural Description

Throughout this paper, we let G stand for a counterexample with the smallest number of vertices. For each vertex v of G, let $N(v)$ denote the set of all the neighbors of v in G and let $U(v)$ denote the set of all the vertices outside $N(v) \cup \{v\}$. For each vertex subset X of G, let $G[X]$ stand for the subgraph of G induced by X. We also use $\{abc; x, y, z\}$ to denote an F or $\bar{F}$ with the vertex set $\{a, b, c, x, y, z\}$, in which x, y, z are pairwise nonadjacent and abc is a triangle.

Lemma 1 *Let v be a vertex of G. Then the following two statements hold.*
(i) Each P_4 in $G[N(v)]$ extends into an A in $G[N(v)]$;
(ii) Each P_4 in $G[U(v)]$ extends into an A in $G[U(v)]$.

Proof. (i) Let $abcd$ be an arbitrary P_4 in $G[N(v)]$ and let w be the fifth vertex of an A that contains $abcd$. Then $w \in N(v)$, for otherwise $\{bcv; a, d, w\}$ would induce an $\bar{F}$ in G, a contradiction.

(ii) Let $abcd$ be an arbitrary P_4 in $G[U(v)]$ and let w be the fifth vertex of an A that contains $abcd$. Then $w \in U(v)$, for otherwise $\{bcw; a, d, v\}$ would induce an F in G, a contradiction. □

Lemma 2 *Let S be a maximal stable set and let C be a maximal clique of G. If S and C share no vertex, then either the following case 1 or case 2 occurs:*

Case 1. *S can be partitioned into S_1, S_2 and C can be partitioned into C_1, C_2 such that*

- *none of S_1, S_2, C_1, C_2 is empty;*
- *each vertex in S_i is adjacent to each vertex in C_i in G for $i = 1, 2$;*
- *no vertex in S_i is adjacent to any vertex in C_{i+1} in G for $i = 1, 2$, where the subscript is taken modulo 2.*

Case 2. *S can be partitioned into S_1, S_2, S_3, S_4, and C can be partitioned into C_1, C_2, C_3, C_4 such that*

- *neither S_i nor C_i is empty for $i = 1, 2, 3, 4$;*
- *each vertex in S_i is adjacent to each vertex in $C_i \cup C_{i+1}$ in G for $i = 1, 2, 3, 4$, where the subscript is taken modulo 4;*
- *no vertex in S_i is adjacent to any vertex in $C_{i+2} \cup C_{i+3}$ in G for $i = 1, 2, 3, 4$, where the subscript is taken modulo 4.*

Proof. Let us make some simple observations first.

(2.1) For any $x \in S$, there exists $y \in S - \{x\}$ such that $C \subseteq N(x) \cup N(y)$.

To justify (2.1), note that by Lemma 1(ii) and the minimality of G, each maximal stable set in $G[U(x)]$ meets each maximal clique in $G[U(x)]$. Since $S - \{x\}$ is a maximal stable set in $G[U(x)]$ and $C - N(x)$ is a clique in $G[U(x)]$, there must exist a $y \in S - \{x\}$ which is adjacent to all the vertices in $C - N(x)$. So (2.1) holds.

(2.2) Let x, y be any two vertices in S with $C \subseteq N(x) \cup N(y)$. Then $C \cap N(x) \cap N(y) = \emptyset$.

To justify (2.2), assume to the contrary that a is a vertex in $C \cap N(x) \cap N(y)$. Let us turn to consider $\bar{G}$. Since each P_4 extends into an A in G and since both P_4 and A are self-complement, it is easy to see that each P_4 in $\bar{G}$ extends into an A in $\bar{G}$. So (2.1) is valid with respect to $\bar{G}$ with S in place of C, C in place of S, and a in place of x. Hence there exists $b \in C - \{a\}$ such that each vertex in S is adjacent to at least one of a and b in $\bar{G}$. However, if $b \in C \cap N(x)$ then x is adjacent to neither of a and b in $\bar{G}$; if $b \in C \cap N(y)$ then y is adjacent to neither of a and b in $\bar{G}$. So we reach a contradiction in either case.

(2.3) There exist no vertices x, y in S such that $N(x) \cap C$ is a *proper* subset of $N(y) \cap C$.

To prove (2.3), assume to the contrary that $N(x) \cap C$ is a proper subset of $N(y) \cap C$. According to (2.1), there exists $z \in S - \{x\}$ such that $C \subseteq N(x) \cup N(z)$. Thus $C \subseteq N(y) \cup N(z)$ and $C \cap N(y) \cap N(z) \neq \emptyset$, the existence of the pair y, z contradicts (2.2).

(2.4) For any $a \in C$, there exists $b \in C - \{a\}$ such that $S \subseteq N(a) \cup N(b)$ and that $S \cap N(a) \cap N(b) = \emptyset$.

To justify (2.4), applying (2.1) to $\bar{G}$ with C in place of S and with S in place of C, we see that for any $a \in C$, there exists $b \in C - \{a\}$ such that $S \subseteq U(a) \cup U(b)$. For this pair a, b, it can be seen from (2.2) with respect to $\bar{G}$ that $S \cap U(a) \cap U(b) = \emptyset$. Hence each vertex in S is nonadjacent to precisely one of a and b in G, which implies that each vertex in S is adjacent to precisely one of a and b in G, so (2.4) follows.

Throughout the remainder of the proof of this lemma, let us reserve x, y for two vertices in S and a, b for two vertices in C such that each vertex in S is adjacent to precisely one of a and b, and each vertex in C is adjacent to precisely one of x and y. Note that the existence of x, y, a, b is guaranteed by (2.1), (2.2) and (2.4). Rename the vertices if necessary, we may assume that $a \in N(x) \cap C$ and $b \in N(y) \cap C$. For convenience, set $X = N(x) \cap C$, $Y = N(y) \cap C$, $A = N(a) \cap S$, and $B = N(b) \cap S$. Since $x \in A$, $y \in B$, $a \in X$, and $b \in Y$, we have

(2.5) None of A, B, X, Y is empty.

(2.6) Let u, v be any two vertices in A. Then either $N(u) \cap Y \subseteq N(v) \cap Y$ or $N(v) \cap Y \subseteq N(u) \cap Y$.

To justify (2.6), assume to the contrary that p is a vertex in $((N(u) - N(v)) \cap Y$ and q is a vertex in $((N(v) - N(u)) \cap Y$. Then $\{apq; u, v, y\}$ induces an $\bar{F}$ in G, a contradiction.

Similarly, we have the following statement.

(2.7) Let u, v be any two vertices in B. Then either $N(u) \cap X \subseteq N(v) \cap X$ or $N(v) \cap X \subseteq N(u) \cap X$.

Let $U = \cup_{p \in B} N(p) \cap X$ and let $V = \cup_{p \in A} N(p) \cap Y$. Then (2.6) and (2.7) guarantee the existence of a vertex $u \in B$ and a vertex $v \in A$ such that $U = N(u) \cap X$ and $V = N(v) \cap Y$; let us reserve u, v for these two vertices throughout the remainder of the proof of this lemma.

(2.8) We have $X - U \neq \emptyset$ and $Y - V \neq \emptyset$.

Suppose to the contrary that $X - U = \emptyset$, then $N(x) \cap C$ is a proper subset of $N(u) \cap C$, contradicting (2.3). Similarly, we obtain $Y - V \neq \emptyset$.

(2.9) Each vertex in A is adjacent to each vertex in $X - U$; each vertex in B is adjacent to each vertex in $Y - V$.

We only consider the case for A, as the case for B is a mirror image. Suppose to the contrary that a vertex p in A is not adjacent to some vertex w in $X - U$. Let q be a vertex in $S - \{p\}$ such that $C \subseteq N(p) \cup N(q)$ (q exists by (2.1)). Then $q \in B$ as b is adjacent to no vertex in A.

Hence w is adjacent to neither of p and q (recall the definition of U), a contradiction.

(2.10) We have $N(v) \cap U = \emptyset$ and $N(u) \cap V = \emptyset$.

For otherwise, it follows from (2.9) and the definitions of u and v that $C \subseteq N(u) \cup N(v)$ and $C \cap N(u) \cap N(v) \neq \emptyset$, the existence of the pair u, v contradicts (2.2).

(2.11) For any vertex p in S, either $N(p) \cap U = \emptyset$ or $N(p) \cap V = \emptyset$.

By virtue of (2.10), we may assume that $p \neq u, v$. Suppose to the contrary that p is adjacent to both q in U and r in V. Symmetry allows us to assume $p \in A$. Thus $\{qrb; p, u, y\}$ would induce an $\bar{F}$ in G, a contradiction.

(2.12) For any vertex p in S, if $N(p) \cap U = \emptyset$ then $V \subseteq N(p)$; if $N(p) \cap V = \emptyset$ then $U \subseteq N(p)$.

We only consider the case $p \in A$, as the case $p \in B$ is a mirror image. If $N(p) \cap U = \emptyset$ then $V \subseteq N(p)$, for otherwise $N(p) \cap C$ would be a proper subset of $N(v) \cap C$ according to (2.9) and the definition of v, contradicting (2.3); if $N(p) \cap V = \emptyset$ then $U \subseteq N(p)$, for otherwise $N(p) \cap C$ would be a proper subset of $N(x) \cap C$, contradicting (2.3).

(2.13) If one of U and V is empty, then the other is also empty.

Suppose the contrary: without loss of generality, we may assume $U = \emptyset$ however $V \neq \emptyset$. Then by (2.7) and the definition of V, $N(x) \cap C$ is a proper subset of $N(v) \cap C$, a contradiction.

Now (2.13) allows us to distinguish between two cases.

Case 1. $U = V = \emptyset$. Set $S_1 = A$, $S_2 = B$, $C_1 = X$, and $C_2 = Y$. The selection of a, b, x, y implies that S_1, S_2 form a partition of S, and C_1, C_2 form a partition of C. By (2.5), none of S_1, S_2, C_1, C_2 is empty; from (2.9), it can be seen that each vertex in S_i is adjacent to each vertex in C_i in G for $i = 1, 2$; from the definitions of U and V, it can be deduced that no vertex in S_i is adjacent to any vertex in C_{i+1} in G for $i = 1, 2$, where the subscript is taken modulo 2. Hence the present case coincides with Case 1 as described in our lemma.

Case 2. $U \neq \emptyset \neq V$. Set $C_1 = X - U$, $C_2 = U$, $C_3 = Y - V$, $C_4 = V$, and set $S_1 = \{p \in A : U \subseteq N(p)\}$, $S_2 = \{p \in B : U \subseteq N(p)\}$, $S_3 = \{p \in B : V \subseteq N(p)\}$, $S_4 = \{p \in A : V \subseteq N(p)\}$. Clearly, C_1, C_2, C_3, C_4 form a partition of C. In view of (2.11) and (2.12), we see that S_1, S_2, S_3, S_4 form a partition of S. From (2.8), it follows that C_i is nonempty for $i = 1, 2, 3, 4$. Since $x \in S_1$, $u \in S_2$, $y \in S_3$, $v \in S_4$, none of S_1, S_2, S_3, S_4 is empty. From (2.9), (2.11), (2.12) and the definition of S_i, we conclude that each vertex in S_i is adjacent to each vertex in $C_i \cup C_{i+1}$ in G, and that no vertex in S_i is adjacent to any vertex in $C_{i+2} \cup C_{i+3}$ in G for $i = 1, 2, 3, 4$, where the subscript is taken modulo 4. Hence the present case coincides with Case 2 as described in our lemma.

This completes the proof of Lemma 2. □

Corollary 1 *There exist a maximal stable set S and a maximal clique C in G such that S and C are vertex disjoint and that the configuration between S and C is as described in Case 2 of Lemma 2.*

Proof. Since G is a counterexample, there exist a maximal stable set S and a maximal clique C in G such that S and C are vertex disjoint. If the configuration between S and C is as described in Case 2 of Lemma 2, then we are done; so we assume that the configuration is as described in Case 1. Let s_i be a vertex in S_i (as in the statement of Case 1) and let c_i be a vertex in C_i for $i = 1, 2$. Then, by hypothesis, the path $s_1c_1c_2s_2$ is contained in an A. Let w be the fifth vertex of this A and let K be an arbitrary maximal clique that contains $\{c_1, c_2, w\}$. Then S and K are vertex disjoint since no vertex in S is adjacent to both c_1 and c_2. In view of the structure of $\{s_1, c_1, c_2, s_2; w\}$, we can see that the configuration between S and K is not as described in Case 1 of Lemma 2. Hence, by Lemma 2, Case 2 must occur. □

Lemma 3 *Suppose S is a maximal stable set in G, $\{S_1, S_2, S_3, S_4\}$ is a partition of S, and $\{c_1, c_2, c_3, c_4\}$ is a clique with four vertices outside S such that*

- *none of S_1, S_2, S_3, S_4 is empty;*
- *each vertex in S_i is adjacent to both c_i and c_{i+1} for $i = 1, 2, 3, 4$, where the subscript is taken modulo 4;*
- *no vertex in S_i is adjacent to c_{i+2} or c_{i+3} in G for $i = 1, 2, 3, 4$, where the subscript is taken modulo 4.*

Let s_i be a vertex in S_i for $i = 1, 2, 3, 4$, let $\{s_1, c_1, c_3, s_3; x\}$ be an arbitrary A that contains the path $s_1c_1c_3s_3$, and let $\{s_4, c_1, c_3, s_2; y\}$ be an arbitrary A that contains the path $s_4c_1c_3s_2$. Then the following statements hold:

- *x is adjacent to all the vertices in $S_2 \cup S_4$ and adjacent to no vertex in $S_1 \cup S_3 \cup \{c_2, c_4\}$;*
- *y is adjacent to all the vertices in $S_1 \cup S_3$ and adjacent to no vertex in $S_2 \cup S_4 \cup \{c_2, c_4\}$.*
- *x and y are adjacent.*

Proof. Let us first prove that x is adjacent to all the vertices in $S_2 \cup S_4$.

Suppose to the contrary that x is nonadjacent to some vertex z in $S_2 \cup S_4$; symmetry allows us to assume that $z \in S_2$. Then x is adjacent to c_2 for otherwise $\{c_1c_2c_3; s_1, x, z\}$ would induce an $\bar{F}$ in G, a contradiction. Let C be an arbitrary maximal clique in G that contains $\{c_1, c_2, c_3, x\}$. Then S and C are vertex disjoint since, by assumption, no vertex in S is adjacent to all three of c_1, c_2, c_3. Furthermore, it can be seen from the structure between $S_1 \cup S_2 \cup S_3$ and $\{c_1, c_2, c_3\}$

that the configuration between S and C is not as described in Case 1 of Lemma 2. Hence, by Lemma 2, Case 2 must occur; let S'_1, S'_2, S'_3, S'_4 be the partition of S and let C_1, C_2, C_3, C_4 be the partition of C as described there. It is a routine matter to check that $N(x) \cap S$, $N(c_i) \cap S$ for $i = 1, 2, 3$, are all distinct (note the existence of z). So each C_j contains precisely one vertex from $\{c_1, c_2, c_3, x\}$ for $j = 1, 2, 3, 4$. Hence, it follows from the configuration between S and C that each vertex in S is adjacent to precisely two vertices in $\{c_1, c_2, c_3, x\}$, however s_3 is adjacent to no vertex in $\{c_1, c_2, x\}$, a contradiction.

Similarly, we can prove that y is adjacent to all the vertices in $S_1 \cup S_3$.

Since x is adjacent to each vertex in $S_2 \cup S_4$, xs_2 and xs_4 are two edges in G. It follows that x and y are adjacent, for otherwise $\{c_1c_3x; s_2, s_4, y\}$ would induce an $\bar{F}$ in G, a contradiction.

The preceding statement implies that $\{c_1, c_3, x, y\}$ is a clique in G. Let C be any maximal clique that contains $\{c_1, c_3, x, y\}$. Then S and C are vertex disjoint since, by assumption, no vertex in S is adjacent to both c_1 and c_3. By virtue of vertices s_1, s_2, s_3, s_4, we can see that $N(c_1) \cap S$, $N(c_3) \cap S$, $N(x) \cap S$, and $N(y) \cap S$ are all distinct. Hence, by Lemma 2, the configuration between S and C is not as described in Case 1. So Case 2 must occur. Furthermore, let C_1, C_2, C_3, C_4 be the partition as described in Case 2. Then each C_i contains precisely one vertex in $\{c_1, c_3, x, y\}$, and therefore each vertex in S is adjacent to precisely two vertices in $\{c_1, c_3, x, y\}$ (as in the statement of Case 2), which implies that x is adjacent to no vertex in $S_1 \cup S_3$, and y is adjacent to no vertex in $S_2 \cup S_4$.

Let us now prove that there is no edge between $\{x, y\}$ and $\{c_2, c_4\}$ in G.

Suppose to the contrary that x is adjacent to c_2 or c_4. Then symmetry allows us to assume that c_2x is an edge of G. It follows that $\{c_2c_3x; s_1, s_3, s_4\}$ induces an F in G, a contradiction.

Suppose to the contrary that y is adjacent to c_2 or c_4. Then symmetry allows us to assume that c_2y is an edge of G. It follows that $\{c_2c_3y; s_1, s_2, s_3\}$ induces an $\bar{F}$ in G, a contradiction.

This completes the proof of Lemma 3. □

Since G is a counterexample, Corollary 1 guarantees the existence of a maximal stable set S and a maximal clique C in G such that S and C are disjoint and that the configuration between S and C is as described in Case 2 of Lemma 2. Renaming the partition S_1, S_2, S_3, S_4 of S given in Case 2, we can find four vertices v_1, v_2, v_3, v_4 in C such that

(II.1) v_1 is adjacent to each vertex in $S_1 \cup S_2$ and adjacent to no vertex in $S_3 \cup S_4$;

(II.2) v_2 is adjacent to each vertex in $S_3 \cup S_4$ and adjacent to no vertex in $S_1 \cup S_2$;

(II.3) v_3 is adjacent to each vertex in $S_1 \cup S_3$ and adjacent to no vertex in $S_2 \cup S_4$;

(II.4) v_4 is adjacent to each vertex in $S_2 \cup S_4$ and adjacent to no vertex in $S_1 \cup S_3$.

For convenience, we reserve a vertex s_i in S_i for $i = 1, 2, 3, 4$ hereafter. Since $s_1v_3v_4s_4$ is a P_4, by hypothesis it extends into an A in G; let v_5 be the fifth vertex of this A. Clearly v_5 is outside $S \cup \{v_1, v_2, v_3, v_4\}$. By Lemma 3, we have

(II.5) v_5 is adjacent to each vertex in $S_2 \cup S_3$ and adjacent to no vertex in $S_1 \cup S_4 \cup \{v_1, v_2\}$.

Since $s_2v_4v_3s_3$ is a P_4 in $G[N(v_5)]$, by Lemma 1 it extends into an A in $G[N(v_5)]$; let v_6 be the the fifth vertex of this A. Clearly v_6 is outside $S \cup \{v_1, v_2, v_3, v_4, v_5\}$. By Lemma 3, we have

(II.6) v_6 is adjacent to each vertex in $S_1 \cup S_4$ and adjacent to no vertex in $S_2 \cup S_3 \cup \{v_1, v_2\}$.

The reader may refer to Figure 2 for the adjacency between S and $\{v_1, v_2, \ldots, v_6\}$.

We aim to prove that G contains a subgraph as depicted in Figure 2. For this purpose, let us inductively construct a sequence $\{v_i\}$, $i = 1, 2, \ldots$, starting with $v_1, v_2, \ldots, v_6$. We intend to show that the sequence contains 12 consecutive vertices which, together with S, generate a subgraph of G as desired.

(II.7) The construction of v_i for $i \geq 7$ goes as follows, here $N_S(v) = N(v) \cap S$.

- If i is odd, then $\{\{s_1, s_2, s_3, s_4\} - N_S(v_{i-6})\} \cup \{v_{i-1}, v_{i-2}\}$ induces a P_4 in G, and v_i is the fifth vertex of an arbitrary A that contains this P_4;
- If i is even, then $\{\{s_1, s_2, s_3, s_4\} - N_S(v_{i-6})\} \cup \{v_{i-2}, v_{i-3}\}$ induces a P_4 in G, and v_i is the fifth vertex of an arbitrary A that contains this P_4.

To show the validity of the above construction, we need to justify that $\{\{s_1, s_2, s_3, s_4\} - N_S(v_{i-6})\} \cup \{v_{i-1}, v_{i-2}\}$ induces a P_4 in G if i is odd, and $\{\{s_1, s_2, s_3, s_4\} - N_S(v_{i-6})\} \cup \{v_{i-2}, v_{i-3}\}$ induces a P_4 in G if i is even. Let us first consider the cases $i = 7, 8, \ldots, 12$.

From the structure between S and $\{v_1, v_2, \ldots, v_6\}$ (recall (II.1)-(II.6)), we see that $\{\{s_1, s_2, s_3, s_4\} - N_S(v_1)\} \cup \{v_6, v_5\}$ induces a P_4, $s_3v_5v_6s_4$, in G, and $\{\{s_1, s_2, s_3, s_4\} - N_S(v_2)\} \cup \{v_6, v_5\}$ induces a P_4, $s_1v_6v_5s_2$, in G. Since every P_4 extends into an A in G, v_7 and v_8 exist. By virtue of Lemma 3 (with $\{v_3, v_4, v_5, v_6\}$ in place of $\{c_1, c_2, c_3, c_4\}$), we get

(II.8)
- v_7 is adjacent to all the vertices in $S_1 \cup S_2$ and adjacent to no vertex in $S_3 \cup S_4 \cup \{v_3, v_4\}$, implying $N_S(v_7) = N_S(v_1)$ (recall (II.1));
- v_8 is adjacent to all the vertices in $S_3 \cup S_4$ and adjacent to no vertex in $S_1 \cup S_2 \cup \{v_3, v_4\}$, implying $N_S(v_8) = N_S(v_2)$ (recall (II.2));
- v_7 and v_8 are adjacent.

In view of (II.8), we see that $\{\{s_1, s_2, s_3, s_4\} - N_S(v_3)\} \cup \{v_8, v_7\}$ induces a P_4, $s_2v_7v_8s_4$, in G, and $\{\{s_1, s_2, s_3, s_4\} - N_S(v_4)\} \cup \{v_8, v_7\}$ induces a P_4, $s_1v_7v_8s_3$, in G. Thus v_9 and v_{10} are well defined. Using Lemma 3, we have

(II.9)
- $N_S(v_9) = N_S(v_3)$ and $N_S(v_{10}) = N_S(v_4)$;
- There is no edge between $\{v_9, v_{10}\}$ and $\{v_5, v_6\}$;
- v_9 and v_{10} are adjacent.

Based on (II.9), we see that $\{\{s_1, s_2, s_3, s_4\} - N_S(v_5)\} \cup \{v_{10}, v_9\}$ induces a P_4, $s_1v_9v_{10}s_4$, in G, and $\{\{s_1, s_2, s_3, s_4\} - N_S(v_6)\} \cup \{v_{10}, v_9\}$ induces a P_4, $s_2v_{10}v_9s_3$, in G. Thus v_{11} and v_{12} are well defined. By Lemma 3, we obtain

(II.10)
- $N_S(v_{11}) = N_S(v_5)$ and $N_S(v_{12}) = N_S(v_6)$;
- There is no edge between $\{v_{11}, v_{12}\}$ and $\{v_7, v_8\}$;
- v_{11} and v_{12} are adjacent.

From (II.8)-(II.10), it can be seen that the subgraph of G induced by $S \cup \{v_1, v_2, \ldots, v_6\}$ is isomorphic to the subgraph induced by $S \cup \{v_7, v_8, \ldots, v_{12}\}$, with the isomorphism $f(v_i) = v_{i+6}$ for $1 \le i \le 6$. Since we have established the validity of (II.7) for $i \le 12$, the validity for general i follows from periodicity.

Lemma 4 *The sequence $\{v_i\}$ constructed in (II.7) enjoys the following properties:*

(i) $N_S(v_i) = N_S(v_j)$ *whenever* $i - j \equiv 0 \pmod 6$*;*

(ii) $\{v_{2i-1}, v_{2i}, v_{2i+1}, v_{2i+2}\}$ *induces a clique in G for any $i \ge 1$;*

(iii) There is no edge between $\{v_{2i-1}, v_{2i}\}$ and $\{v_{2i+3}, v_{2i+4}\}$ in G for any $i \ge 1$;

(iv) v_i and v_{i+6} are nonadjacent in G for any $i \ge 1$;

(v) v_i and v_j are distinct vertices in G whenever $0 < |i - j| < 12$;

(vi) For any $v \notin S$ and $i \ge 1$, if $N_S(v_{2i-1}) \subseteq N_S(v)$ and $N_S(v_{2i}) \cap N_S(v) \ne \emptyset$, or if $N_S(v_{2i}) \subseteq N_S(v)$ and $N_S(v_{2i-1}) \cap N_S(v) \ne \emptyset$, then v is nonadjacent to v_{2i-1} or v_{2i};

(vii) For any $v \notin S$ and $i \ge 1$, if $N_S(v) \ne N_S(v_j)$ holds for each j with $2i - 1 \le j \le 2i + 2$, then none of $\{v, v_{2i-1}, \ldots, v_{2i+2}\} - \{v_j\}$ induces a clique in G, for any j with $2i - 1 \le j \le 2i + 2$.

Proof. (i) It follows from (II.8)-(II.10) that $N_S(v_i) = N_S(v_{i-6})$ for each i with $7 \le i \le 12$. This periodicity together with (II.7) imply that $N_S(v_i) = N_S(v_{i-6})$ for each $i \ge 7$, and so (i) is proved.

(ii) From (II.8)-(II.10) and the construction of the sequence $\{v_i\}$, we conclude that $\{v_{2i-1}, v_{2i}, v_{2i+1}, v_{2i+2}\}$ induces a clique in G for each i with $1 \le i \le 5$. Suppose (ii) holds for $i \le k$. Let us proceed to the case $i = k+1$. Since $\{v_{2k-1}, v_{2k}, v_{2k+1}, v_{2k+2}\}$ is a clique, in view of statement (i), (II.7) and Lemma 3, we see that v_{2k+3} is adjacent to v_{2k+4}. Thus $\{v_{2k+1}, v_{2k+2}, v_{2k+3}, v_{2k+4}\}$ induces a clique in G, completing the proof of (ii).

(iii) From (II.5), (II.6) and (II.8)-(II.10), it can be seen that there is no edge between $\{v_{2i-1}, v_{2i}\}$ and $\{v_{2i+3}, v_{2i+4}\}$ in G for any $1 \le i \le 4$. Suppose (iii) holds for $i \le k$. Let us proceed to the case $i = k+1$. Since $\{v_{2k+1}, v_{2k+2}, v_{2k+3}, v_{2k+4}\}$ is a clique (by (ii)), in view of statement (i), (II.7) and Lemma 3, we see that there is no edge between $\{v_{2k+5}, v_{2k+6}\}$ and $\{v_{2k+1}, v_{2k+2}\}$. So (iii) holds.

(iv) Assume the contrary: $v_i v_{i+6}$ is an edge of G for some i. Consider the subgraph induced by $\{v_{i+4}v_{i+5}v_{i+6}; v_i, \{s_1, s_2, s_3, s_4\} - N_S(v_i)\}$ if i is odd and by $\{v_{i+3}v_{i+4}v_{i+6}; v_i, \{s_1, s_2, s_3, s_4\} - N_S(v_i)\}$ if i is even. In either case, by statements (i)-(iii), the subgraph is isomorphic to an F (we only need to check the case $i \le 6$), a contradiction.

(v) Suppose to the contrary that $v_i = v_j$ for some i and j with $0 < j - i \le 12$. Then by statement (i) and (II.1)-(II.6), we have $j = i + 6$. From statements (ii) and (iii), it follows that v_{i+6} is adjacent to v_{i+4}, while v_i is nonadjacent to v_{i+4}, contradicting the assumption $v_i = v_{i+6}$.

(vi) By (II.1)-(II.6) and statement (i), we have $N_S(v_{2i-1}) \cap N_S(v_{2i}) = \emptyset$, which, combined with the hypothesis, implies that $v \notin \{v_{2i-1}, v_{2i}\}$. Suppose to the contrary that $\{v_{2i-1}, v_{2i}, v\}$ is a clique of G. Then the configuration between S and any maximal clique containing $\{v, v_{2i-1}, v_{2i}\}$ is as described in Case 2 of Lemma 2; let S'_1, S'_2, S'_3, S'_4 be the corresponding partition of S. Then each of v, v_{2i-1} and v_{2i} is adjacent to precisely two of these sets, which implies that there exist two of S'_1, S'_2, S'_3, S'_4 whose union is $N_S(v_{2i-1})$, and the union of remaining two is $N_S(v_{2i})$. Now from the hypothesis, it can be seen that v has neighbors in at least three of S'_1, S'_2, S'_3, S'_4, contradicting Lemma 2.

(vii) Suppose the contrary: $\{v, v_{2i-1}, \dots, v_{2i+2}\} - \{v_j\}$ is a clique for some j with $2i-1 \le j \le 2i+2$. Let K be a maximal clique containing $\{v, v_{2i-1}, \dots, v_{2i+2}\} - \{v_j\}$. Since each vertex in S is adjacent to at most two of $\{v_{2i-1}, v_{2i}, v_{2i+1}, v_{2i+2}\}$, S and K are disjoint. Note that $N_S(v_k)$, for $2i-1 \le k \le 2i+2$, are distinct, so the configuration between S and K is as described in Case 2 of Lemma 2, which forces $N_S(v) = N_S(v_k)$ for some k with $2i-1 \le k \le 2i+2$, a contradiction. □

Recall that we construct the sequence $\{v_i\}$ in order to obtain a subgraph of G as depicted in Figure 2. To this end, let us further impose some restrictions on $\{v_i\}$.

(II.11) The construction of v_i for $i \geq 7$ goes as follows.

- If i is odd, then v_i is selected so that

  (a) $\{\{s_1, s_2, s_3, s_4\} - N_S(v_{i-6})\} \cup \{v_i, v_{i-1}, v_{i-2}\}$ induces an A in G, and

  (b) subject to (a), v_i is taken from $\{v_1, v_2, \ldots, v_{i-1}\}$ whenever possible;

- If i is even, then v_i is selected so that

  (a) $\{\{s_1, s_2, s_3, s_4\} - N_S(v_{i-6})\} \cup \{v_i, v_{i-2}, v_{i-3}\}$ induces an A in G, and

  (b) subject to (a), v_i is taken from $\{v_1, v_2, \ldots, v_{i-1}\}$ whenever possible.

Note that since Lemma 4 holds for any sequence constructed in (II.7), it remains valid for a sequence $\{v_i\}$ output by (II.11).

Lemma 5 *Every sequence $\{v_i\}$ constructed in (II.11) contains a set of 12 consecutive vertices $P = \{v_k, v_{k+1}, \ldots, v_{k+11}\}$, such that the subgraph induced by $S \cup P$ in G, denoted by Σ', can be obtained from Σ (depicted in Figure 2) by possibly adding some edges from $Q = \{v_{2i}v_{2i+5} : 1 \leq i \leq 6\}$, where the subscripts are taken modulo 12.*

Remark. We shall show in the next section that v_2 and v_7 are nonadjacent in G. Similarly, we can prove that no element of Q is an edge of G. Thus Σ' is nothing but Σ itself. However, we do not need this result in our proof.

Proof. Let us make some observations about sequence $\{v_i\}$. We propose to prove that

(5.1) There exist some i and j with $1 \leq i \neq j \leq 18$ such that $v_i = v_j$.

To verify (5.1), suppose the contrary: v_1, v_2, ..., v_{18} are distinct. Let E stand for the edge set of G. Consider v_{17}, we have

(5.2) $v_{17}v_8 \in E$, $v_{17}v_9 \notin E$, and $v_{17}v_{10} \notin E$.

Assume that $v_{17}v_9 \in E$. By Lemma 4(i), $\{s_2v_{17}, s_3v_{17}, s_3v_{14}\} \subseteq E$, and v_{14} is adjacent to neither s_1 nor s_2; by Lemma 4(iii), v_{14} is adjacent to neither v_9 nor v_{17}. Thus $\{v_{17}s_3v_9; s_2, v_{14}, s_1\}$ induces an F in G, a contradiction. Next, assume that $v_{17}v_8 \notin E$. By Lemma 4(i), v_{17} is adjacent to neither s_1 nor s_4, and $v_{17}s_3 \in E$. So $\{s_3v_8v_9; v_{17}, s_4, s_1\}$ is an F in G, a contradiction. Finally, assume that $v_{17}v_{10} \in E$. By Lemma 4(i), v_{17} is adjacent to both s_2 and s_3 but not to s_4, and v_{13} is adjacent to s_2 but not to s_3 or s_4; by Lemma 4(iii), $v_{13}v_{10} \notin E$ and $v_{13}v_{17} \notin E$. Thus $\{v_{17}s_2v_{10}; s_3, v_{13}, s_4\}$ induces an F in G, a contradiction.

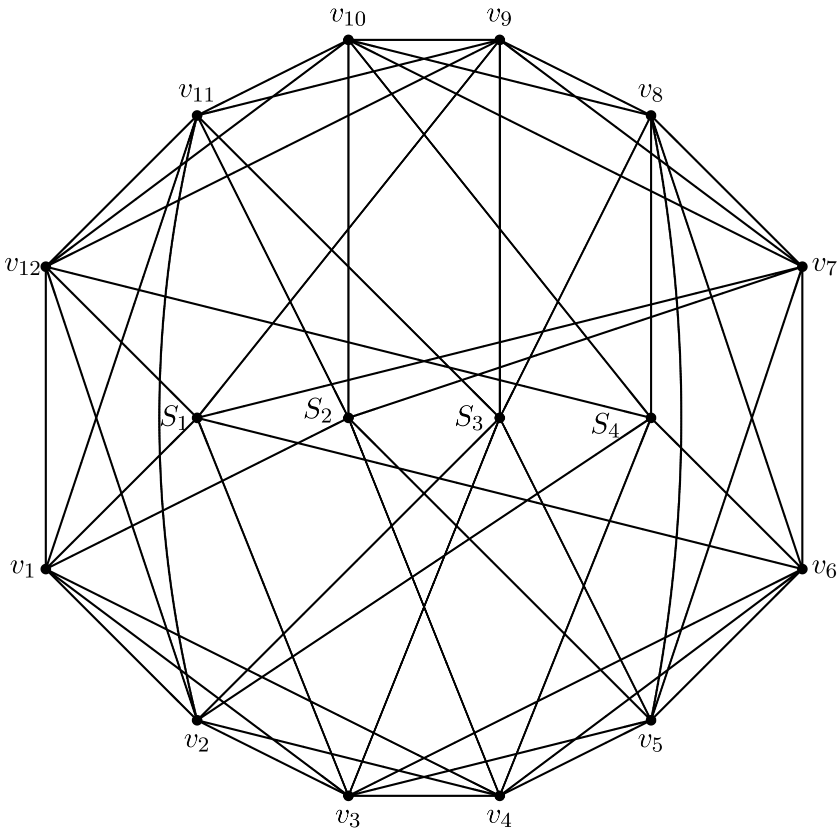

Figure 2. The Graph Σ

(5.3) v_{17} is adjacent to each of v_3, v_4, v_6 and v_7.

In view of (5.2), $v_{17}v_8 \in E$. By Lemma 4(i), v_{17} is adjacent to both s_2 and s_3 and nonadjacent to s_4; by Lemma 4(iii), $v_3v_8 \notin E$. Since $\{v_{17}s_3v_8; s_2, v_3, s_4\}$ does not induce an F in G, we have $v_{17}v_3 \in E$. Next, since $\{v_3s_1v_6; v_{17}, v_9, s_4\}$ is not isomorphic to an F, by (5.2) and Lemma 4, we obtain $v_{17}v_6 \in E$. Then, since $\{v_6v_7v_8; v_{10}, v_{17}, s_1\}$ is not an $\bar{F}$ in G, by (5.2) and Lemma 4, we get $v_{17}v_7 \in E$. Finally, since $\{v_{17}s_2v_7; s_3, v_4, s_1\} \neq F$, we have $v_{17}v_4 \in E$.

Now let us turn to consider v_{13}.

(5.4) v_{13} is adjacent to each of v_2, v_3 and v_4, but not adjacent to v_5.

Since $\{v_{17}s_2v_4; s_3, v_{13}, s_4\} \neq F$, Lemma 4 and (5.3) imply that $v_{13}v_4 \in E$. Notice that $\{v_{13}s_2v_5; s_1, v_{10}, s_3\} \neq F$, so we have $v_{13}v_5 \notin E$. From $\{s_2v_4v_5; v_3, v_7, v_{13}\} \neq \bar{F}$, it follows that $v_{13}v_3 \in E$. Since $\{v_2v_3v_4; v_{13}, s_4, s_3\} \neq \bar{F}$, we get $v_{13}v_2 \in E$.

Next let us derive some adjacency properties concerning v_{11}.

(5.5) $v_{11}v_2 \in E$ and $v_{11}v_3 \notin E$.

Using $\{v_{11}v_3s_3; s_2, s_1, v_8\} \neq F$, we get $v_{11}v_3 \notin E$. Since $\{v_{13}v_2v_3; v_{11}, s_4, v_5\} \neq F$, by virtue of Lemma 4 and (5.4), we have $v_{11}v_2 \in E$.

Now we are ready to complete the proof of (5.1).

Observe that $v_{12}v_2 \notin E$; otherwise, since $\{\{s_1, s_2, s_3, s_4\} - N_S(v_8)\} \cup \{v_2, v_{12}, v_{11}\}$ induces an A in G, by (II.11), v_2 is a candidate for v_{14}, and thus v_{14} is identical with some vertex v_i with $1 \leq i \leq 13$, contradicting our assumption that v_1, v_2, $\ldots$, v_{18} are distinct. Since $\{v_{10}v_{11}v_{12}; v_8, v_2, s_1\} \neq F$, we see that v_{10} is adjacent to v_2, which, together with Lemma 4, implies that $\{v_{10}v_2s_4; s_2, s_3, v_6\}$ induces an F in G, a contradiction. So (5.1) is established.

Now (5.1) and Lemma 4(i) and (v) guarantee the existence of some subscript i, $1 \leq i \leq 6$, such that $v_i = v_{i+12}$. Without loss of generality, we may assume $i = 1$ or 2. (To see this, we may resort to the following plain trick: in case $i = 3$ or 4, rename original S_1 as S_2, S_2 as S_3, S_3 as S_1, keep S_4 unchanged, and rename the original v_t as v_{t-2} for $t \geq 3$. The case when $i = 5$ or 6 can be handled likewise.) Now let us proceed to establish the following stronger result.

(5.6) There exists an *odd* subscript k, $1 \leq k \leq 6$, such that $v_k = v_{k+12}$ and $v_{k+1} = v_{k+13}$.

Assume the contrary: no such k exists. Let us distinguish between two cases according to the value of i.

Case 5.1. $i = 1$, that is $v_{13} = v_1$. By assumption, we have $v_{14} \neq v_2$.

(5.7) v_{14} is adjacent to v_4.

Assume $v_{14}v_4 \notin E$. In view of Lemma 4 and the hypothesis $v_{13} = v_1$, no element in $\{v_1v_6, v_1v_{10}, v_4v_{10}, v_6v_{10}, v_{10}v_{14}, s_2v_6, s_2v_{14}\}$ is an edge of G. Since $\{v_1v_4s_2; v_{14}, v_6, v_{10}\} \neq F$, v_{14} is adjacent to v_6. Next, since $\{v_5v_6v_8; s_2, v_{14}, v_9\}$ is not isomorphic to an F, Lemma 4 implies that $v_{14}v_5 \in E$. Thus by Lemma 4, $\{s_3v_{11}v_{14}; v_1, v_5, v_9\}$ induces an $\bar{F}$ in G, a contradiction.

(5.8) v_{14} is nonadjacent to v_3.

Suppose the contrary, then v_{14} is adjacent to both v_3 and v_4 (by (5.7)). Since each of $\{\{s_1, s_2, s_3, s_4\} - N_S(v_9)\} \cup \{v_3, v_{14}, v_1\}$ and $\{\{s_1, s_2, s_3, s_4\} - N_S(v_{10})\} \cup \{v_4, v_{14}, v_1\}$ induces an A in G. According to (II.11), $v_{15} = v_3$ and $v_{16} = v_4$. So (5.6) holds for $k = 3$, contradicting our assumption.

Based on (5.7), (5.8), Lemma 4 and the fact $\{v_1v_3v_4; v_5, v_{14}, s_1\} \neq \bar{F}$, we get $v_{14}v_5 \in E$. Since $\{s_3v_5v_8; v_6, v_9, v_{14}\}$ is not isomorphic to an $\bar{F}$, v_{14} is adjacent to v_6. From the fact $\{v_3v_4v_6; v_{14}, s_1, v_2\} \neq \bar{F}$, it follows that $v_{14}v_2 \in E$. Hence $\{v_5v_6v_{14}; s_2, s_1, v_2\}$ induces an F in G, a contradiction.

Case 5.2. $i = 2$, that is $v_{14} = v_2$. By assumption, we have $v_{13} \neq v_1$.

(5.9) v_{13} is adjacent to v_4

Suppose to the contrary that $v_{13}v_4 \notin E$. Since $\{s_2v_4v_5; v_{13}, s_4, s_3\} \neq F$, we have $v_{13}v_5 \in E$. Now from Lemma 4, we can see that $\{s_2v_5v_{13}; v_{10}, s_3, s_1\}$ induces an F in G, a contradiction.

(5.10) v_{13} is nonadjacent to v_3.

Otherwise, imitating the proof of (5.8), we obtain $v_{15} = v_3$ and $v_{16} = v_4$, contradicting our assumption.

Now (5.9), (5.10) and Lemma 4, together with the fact $\{v_2v_3v_4; v_6, v_{13}, s_3\} \neq \bar{F}$, imply that $v_{13}v_6 \in E$. Since $\{v_{13}s_2v_5; s_1, v_{10}, s_3\} \neq F$, $v_{13}v_5 \notin E$. It follows that $\{v_5v_6v_7; s_3, v_{13}, v_{10}\}$ induces an F in G, a contradiction. So the proof of (5.6) is complete.

From (5.6) and Lemma 4, our lemma follows. □

3 Further Preparation

Let Σ' be the graph specified in Lemma 5. Recall that Σ' can be obtained from Σ depicted in Figure 2 by adding possibly some edges from $Q = \{v_{2i}v_{2i+5} : 1 \leq i \leq 6\}$. For convenience, the reader may use Σ in place of Σ' in our proof, except when adjacency concerning the set Q is involved. Thus we need not refer to Lemma 4 again and again.

Lemma 6 *There exists a vertex α outside Σ' such that*
(i) α is adjacent to each vertex in $S_1 \cup S_2 \cup S_3 \cup \{v_{2i-1} : 1 \leq i \leq 6\}$, and
(ii) α is adjacent to no vertex in $\{v_{2i} : 1 \leq i \leq 6\}$.

Proof. Since $v_6v_3v_1v_{12}$ is a P_4 in G, by hypothesis it is contained in an A; let α be the fifth vertex of this A. We aim to prove that α satisfies both (i) and (ii). To this end, note that clearly α is outside Σ'. Moreover, α is adjacent to each vertex in S_1, for otherwise let s_1 be a vertex in S_1 with $\alpha s_1 \notin E$. Then $\{s_1v_1v_3; \alpha, v_6, v_{12}\}$ would induce an $\bar{F}$ in G, a contradiction.

The proof is by contradiction. Assume the contrary: α is not as desired. Since $\{s_1v_7v_9; \alpha, s_2, s_3\} \neq F$ for any $s_2 \in S_2$ and $s_3 \in S_3$, one of the following four cases must occur:

- Case 6.1. α is adjacent to each vertex in S_2;
- Case 6.2. α is adjacent to each vertex in S_3;
- Case 6.3. α is adjacent to v_7;
- Case 6.4. α is adjacent to v_9.

We shall reach a contradiction in each case.

Case 6.1. α is adjacent to each vertex in S_2. Observe that in this case α is adjacent to v_7 or v_{10} for otherwise $\{s_2v_7v_{10}; \alpha, v_6, v_{12}\}$ would be an F, a contradiction. Let us distinguish between two subcases.

Subcase 6.1.a) α is adjacent to v_7. Since $\{v_7v_9v_{10}; \alpha, s_3, s_4\} \neq F$ for any $s_3 \in S_3$ and $s_4 \in S_4$, we have four possibilities to consider:

(6.1.1) α is adjacent to each vertex in S_3;

(6.1.2) α is adjacent to each vertex in S_4;

(6.1.3) α is adjacent to v_9;

(6.1.4) α is adjacent to v_{10}.

Now let us analyse them one by one.

(6.1.1) α is adjacent to each vertex in S_3. By Lemma 4(vi), α is adjacent to none of v_2, v_4, and v_8 in G, and so α is adjacent to v_9 since $\{s_3v_8v_9; \alpha, v_6, v_{12}\} \neq F$. Now Lemma 4(vi) ensures that α is nonadjacent to v_{10}. It follows that α is adjacent to v_{11} since $\{s_2v_7v_{10}; v_8, v_{11}, \alpha\} \neq \bar{F}$, and α is adjacent to v_5 since $\{s_3v_5v_8; \alpha, v_4, v_{10}\} \neq F$. Since α is the fifth vertex of an A that contains $v_6v_3v_1v_{12}$, we conclude that α is adjacent to each vertex in $S_1 \cup S_2 \cup S_3 \cup \{v_{2i-1} : \ 1 \leq i \leq 6\}$, but to no vertex in $\{v_{2i} : \ 1 \leq i \leq 6\}$, contradicting our assumption that α is not as desired.

So we may assume that α is nonadjacent to some vertex $s_3 \in S_3$.

(6.1.2) α is adjacent to each vertex in S_4. By Lemma 4(vi), α is adjacent to neither v_4 nor v_8. Thus α is adjacent to v_{10} since $\{v_7v_8v_{10}; \alpha, s_3, v_{12}\} \neq F$. It follows that $\{s_2v_{10}\alpha; v_4, v_8, s_1\}$ induces an F in G, a contradiction.

So we may assume that α is nonadjacent to some vertex $s_4 \in S_4$.

(6.1.3) α is adjacent to v_9. Note that α is adjacent to v_8 or v_{10} since $\{v_7v_8v_{10}; \alpha, s_3, v_{12}\} \neq F$. Consider the subgraph of G induced by $\{\alpha v_8v_9; s_2, v_6, v_{12}\}$ in the former case and by $\{\alpha v_9v_{10}; v_1, s_3, s_4\}$ in the latter. We have an F in either case, a contradiction.

So we may assume that α is nonadjacent to v_9.

(6.1.4) α is adjacent to v_{10}. We have $\{s_1v_9v_{12}; \alpha, s_3, s_4\} = F$ as α is adjacent to none of s_3, s_4 and v_9, a contradiction.

Subcase 6.1.b) α is adjacent to v_{10}. Clearly we may assume that αv_7 is not an edge of G in this subcase.

Observe that α is nonadjacent to some vertex $s_3 \in S_3$, for otherwise α is adjacent to each vertex in $S_1 \cup S_2 \cup S_3$. Hence, by Lemma 4(vi), α is nonadjacent to v_4, and thus $\{\alpha s_2v_{10}; s_3, v_4, v_{12}\}$

induces an F in G, a contradiction. The above observation together with the fact $\{v_7v_9v_{10}; v_6, s_3, \alpha\} \neq F$, for any $s_3 \in S_3$, yield that α is adjacent to v_9. Since $\{\alpha, v_9, v_{10}\}$ is a clique, by Lemma 4(vi), α is nonadjacent to some vertex $s_4 \in S_4$. We can thus deduce that $\{\alpha v_9v_{10}; v_1, s_3, s_4\}$ is isomorphic to an F, a contradiction.

So we can assume α is nonadjacent to some vertex $s_2 \in S_2$ hereafter.

Case 6.2. α is adjacent to each vertex in S_3. In view of the existence of s_2, we conclude that $\{\alpha v_1s_1; s_3, s_2, v_6\}$ induces an F, a contradiction.

So we can assume α is nonadjacent to some vertex $s_3 \in S_3$ hereafter.

Case 6.3. α is adjacent to v_7. Note that in this case α is adjacent to v_8 or v_{10}, for otherwise $\{v_7v_8v_{10}; \alpha, s_3, v_{12}\}$ would be an F in G, a contradiction. Let us distinguish between two subcases.

Subcase 6.3.a) α is adjacent to v_8. Since $\{\alpha v_8v_{10}; s_1, s_3, s_2\} \neq F$, we deduce that α is nonadjacent to v_{10}. Next, since $\{v_7v_8v_{10}; s_4, s_2, \alpha\} \neq \bar{F}$ for any $s_4 \in S_4$, we conclude that α is adjacent to each vertex in S_4. Thus $N_S(\alpha) \neq N_S(v_i)$ for any i with $1 \leq i \leq 4$ and hence, by Lemma 4(vii), $\{\alpha, v_1, v_3, v_4\}$ is not a clique, so α is nonadjacent to v_4. It follows that $\{\alpha s_4v_8; s_1, v_4, s_3\}$ is an F, a contradiction.

Subcase 6.3.b) α is adjacent to v_{10}. Clearly we may assume that αv_8 is not an edge of G. Since $\{v_2v_3s_3; v_{12}, \alpha, v_8\} \neq F$, we see that α is adjacent to v_2. Since $\{\alpha v_1v_3; v_9, s_2, v_6\} \neq F$, we conclude that α is nonadjacent to v_9. Hence α is adjacent to each vertex $s_4 \in S_4$ for otherwise $\{v_2s_4v_{12}; \alpha, v_6, v_9\}$ would be an F in G. Therefore $\{s_4v_8v_{10}; v_9, \alpha, v_6\}$ induces an $\bar{F}$ in G, a contradiction.

Case 6.4. α is adjacent to v_9. Since α is nonadjacent to s_2, $\{\alpha v_1v_3; v_9, s_2, v_6\}$ induces an F in G, a contradiction.

This completes the proof of Lemma 6. □

Lemma 7 *There exists a vertex β outside $\Sigma' \cup \{\alpha\}$ such that*
(i) β is adjacent to each vertex in $S_1 \cup S_2 \cup S_4 \cup \{v_1, v_4, v_6, v_7, v_{10}, v_{12}\}$, and
(ii) β is adjacent to no vertex in $\{v_2, v_3, v_5, v_8, v_9, v_{11}, \alpha\}$.

Proof. Let β be the fifth vertex of an A that contains $v_5v_4v_1v_{11}$. Then it can be shown by case analysis that β is as desired. However, we have a quick proof of the present lemma.

Rename the vertices of Σ' so that

- the original v_1 becomes v_{11}, original v_2 becomes v_{12}, v_7 becomes v_5, v_8 becomes v_6;
- each of the remaining original v_i becomes v_{13-i}, and that
- the original S_j becomes S_{j+1}, where $1 \leq j \leq 4$ and the subscript is taken modulo 4.

Then all the statements except the one concerning the adjacency between α and β can be deduced from Lemma 6. From $\{s_1\alpha\beta; s_2, v_{12}, v_3\} \neq \bar{F}$, it follows that α and β are nonadjacent, completing the proof. □

Lemma 8 *Vertex v_2 is nonadjacent to v_7 in G.*

Proof. Suppose the contrary: v_2v_7 is an edge of G, we aim to reach a contradiction. Let η be the fifth vertex of an A that contains $v_9v_7v_2v_1$. It is easy to see that $\eta \notin \Sigma' \cup \{\alpha, \beta\}$. Since $\{v_7v_9v_{10}; \eta, s_3, s_4\} \neq F$ for any $s_3 \in S_3$ and $s_4 \in S_4$, one of the following three cases must occur:

- Case 8.1. η is adjacent to each vertex in S_3;
- Case 8.2. η is adjacent to each vertex in S_4;
- Case 8.3. η is adjacent to v_{10}.

Let us deal with these cases separately.

Case 8.1. η is adjacent to each vertex in S_3.

Since $\{s_3v_2v_3; v_1, v_5, \eta\} \neq \bar{F}$, η is adjacent to v_5 or to v_3. Let us distinguish between two subcases.

Subcase 8.1.a) η is adjacent to v_5.

We claim that η is nonadjacent to v_6. To justify this, assume the contrary, then by Lemma 4(vii), η is nonadjacent to some $s_1 \in S_1$ since $\{v_5, v_6, v_7, \eta\}$ induces a clique in G. Since $\{s_1v_6v_7; \eta, v_9, v_3\} \neq \bar{F}$, η is adjacent to v_3. Note that $\{\eta, v_3, v_5, v_6\}$ is a clique and $N_S(\eta) \neq N_S(v_i)$ for any $i \in \{3, 4, 6\}$, by virtue of Lemma 4(vii), we have $N_S(\eta) = N_S(v_5) = S_2 \cup S_3$. It follows that $N_S(\eta) \neq N_S(v_i)$ for any i with $1 \leq i \leq 4$. By Lemma 4(vii), $\{\eta, v_2, v_3, v_4\}$ is not a clique, and so η is nonadjacent to v_4. Hence we have $\{\eta s_2v_7; s_3, v_4, s_1\} = F$, a contradiction and thus the claim is justified. Since $\{s_3v_5v_8; v_6, v_9, \eta\} \neq \bar{F}$, we see that η is adjacent to v_8. Next, observe that η is nonadjacent to some $s_2 \in S_2$; otherwise, since $\{\eta s_3v_2; s_2, v_9, s_4\} \neq F$ for any $s_4 \in S_4$, η is adjacent to each vertex in S_4. Thus η is adjacent to each vertex in $S_2 \cup S_3 \cup S_4$, while $\{\eta, v_7, v_8\}$ is a clique, contradicting Lemma 4(vi). Since $\{\eta s_3v_5; s_4, v_9, s_2\} \neq F$ for any

$s_4 \in S_4$, η is adjacent to no vertex in S_4. Since $\{v_6v_7v_8; \eta, s_4, s_1\} \neq \bar{F}$ for any $s_1 \in S_1$, η is adjacent to each vertex in S_1; since $\{\eta s_3v_8; s_1, v_{11}, s_4\} \neq F$, η is adjacent to v_{11}; and since $\{\eta s_1v_7; s_3, v_{12}, s_2\} \neq F$, η is adjacent to v_{12}. Thus $\{\eta, v_2, v_{11}, v_{12}\}$ is a clique and $N_S(\eta) \neq N_S(v_i)$ for any $i \in \{11, 12, 1, 2\}$, contradicting Lemma 4(vii).

Subcase 8.1.b) η is adjacent to v_3, but nonadjacent to v_5. Notice that η is adjacent to no vertex in S_4 since $\{\eta s_3v_3; s_4, v_9, v_1\} \neq F$ for any $s_4 \in S_4$. Thus η is adjacent to no vertex in S_1 since $\{\eta s_3v_2; s_1, v_5, s_4\} \neq F$ for any $s_1 \in S_1$ and $s_4 \in S_4$. Next, note that η is adjacent to v_{11} since $\{v_1v_2v_3; \eta, s_1, v_{11}\} = \bar{F}$, where $s_1 \in S_1$. We claim that η is nonadjacent to v_8, for otherwise, η is adjacent to v_6 since $\{v_6v_7v_8; \eta, s_4, s_1\} \neq \bar{F}$. Then η is adjacent to each vertex in S_2 since $\{\eta v_6v_7; s_3, s_4, s_2\} \neq F$ for any $s_2 \in S_2$. But then we get $\{\eta s_3v_2; s_2, v_9, s_4\} = F$, a contradiction and thus our claim is proved. Since $\{s_3v_9v_{11}; v_{12}, \eta, v_8\} \neq \bar{F}$, η is adjacent to v_{12}. Thus η is adjacent to each vertex in S_2 since $\{v_1v_{11}v_{12}; \eta, s_1, s_2\} \neq \bar{F}$ for any $s_2 \in S_2$. It follows that $\{\eta s_3v_2; s_2, v_9, s_4\}$ induces an F in G, where $s_2 \in S_2$ and $s_4 \in S_4$, a contradiction.

So we may assume that η is nonadjacent to some vertex $s_3 \in S_3$.

Case 8.2. η is adjacent to each vertex in S_4.

Since $\{v_2s_4v_{12}; v_{10}, v_1, \eta\} \neq \bar{F}$, where $s_4 \in S_4$, we see that η is adjacent to v_{10} or to v_{12}. Let us distinguish between two subcases.

Subcase 8.2.a) η is adjacent to v_{10}.

In this subcase, η is adjacent to each vertex in S_1 or to v_{11} since $\{v_7v_9v_{10}; v_{11}, \eta, s_1\} \neq \bar{F}$ for any $s_1 \in S_1$.

(8.2.1) η is adjacent to each vertex in S_1. We claim that η is nonadjacent to some vertex $s_2 \in S_2$, for otherwise by Lemma 4(vi) with respect to $\{v_7, v_8\}$, we can see that η is nonadjacent to v_8. Note that η is nonadjacent to v_5 since $\{\eta s_2v_5; s_4, v_1, s_3\} \neq F$, so η is adjacent to v_4 since $\{s_2v_5v_7; v_8, \eta, v_4\} \neq \bar{F}$. Since η is adjacent to each vertex in $S_1 \cup S_2 \cup S_4$, by Lemma 4(vi), η is nonadjacent to v_3. Since $\{v_4v_6s_4; v_8, \eta, v_3\} \neq \bar{F}$, η is adjacent to v_6. But then $\{s_1v_3v_6; v_5, \eta, v_1\} = \bar{F}$, a contradiction. So the claim is justified. From $\{\eta s_1v_7; s_4, v_3, s_2\} \neq F$, it follows that η is adjacent to v_3.

Since η is adjacent to each vertex in $S_1 \cup S_4$, we have $N_S(\eta) \neq N_S(v_i)$ for any i with $1 \leq i \leq 4$ or with $7 \leq i \leq 10$. Thus, by Lemma 4(vii), η is adjacent to neither v_8 nor v_4. Thus η is adjacent to v_{12} since $\{s_1v_7v_9; v_8, v_{12}, \eta\} \neq \bar{F}$. It follows that η is adjacent to v_{11} since $\{v_1v_{11}v_{12}; v_4, s_3, \eta\} \neq F$. In summary, we have

- η is adjacent to each vertex in $\{s_1, s_4, v_3, v_{10}, v_{11}, v_{12}\}$, and
- η is adjacent to no vertex in $\{s_2, s_3, v_1, v_4, v_8\}$.

Now let α and β be as specified in Lemma 6 and Lemma 7, respectively. Recall that α is nonadjacent to β. Since $\{v_{11}\alpha\eta; s_1, v_{10}, s_3\} \neq \bar{F}$ and $\{s_4\eta\beta; v_8, v_3, s_2\} \neq F$, we conclude that neither of $\alpha\eta$ and $\beta\eta$ is an edge of G. Thus $\{v_1v_{11}v_{12}; \eta, \beta, \alpha\} = \bar{F}$, a contradiction.

So we may assume that η is nonadjacent to some vertex $s_1 \in S_1$.

(8.2.2) η is adjacent to v_{11}. We claim that η is nonadjacent to v_6, otherwise, since η is adjacent to neither of s_1 and s_3, we have $N_S(\eta) \neq N_S(v_i)$ for any i with $5 \leq i \leq 8$. It follows from Lemma 4(vii) that η is adjacent to neither v_5 nor v_8. Thus $\{v_8v_9v_{10}; v_5, s_1, \eta\}$ induces an F in G, a contradiction and thus our claim is verified. Since $\{s_4v_8v_{10}; v_9, \eta, v_6\} \neq \bar{F}$, η is adjacent to v_8. Note that $\{\eta, v_7, v_8, v_{10}\}$ is a clique and $N_S(\eta) \neq N_S(v_i)$ for any i with $7 \leq i \leq 9$, by Lemma 4(vii), we have $N_S(\eta) = N_S(v_{10}) = S_2 \cup S_4$. Hence $N_S(\eta) \neq N_S(v_i)$ for any $i \in \{11, 12, 1, 2\}$. Once again, by Lemma 4(vii), η is nonadjacent to v_{12}. Thus $\{\eta s_4v_8; s_2, v_{12}, s_3\} = F$, a contradiction.

Subcase 8.2.b) η is adjacent to v_{12}.

In view of Subcase 8.2.a), we may assume that η is nonadjacent to v_{10}. Since $\{v_9v_{10}v_{12}; s_3, s_2, \eta\} \neq F$ for any $s_2 \in S_2$, η is adjacent to each vertex in S_2. So η is adjacent to each vertex in $S_2 \cup S_4$ and thus $N_S(\eta) \neq N_S(v_i)$ for any $i \in \{11, 12, 1, 2\}$. By Lemma 4(vii), η is nonadjacent to v_{11}. Since $\{\eta s_2v_7; s_4, v_{11}, s_1\} \neq F$ for any $s_1 \in S_1$, η is adjacent to each vertex in S_1. It follows that $\{\eta s_4v_2; s_1, v_{10}, s_3\} = F$, a contradiction.

So we may assume that η is nonadjacent to some vertex $s_4 \in S_4$.

Case 8.3. η is adjacent to v_{10}.

Since $\{v_7v_9v_{10}; v_{11}, \eta, s_1\} \neq \bar{F}$ for any $s_1 \in S_1$, η is adjacent to each vertex in S_1, or to v_{11}. We distinguish between two subcases.

Subcase 8.3.a) η is adjacent to each vertex in S_1.

We claim that η is nonadjacent to v_8; otherwise, since $\{\eta, v_7, v_8, v_{10}\}$ is a clique, by Lemma 4(vii), $N_S(\eta) = N_S(v_i)$ for some i with $7 \leq i \leq 10$. Since η is adjacent to neither of s_3 and s_4, $N_S(\eta) = N_S(v_7) = S_1 \cup S_2$. From $\{s_2v_{10}v_{11}; v_9, v_1, \eta\} \neq \bar{F}$ and $\{v_8v_9v_{10}; v_{12}, \eta, s_3\} \neq \bar{F}$, we deduce that η is adjacent to both of v_{11} and v_{12}. Thus $\{\eta, v_{10}, v_{11}, v_{12}\}$ is a clique and $N_S(\eta) \neq N_S(v_i)$ for any i with $9 \leq i \leq 12$, contradicting Lemma 4(vii) and so the claim is proved. Since $\{s_1v_7v_9; v_8, v_{12}, \eta\} \neq \bar{F}$, η is adjacent to v_{12}. Note that $N_S(\eta) \neq N_S(v_i)$ for any i

with $9 \leq i \leq 12$ as η is adjacent to none of s_3 and s_4, by Lemma 4(vii) η is nonadjacent to v_{11}, and so $\{v_7v_8v_9; \eta, s_4, v_{11}\} = F$, a contradiction.

Subcase 8.3.b) η is adjacent to v_{11} but nonadjacent to some vertex $s_1 \in S_1$.

Since $\{s_3v_2v_{11}; \eta, v_9, v_3\} \neq \bar{F}$, η is adjacent to v_3. Note that $N_S(\eta) \neq N_S(v_i)$ for any i with $1 \leq i \leq 4$ as η is adjacent to none of s_1, s_3 and s_4. By Lemma 4(vii), η is nonadjacent to v_4. It follows that η is adjacent to v_5 as $\{v_2, v_3, v_4; v_5, s_4, \eta\} \neq \bar{F}$, and then η is adjacent to v_6 as $\{v_5, v_6, v_7; s_1, \eta, v_4\} \neq \bar{F}$. Therefore, $\{\eta, v_5, v_6, v_7\}$ is clique and $N_S(\eta) \neq N_S(v_i)$ for any i with $5 \leq i \leq 8$, contradicting Lemma 4(vii).

The proof of Lemma 8 is complete. □

4 Proof of The Theorem

Let s_4 be a vertex in S_4. From Lemmas 7 and 8 we deduce that $v_2s_4\beta v_7$ is a P_4, so by hypothesis it is contained in an A; let π be the fifth vertex of this A. It is easy to see that $\pi \notin \Sigma' \cup S \cup \{\alpha, \beta\}$. Since $\{s_4v_8v_{10}; \pi, s_3, s_2\} \neq F$ for any $s_2 \in S_2$ and $s_3 \in S_3$, one of the following four cases must occur:

- Case 1. π is adjacent to each vertex in S_2;
- Case 2. π is adjacent to each vertex in S_3;
- Case 3. π is adjacent to v_8;
- Case 4. π is adjacent to v_{10}.

We shall reach a contradiction in each case.

Case 1. π is adjacent to each vertex in S_2.

Since $\{s_2v_{11}v_1; \pi, s_3, s_1\} \neq F$ for any $s_1 \in S_1$ and $s_3 \in S_3$, we can distinguish among the following four subcases:

- Subcase 1.a) π is adjacent to each vertex in S_1;
- Subcase 1.b) π is adjacent to each vertex in S_3;
- Subcase 1.c) π is adjacent to v_{11};
- Subcase 1.d) π is adjacent to v_1.

Let us handle these subcases separately.

Subcase 1.a) π is adjacent to each vertex in S_1.

Since $\{s_1v_9v_{12}; \pi, v_8, v_2\} \neq F$ where $s_1 \in S_1$, we see that π is adjacent to v_8, or to v_9, or to v_{12}.

(1) π is adjacent to v_8. Note that π is adjacent to each vertex in S_3, or to v_4 since $\{\pi s_4 v_8; s_1, v_4, s_3\} \neq F$ for any $s_3 \in S_3$.

If π is adjacent to each vertex in S_3, then π is adjacent to each vertex in $S_1 \cup S_2 \cup S_3$, and by Lemma 4(vi), π is nonadjacent to at least one of v_{2i-1} and v_{2i} for any i with $1 \leq i \leq 6$; this observation will be used repeatedly in this paragraph. We claim that π is nonadjacent to v_3, for otherwise, according to the above observation π is nonadjacent to v_4. So π is adjacent to v_{11} for $\{s_3 v_2 v_3; v_4, \pi, v_{11}\} \neq \bar{F}$; again, by the observation, π is nonadjacent to v_{12}. Thus $\{\pi s_2 v_{11}; v_8, v_4, v_{12}\} = F$, a contradiction. So the claim is justified. Since $\{s_3 v_8 v_5; v_7, v_3, \pi\} \neq \bar{F}$, π is adjacent to v_5; once again by the observation, π is nonadjacent to v_6. Thus $\{v_3 v_5 s_3; \pi, v_2, v_6\} = \bar{F}$, a contradiction.

Suppose π is adjacent to v_4 but nonadjacent to some vertex $s_3 \in S_3$. Note that π is nonadjacent to v_3 for $\{\pi s_1 v_3; s_4, v_7, s_3\} \neq F$. Thus π is adjacent to v_{12} for $\{v_2 v_4 s_4; \pi, v_{12}, v_3\} \neq \bar{F}$. But then $\{v_{12} s_4 \pi; v_8, s_1, v_2\} = \bar{F}$, a contradiction.

So we may assume that π is nonadjacent to v_8.

(2) π is adjacent to v_9. Since $\{\pi s_1 v_9; s_2, v_3, v_8\} \neq F$, π is adjacent to v_3. So π is adjacent to each vertex in S_3 since $\{\pi s_1 v_3; s_4, v_7, s_3\} \neq F$ for any $s_3 \in S_3$. Hence π is adjacent to each vertex in $S_1 \cup S_2 \cup S_3$, and Lemma 4(vi) ensures that π is nonadjacent to v_{10}. Since $\{s_1 v_7 v_9; v_{10}, \pi, v_6\} \neq \bar{F}$, π is adjacent to v_6, and thus, by Lemma 4(vi), π is nonadjacent to v_5. Hence $\{s_4 v_6 v_8; v_5, v_{10}, \pi\} = \bar{F}$, a contradiction.

So we may assume that π is nonadjacent to v_9.

(3) π is adjacent to v_{12}. Since $\{v_{12} s_1 v_9; v_7, v_{11}, \pi\} \neq \bar{F}$, π is adjacent to v_{11}. Note that π is adjacent to each vertex in $S_1 \cup S_2 \cup \{v_{11}, v_{12}\}$; by Lemma 4(vi), π is nonadjacent to some vertex $s_3 \in S_3$. Thus $\{\pi s_2 v_{11}; s_4, v_7, s_3\} = F$, a contradiction.

So we may assume that π is nonadjacent to some vertex $s_1 \in S_1$.

Subcase 1.b) π is adjacent to each vertex in S_3.

Since $\{\pi s_4 v_6; s_2, v_2, s_1\} \neq F$, π is nonadjacent to v_6. We claim that π is adjacent to v_{12}; otherwise, since $\{s_3 v_2 v_3; \pi, v_{12}, v_6\} \neq F$, π is adjacent to v_3. Notice that $\{\pi s_3 v_3; s_4, v_{11}, s_1\} \neq F$, so π is adjacent to v_{11}. Thus, by Lemma 7, we get $\{s_2 \beta \pi; s_4, v_{11}, v_7\} = \bar{F}$, a contradiction. So our claim is justified. Since π is adjacent to each vertex in $S_2 \cup S_3 \cup \{s_4\}$, Lemma 4(vi) implies that π is nonadjacent to v_{11}. So π is adjacent to v_4 since $\{v_{12} v_2 s_4; v_4, \pi, v_{11}\} \neq \bar{F}$. It follows that π is adjacent to each vertex $s_4' \in S_4$ since $\{\pi s_2 v_4; s_3, v_7, s_4'\} \neq F$. Hence, by Lemma 4(vi) with respect to $\{v_3, v_4\}$, π is nonadjacent to v_3 since π is adjacent to each vertex in

$S_2 \cup S_3 \cup S_4$. Thus π is adjacent to v_8 since $\{s_4v_4v_6; v_3, v_8, \pi\} \neq \bar{F}$, and so π is adjacent to v_5 since $\{s_3v_8v_5; v_7, v_3, \pi\} \neq \bar{F}$. It follows that $\{s_3v_3v_5; v_6, \pi, v_2\} = \bar{F}$, a contradiction.

So we may assume that π is nonadjacent to some $s_3 \in S_3$.

Subcase 1.c) π is adjacent to v_{11}.

Since π is nonadjacent to s_3, we have $\{\pi s_2v_{11}; s_4, v_7, s_3\} = F$, a contradiction.

So we may assume that π is nonadjacent to v_{11}.

Subcase 1.d) π is adjacent to v_1. Since $\{\pi s_2v_1; s_4, v_5, s_1\} \neq F$, π is adjacent to v_5. Since $\{\pi s_4v_6; s_2, v_2, s_1\} \neq F$, π is nonadjacent to v_6. So π is adjacent to v_{10} since $\{s_2v_5v_7; v_6, v_{10}, \pi\} \neq \bar{F}$. It follows that π is adjacent to v_8 since $\{v_7v_8v_{10}; s_1, s_3, \pi\} \neq F$, and thus π is adjacent to v_{12} since $\{\pi s_4v_8; s_2, v_{12}, s_3\} \neq F$. Since $\{s_4v_{12}v_2; v_{11}, v_4, \pi\} \neq \bar{F}$, π is adjacent to v_4, and thus we have $\{\pi v_5v_8; s_3, v_{10}, v_4\} = \bar{F}$, a contradiction.

So we may assume hereafter that π is nonadjacent to some $s_2 \in S_2$.

Case 2. π is adjacent to each vertex in S_3.

Since $\{s_3v_3v_5; \pi, s_1, s_2\} \neq F$ for any $s_1 \in S_1$, we can distinguish among the following three subcases:

- Subcase 2.a) π is adjacent to each vertex in S_1;
- Subcase 2.b) π is adjacent to v_3;
- Subcase 2.c) π is adjacent to v_5.

Let us deal with them separately.

Subcase 2.a) π is adjacent to each vertex in S_1.

Since $\{s_3v_5\pi; v_2, s_2, s_1\} \neq F$, π is nonadjacent to v_5. Observe that π is nonadjacent to v_{11}, for otherwise, π is adjacent to v_8 since $\{s_3v_{11}\pi; v_8, s_2, s_1\} \neq F$. Thus $\{s_4v_8\pi; v_2, v_5, s_1\} = F$, a contradiction. Since $\{s_1v_7v_9; \pi, v_5, v_{11}\} \neq F$, π is adjacent to v_9. Since $\{s_1v_9\pi; v_3, v_{11}, s_4\} \neq F$, π is adjacent to v_3. Since π is adjacent to each vertex in $S_1 \cup S_3 \cup \{s_4\}$, $N_S(\pi) \neq N_S(v_i)$ for any i with $1 \leq i \leq 4$. By Lemma 4(vii), π is nonadjacent to v_4, and then we have $\{s_3v_3v_2; v_4, v_{11}, \pi\} = \bar{F}$, a contradiction.

So we may assume that π is nonadjacent to some $s_1 \in S_1$.

Subcase 2.b) π is adjacent to v_3.

Observe that π is nonadjacent to v_4, for otherwise, π is adjacent to v_{12} since $\{s_4v_4\pi; v_{12}, s_2, s_3\} \neq F$, and by Lemma 7 we have $\{v_4\pi\beta; v_{12}, s_2, v_3\} = \bar{F}$, a contradiction. We claim that π is adjacent to v_8; otherwise, since $\{s_3v_3v_5; v_4, v_8, \pi\} \neq \bar{F}$, π is adjacent to v_5. It follows that $\{v_3v_4v_5; s_2, \pi, v_2\} = \bar{F}$, a contradiction. So the claim is proved. Note that $\{s_4v_8v_6; v_7, v_4, \pi\} \neq \bar{F}$, so π is adjacent to v_6. Since $N_S(\pi) \neq N_S(v_i)$ for any i with $3 \leq i \leq 6$ and π is adjacent to both v_3 and v_6, by Lemma 4(vii), π is adjacent to neither of v_4 and v_5. Thus we get $\{s_4v_4v_6; v_5, \pi, v_2\} = \bar{F}$, a contradiction.

So we may assume that π is nonadjacent to v_3.

Subcase 2.c) π is adjacent to v_5.

Since $\{s_3v_5\pi; v_9, s_2, s_4\} \neq F$, π is adjacent to v_9, and thus we have $\{s_3v_9\pi; v_3, v_7, s_4\} = F$, a contradiction.

So we assume hereafter that π is nonadjacent to some vertex $s_3 \in S_3$.

Case 3. π is adjacent to v_8.

Since $\{s_4v_8v_6; v_7, v_4, \pi\} \neq \bar{F}$, π is adjacent to v_4 or to v_6. We distinguish between two subcases accordingly.

Subcase 3.a) π is adjacent to v_4.

We claim that π is nonadjacent to v_3; otherwise, since $N_S(\pi) \neq N_S(v_i)$ for any i with $1 \leq i \leq 4$, Lemma 4(vii) ensures that π is nonadjacent to v_1. Since $\{v_1v_3v_4; \pi, s_2, s_1\} \neq \bar{F}$ for any $s_1 \in S_1$, π is adjacent to each vertex in S_1, and thus $\{\pi s_1v_3; s_4, v_7, s_3\} = F$, a contradiction. So the claim is justified. In view of $\{v_3v_4v_6; s_3, \pi, v_7\} \neq F$, we see that π is adjacent to v_6, and thus π is adjacent to each vertex in S_1 as $\{v_3v_4v_6; \pi, s_1, v_2\} \neq \bar{F}$ for any $s_1 \in S_1$. Since $\{s_4v_2v_4; v_3, \pi, v_{12}\} \neq \bar{F}$, π is adjacent to v_{12}. It follows that $\{v_{12}s_4\pi; v_8, s_1, v_2\} = \bar{F}$, a contradiction.

Subcase 3.b) π is adjacent to v_6, but nonadjacent to v_4.

Since $\{s_4v_4v_6; v_5, \pi, v_2\} \neq \bar{F}$, π is adjacent to v_5. Note that $\{\pi, v_5, v_6, v_8\}$ induces a clique in G and that $N_S(\pi) \neq N_S(v_i)$ for any $i \in \{5, 7, 8\}$. By Lemma 4(vii), we have $N_S(\pi) = N_S(v_6) = S_1 \cup S_4$. Thus $\{s_4v_8\pi; v_4, s_3, s_1\} = F$, a contradiction.

So we may assume hereafter that π is nonadjacent to v_8.

Case 4. π is adjacent to v_{10}.

Since $\{s_4v_{10}\pi; v_2, s_2, s_1\} \neq F$ for any $s_1 \in S_1$, π is adjacent to no vertex in S_1. Observe that π is nonadjacent to v_9; otherwise, since $N_S(\pi) \neq N_S(v_i)$ for any i with $9 \leq i \leq 12$, by Lemma

4(vii), π is adjacent to none of v_{11} and v_{12}. Thus $\{v_9v_{10}v_{11}; s_2, s_3, \pi\} = \bar{F}$, a contradiction. Since $\{s_4v_{10}v_8; v_9, v_6, \pi\} \neq \bar{F}$, π is adjacent to v_6. It follows that π is nonadjacent to v_4, for otherwise, π is adjacent to v_5 as $\{v_4v_5v_6; v_8, \pi, s_2\} \neq \bar{F}$. So $\{\pi, v_4, v_5, v_6\}$ is a clique and $N_S(\pi) \neq N_S(v_i)$ for any i with $3 \leq i \leq 6$, contradicting Lemma 4 (vii). Hence π is adjacent to v_5 since $\{s_4v_4v_6; v_5, \pi, v_2\} \neq \bar{F}$. Since $N_S(\pi) \neq N_S(v_i)$ for any i with $3 \leq i \leq 6$, by Lemma 4(vii) π is nonadjacent to v_3. Hence $\{v_3v_5v_6; \pi, s_1, s_3\}$ induces an $\bar{F}$ in G, a contradiction.

This completes the proof of our theorem. □

Acknowledgment. We are indebted to an anonymous referee for discovering an error in our original proof. We greatly appreciate his/her insightful skepticism and invaluable comments and suggestions.

References

[1] C. Berge, Problems 9.11 and 9.12, in: I. Rival (ed.), *Graphs and Order*, Reidel, Dordrecht, 1985, pp. 583-584.

[2] V. Chvátal, Oral Communication, 1992.

[3] T. Gallai, Transitive orienterbare graphen, *Acta Sci. Math. Hung.* **18** (1967), 25-66.

[4] A. Ghouila-Houri, Caractérisation des graphes non orientés dont on peut orienter les arêtes de manière à obtenir le graphe d'une relation d'ordre, *C. R. Acad. Sci. Paris* **254** (1962), 1370-1371.

[5] P. C. Gilmore and A. J. Hoffman, A Characterization of Comparability Graphs and Interval Graphs, *Canad. J. Math.* **16** (1964), 539-548.

[6] P. Grillet, Maximal Chains and Antichains, *Fund. Math.* **65** (1969), 157-167.

[7] W. Zang, Generalizations of Grillet's Theorem on Maximal Stable Sets and Maximal Cliques in Graphs, *Discrete Math.* **143** (1995), 259-268.