Thompson's conjecture for real semi-simple Lie groups

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Abstract

A proof of Thompson's conjecture for real semi-simple Lie groups has been given by Kapovich, Millson, and Leeb. In this note, we give another proof of the conjecture by using a theorem of Alekseev, Meinrenken, and Woodward from symplectic geometry.

1 Introduction

Thompson's conjecture [T] for the group $GL(n, \mathbb{C})$, which relates eigenvalues of matrix sums and singular values of matrix products, was first proved by Klyachko in [Kl]. In [Al-Me-W], by applying a Moser argument to certain symplectic structures, Alekseev, Meinrenken, and Woodward gave a proof of Thompson's conjecture for all quasi-split real semi-simple Lie groups. In [Ka-Le-M1], Kapovich, Millson, and Leeb have, among other things, proved Thompson's conjecture for an arbitrary semi-simple Lie group G_0. A direct geometric argument can also be found in Section 4.2.4 of [Le-M]. In this note, we give a different proof of Thompson's conjecture for arbitrary semi-simple real groups by extending the proof of Alekseev, Meinrenken, and Woodward for quasi-split groups. In fact, we prove a stronger result, Theorem 2.2, which implies Thompson's conjecture. Let $G_0 = K_0 A_0 N_0$ be an Iwasawa decomposition of G_0 and let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be a compatible Cartan decomposition of the Lie algebra of $\mathfrak{g}_0$. Theorem 2.2 asserts that for each $l \geq 1$, there is a diffeomorphism $L : (A_0 N_0)^l \to (\mathfrak{p}_0)^l$ which relates the addition on $\mathfrak{p}_0$ with the multiplication on $A_0 N_0$ and intertwines naturally defined K_0-actions. When G_0 is quasi-split, Theorem 2.2 follows from results in [Al-Me-W]. The key step in our proof of

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Theorem 2.2 for an arbitrary G_0 is to relate an arbitrary real semi-simple Lie algebra $\mathfrak{g}_0$ to a quasi-split real form in its complexification.

In Section 2, we state Theorem 2.2 and show that it implies Thompson's conjecture. Inner classes of real forms and quasi-split real forms are reviewed in Section 3. The proof of Theorem 2.2 is given in Section 4. Since the version of the Alekseev-Meinrenken-Woodward theorem we present in this paper is not explicitly stated in [Al-Me-W], we give an outline of its proof in the Section 5, the Appendix.

Acknowledgments. Although we do not explicitly use any results from [Fo], our paper is very much inspired by [Fo]. We thank P. Foth for showing us a preliminary version of [Fo] and for helpful discussions. We also thank J. Millson and M. Kapovich for sending us the preprint [Ka-Le-M1] and R. Sjamaar for answering some questions. Thanks also go to the referee for helpful comments.

2 Thompson's conjecture

Let G be a complex connected reductive algebraic group with an anti-holomorphic involution τ. Let G_0 be a subgroup of the fixed point set G^τ of τ containing the identity connected component. Then G_0 is a real reductive Lie group in the sense of [Wa] pp. 42–45, which implies that G_0 has Cartan and Iwasawa decompositions. Let $\mathfrak{g}_0$ be the Lie algebra of G_0. Fix a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$, and let $G_0 = P_0 K_0$ be the corresponding Cartan decomposition of G_0. Let $\mathfrak{a}_0 \subset \mathfrak{p}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$. Fix a choice Δ_{res}^+ of positive roots in the restricted root system Δ_{res} for $(\mathfrak{g}_0, \mathfrak{a}_0)$, and let $\mathfrak{n}_0$ be the subspace of $\mathfrak{g}_0$ spanned by the root vectors for roots in Δ_{res}^+. Then $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$ is an Iwasawa decomposition for $\mathfrak{g}_0$. Let $A_0 = \exp(\mathfrak{a}_0)$ and $N_0 = \exp(\mathfrak{n}_0)$. Then we have the Iwasawa decomposition $G_0 = K_0 A_0 N_0$ for G_0.

Consider now the space $X_0 := G_0/K_0$ with the left G_0–action given by

$$G_0 \times (G_0/K_0) \longrightarrow G_0/K_0 : \; (g_1, gK_0) \longmapsto g_1 g K_0, \quad g_1, g \in G_0. \tag{2.1}$$

Thompson's conjecture is concerned with K_0–orbits in X_0. Identify

$$\mathfrak{p}_0 \overset{\exp}{\cong} P_0 \cong G_0/K_0 = X_0$$

via the Cartan decomposition $G_0 = P_0 K_0$. The K_0–action on X_0 in (2.1) becomes the adjoint action of K_0 on $\mathfrak{p}_0$. Orbits of K_0 in $\mathfrak{p}_0$ are called (real) flag manifolds. Let $\mathfrak{a}_0^+ \subset \mathfrak{a}_0$ be the Weyl chamber determined by Δ_{res}^+. It is well-known that every K_0–orbit in $\mathfrak{p}_0$ goes through a unique element $\lambda \in \mathfrak{a}_0^+$.

On the other hand, we can also identify

$$A_0 N_0 \cong G_0/K_0 = X_0$$

via the Iwasawa decomposition $G_0 = A_0 N_0 K_0$. Then the K_0–action on X_0 becomes the following action of K_0 on $A_0 N_0$:

$$k \cdot b := p(kb) = p(kbk^{-1}), \quad \text{for} \quad k \in K_0, \, b \in A_0 N_0, \tag{2.2}$$

where $p: G_0 \to A_0N_0$ is the projection $b_1k_1 \mapsto b_1$ for $k_1 \in K$ and $b_1 \in A_0N_0$. Let $E_0 : \mathfrak{p}_0 \to A_0N_0$ be the composition of the identifications:

$$E_0 : \ \mathfrak{p}_0 \overset{\exp}{\cong} P_0 \cong G_0/K_0 \cong A_0N_0. \tag{2.3}$$

Then E_0 is K_0–equivariant, and $E_0(\mathfrak{a}_0) = A_0$. Thus every K_0–orbit in A_0N_0 goes through a unique point $a = \exp\lambda \in A_0^+ := \exp\mathfrak{a}_0^+$.

Thompson's conjecture for G_0 is concerned with the sum of K_0–orbits in $\mathfrak{p}_0$ and the product of K_0–orbits in A_0N_0. To further prepare for the statement of the conjecture, let $l \geq 1$ be an integer, and consider the two maps

$$\begin{aligned}
&\mathbf{a} : \ \mathfrak{p}_0 \times \mathfrak{p}_0 \times \cdots \times \mathfrak{p}_0 \longrightarrow \mathfrak{p}_0 : \ (\xi_1, \xi_2, \cdots, \xi_l) \longmapsto \xi_1 + \xi_2 + \cdots + \xi_l, \\
&\mathbf{m} : \ A_0N_0 \times A_0N_0 \times \cdots \times A_0N_0 \longrightarrow A_0N_0 : \ \ (b_1, b_2, \cdots, b_l) \longmapsto b_1b_2\cdots b_l.
\end{aligned}$$

Clearly, $\mathbf{a}$ is K_0–equivariant for the diagonal action of K_0 on $(\mathfrak{p}_0)^l$. On the other hand, define the *twisted diagonal action* $\mathcal{T}$ of K_0 on $(A_0N_0)^l$ by

$$k \longmapsto \mathcal{T}_k := \nu^{-1} \circ \delta_k \circ \nu : \ (A_0N_0)^l \longrightarrow (A_0N_0)^l, \tag{2.4}$$

where δ_k is the diagonal action of $k \in K_0$ on $(A_0N_0)^l$, and $\nu : (A_0N_0)^l \longrightarrow (A_0N_0)^l$ is the diffeomorphism given by

$$\nu(b_1, b_2, \cdots, b_l) \longmapsto (b_1, \ b_1b_2, \ \cdots, \ b_1b_2\cdots b_l). \tag{2.5}$$

See Remark 2.4 for motivation of the twisted diagonal action. Let e be the identity element of A_0N_0 and identify $T_e(A_0N_0) \cong \mathfrak{a}_0 + \mathfrak{n}_0 \cong \mathfrak{g}_0/\mathfrak{k}_0 \cong \mathfrak{p}_0$. We will regard the map $\mathbf{a}$, respectively the diagonal K_0–action on $(\mathfrak{p}_0)^l$, as the linearization of the map $\mathbf{m}$, respectively the twisted diagonal K_0–action on $(A_0N_0)^l$, at the point $(e, e, \cdots, e)$.

Notation 2.1 For $\lambda \in \mathfrak{a}_0$, we will use O_λ to denote the K_0–orbit in $\mathfrak{p}_0$ through λ. For $a \in A_0$, we will use D_a to denote the K_0–orbit in A_0N_0 through the point a. If $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \in (\mathfrak{a}_0)^l$, we set $a_j = \exp(\lambda_j)$ for $1 \leq j \leq l$, and

$$O_\Lambda = O_{\lambda_1} \times O_{\lambda_2} \times \cdots \times O_{\lambda_l} \quad \text{and} \quad D_\Lambda = D_{a_1} \times D_{a_2} \times \cdots \times D_{a_l}.$$

In this paper, we will prove the following theorem.

Theorem 2.2 *For every integer $l \geq 1$, there is a K_0–equivariant diffeomorphism $L : (A_0N_0)^l \to (\mathfrak{p}_0)^l$ such that $\mathbf{m} = E_0 \circ \mathbf{a} \circ L$ and $L(D_\Lambda) = O_\Lambda$ for every $\Lambda \in (\mathfrak{a}_0)^l$.*

Theorem 2.2 now readily implies the following Thompson's conjecture for G_0.

Corollary 2.3 (Thompson's conjecture) *For each $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \in (\mathfrak{a}_0)^l$, the two spaces*

$$(\mathbf{m}^{-1}(e) \cap D_\Lambda)/K_0 = \{(b_1, b_2, \cdots, b_l) \in D_\Lambda : b_1b_2\cdots b_l = e\}/K_0$$

and

$$(\mathbf{a}^{-1}(0) \cap O_\Lambda)/K_0 = \{(\xi_1, \xi_2, \cdots, \xi_l) \in O_\Lambda : \xi_1 + \xi_2 + \cdots + \xi_l = 0\}/K_0$$

are homeomorphic. In particular, one is non-empty if and only if the other is.

Proof. Let $L : (A_0N_0)^l \to (\mathfrak{p}_0)^l$ be the diffeomorphism in Theorem 2.2. Then L induces a homeomorphism $L : \mathbf{m}^{-1}(e) \to \mathbf{a}^{-1}(0)$. Since $L(D_\Lambda) = O_\Lambda$ and L is K_0–equivariant, it induces a homeomorphism from $(\mathbf{m}^{-1}(e) \cap D_\Lambda)/K_0$ to $(\mathbf{a}^{-1}(0) \cap O_\Lambda)/K_0$.

Q.E.D.

Remark 2.4 Equip $X_0 = G_0/K_0$ with the G_0–invariant Riemannian structure defined by the restriction of the Killing form of $\mathfrak{g}_0$ on $\mathfrak{p}_0$. For $x_1, x_2 \in X_0$, let $\overline{x_1x_2}$ be the unique geodesic in X_0 connecting x_1 and x_2. Then there is a unique $\lambda \in \mathfrak{a}_0^+$ such that $g \cdot x_1 = *$ and $g \cdot x_2 = (\exp \lambda) \cdot *$ for some $g \in G_0$, where $* = eK_0 \in X_0$ is the base point. The element $\lambda \in \mathfrak{a}_0^+$ is called [Ku-Le-M] the $\mathfrak{a}_0^+$–length of $\overline{x_1x_2}$. Representing the vertices of an $*$–based l–gon in X_0 by

$$(*,\ b_1 \cdot *,\ b_1b_2 \cdot *,\ \cdots,\ b_1b_2\cdots b_l \cdot *)$$

for some $b_1, b_2, \cdots, b_l \in A_0N_0$, we can regard $(b_1, b_2, \cdots, b_l)$ as the set of edges of the l–gon. Then for $\Lambda \in (\mathfrak{a}_0^+)^l$, the space

$$\{(b_1, b_2, \cdots, b_l) \in D_\Lambda :\ b_1b_2\cdots b_l = e\}/K_0$$

can be identified with the space of G_0–equivalence classes of *closed* l–gons in X_0 with fixed "side length" Λ. Similarly, the space

$$\{(\xi_1, \xi_2, \cdots, \xi_l) \in O_\Lambda : \xi_1 + \xi_2 + \cdots + \xi_l = 0\}/K_0$$

can be identified with the space of equivalent l–gons with fixed side length in the "infinitesimal symmetric space" $\mathfrak{p}_0$. Using the right A_0N_0–action on K_0 given by

$$k^b := q(kb), \qquad b \in A_0N_0,\ k \in K_0, \tag{2.6}$$

where $q : G_0 \to K_0$ is the projection $q(b_1k_1) = k_1$ for $b_1 \in A_0N_0$ and $k_1 \in K_0$, it is easy to see that $\mathcal{T}_k$ as defined in 2.4 is also given by

$$\mathcal{T}_k(b_1, b_2, \cdots, b_l) := (k_1 \cdot b_1,\ k_2 \cdot b_2,\ \cdots,\ k_l \cdot b_l), \tag{2.7}$$

where $k_1 = k, k_j = k^{b_1b_2\cdots b_{j-1}}$ for $2 \le j \le l$. In the Appendix, we will see that this formula naturally arises in the context of Poisson Lie group actions.

Remark 2.5 When $G_0 = GL(n, \mathbb{R})$, recall that the singular values of $g \in G_0$ are by definition the eigenvalues of $\sqrt{gg^t}$. Thompson's conjecture for $GL(n, \mathbb{R})$ says that, for any collection $(\lambda_1, \lambda_2, \cdots, \lambda_l)$ of real diagonal matrices, the following two statements are equivalent (see Section 4.2 of [Al-Me-W]):

1) there exist matrices $g_j \in GL(n, \mathbb{R})$ whose singular values are entries of $a_j = \exp(\lambda_j)$ and $g_1g_2\cdots g_l = e$;

2) there exist symmetric matrices ξ_j whose eigenvalues are entries of λ_j and such that $\xi_1 + \xi_2 + \cdots + \xi_l = 0$.

Remark 2.6 By a theorem of O'Shea and Sjamaar [O-S], the set $\mathbf{a}^{-1}(0) \cap O_\Lambda$ is non-empty if and only if $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ lies in a certain polyhedral cone $\mathcal{P}$ in $(\mathfrak{a}_0^+)^l$. For $G_0 = SL(2,\mathbb{R})$, we have $\mathfrak{a}_0^+ = \mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda > 0\}$. In this case $\mathfrak{p}_0$ may be identified with $\mathbb{R}^2$ and the K_0–action may be identified with rotation. We know from classical triangle inequalities that $\mathcal{P}$ for $l = 3$ is given by the inequalities

$$|\lambda_1 - \lambda_2| \leq \lambda_3 \leq \lambda_1 + \lambda_2. \tag{2.8}$$

The fact that these triangle inequalities are also the necessary and sufficient conditions for λ_1, λ_2 and λ_3 to be the sides of a geodesic triangle in the hyperbolic 2-space $SL(2,\mathbb{R})/SO(2)$ follows from the *law of cosines* in this space, which says that

$$\cos\theta = \frac{\cosh\lambda_1 \cosh\lambda_2 - \cosh\lambda_3}{\sinh\lambda_1 \sinh\lambda_2}$$

where θ is the angle between the two sides with lengths λ_1 and λ_2.

For any arbitrary real semi-simple Lie group G_0, there have been intensive research activities on the inequalities that describe the polyhedral cone $\mathcal{P}$. We refer to [Ku-Le-M] for explicit examples of these polyhedral cones when X_0 has rank 3 and to [Fu] [Ka-Le-M1] [Ka-Le-M2] [Le-M] for an account on the history and the connections between this problem and others fields such as Schubert calculus, representation theory, symmetric spaces, geometric invariant theory, and integrable systems. See also Remark 4.9.

Remark 2.7 Finally, we remark that it is enough to prove Theorem 2.2 for G with trivial center. Indeed, let Z be the center of G, let $Z_0 = G_0 \cap Z$, and let $G_0' = G_0/Z_0$ with Lie algebra $\mathfrak{g}_0' = \mathfrak{g}_0/\mathfrak{z}_0$, where the Lie algebra $\mathfrak{z}_0$ of Z_0 is the center of $\mathfrak{g}_0$. Let $j : G_0 \to G_0'$ and $\mathfrak{g}_0 \to \mathfrak{g}_0'$ be the natural projections. Let $\mathfrak{k}_0' = j(\mathfrak{k}_0)$ and $\mathfrak{p}_0' = j(\mathfrak{p}_0)$, and let $K_0' = j(K_0)$, $A_0' = j(A_0)$, and $N_0' = j(N_0)$. By Corollary 1.3 from [Kn1], $Z_0 = (K_0 \cap Z_0)(A_0 \cap Z_0)$, and $\exp : \mathfrak{z}_0 \cap \mathfrak{a}_0 \to Z_0 \cap A_0$ is an isomorphism. Thus $G_0' = K_0'A_0'N_0'$ is an Iwasawa decomposition for G_0', and $\mathfrak{g}_0' = \mathfrak{k}_0' + \mathfrak{p}_0'$ is a Cartan decomposition for $\mathfrak{g}_0'$. Moreover, $\mathfrak{p}_0 \cong \mathfrak{p}_0' \oplus (\mathfrak{a}_0 \cap \mathfrak{z}_0)$ and $A_0 \cong A_0' \times (A_0 \cap Z_0)$ and $N_0' \cong N_0$. Let

$$\begin{array}{llll} \mathbf{a} : & (\mathfrak{p}_0')^l \longrightarrow \mathfrak{p}_0', & \mathbf{m} : & (A_0'N_0')^l \longrightarrow A_0N_0, \\ \mathbf{a} : & (\mathfrak{a}_0 \cap \mathfrak{z}_0)^l \longrightarrow \mathfrak{a}_0 \cap \mathfrak{z}_0, & \mathbf{m} : & (A_0 \cap Z_0)^l \longrightarrow A_0 \cap Z_0 \end{array}$$

be respectively the addition and multiplication maps. If $L' : (A_0'N_0')^l \to (\mathfrak{p}_0')^l$ is a diffeomorphism satisfying the requirements in Theorem 2.2 for the group G_0', then $L = (L', (\log)^l)$ will be a diffeomorphism from $(A_0N_0)^l$ to $(\mathfrak{p}_0)^l$ satisfying the requirements in Theorem 2.2 for the group G_0 , where we use the obvious identifications between $(A_0N_0)^l \cong (A_0'N_0')^l \times (A_0 \cap Z_0)^l$ and $(\mathfrak{p}_0)^l \cong (\mathfrak{p}_0')^l \times (\mathfrak{a}_0 \cap \mathfrak{z}_0)^l$.

3 Inner classes of real forms and quasi-split real forms

Let $\mathfrak{g}$ be a semi-simple complex Lie algebra. Recall that real forms of $\mathfrak{g}$ are in one to one correspondence with complex conjugate linear involutive automorphisms of $\mathfrak{g}$. For such an involution τ, the corresponding real form is the fixed point set $\mathfrak{g}^\tau$ of τ. We will refer to both $\mathfrak{g}^\tau$ and τ as the real form. Throughout this paper, if V is a set and σ in an involution on V, we will use V^σ to denote the set of σ-fixed points in V. Let G be the adjoint group of $\mathfrak{g}$.

Definition 3.1 *(Definitions 2.4 and 2.6 of [A-B-V])* Two real forms τ_1 and τ_2 of $\mathfrak{g}$ are said to be inner to each other if there exists $g \in G$ such that $\tau_1 = \mathrm{Ad}_g \tau_2$. A real form τ of $\mathfrak{g}$ is said to be *quasi-split* if there exists a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\tau(\mathfrak{b}) = \mathfrak{b}$.

Inner classes of real forms are classified by involutive automorphisms of the Dynkin diagram $D(\mathfrak{g})$ of $\mathfrak{g}$. Indeed, let $\mathrm{Aut}_\mathfrak{g}$ be the group of complex linear automorphisms of $\mathfrak{g}$. Its identity component is the adjoint group G. Let $\mathrm{Aut}_{D(\mathfrak{g})}$ be the automorphism group of the Dynkin diagram $D(\mathfrak{g})$ of $\mathfrak{g}$. There is a split short exact sequence ([A-B-V], Proposition 2.11),

$$1 \longrightarrow G \longrightarrow \mathrm{Aut}_\mathfrak{g} \stackrel{\varrho}{\longrightarrow} \mathrm{Aut}_{D(\mathfrak{g})} \longrightarrow 1. \tag{3.1}$$

Denote by $\mathbf{R}$ the set of all real forms of $\mathfrak{g}$. Let θ be any compact real form of $\mathfrak{g}$. Define

$$\varpi : \mathbf{R} \longrightarrow \mathrm{Aut}_{D(\mathfrak{g})} : \varpi(\tau) \;=\; \varrho(\tau\theta). \tag{3.2}$$

Then $\varpi(\tau)^2 = 1$ for each τ, and τ_1 and τ_2 are inner to each other if and only if $\varpi(\tau_1) = \varpi(\tau_2)$. Conversely, for every involutive $d \in \mathrm{Aut}_{D(\mathfrak{g})}$, we can construct $\gamma_d \in \mathrm{Aut}_\mathfrak{g}$ such that $\tau := \gamma_d\theta$ is a real form with $\varpi(\tau) = d$ (see (3.4) below). Thus the map ϖ gives a bijection between inner classes of real forms of $\mathfrak{g}$ and involutive elements in $\mathrm{Aut}_{D(\mathfrak{g})}$ (Proposition 2.12 of [A-B-V]).

Definition 3.2 Let d be an involutive automorphism of the Dynkin diagram $D(\mathfrak{g})$ of $\mathfrak{g}$. We say that a real form τ of $\mathfrak{g}$ is *of inner class* d or *in the* d*–inner class* if $\varpi(\tau) = d$.

By Proposition 2.7 of [A-B-V], every inner class of real forms of $\mathfrak{g}$ contains a quasi-split real form that is unique up to G–conjugacy. In the following, for each involutive $d \in \mathrm{Aut}_{D(\mathfrak{g})}$, we will construct an explicit quasi-split real form τ_d in the d–inner class. We will then show that, up to G–conjugacy, every real form in the d–inner class is of the form $\tau = \mathrm{Ad}_{\dot{w}_0}\tau_d$, where w_0 is a certain Weyl group element and $\dot{w}_0$ a representative of w_0 in G. We first fix once and for all the following data for $\mathfrak{g}$:

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let Δ be the corresponding root system. Fix a choice of positive roots Δ^+in Δ, and let Σ be the basis of simple roots. Let $\ll \cdot, \cdot \gg$ be the Killing form on $\mathfrak{g}$. For each $\alpha \in \Delta$, let $E_{\pm\alpha}$ be root vectors such that $[E_\alpha, E_{-\alpha}] = H_\alpha$ for all $\alpha \in \Delta^+$, where H_α is the unique element of $\mathfrak{h}$ defined by $\ll H, H_\alpha \gg = \alpha(H)$ for all $H \in \mathfrak{h}$, and the numbers $m_{\alpha,\beta}$ for $\alpha, \beta \in \Delta$ defined for $\alpha \neq -\beta$ by

$$\begin{array}{rcll} [E_\alpha, E_\beta] & = & m_{\alpha,\beta}E_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta \\ & = & 0 & \text{otherwise} \end{array}$$

satisfy $m_{-\alpha,-\beta} = -m_{\alpha,\beta}$. We will refer to the set $\{E_\alpha, E_{-\alpha} : \alpha \in \Delta^+\}$ as (part of) a Weyl basis. Using a Weyl basis $\{E_\alpha, E_{-\alpha} : \alpha \in \Delta^+\}$, we can define a compact real form $\mathfrak{k}$ of $\mathfrak{g}$ as

$$\mathfrak{k} = \text{span}_{\mathbb{R}}\{\sqrt{-1}H_\alpha, X_\alpha := E_\alpha - E_{-\alpha}, Y_\alpha := \sqrt{-1}(E_\alpha + E_{-\alpha}) : \alpha \in \Delta^+\}. \tag{3.3}$$

Let θ be the complex conjugation of $\mathfrak{g}$ defining $\mathfrak{k}$. We also define a split real form η_0 of $\mathfrak{g}$ by setting $\eta_0|_{\mathfrak{a}} = id$, and $\eta_0(E_\alpha) = E_\alpha$ for every $\alpha \in \Delta$. Clearly $\theta\eta_0 = \eta_0\theta$. The inner class for η_0 is easily seen to be the automorphism of the simple roots given by $-w^0$, where w^0 is the longest element in the Weyl group W of $(\mathfrak{g}, \mathfrak{h})$.

An explicit splitting of the short exact sequence (3.1) can be constructed using the Weyl basis. Indeed, for any $d \in \text{Aut}_{D(\mathfrak{g})}$, define $\gamma_d \in \text{Aut}_{\mathfrak{g}}$ by requiring

$$\gamma_d(H_\alpha) = H_{d\alpha}, \quad \text{and} \quad \gamma_d(E_\alpha) = E_{d\alpha} \tag{3.4}$$

for each simple root α. Then $d \mapsto \gamma_d$ is a group homomorphism from $\text{Aut}_{D(\mathfrak{g})}$ to $\text{Aut}_{\mathfrak{g}}$ and is a section of ϱ in (3.1). Moreover, every γ_d commutes with both θ and η_0 because they commute on a set of generators of $\mathfrak{g}$.

Lemma 3.3 *For an involutive element $d \in \text{Aut}_{D(\mathfrak{g})}$, let $\gamma_{-w^0d} \in \text{Aut}_{\mathfrak{g}}$ be the lifting of $-w^0d \in \text{Aut}_{D(\mathfrak{g})}$ as defined in (3.4). Define*

$$\tau_d = \eta_0\gamma_{-w^0d}. \tag{3.5}$$

Then τ_d is a quasi-split real form of $\mathfrak{g}$ in the d–inner class.

Proof. We know that $(\tau_d)^2 = 1$ because $-w^0 \in \text{Aut}_{D(\mathfrak{g})}$ is in the center. Since τ_d maps every positive root vector to another positive root vector, it is a quasi-split real form. Finally, since $\varpi(\tau_d) = \varrho((\gamma_{-w^0})(\gamma_{-w^0d})) = d$, we see that τ_d is in the d–inner class.

Q.E.D.

To relate an arbitrary real form in the d–inner class with the quasi-split real form τ_d, we recall some definitions from [Ar]. Note first that $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition, where $\mathfrak{a} = \text{span}_{\mathbb{R}}\{H_\alpha : \alpha \in \Delta\}$ and $\mathfrak{n} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$.

Definition 3.4 A real form τ of $\mathfrak{g}$ is said to be *normally related to* $(\mathfrak{k}, \mathfrak{a})$ and *compatible with* Δ^+ if

1) $\tau\theta = \theta\tau$, and $\tau(\mathfrak{h}) = \mathfrak{h}$;

2) $\mathfrak{a}^\tau \subset (\sqrt{-1}\mathfrak{k})^\tau$ is maximal abelian in $(\sqrt{-1}\mathfrak{k})^\tau$;

3) if $\alpha \in \Delta^+$ is such that $\alpha|_{\mathfrak{a}^\tau} \neq 0$, then $\tau(\alpha) \in \Delta^+$, where $\tau(\alpha) \in \mathfrak{a}^*$ is defined by $\tau(\alpha)(\lambda) = \alpha(\tau(\lambda))$ for $\lambda \in \mathfrak{a}$.

We will call a real form with Properties 1)-3) an *Iwasawa real form* relative to $(\mathfrak{k}, \mathfrak{a}, \mathfrak{n})$.

Remark 3.5 Once 1) and 2) in Definition 3.4 are satisfied, 3) is equivalent to the set $r(\Delta^+)\backslash\{0\} \subset \Delta_{\rm res}$ being a choice of positive roots for the restricted root system $\Delta_{\rm res}$ of $(\mathfrak{g}^\tau, \mathfrak{a}^\tau)$, where r is the map dual to the inclusion $\mathfrak{a}^\tau \hookrightarrow \mathfrak{a}$. If τ is an Iwasawa real form relative to $(\mathfrak{k}, \mathfrak{a}, \mathfrak{n})$, then so is ${\rm Ad}_t \tau {\rm Ad}_t^{-1}$ for any $t \in \exp(i\mathfrak{a})$.

Proposition 3.6 *1) Every real form of* $\mathfrak{g}$ *is conjugate by an element in* G *to a real form that is Iwasawa relative to* $(\mathfrak{k}, \mathfrak{a}, \mathfrak{n})$*;*

2) Suppose that τ *is an Iwasawa real form relative to* $(\mathfrak{k}, \mathfrak{a}, \mathfrak{n})$ *and suppose that* τ *is in the d–inner class. Let* w_0 *be the longest element of the subgroup of* W *generated by the reflections corresponding to roots in the set*

$$\Delta_0 = \{\alpha \in \Delta : \alpha|_{\mathfrak{a}^\tau} = 0\} = \{\alpha \in \Delta : \tau(\alpha) = -\alpha\}.$$

Then there is a representative $\dot{w}_0$ *of* w_0 *in* G *such that*

$$\tau = {\rm Ad}_{\dot{w}_0} \tau_d. \tag{3.6}$$

Proof. Statement 1) follows from Proposition 1.2, Section 2.8, and Corollary 2.5 of [Ar].

2) Assume now that τ is an Iwasawa real form relative to $(\mathfrak{k}, \mathfrak{a}, \mathfrak{n})$ and that τ is in the d–inner class. Consider $\tau_d \tau$. Since both τ and τ_d are in the d-inner class, $\tau_d \tau = {\rm Ad}_g$ for some $g \in G$. Since both τ and τ_d leave $\mathfrak{h}$ invariant, the element g represents an element in the Weyl group W. Let $\Sigma_0 = \Sigma \cap \Delta_0$. By Section 2.8 of [Ar], $\alpha \in \Delta_0$ if and only if α is in the linear span of Σ_0. For every $\alpha \in \Sigma_0$, we have $\tau_d \tau(\alpha) = -\tau_d(\alpha) \in -\Delta^+$, and for every $\alpha \in \Delta^+ - \Delta_0$, since $\tau(\alpha) \in \Delta^+$, we have $(\tau_d \tau)(\alpha) \in \Delta^+$. Thus g represents the element $w_0 = (w_0)^{-1}$.

Q.E.D.

Remark 3.7 Recall from [Ar] that the Satake diagram of τ is the Dynkin diagram of $\mathfrak{g}$ with simple roots in Σ_0 painted black, simple roots in $\Sigma - \Sigma_0$ painted white, and a two sided arrow drawn between two white simple roots α and α' if $\tau(\alpha) = \alpha' + \beta$ for some $\beta \in \Delta_0$. From (3.6) we see that $\alpha' = -w^0 d(\alpha)$ if α is a white simple root. Conversely, given a Satake diagram for a real form of $\mathfrak{g}$, let $c \in {\rm Aut}_{D(\mathfrak{g})}$ be defined by

$$c(\alpha) = \begin{cases} -w_0 \alpha & \text{if } \alpha \text{ is black} \\ \alpha' & \text{if } \alpha \text{ is white,} \end{cases} \tag{3.7}$$

where w_0 is the longest element in the subgroup of the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ generated by the black dots in the Satake diagram, and $\alpha \mapsto \alpha'$ is the order 2 involution on the set of white dots in the diagram. Then c is involutive, and the inner class of the real form is $d = -w^0 c$.

We now return to the real form τ in Proposition 3.6. Set

$$\Delta_1^+ = \{\alpha \in \Delta^+ : \alpha|_{\mathfrak{a}^\tau} \neq 0\}, \quad (\Delta_1^+)' = \Delta^+ \cap \Delta_0 = \{\alpha \in \Delta^+ : \alpha|_{\mathfrak{a}^\tau} = 0\}.$$

Then $\tau(\Delta_1^+) \subset \Delta_1^+$. Set

$$\mathfrak{n}_1 = \sum_{\alpha \in \Delta_1^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_1' = \sum_{\alpha \in (\Delta_1^+)'} \mathfrak{g}_\alpha. \tag{3.8}$$

Then $\mathfrak{n}_1$ is τ-invariant, and, since there are no non-compact imaginary roots for the Cartan subalgebra $\mathfrak{h}^\tau$ of $\mathfrak{g}^\tau$, we have $\tau|_{\mathfrak{n}_1'} = \theta|_{\mathfrak{n}_1'}$ (Proposition 1.1 of [Ar]). Since the restriction of Δ^+ to $\mathfrak{a}^\tau$ gives a choice of positive restricted roots for $(\mathfrak{g}^\tau, \mathfrak{a}^\tau)$, we know that

$$\mathfrak{g}^\tau = \mathfrak{k}^\tau + \mathfrak{a}^\tau + (\mathfrak{n}_1)^\tau \tag{3.9}$$

is an Iwasawa decomposition of $\mathfrak{g}^\tau$.

4 The proof of Theorem 2.2

By Remark 2.7, it is enough to prove Theorem 2.2 when G of adjoint type. Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, the Lie algebra of G. Then by Proposition 3.6, we can assume that $\mathfrak{g}_0 = \mathfrak{g}^\tau$, where τ is the involution on $\mathfrak{g}$ given by (3.6), and d is the inner class of $\mathfrak{g}_0$. The lifting of τ to G will also be denoted by τ. Let G_0 contain the connected component of the identity of the subgroup G^τ. In this section, we will prove Theorem 2.2 for G_0.

The first step in out proof of Theorem 2.2 for G_0 is to realize the various objects associated to G_0 as fixed point sets of involutions on the corresponding objects related to G. We will then apply a theorem of Alekseev-Meinrenken-Woodward, stated as Theorem 4.7 below, whose proof using Poisson geometry will be outlined in Section 5 the Appendix.

We will keep all the notation from Section 3. In particular, set

$$\mathfrak{k}_0 = \mathfrak{k}^\tau, \quad \mathfrak{p}_0 = (\sqrt{-1}\mathfrak{k})^\tau, \quad \mathfrak{a}_0 = \mathfrak{a}^\tau, \quad \text{and} \quad \mathfrak{n}_0 = (\mathfrak{n}_1)^\tau.$$

Then $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$, and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$ an Iwasawa decomposition of $\mathfrak{g}_0$. Let K be the connected subgroup of G with Lie algebra $\mathfrak{k}$. Let $K_0 = K \cap G_0$, $K^\tau = K \cap G^\tau$, and let

$$N_1 = \exp(\mathfrak{n}_1), \quad A_0 = \exp(\mathfrak{a}_0), \quad \text{and} \quad N_0 = N_1 \cap G^\tau = \exp(\mathfrak{n}_0).$$

Lemma 4.1 *$G_0 = K_0 A_0 N_0$ is an Iwasawa decomposition of G_0, and $G^\tau = K^\tau A_0 N_0$ is an Iwasawa decomposition of G^τ.*

Proof. The statements follow from Lemma 2.1.7 in [Wa] and the facts that K_0 and K^τ are maximally compact subgroups of G_0 and of G^τ respectively.

Q.E.D.

Let $A = \exp \mathfrak{a}$ and $N = \exp \mathfrak{n}$ so that $G = KAN$ is an Iwasawa decomposition of G. We will now identify $A_0 N_0$ with the fixed point set of an involution on AN, although we note that the involution τ does not leave AN invariant unless $\tau = \tau_d$. Define

$$\sigma : AN \longrightarrow AN : \sigma(b) = p(\tau(b)), \tag{4.1}$$

where $p : G = KAN \to AN$ is the projection $b_1 k_1 \mapsto b_1$ for $k_1 \in K$ and $b_1 \in AN$. Recall from (3.6) that $\tau = \mathrm{Ad}_{\dot{w}_0} \tau_d$ on $\mathfrak{g}$. Since both τ and τ_d commute with θ, the element $\dot{w}_0$ is in K. As for the case of G_0, we can use the Iwasawa decomposition $G = KAN$ to define a left action of K on $AN \cong G/K$ by

$$k \cdot b := p(kb), \qquad k \in K,\ b \in AN. \tag{4.2}$$

Then $\sigma : AN \to AN$ is also given by

$$\sigma(b) = \dot{w}_0 \cdot \tau_d(b), \quad \text{for} \quad b \in AN. \tag{4.3}$$

Lemma 4.2 *$\sigma : AN \to AN$ is an involution, and $A_0 N_0 = (AN)^\sigma$, the fixed point set of σ in AN.*

Proof. The fact that $\sigma^2 = 1$ follows from the fact that $\dot{w}_0 \tau_d(\dot{w}_0) = 1$. Recall that $A_0 N_0$ is the fixed point set of τ in AN_1, where $N_1 = \exp(\mathfrak{n}_1)$ and $\mathfrak{n}_1 \subset \mathfrak{n}$ is given in (3.8). Since σ coincides with τ on AN_1, we have $A_0 N_0 \subset (AN)^\sigma$. Suppose now that $b \in AN$ is such that $\sigma(b) = b$. Then there exists $k \in K$ such that $\tau(b) = bk$. By Proposition 7.1.3 in [Sl], we can decompose b as $b = gak_1$ for some $k_1 \in K, a \in A$ and $g \in G^\tau$. Then $\tau(b) = g\tau(a)\tau(k_1)$, and thus $g\tau(a)\tau(k_1) = gak_1 k$, from which it follows that $\tau(a) = a$ and $\tau(k_1) = k_1 k$. Thus $k = k_1^{-1}\tau(k_1)$, and hence $\tau(bk_1^{-1}) = bk_1^{-1}$. Write $bk_1^{-1} = b_2 k_2$ with $k_2 \in K^\tau$ and $b_2 \in A_0 N_0$ using the Iwasawa decomposition of G^τ. It follows then that $k_2 = k_1^{-1}$ and $b_2 = b$, so $b \in A_0 N_0$.

Q.E.D.

Let now $l \geq 1$ be an integer. As in Section 2, we have the twisted diagonal action $k \mapsto \mathcal{T}_k$ of K on $(AN)^l$ given by

$$\mathcal{T}_k = \nu^{-1} \circ \delta_k \circ \nu : \ (AN)^l \longrightarrow (AN)^l, \tag{4.4}$$

where $\nu : (AN)^l \to (AN)^l$ is as in (2.5) with $A_0 N_0$ replaced by AN, and δ_k denotes the diagonal action of $k \in K$ on AN. Set $(\tau_d)^l = (\tau_d, \tau_d, \cdots, \tau_d) : (AN)^l \to (AN)^l$.

Lemma 4.3 *For an integer $l \geq 1$, define*

$$\sigma^{(l)} = \mathcal{T}_{\dot{w}_0} \circ (\tau_d)^l : \ (AN)^l \longrightarrow (AN)^l.$$

Then $\sigma^{(l)}$ is an involution, and the fixed point set of $\sigma^{(l)}$ is $(A_0 N_0)^l$.

Proof. Let $\sigma^l = (\sigma, \sigma, \cdots, \sigma) : (AN)^l \to (AN)^l$, where $\sigma : AN \to AN$ is as in Lemma 4.2. Since τ_d is a group automorphism of AN, we have

$$\sigma^{(l)} = \nu^{-1} \circ \delta_{\dot{w}_0} \circ \nu \circ (\tau_d)^l = \nu^{-1} \circ (\delta_{\dot{w}_0} \circ (\tau_d)^l) \circ \nu = \nu^{-1} \circ \sigma^l \circ \nu.$$

Thus $(\sigma^{(l)})^2 = 1$. Moreover, let $b = (b_1, b_2, \cdots, b_l) \in (AN)^l$, and let $b' = \nu(b)$. Then $\sigma^{(l)}(b) = b$ if and only if $\sigma^l(b') = b'$, which in turn is equivalent to $b_j \in A_0 N_0$ for each $1 \leq j \leq l$ because of Lemma 4.2 and because of the fact that $A_0 N_0$ is a subgroup of AN.

Q.E.D.

Let $\mathfrak{p} = \sqrt{-1}\mathfrak{k}$, so $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. Let $E : \mathfrak{p} \to AN$ be the composition of the identifications

$$E: \ \mathfrak{p} \stackrel{\exp}{\cong} \exp(\mathfrak{p}) \cong G/K \cong AN. \tag{4.5}$$

Then E is K–equivariant with respect to the action of K on $\mathfrak{p}$ by conjugation and the action of K on AN given in (4.2).

Lemma 4.4 *$E \circ (\tau|_{\mathfrak{p}}) = \sigma \circ E$, and $E_0 = E|_{\mathfrak{p}_0} : \mathfrak{p}_0 \to A_0N_0$.*

Proof. Consider $E^{-1} : AN \to \mathfrak{p}$. For $g \in G$, define $g^* = \theta(g^{-1})$. Then for every $b \in AN$, $E^{-1}(b) = \frac{1}{2}\log(bb^*)$ for all $b \in AN$, and thus

$$\begin{aligned} E^{-1}(\sigma(b)) &= \frac{1}{2}\log(\dot{w}_0\tau_d(b)\tau_d(b)^*\dot{w}_0^{-1}) \\ &= \mathrm{Ad}_{\dot{w}_0}\left(\frac{1}{2}\log(\tau_d(b)\tau_d(b)^*)\right) = \mathrm{Ad}_{\dot{w}_0}\tau_d(E^{-1}(b)) = \tau(E^{-1}(b)). \end{aligned}$$

Thus $E \circ (\tau|_{\mathfrak{p}}) = \sigma \circ E$, and $E(\mathfrak{p}_0) = A_0N_0$ by Lemma 4.2. It also follows that $E|_{\mathfrak{p}_0} = E_0$.

Q.E.D.

Notation 4.5 For $\lambda \in \mathfrak{a} \subset \mathfrak{p}$, let $\mathcal{O}_\lambda$ be the K–orbit in $\mathfrak{p}$ through λ, and let $\mathcal{D}_a$ be the K–orbit in AN through $a = \exp\lambda \in A$. For $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \in (\mathfrak{a})^l$, set

$$\mathcal{O}_\Lambda = \mathcal{O}_{\lambda_1} \times \mathcal{O}_{\lambda_2} \times \cdots \times \mathcal{O}_{\lambda_l}, \quad \text{and} \quad \mathcal{D}_\Lambda = \mathcal{D}_{a_1} \times \mathcal{D}_{a_2} \times \cdots \times \mathcal{D}_{a_l}. \tag{4.6}$$

Recall from Section 2 that, for $\lambda \in \mathfrak{a}_0$, O_λ denotes the K_0–orbit in $\mathfrak{p}_0$ through λ, and D_a denotes the K_0–orbit in A_0N_0 through $a = \exp\lambda \in A_0$.

Lemma 4.6 *Let $\lambda \in \mathfrak{a}_0$ and $a = \exp\lambda \in A_0$. Then*

1). $\tau_d(\lambda) = \lambda$, so both $\mathcal{O}_\lambda \subset \mathfrak{p}$ and $\mathcal{D}_a \subset AN$ are τ_d-invariant;

2). $\mathcal{O}_\lambda$ is τ-invariant, and $(\mathcal{O}_\lambda)^\tau = O_\lambda$;

3). $\mathcal{D}_a$ is σ-invariant and $(\mathcal{D}_a)^\sigma = D_a$.

Proof. 1). Let α be a simple root such that $\tau(\alpha) = -\alpha$. Then for any $\lambda \in \mathfrak{a}_0$,

$$\alpha(\lambda) = \alpha(\tau(\lambda)) = (\tau(\alpha))(\lambda) = -\alpha(\lambda) = 0.$$

Thus $r_\alpha(\lambda) = \lambda$, where r_α is the reflection in $\mathfrak{a}$ defined by α. Since w_0 is a product of such reflections (see Proposition 3.6), we see that w_0 acts trivially on $\mathfrak{a}_0$. Thus every $\lambda \in \mathfrak{a}_0$ is also fixed by τ_d.

2). This is a standard fact. See, for example, Example 2.9 in [O-S]. We remark that the key point is to show that $(\mathcal{O}_\lambda)^\tau$ is connected. 3) follows from 2) and Lemma 4.4.

Q.E.D.

We can now state the Alekseev-Meinrenken-Woodward theorem. Set

$$\mathbf{a} : \mathfrak{p} \times \mathfrak{p} \times \cdots \times \mathfrak{p} \longrightarrow \mathfrak{p} : \ (x_1, x_2, \cdots, x_l) \longmapsto x_1 + \cdots + x_l,$$
$$\mathbf{m} : \ AN \times AN \times \cdots \times AN \longrightarrow AN : \ \ (b_1, b_2, \cdots, b_l) \longmapsto b_1 b_2 \cdots b_l.$$

As for the case of G_0, we will equip $\mathfrak{p}^l$ with the diagonal K–action by conjugation, and we will equip $(AN)^l$ with the twisted diagonal action $\mathcal{T}$ given by (4.4).

Theorem 4.7 (Alekseev-Meinrenken-Woodward) *[Al-Me-W] For every quasi-split real form τ_d given in (3.5) and for every integer $l \geq 1$, there is a K–equivariant diffeomorphism $L : (AN)^l \to \mathfrak{p}^l$ such that*

$$\mathbf{m} = E \circ \mathbf{a} \circ L, \quad \text{and} \quad (\tau_d)^l \circ L = L \circ (\tau_d)^l. \tag{4.7}$$

Moreover, $L(\mathcal{D}_\Lambda) = \mathcal{O}_\Lambda$ for every $\Lambda \in \mathfrak{a}^l$.

Remark 4.8 Theorem 4.7, whose proof will be outlined in the Appendix, is a consequence of a Moser Isotopy Lemma for Hamiltonian K–actions on Poisson manifolds, a rigidity theorem for such spaces. More precisely, we will see in the Appendix that the map $E^{-1} \circ \mathbf{m} : (AN)^l \to \mathfrak{p}$ is a moment map for the twisted diagonal K–action $\mathcal{T}$ on $(AN)^l$ with respect to a Poisson structure π_1 on $(AN)^l$, and $(\tau_d)^l$ is an anti–Poisson involution for π_1. Moreover, the symplectic leaves of π_1 are precisely all the orbits $\mathcal{O}_\Lambda$ for $\Lambda \in \mathfrak{a}^l$. The quintuple

$$\mathcal{Q}_1 = ((AN)^l, \ \pi_1, \ \mathcal{T}, \ E^{-1} \circ \mathbf{m}, \ (\tau_d)^l)$$

will be referred to as a Hamiltonian Poisson K–space with anti–Poisson involution. In fact, $\mathcal{Q}_1$ belongs to a smooth family

$$\mathcal{Q}_s = ((AN)^l, \ \pi_s, \ \mathcal{T}_s, \ E^{-1} \circ \mathbf{m}_s, \ (\tau_d)^l)$$

as the case for $s = 1$, and when $s = 0$, $\mathcal{T}_s$ is the diagonal K–action on $(AN)^l$ and $E^{-1} \circ \mathbf{m}_0 = \mathbf{a} \circ (E^{-1})^l$. The Moser Isotopy Lemma, Proposition 5.1 in Section 5, implies that $\mathcal{Q}_s$ is isomorphic to $\mathcal{Q}_0$ by a diffeomorphism ψ_s of $(AN)^l$ for every $s \in \mathbb{R}$. The map L in Theorem 4.7 is then taken to be $(E^{-1})^l \circ \psi_1$.

We will assume Theorem 4.7 for now and prove Theorem 2.2 for G_0.

Proof of Theorem 2.2. Let $L : (AN)^l \to \mathfrak{p}^l$ be as in Theorem 4.7. Since L is K–equivariant and intertwines $(\tau_d)^l : (AN)^l \to (AN)^l$ and $(\tau_d)^l : \mathfrak{p}^l \to \mathfrak{p}^l$, it also intertwines

$$\sigma^{(l)} = \mathcal{T}_{\dot{w}_0} \circ (\tau_d)^l : \ (AN)^l \longrightarrow (AN)^l \quad \text{and} \quad \tau^l = \delta_{\dot{w}_0} \circ (\tau_d)^l : \ \mathfrak{p}^l \longrightarrow \mathfrak{p}^l.$$

Thus by Lemma 4.3, we know that $L((A_0N_0)^l) = (\mathfrak{p}_0)^l$. Denote $L|_{(A_0N_0)^l} : (A_0N_0)^l \to (\mathfrak{p}_0)^l$ also by L. Then clearly L is K_0–equivariant, and since $E_0 : \mathfrak{p}_0 \to A_0N_0$ coincides with the restriction of $E : \mathfrak{p} \to AN$ to $\mathfrak{p}_0$, we see that $\mathbf{m} = E_0 \circ \mathbf{a} \circ L$. Finally, let $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \in (\mathfrak{a}_0)^l$. Then by Lemma 4.6, we see that

$$L(D_\Lambda) = L(\mathcal{D}_\Lambda \cap (A_0N_0)^l) = \mathcal{O}_\Lambda \cap (\mathfrak{p}_0)^l = O_\Lambda.$$

Q.E.D.

Remark 4.9 Let $\mathfrak{a}^+ = \{x \in \mathfrak{a} : \alpha(x) \geq 0, \forall \alpha \in \Delta^+\}$ so that $\mathfrak{a}_0^+ = \mathfrak{a}_0 \cap \mathfrak{a}^+$. By Kirwan's convexity theorem, there exists a polyhedral cone $\mathcal{P} \subset (\mathfrak{a}^+)^l$ such that $\Lambda \in \mathcal{P}$ if and only if $\mathbf{a}^{-1}(0) \cap \mathcal{O}_\Lambda$ is non-empty. The cone $\mathcal{P}$ is explicitly computed in [B-S]. Set $\mathcal{P}_0 = \mathcal{P} \cap (\mathfrak{a}_0)^l \subset (\mathfrak{a}_0^+)^l$. Then by a theorem of O'Shea-Sjamaar [O-S], the set $\mathbf{a}^{-1}(0) \cap O_\Lambda$ is non-empty if and only if $\Lambda \in \mathcal{P}_0$. If we use $\mathcal{P}_d \subset (\mathfrak{a}^{\tau_d})^l$ to denote the polyhedral cone $\mathcal{P} \cap (\mathfrak{a}^{\tau_d})^l$ for the quasi-split form τ_d, it follows from $\mathfrak{a}_0 \subset \mathfrak{a}^{\tau_d}$ that

$$\mathcal{P}_0 = \mathcal{P}_d \cap (\mathfrak{a}_0)^l. \tag{4.8}$$

A statement related to this fact is given in [Fo].

5 Appendix: Alekseev-Meinrenken-Woodward theorem

In this appendix, we give an outline of the proof of the Alekseev-Meinrenken-Woodward theorem, stated as Theorem 4.7 in this paper. Theorem 3.5 of [Al-Me-W] shows the existence of a diffeomorphism $L_\Lambda : \mathcal{D}_\Lambda \to \mathcal{O}_\Lambda$ satisfying (4.7) for each $\Lambda \in \mathfrak{a}^l$. To show that all the L_Λ's come from a globally defined L on all of $(AN)^l$, one uses the Moser Isotopy Lemma for Hamiltonian Poisson K–spaces proved in [Al-Me]. What we present here is a collection of arguments from [Al] [Al-Me-W] [Ka-M-T] and [Al-Me].

5.1 Gauge transformation for Poisson structures

Recall that a Poisson structure on a manifold M is a smooth section π of $\wedge^2 TM$ such that $[\pi, \pi] = 0$, where $[\,,\,]$ is the Schouten bracket on the space of multi-vector fields on M. For a smooth section π of $\wedge^2 TM$, we will use $\pi^\#$ to denote the bundle map $\pi^\# : T^*M \to TM : \pi^\#(\alpha) = \pi(\cdot, \alpha)$ for all cotangent vectors α. Similarly, for a 2-form γ on M, we will set $\gamma^\# : TM \to T^*M : \gamma^\#(v) = \gamma(\cdot, v)$ for all tangent vectors v. Suppose now that π is a Poisson structure on M and that γ is a closed 2–form on M. If the bundle map $1 + \gamma^\# \pi^\# : T^*M \to T^*M$ is invertible, the section π' of $\wedge^2 TM$ given by

$$(\pi')^\# = \pi^\# (1 + \gamma^\# \pi^\#)^{-1} : \; T^*M \longrightarrow TM \tag{5.1}$$

is then a Poisson structure on M. The Poisson structure π' will be called the *gauge transformation* of π by the closed 2-from γ, and we write $\pi' = \mathcal{G}_\gamma(\pi)$. It is clear from (5.1) that π and π' have the same symplectic leaves. If S is a common symplectic leaf, then the symplectic 2-forms ω' and ω coming from π' and π differ by $i_S^* \gamma$, where $i_S : S \to M$ is the inclusion map. See [Se-W] for more detail.

5.2 The Poisson Lie groups $(K, s\pi_K)$ and $(AN, \bullet_s, \pi_{AN,s})$

The group AN carries a distinguished Poisson structure π_{AN}. Indeed, let $\langle\,,\,\rangle$ be the imaginary part of the Killing form of $\mathfrak{g}$ and identify $\mathfrak{k}$ with $(\mathfrak{a} + \mathfrak{n})^*$ via $\langle\,,\,\rangle$. For $x \in \mathfrak{k}$, denote by $\bar{x}$ the right invariant 1-form on AN defined by x. Let x_{AN} be the generator of the action of $\exp(tx)$ on AN according to (4.2). Then the unique section π_{AN} of $\wedge^2 T(AN)$ such that

$$\pi_{AN}^{\#}(\bar{x}) = x_{AN}, \qquad \forall x \in \mathfrak{k} \tag{5.2}$$

is a Poisson bi-vector field on AN. The Poisson structure π_{AN} makes AN into a *Poisson Lie group* in the sense that the group multiplication map $AN \times AN \to AN : (b_1, b_2) \mapsto b_1 b_2$ is a Poisson map, where $AN \times AN$ is equipped with the product Poisson structure $\pi_{AN} \times \pi_{AN}$. We refer to [L-W] and [L] for details on Poisson Lie groups and Poisson Lie group actions and to [L-Ra]) for details on π_{AN}. In particular, the dual Poisson Lie group of (AN, π_{AN}) is K together with the Poisson structure π_K, explicitly given by $\pi_K = \Lambda_0^r - \Lambda_0^l$, where

$$\Lambda_0 = \frac{1}{2} \sum_{\alpha \in \Delta^+} X_\alpha \wedge Y_\alpha \in \wedge^2 \mathfrak{k}$$

with $X_\alpha, Y_\alpha \in \mathfrak{k}$ given in (3.3) and Λ_0^r and Λ_0^l being respectively the right and left invariant bi-vector fields on K determined by Λ_0. It follows from (5.2) that the symplectic leaves of π_{AN} are precisely the orbits of the K–action on AN given in (4.2).

Let now d be an involutive automorphism of the Dynkin diagram of $\mathfrak{g}$, and let τ_d be the quasi–split real form of $\mathfrak{g}$ given in (3.5). Recall that τ_d leaves AN invariant and defines a group isomorphism on AN. It is easy to check (see also Section 2.3 of [Al-Me-W]) that $\tau_d : (AN, \pi_{AN}) \to (AN, \pi_{AN})$ is anti-Poisson, i.e., $\tau_{d*}\pi_{AN} = -\pi_{AN}$. Similarly, $\tau_d : K \to K$ is anti–Poisson for π_K. We will denote the restrictions of τ_d to K and to AN both by τ_d, and we will refer to (K, π_K, τ_d) and (AN, π_{AN}, τ_d) as a *dual pair* of *Poisson Lie groups with anti–Poisson involutions*. In the context of Poisson Lie groups, the K–action on AN given in (4.2) and the AN–action on K given in (2.6) (with AN replacing A_0N_0 and K replacing K_0) are respectively called the *dressing actions*.

As is noticed in [Al] and [Ka-M-T], we in fact have a smooth family of Poisson Lie groups $(AN, \bullet_s, \pi_{AN,s})$ for $s \in \mathbb{R}$. Indeed, for $s \in \mathbb{R} - \{0\}$, let $F_s : \mathfrak{p} \to \mathfrak{p}$ be the diffeomorphism $x \mapsto sx$, and let $I_s = E \circ F_s \circ E^{-1} : AN \to AN$. Let $\pi_{AN,s}$ be the Poisson bi-vector field on AN such that $(I_s)_* \pi_{AN,s} = s\pi_{AN}$, or,

$$\pi_{AN,s}(b) = s(I_s^{-1})_* \left(\pi_{AN}(I_s(b))\right), \qquad b \in AN. \tag{5.3}$$

Define the group structure $\bullet_s : AN \times AN \to AN$ by

$$b_1 \bullet_s b_2 := I_s^{-1}(I_s(b_1) I_s(b_2)), \qquad b_1, b_2 \in AN.$$

Then since $s \neq 0$, the map $(AN \times AN,\ \pi_{AN,s} \times \pi_{AN,s}) \to (AN, \pi_{AN,s}) : (b_1, b_2) \mapsto b_1 b_2$ is a Poisson map, so $(AN, \bullet_s, \pi_{AN,s})$ is a Poisson Lie group for each $s \in \mathbb{R}$, $s \neq 0$. On the other hand, $\mathfrak{p} \cong \mathfrak{k}^*$ has the linear Poisson structure $\pi_{\mathfrak{p},0}$ defined by the Lie algebra $\mathfrak{k}$. Let $\bullet_0$ and $\pi_{AN,0}$ be the pullbacks to AN by $E^{-1} : AN \to \mathfrak{p}$ of the abelian group structure on $\mathfrak{p}$ and the Poisson structure $\pi_{\mathfrak{p},0}$ on $\mathfrak{p}$. Then we get a smooth family of Poisson Lie group structures $(\bullet_s, \pi_{AN,s})$ on AN for every $s \in \mathbb{R}$ (see [Al] and [Ka-M-T]). The dual Poisson Lie group of $(AN, \bullet_s, \pi_{AN,s})$ is again the Lie group K (with group structure independent on

s) but with the Poisson structure $s\pi_K$, if we identify again $\mathfrak{k} \cong (\mathfrak{a}+\mathfrak{n})^*$ via the imaginary part of the Killing form of $\mathfrak{g}$. It is also clear that τ_d is a group isomorphism for $\bullet_s$ and is anti–Poisson for $\pi_{AN,s}$. Thus we get a dual pair of Poisson Lie groups $(K, s\pi_K, \tau_d)$ and $(AN, \bullet_s, \pi_{AN,s}, \tau_d)$ with anti–Poisson involutions for each $s \in \mathbb{R}$.

5.3 The Poisson structure π_s on $(AN)^l$

As is noted in [Al], for each $s \in \mathbb{R}$, the Poisson structure $\pi_{AN,s}$ on AN is related to π_{AN} by a gauge transformation. Recall that for $x \in \mathfrak{k} \cong (\mathfrak{a}+\mathfrak{n})^*$, $\bar{x}$ is the right invariant 1–form on AN defined by x. Let l_x be the differential of the linear function $\xi \mapsto \langle x, \xi\rangle$ on $\mathfrak{p}$, and let $x_{\mathfrak{p}}$ be the vector field on $\mathfrak{p}$ generating the adjoint action of $\exp(tx) \in K$ on $\mathfrak{p}$. By Proposition 3.1 of [Al-Me-W], there is a 1-form β on $\mathfrak{p}$ such that $\beta(0) = 0$ and

$$-(d\beta)^{\#}(x_{\mathfrak{p}}) = E^*\bar{x} - l_x, \qquad \forall x \in \mathfrak{k},$$

Moreover, $(\tau_d)^*\beta = -\beta$ for every quasi-split real form τ_d given in (3.5). For $s \in \mathbb{R} - \{0\}$, let $\beta_s = \frac{1}{s}(F_s^*\beta)$. Since $\beta(0) = 0$, the family β_s extends smoothly to $\beta_0 = 0$. Let

$$\alpha = (E^{-1})^*\beta, \quad \text{and} \quad \alpha_s = (E^{-1})^*\beta_s = \frac{1}{s}(I_s^*\alpha)$$

for all $s \in \mathbb{R}$. Then it is easy to show that, for every $s \in \mathbb{R}$,

$$\pi_{AN,s} = \mathcal{G}_{d(\alpha-\alpha_s)}(\pi_{AN}) = \mathcal{G}_{-d\alpha_s}(\pi_{AN,0}).$$

Assume now that $l \geq 1$ is an integer. For each $s \in \mathbb{R}$, set

$$\mathbf{m}_s : (AN)^l \longrightarrow AN : (b_1, b_2, \cdots, b_l) \longmapsto b_1 \bullet_s b_2 \bullet_s \cdots \bullet_s b_l.$$

By generalizing the "linearization" procedure of Hamiltonian symplectic $(K, s\pi_K)$–spaces described in [Al-Me-W] to the case of Poisson manifolds, one can show that

$$\pi_s := \mathcal{G}_{d\mathbf{m}_s^*\alpha_s}(\pi_{AN,s} \times \pi_{AN,s} \times \cdots \times \pi_{AN,s}) \tag{5.4}$$

is a well-defined Poisson structure on $(AN)^l$ for each $s \in \mathbb{R}$. Define the twisted diagonal action $\mathcal{T}_s$ of K on $(AN)^l$ by

$$k \longmapsto \mathcal{T}_{s,k} := \nu_s^{-1} \circ \delta_k \circ \nu_s, \tag{5.5}$$

where, again, δ_k denotes the diagonal action of k on $(AN)^l$ for the action of K on AN given in (4.2), and $\nu_s \in \mathrm{Diff}((AN)^l)$ is given by

$$\nu_s(b_1, b_2, \cdots, b_l) = (b_1,\ b_1 \bullet_s b_2,\ \cdots,\ b_1 \bullet_s b_2 \bullet_s \cdots \bullet_s b_l).$$

Note that $\mathcal{T}_{s,k}$ is $\mathcal{T}_k$ when $s = 1$ and is δ_k when $s = 0$. Then again it follows from [Al-Me-W] that for each $s \in \mathbb{R}$, the action $\mathcal{T}_s$ of K on $(AN)^l$ is Hamiltonian with respect to the Poisson structure π_s with the map $E^{-1} \circ \mathbf{m}_s : (AN)^l \to \mathfrak{p} \cong \mathfrak{k}^*$ as a moment map. Moreover, for every quasi–split real form τ_d defined in (3.5), the Cartesian product $(\tau_d)^l = \tau_d \times \tau_d \times \cdots \times \tau_d$ is anti–Poisson for π_s for every $s \in \mathbb{R}$.

5.4 The Moser Isotopy Lemma

Let U be a connected Lie group with Lie algebra $\mathfrak{u}$. Suppose that σ_U is an involutive automorphism of U with the corresponding involution on $\mathfrak{u}$ denoted by $\sigma_{\mathfrak{u}}$. Define $\sigma_{\mathfrak{u}^*} = -(\sigma_{\mathfrak{u}})^*$. If (M, π_M, Φ) is a Hamiltonian Poisson U–space, an anti-Poisson involution σ_M of (M, π_M) is said to be compatible with σ_U if $\Phi \circ \sigma_M = \sigma_{\mathfrak{u}^*} \circ \Phi$. The following Moser Isotopy Lemma for Hamiltonian Poisson U–spaces with anti–Poisson involutions is proved in [Al-Me]. See [Al-Me-W] for the symplectic case.

Proposition 5.1 *Let U be a connected compact semi-simple Lie group with Lie algebra $\mathfrak{u}$, and let (M, π_s, Φ_s), $s \in \mathbb{R}$, be a smooth family of Hamiltonian Poisson U–spaces. Suppose that there exists a smooth family of 1-forms ϵ_s on M with $\epsilon_0 = 0$ such that $\pi_s = \mathcal{G}_{d\epsilon_s}\pi_0$ for every $s \in \mathbb{R}$. Assume also that π_0 has compact symplectic leaves. Then (M, π_s, Φ_s) is isomorphic to (M, π_0, Φ_s) for every $s \in \mathbb{R}$ as Hamiltonian Poisson U–spaces, i.e., there exists $\psi_s \in \mathrm{Diff}(M)$ for $s \in \mathbb{R}$ with $\psi_0 = \mathrm{id}$, such that for every $s \in \mathbb{R}$,*

1) $\pi_s = \psi_{s*}\pi_0$; 2) $\Phi_s \circ \psi_s = \Phi_0$.

If σ_U is an involutive automorphism on U, and if for each $s \in \mathbb{R}$, $\sigma_{M,s}$ is an anti-Poisson involution for π_s compactible with σ_U and such that $\sigma_{M,s}^*\dot{\epsilon}_s = -\dot{\epsilon}_s$, then ψ_s can be chosen such that $\psi_s \circ \sigma_{M,0} = \sigma_{M,s} \circ \psi_s$ for all $s \in \mathbb{R}$.

5.5 Proof of Theorem 4.7

Consider the Hamiltonian Poisson K–space $(M = (AN)^l, \pi_s, \Phi_s)$ with $\Phi_s = E^{-1} \circ \mathbf{m}_s$. The action of K on $(AN)^l$ induced by (π_s, Φ_s) is the twisted diagonal action $\mathcal{T}_s$ given in (5.4). From the definition of π_s, we know that $\pi_s = \mathcal{G}_{d\epsilon_s}\pi_0$, where

$$\epsilon_s = \mathbf{m}_s^*\alpha_s - \sum_{j=1}^{l} p_j^*\alpha_s$$

with $p_j : (AN)^l \to AN$ denoting the projection to the j'th factor. For every quasi-split real form τ_d given in (3.5), since τ_d is a group isomorphism for $(AN, \bullet_s)$, we have $\tau_d^*\epsilon_s = -\epsilon_s$, and thus $\tau_d^*\dot{\epsilon}_s = -\dot{\epsilon}_s$ for every $s \in \mathbb{R}$. Let $\sigma_{M,s} = (\tau_d)^l$ and let $\psi_s \in \mathrm{Diff}((AN)^l)$ be as in Proposition 5.1. Then $L := (E^{-1})^l \circ \psi_1^{-1} : (AN)^l \to (\mathfrak{p})^l$ is the diffeomorphism in Theorem 4.7. Indeed,

$$E \circ a \circ L = E \circ a \circ (E^{-1})^l \circ \psi_1^{-1} = E \circ \Phi_0 \circ \psi_1^{-1} = E \circ \Phi_1 = m,$$

where the second equality follows from the identity

$$a \circ (E^{-1})^l = E^{-1} \circ m_0,$$

which is a trivial consequence of the fact that m_0 is the pullback of addition by the map E.

Q.E.D.

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