POISSON STRUCTURES ON COMPLEX FLAG MANIFOLDS
ASSOCIATED WITH REAL FORMS

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Abstract. For a complex semisimple Lie group $G$ and a real form $G_0$ we define a
Poisson structure on the variety of Borel subgroups of $G$ with the property that all $G_0$-
orbits in $X$ as well as all Bruhat cells (for a suitable choice of a Borel subgroup of $G$)
are Poisson submanifolds. In particular, we show that every non-empty intersection of a
$G_0$-orbit and a Bruhat cell is a regular Poisson manifold and we compute the dimension
of its symplectic leaves.

Dedicated to Alan Weinstein on
the occasion of his 60th Birthday.

1. Introduction.

Let $G$ be a connected and simply connected complex semisimple Lie group with Lie
algebra $\mathfrak{g}$, and let $X$ be the variety of Borel subalgebras of $\mathfrak{g}$. In this paper we use a
real form $\mathfrak{g}_0$ of $\mathfrak{g}$ to define a Poisson structure on $X$. This Poisson structure depends on
a choice of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{g}_0 \cap \mathfrak{b}$ is a maximally compact Cartan
subalgebra of $\mathfrak{g}_0$. Instead of dealing with each real form individually, we fix a Borel
subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Then, as is shown in [6], a real from $\mathfrak{g}_v$
of $\mathfrak{g}$ can be constructed from each Vogan diagram $v$ for $\mathfrak{g}$ such that $\mathfrak{g}_v \cap \mathfrak{b}$ is a maximally
compact Cartan subalgebra of $\mathfrak{g}_v$. The corresponding Poisson structure on $X$ is denoted
by $\Pi_v$.

Let $G_v$ be the real form of $G$ corresponding to $\mathfrak{g}_v$, and let $B$ be the Borel subgroup of $G$
with Lie algebra $\mathfrak{b}$. The Poisson structure $\Pi_v$ has the property that each $G_v$-orbit as well
as each $B$-orbit in $X$ is a Poisson submanifold. The $B$-orbits in $X$ will be referred to as
the Bruhat cells. We compute the rank of $\Pi_v$. In particular, if $G_v$-orbit $\mathcal{O}$ meets a Bruhat
cell $\mathcal{C}$, they intersect transversally, and we find that all the symplectic leaves in $\mathcal{O} \cap \mathcal{C}$
have the same dimension, so $\mathcal{O} \cap \mathcal{C}$ is a regular Poisson manifold. Moreover, we show that

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all symplectic leaves in each connected component of $\mathcal{O} \cap \mathcal{C}$ are translates of each other by elements of a Cartan subgroup of $G_v$. We also show that the $G_v$-invariant Poisson cohomology for each open $G_v$-orbit in $X$ is isomorphic to the de Rham cohomology of $X$.

Throughout this paper, if $V$ is a set and $\sigma$ is an involution on $V$, we will use $V^\sigma$ to denote the fixed point set of $\sigma$ in $V$.

2. Real forms of $g$ and Vogan diagrams

Let $g$ be a complex simple Lie algebra. In this section we recall the classification of real forms of $g$ by Vogan diagrams. Details can be found in [6, Chapter 6].

Suppose that $g_0$ is a real form of $g$ and that $\tau_0$ is the corresponding complex-conjugate linear involution on $g$. Let $\theta_0$ be a Cartan involution of $g_0$, and let $h_0$ be a $\theta_0$-stable maximally compact Cartan subalgebra of $g_0$. Set $t_0 = h_0^\theta$ and $a_0 = h_0^{-\theta_0}$ so that $h_0 = t_0 + a_0$. Let $\gamma_0$ be the complexification of $\theta_0$. Then the Cartan subalgebra $h = h_0 + i h_0$ of $g$ is $\gamma_0$-stable. Let $\Delta$ be the root system for $(g, h)$. Since $h_0$ is a maximally compact Cartan subalgebra of $g_0$, there exists $x_0 \in i t_0$ that is regular for $\Delta$. Define the subset $\Delta^+ \subset \Delta$ of positive roots in $\Delta$ by $\alpha \in \Delta^+$ if and only if $\alpha(x_0) > 0$. Then $\gamma_0(\Delta^+) = \Delta^+$. Let $\Sigma \subset \Delta^+$ be the set of simple roots in $\Delta^+$. Then $\gamma_0(\Sigma) = \Sigma$, so $\gamma_0$ gives rise to an involutive automorphism of the Dynkin diagram of $g$. Let $I$ be the set of non-compact imaginary simple roots. The Vogan diagram of $g_0$ associated to the triple $(\theta_0, h_0, \Delta^+)$ is the Dynkin diagram $D(g)$ of $g$ together with an involutive automorphism $\gamma_0$ on $D(g)$ and the vertices corresponding to the simple roots in $I$ painted black.

In general, a Vogan diagram for $g$ is defined to be a triple $(D(g), d, I)$, where $D(g)$ is the Dynkin diagram of $g$, $d$ is an involutive automorphism of $D(g)$, and $I$ is a subset of vertices of $D(g)$ such that $d(\alpha) = \alpha$ for each $\alpha \in I$. Every Vogan diagram for $g$ comes from a real form of $g$ (see below), although two different Vogan diagrams can come from isomorphic real forms. A non-redundant list of Vogan diagrams with the corresponding isomorphism class of real forms for all simple Lie algebras is given in [6]. Every Vogan diagram in the list in [6] is normalized in the sense that at most one vertex is painted black.

For the purpose of defining Poisson structures on the variety of Borel subalgebras of $g$, we now recall the explicit construction of a real form of $g$ from a Vogan diagram [6, Theorem 6.88]. We need to fix the following data for $g$.

Choose a Cartan subalgebra $h$ of $g$ and let $\Delta$ be the root system for $(g, h)$. Fix a choice of positive roots $\Delta^+$ and let $\Sigma$ be the basis of simple roots. Let $\ll, \gg$ be the Killing form of $g$ and let root vectors $\{E_\alpha : \alpha \in \Delta\}$ be chosen such that $[E_\alpha, E_{-\alpha}] = H_\alpha$ for each $\alpha \in \Delta^+$, where $H_\alpha$ is the unique element of $h$ defined by $\ll H, H_\alpha \gg = \alpha(H)$ for all $H \in h$, and such that the numbers $m_{\alpha, \beta}$ given by $[E_\alpha, E_\beta] = m_{\alpha, \beta} E_{\alpha + \beta}$ when $\alpha + \beta \in \Delta$
are real. Define a compact real form \( \mathfrak{k} \) of \( \mathfrak{g} \) as
\[
\mathfrak{k} = \text{span}_\mathbb{R}\{iH_\alpha, X_\alpha := E_\alpha - E_{-\alpha}, Y_\alpha := i(E_\alpha + E_{-\alpha})\},
\]
and let \( \theta \) be the complex conjugation of \( \mathfrak{g} \) defining \( \mathfrak{k} \). If \( d \) is an involutive automorphism of the Dynkin diagram of \( \mathfrak{g} \), define \( \gamma_d \) to be the unique automorphism of \( \mathfrak{g} \) satisfying \( \gamma_d(H_\alpha) = H_{d(\alpha)} \) and \( \gamma_d(E_\alpha) = E_{d(\alpha)} \) for each simple root \( \alpha \).

Given a Vogan diagram \( v \) for \( \mathfrak{g} \), not necessarily normalized, with the involutive diagram automorphism \( d \), let \( t_v \) be the unique element in the adjoint group of \( \mathfrak{g} \) such that
\[
\text{Ad}_{t_v}(E_\alpha) = \begin{cases} E_\alpha & \text{if } \alpha \text{ is a blank vertex in } v \\ -E_\alpha & \text{if } \alpha \text{ is a painted vertex in } v \end{cases}
\]
Define a complex conjugate linear involution
\[
\tau_v := \text{Ad}_{t_v} \circ \gamma_d \circ \theta.
\]

**Notation 2.1.** We use \( \mathfrak{g}_v = \mathfrak{g}^{\tau_v} \) to denote the real form of \( \mathfrak{g} \) defined by \( \tau_v \). Set \( \theta_v = \theta|_{\mathfrak{g}_v} \). Then \( \theta_v \) is a Cartan involution of \( \mathfrak{g}_v \), and \( \mathfrak{h}^{\tau_v} \) is a \( \theta_v \)-stable maximally compact Cartan subalgebra of \( \mathfrak{g}_v \), with \( \mathfrak{h} = \mathfrak{h}^{\tau_v} + i\mathfrak{h}^{\tau_v} \). The complexification of \( \tau_v \) is
\begin{equation}
\gamma_v := \tau_v \theta = \theta \tau_v = \text{Ad}_{t_v} \gamma_d.
\end{equation}
Since \( \gamma_v(\Delta^+) = \Delta^+ \), the Vogan diagram of \( \mathfrak{g}_v \) associated to the triple \((\theta_v, \mathfrak{h}^{\tau_v}, \Delta^+) = v\).

One of the advantages of introducing the real form \( \mathfrak{g}_v \) is as follows. We say that a real subalgebra \( \mathfrak{l} \) of \( \mathfrak{g} \) is Lagrangian if its real dimension is equal to the complex dimension of \( \mathfrak{g} \) and if \( \text{Im} x_1, x_2 \gg 0 \) for all \( x_1, x_2 \in \mathfrak{l} \). A decomposition \( \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2 \) is called a Lagrangian splitting if both \( \mathfrak{l}_1 \) and \( \mathfrak{l}_2 \) are Lagrangian. Let \( \mathfrak{n} \) be the subalgebra of \( \mathfrak{g} \) spanned by the set of all positive root vectors for \( \Delta^+ \). The following fact is easy to prove.

**Lemma 2.2.** Let \( \mathfrak{l}_d := \mathfrak{h}^{-\tau_v} + \mathfrak{n} \). Then \( \mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d \) is a Lagrangian splitting of \( \mathfrak{g} \).

Let \( \mathfrak{a} = \text{span}_\mathbb{R}\{iH_\alpha : \alpha \in \Sigma\} \), and let \( \mathfrak{k} = i\mathfrak{a} \). We note that since
\[
\mathfrak{h}^{-\tau_v} = \mathfrak{h}^{\gamma_d \circ \theta} = \mathfrak{t}^{\gamma_d} + \mathfrak{a}^{\gamma_d},
\]
the Lagrangian complement \( \mathfrak{l}_d \) of \( \mathfrak{g}_v \) depends only on \( d \), and in the case when \( d = 1 \), we have \( \mathfrak{l}_d = \mathfrak{a} + \mathfrak{n} \). Note that \( \mathfrak{h}^{\tau_v} = \mathfrak{h}^{\gamma_d \circ \theta} = \mathfrak{t}^{\gamma_d} + \mathfrak{a}^{\gamma_d} \) also depends only on \( d \).

**Remark 2.3.** Recall [2, Definition 6.10] that two real forms \( \tau_1 \) and \( \tau_2 \) are said to be in the same inner class if there exists \( q \in \text{Int}(\mathfrak{g}) \), the adjoint group of \( \mathfrak{g} \), such that \( \tau_1 = \text{Ad}_q \tau_2 \). Inner classes of real forms are in one-to-one correspondence with involutive automorphisms of the Dynkin diagram of \( \mathfrak{g} \) [2, Proposition 6.12]. Let \( d \) be an involutive automorphism of \( D(\mathfrak{g}) \). Then as \( v \) runs over the collection of all Vogan diagrams with \( d \) as the diagram automorphism, the real form \( \mathfrak{g}_v \) runs over all \( \text{Int}(\mathfrak{g}) \)-conjugacy classes of real forms of \( \mathfrak{g} \) in the inner class corresponding to \( d \).
3. The Poisson structure $\Pi_v$ on $X$.

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, and let $X$ be the variety of all Borel subalgebras of $\mathfrak{g}$. We keep the notation from Section 2. Let $v$ be a Vogan diagram for $\mathfrak{g}$ and $\mathfrak{g}_v = \mathfrak{g}^{\tau_v}$ be the real form of $\mathfrak{g}$ constructed in Section 2. Let $G$ be the connected and simple connected Lie group with Lie algebra $\mathfrak{g}$. Without any risk of confusion, we shall also denote by $\tau_v$ the lift of $\tau_v$ from $G$ to $G$, and we set $G_v = G^{\tau_v}$. It follows from [5, Theorem 8.2, p. 320] that the group $G_v$ is connected.

In this section, we will start with a Vogan diagram $v$ for $\mathfrak{g}$ and define a Poisson structure $\Pi_v$ on $X$ such that every $G_v$-orbit in $X$ is a Poisson submanifold. This Poisson structure comes from an identification of $X$ with the $G$-orbit through $t + \mathfrak{n}$ inside the variety $\mathcal{L}$ of Lagrangian subalgebras of $\mathfrak{g}$, which was studied in [3]. We now recall the relevant details.

Set $n = \dim \mathfrak{g}$ and let $\text{Gr}_R(n, \mathfrak{g})$ be the Grassmannian of real $n$-dimensional subspaces of $\mathfrak{g}$. The set $\mathcal{L}$ of all Lagrangian subalgebras of $\mathfrak{g}$ is naturally a real subvariety of $\text{Gr}_R(n, \mathfrak{g})$. The natural action of $G$ on $\text{Gr}_R(n, \mathfrak{g})$ gives rise to a Lie algebra anti-homomorphism $\kappa$ from $\mathfrak{g}$ to the Lie algebra of vector fields on $\text{Gr}_R(n, \mathfrak{g})$, whose extension from $\wedge^2 \mathfrak{g}$ to the space of bi-vector fields on $\text{Gr}_R(n, \mathfrak{g})$ will also be denoted by $\kappa$. Given a Lagrangian splitting $\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$, we define the element $R_{\mathfrak{l}_1, \mathfrak{l}_2} \in \wedge^2 \mathfrak{g}$ by:

\[
(3.1) \quad \langle R_{\mathfrak{l}_1, \mathfrak{l}_2}, (x_1 + \xi_1) \wedge (x_2 + \xi_2) \rangle = \langle \xi_2, x_1 \rangle - \langle \xi_1, x_2 \rangle, \quad x_1, x_2 \in \mathfrak{l}_1, \xi_1, \xi_2 \in \mathfrak{l}_2,
\]

where $\langle \cdot, \cdot \rangle = \text{Im} \ll \cdot, \cdot \gg$. Set $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2} = \frac{1}{2} \kappa(R_{\mathfrak{l}_1, \mathfrak{l}_2})$. Clearly, $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ is tangent to every $G$-orbit in $\text{Gr}_R(n, \mathfrak{g})$, so it is tangent to $\mathcal{L}$.

**Theorem 3.1.** [3, Theorems 2.14 and 2.18] The bi-vector field $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ restricts to a Poisson structure on $\mathcal{L}$. If $L_1$ and $L_2$ are the connected subgroups of $G$ with Lie algebras $\mathfrak{l}_1$ and $\mathfrak{l}_2$ respectively, then all the $L_1$- as well as $L_2$-orbits in $\mathcal{L}$ are Poisson submanifolds with respect to $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$.

For $\mathfrak{l} \in \mathcal{L}$, let $\mathfrak{n}(\mathfrak{l})$ be the normalizer subalgebra of $\mathfrak{l}$ in $\mathfrak{l}_1$. Let $\mathfrak{m}(\mathfrak{l})$ be the annihilator of $\mathfrak{n}(\mathfrak{l})$ in $\mathfrak{l}$, i.e. $\mathfrak{m}(\mathfrak{l}) = \{ x \in \mathfrak{l} : \langle x, y \rangle = 0 \ \forall y \in \mathfrak{n}(\mathfrak{l}) \} \subset \mathfrak{l}$, and let $\mathcal{V}(\mathfrak{l}) = \mathfrak{n}(\mathfrak{l}) + \mathfrak{m}(\mathfrak{l})$.

**Proposition 3.2.** [3, Theorem 2.21] [9, Corollary 7.3] For each $\mathfrak{l} \in \mathcal{L}$, the space $\mathcal{V}(\mathfrak{l})$ is a Lagrangian subalgebra of $\mathfrak{g}$. The co-dimension of the symplectic leaf of $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ through $\mathfrak{l}$ in the orbit $L_1 \cdot \mathfrak{l}$ is equal to $\text{dim}(\mathcal{V}(\mathfrak{l}) \cap \mathfrak{l})$.

**Notation 3.3.** Let $v$ be a Vogan diagram for $\mathfrak{g}$. We denote by $\Pi_v$ the Poisson structure on $\mathcal{L}$ defined by the Lagrangian splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ in Lemma 2.2. Let $H$, $N$, and $B$ be respectively the connected subgroups of $G$ with Lie algebras $\mathfrak{h}$, $\mathfrak{n}$, and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, so $B = HN$. Identify the $G$-orbit through $t + \mathfrak{n} \in \mathcal{L}$ with $G/B \cong X$. The induced Poisson structure on $X$ will also be denoted by $\Pi_v$. Let $H^{-\gamma_d \omega} = \{ h \in H : \gamma_d \circ \theta(h) = h^{-1} \}$ and let $L_d = H^{-\gamma_d \omega} N$. By the Bruhat lemma, orbits of $L_d$ in $X \cong G/B$, which are the same
as the N-orbits in X, are labeled by the elements in the Weyl group W of Δ. We refer to these N-orbits as the Bruhat cells in X.

By [3, Theorem 2.18], we have

**Proposition 3.4.** Each $G_v$-orbit in X as well as each Bruhat cell in X is a Poisson submanifold with respect to $\Pi_v$.

When $v$ is the Vogan diagram with $d = 1$ and no vertex painted, we have $\tau_v = \theta$, so $g_v = \mathfrak{k}$. The Poisson structure $\Pi_v$ in this case was first introduced in [11] and [13], and it has the property that its symplectic leaves are precisely the Bruhat cells (hence the name “Bruhat Poisson structure” in [11]). In [3] and [10] this Poisson structure was related to some earlier work of Kostant [7] and of Kostant-Kumar [8] on the Schubert calculus on X.

The splitting $g = g_v + l_d$ naturally defines a Lie bialgebra structure on $g_v$ and therefore a Poisson Lie group structure on $G_v$ [11]. All the $G_v$-orbits in $\mathcal{L}$ become $G_v$-Poisson homogeneous spaces [3, 9]. We remark that in [1], Andruskiewitsch and Jancsa classified non-triangular Lie bialgebra structures on $g_v$ using Belavin-Drinfeld triples. The one defined by the splitting $g = g_v + l_d$ comes from the standard Belavin-Drinfeld triple. We refer to [1] for details.

**Example.** Here we take $g = \mathfrak{sl}(2, \mathbb{C})$ and

$$g_v = \mathfrak{su}(1, 1) = \left\{ \begin{pmatrix} ix & y + iz \\ y - iz & -ix \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$ Then $d = 1$ and $l_d = \mathfrak{a} + \mathfrak{n}$ consists of upper triangular matrices in $\mathfrak{sl}(2, \mathbb{C})$ with real diagonal entries. Identify $G/B$ with $\mathbb{P}^1$ via the action

$$\begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix} \cdot [w_0 : w_1] = [aw_0 + bw_1 : cw_0 + dw_1]$$

of $G$ on $\mathbb{P}^1$ and by taking $[1 : 0] \in \mathbb{P}^1$ as the basepoint. There are two Bruhat cells: the zero-dimensional basepoint $[1 : 0]$, and the other being the rest:

$$U_1 = \mathbb{P}^1 \setminus \{[1 : 0]\} = \{[w_0 : w_1], \ w_1 \neq 0\}.$$ In terms of the holomorphic coordinate $z$ on $U_1$ given by $z = w_0/w_1$ the Poisson structure $\Pi_v$, up to a scalar multiple, is given by:

$$\Pi_v = i(1 - |z|^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}.$$ Setting $u = 1/z$, we see that in the $u$-coordinate on the open set

$$U_0 = \{[w_0 : w_1] \in \mathbb{P}^1, \ w_0 \neq 0\} = \{[1 : u], u \in \mathbb{C}\},$$
we have
\[ \Pi_v = i(|u|^2 - 1)|u|^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \bar{u}}. \]

Thus \( \Pi_v \) vanishes precisely at the basepoint \([1 : 0]\) and at every point of the form \([z : 1]\) with \(|z| = 1\). If we identify \( \mathbb{P}^1 \) with the unit sphere \( S^2 \) in \( \mathbb{R}^3 \) via:

\[ (3.2) \quad \mathbb{P}^1 \rightarrow S^2 : [w_0, w_1] \mapsto \left( \frac{2\text{Re}(w_0\bar{w}_1)}{|w_0|^2 + |w_1|^2}, \frac{2\text{Im}(w_0\bar{w}_1)}{|w_0|^2 + |w_1|^2}, \frac{|w_0|^2 - |w_1|^2}{|w_0|^2 + |w_1|^2} \right), \]

then we see that \( \Pi_v \) vanishes at the “North pole” \((0, 0, 1)\) and at every point on the Equator \(x_3 = 0\). Under this identification, there are exactly three orbits of \( SU(1,1) \) on \( S^2 \): the Northern hemisphere, the Equator, and the Southern hemisphere. Each one of these three orbits is clearly a Poisson submanifold.

4. SYMPLECTIC LEAVES OF \( \Pi_v \) IN \( X \).

Suppose that \( \mathcal{O} \) is a \( G_v \)-orbit in \( X \) and \( C \) is a Bruhat cell such that \( \mathcal{O} \cap C \neq \emptyset \). Since \( g = g_v + I_d \), \( \mathcal{O} \) and \( C \) intersect transversally. By Proposition 3.4, \( \mathcal{O} \cap C \) is a Poisson submanifold of \( \Pi_v \). In this section we show that \((\mathcal{O} \cap C, \Pi_v)\) is a regular Poisson manifold and we compute the dimension of its symplectic leaves.

It is well-known [14] that there are only finitely many \( G_v \)-orbits in \( X \). We first recall from [12, Section 6] some facts about these orbits.

Let \( N_G(h) \) be the normalizer subgroup of \( h \) in \( G \). Set

\[ Z = \{ g \in G : g^{-1} \tau_v(g) \in N_G(h) \}. \]

Then \( H \) acts on \( Z \) from the right by right multiplication, and \( G_v \) acts on \( Z \) from the left by left multiplication. Let \( Z \) be the double coset space

\[ Z = G_v \backslash Z / H. \]

For each \( z \in Z \), choose any \( g_z \in Z \) in the double coset \( z \) and define \( \mathcal{O}_z \) to be the \( G_v \)-orbit in \( X \) through \( g_zB \in X \cong G/B \). Clearly, \( \mathcal{O}_z \) is independent of the choice of \( g_z \). According to [12, Theorem 6.1.4], the map \( z \mapsto \mathcal{O}_z \) is a one-to-one correspondence between the set \( Z \) and the set of \( G_v \)-orbits in \( X \). Let \( W = N_G(h)/H \) be the Weyl group. Thus we also have the map

\[ \varphi : Z \rightarrow W : z = G_vg_zH \mapsto g_z^{-1} \tau_v(g_z)H \in W. \]

According to [12, Theorem 6.4.2], the codimension of the \( G_v \)-orbit \( \mathcal{O}_z \) in \( X \) equals \( l(\varphi(z)) \), where \( l \) is the length function on the Weyl group \( W \). We also introduce the map:

\[ \sigma_z = \varphi(z) \tau_v : h \mapsto h. \]

For any \( g_z \) in the double coset \( z \), we also have \( \sigma_z = \text{Ad}^{-1}_{g_z} \circ \tau_v \circ \text{Ad}_{g_z} \), so \( \sigma_z \) is an involution.
Assume now that $z \in Z$ and $w \in W$ are such that $O_z \cap C_w \neq \emptyset$, where $C_w$ is the Bruhat cell in $X$ corresponding to $w$, i.e. the $N$-orbit through $w \in G/B$. Then $\dim_{\mathbb{R}} C_w = 2l(w)$, and since $O_z$ and $C_w$ intersect transversally, we have
\[
\dim(O_z \cap C_w) = 2l(w) - l(\varphi(z)).
\]
Define now
\[
\delta_{z,w} = \dim(\mathfrak{h}^{w\sigma_w} \cap \mathfrak{h}^{-\tau_w}).
\]

**Theorem 4.1.** Each symplectic leaf in the intersection $O_z \cap C_w$ has dimension equal to
\[
\dim(O_z \cap C_w) - \delta_{z,w} = 2l(w) - l(\varphi(z)) - \delta_{z,w}.
\]

**Proof.** We use Proposition 3.2 to compute dimensions of the symplectic leaves in $O_z \cap C_w$.

Let $x = g_z B \in X$ be a point in $O_z \cap C_w$, where $g_z \in Z$ lies in the double coset $z$. Let $\iota_x = \text{Ad}_{g_z}(t + n) \in \mathcal{L}$. Let $n(\iota_x) = g_v \cap \text{Ad}_{g_z}(h + \mathfrak{n})$ be the normalizer subalgebra of $\iota_x$ in $g_v$, $m(\iota_x)$ the annihilator subspace of $n(\iota_x)$ in $\iota_x$, and $V(\iota_x) = n(\iota_x) + m(\iota_x)$. We claim that $V(\iota_x) = \text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})$. Indeed, it follows from the definition of $\sigma_z$ that
\[
\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z}) \subset g_v \cap \text{Ad}_{g_z}(\mathfrak{h} + \mathfrak{n}) = n(\iota_x).
\]
It is also clear that $\text{Ad}_{g_z} \mathfrak{n} \subset m(\iota_x)$, so
\[
\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n}) \subset n(\iota_x) + m(\iota_x) = V(\iota_x).
\]
Since both $\text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})$ and $V(\iota_x)$ have the same dimension, they must coincide.

Let now $S_x$ be the symplectic leaf of $\Pi_v$ in $X$ through $x$. By Proposition 3.2, the codimension of $S_x$ in $O_z$ is equal to $\dim(V(\iota_x) \cap \iota_d)$. Let $\dot{w} \in N_G(\mathfrak{h})$ be a representative of $w \in K$. Since $x \in C_w$, there exist $n \in N$ and $b \in B$ such that $g_z = n\dot{w}b$. Then we have
\[
V(\iota_x) \cap \iota_d = (\text{Ad}_{n\dot{w}b}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})) \cap (\mathfrak{h}^{-\tau_w} + \mathfrak{n}) = \text{Ad}_n \left((\text{Ad}_{\dot{w}}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})) \cap (\mathfrak{h}^{-\tau_w} + \mathfrak{n})\right) = \text{Ad}_n \left(\mathfrak{h}^{w\sigma_w} \cap \mathfrak{h}^{-\tau_w} + (\text{Ad}_{\dot{w}} \mathfrak{n}) \cap \mathfrak{n}\right),
\]
where in the last line we have the direct sum of vector spaces. Since
\[
\dim(\text{Ad}_n \mathfrak{n}) \cap \mathfrak{n} = \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} C_w,
\]
we have
\[
\dim(V(\iota_x) \cap \iota_d) = \delta_{z,w} + \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} C_w,
\]
and thus
\[
\dim S_x = \dim O_z - \dim(V(\iota_x) \cap \iota_d) = \dim(O_z \cap C_w) - \delta_{z,w}.
\]

Note, that the number $\delta_{z,w}$ depends only on $d$ and the two Weyl group elements $\varphi(z)$ and $w$. Define $d : W \to W$ by $d(w) = \gamma_d w \gamma_d$. Following [12], we say that $w \in W$ is a
A $d$-twisted involution if $d(w) = w^{-1}$. Denote by $\mathcal{I}_d$ the set of all $d$-twisted involutions in $W$. Clearly, every $\varphi(z)$ is in $\mathcal{I}_d$. The Weyl group $W$ acts on $\mathcal{I}_d$ by

$$w_1 \ast w = w_1 w d(w_1^{-1}) \text{ for } w_1 \in W, \text{ and } w \in \mathcal{I}_d,$$

and the set $\varphi(Z) \subset \mathcal{I}_d$ is $W$-invariant. In fact, the $W$-action on $G/H$, given by $w \cdot gH = gw^{-1}H$, commutes with the left action of $G_v$ by left multiplication, and thus induces a left action of $W$ on $Z$, which we denote by $w \cdot z$ for $w \in W$ and $z \in Z$. It is also easy to see that $\varphi: Z \to W$ is $W$-equivariant, i.e. $\varphi(w \cdot z) = w \ast \varphi(z)$ for all $w \in W$ and $z \in Z$. Similarly, the involution $\tau_v: G \to G$ gives rise to an involution on $Z$ which depends only on $d$. Since $\varphi: Z \to W$ is $W$-equivariant, we also have $\varphi(d(z)) = d(\varphi(z)) = \varphi(z)^{-1}$. As maps on $\mathfrak{h}$, we see that $w \varphi_1 w^{-1} = (w \ast \varphi(z)) \tau_v$. Thus we also have:

$$\delta_{z,w} = \dim(\mathfrak{h}^{w \ast \varphi(z)} \tau_v \cap \mathfrak{h}^{-\tau_v}).$$

**Corollary 4.2.**  
1) When $w \ast \varphi(z) = 1$, symplectic leaves of $\Pi_v$ in $O_z \cap C_w$ are precisely its connected components.

2) Every open orbit $O_z$ has an open symplectic leaf $O_z \cap C_{w_0}$, where $w_0$ is the longest element in $W$.

3) If $d = 1$, symplectic leaves in an open orbit $O_z$ are precisely the connected components of intersections of Bruhat cells with $O_z$.

**Proof.**

1) When $w \ast \varphi(z) = 1$, we have $\delta_{z,w} = 0$, so every symplectic leaf in $O_z \cap C_w$ is open in $O_z \cap C_w$.

2) Since $C_{w_0}$ is dense in $X$, it intersects with every open orbit $O_z$. Since an orbit $O_z$ is open if and only if $\varphi(z) = 1$, statement 2) follows from 1) and the fact that $w_0$ commutes with $d$. The fact that $C_{w_0} \cap O_z$ is connected follows from the observation that $O_z$ is a connected open complex submanifold of $X$ and thus $O_z \cap (X \setminus C_{w_0})$ is a divisor in $O_z$.

3) follows directly from 1). \qed

Consider now the group $H^{\tau_v} = H \cap G_v$. The Poisson structure $\Pi_v$ on $X$ is $H^{\tau_v}$-invariant. Indeed, let $R \in \wedge^2 \mathfrak{g}$ be the element given in (3.1) for $l_1 = \mathfrak{g}_v$ and $l_2 = l_d$. We can also represent $R$ as $R = \sum_1 \xi_i \wedge y_i$, where $\{y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\xi_i\}$ is the dual basis of $l_d$ with respect to the pairing between $\mathfrak{g}_v$ and $l_d$ given by $\langle , \rangle$, the imaginary part of the Killing form on $\mathfrak{g}$. If $h \in H^{\tau_v}$, then $\{\text{Ad}_h y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\text{Ad}_h \xi_i\}$ is its dual basis. Thus $\text{Ad}_h R = R$. It follows that the Poisson structure $\Pi_v$ on $X$ is $H^{\tau_v}$-invariant.

Assume now that $z \in Z$ and $w \in W$ are such that $O_z \cap C_w \neq \emptyset$. Clearly, $H^{\tau_v}$ leaves $O_z \cap C_w$ invariant. Since the Poisson structure $\Pi_v$ is $H^{\tau_v}$-invariant, if $S_x$ is the symplectic leaf of $\Pi_v$ through $x$, then $hS_x := \{hx_1 : x_1 \in S_x\}$ is the symplectic leaf of $\Pi_v$ through
Let $H_0^{\tau_v}$ be the identity connected component of $H^{\tau_v}$, and for $x \in X$ define:

$$F_x := \bigcup_{h \in H_0^{\tau_v}} hS_x.$$ 

**Proposition 4.3.** For any $x \in X$, the set $F_x$ is a connected component of $O_x \cap C_w$.

**Proof.** It is easy to see that if $F_{x_1} \cap F_{x_2} \neq \emptyset$, then $F_{x_1} = F_{x_2}$. The statement would follow once we prove that $F_x$ is an open subset of $O_x \cap C_w$ for each $x$.

Let $x = g_zB \in O_x \cap C_w$ with $g_z \in Z$ in the double coset $z$. For $y \in \mathfrak{h}^{\tau_v}$, let $X_y$ be the vector field on $X$ generating the action of $\exp(ty) \in H_0^{\tau_v}$ on $X$. We claim that $X_y(x) \in T_xS_x$ if and only if $y \in p(\mathfrak{h}(u^{w*}(z))^{\tau_v})$, where $p : \mathfrak{h} \to \mathfrak{h}^{\tau_v}$ is the projection with respect to the decomposition $\mathfrak{h} = \mathfrak{h}^{\tau_v} + \mathfrak{h}^{-\tau_v}$. Assume the claim. Then since the kernel of the map $p : \mathfrak{h}(u^{w*}(z))^{\tau_v} \to \mathfrak{h}^{\tau_v}$ has dimension $\dim(\mathfrak{h}(u^{w*}(z))^{\tau_v}) + \delta_{z,w} = \delta_{z,w}$. Thus $J_x$ is onto, and the $H_0^{\tau_v}$-orbit in $O_x \cap C_w$ through $x$ is transversal to the symplectic leaf $S_x$. It follows that $F_x$ is open in $O_x \cap C_w$.

It remains to prove the claim. Denote also by $p : \mathfrak{g} \to \mathfrak{g}_v$ the projection with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_v + I_d$, and let $q$ be the projection $q : \mathfrak{g}_v \to \mathfrak{g}_v/\mathfrak{g}_v \cap \text{Ad}_{g_z}b \cong T_xO_x$. Then by [9, Corollary 7.3], we have $T_xS_x = (q \circ p)(\mathcal{V}(I_x))$, where, as in the proof of Theorem 4.1, $\mathcal{V}(I_x) = \text{Ad}_{g_z}(\mathfrak{h}^{\sigma_z} + \mathfrak{n})$. Let $y \in \mathfrak{h}^{\tau_v}$. If $X_y(x) \in T_xS_x$, then there exist $y_1 \in I_d$ and $y_2 \in \mathfrak{g}_v$ with $y_1 + y_2 \in \mathcal{V}(I_x)$ such that $y - y_2 \in \mathfrak{g}_v \cap \text{Ad}_{g_z}b \subset \mathcal{V}(I_x)$. Thus $y + y_1 = y - y_2 + y_1 + y_2 \in \mathcal{V}(I_x)$. Write $y = \xi_1 + u_1$, where $\xi_1 \in \mathfrak{h}^{-\tau_v}$ and $u_1 \in \mathfrak{n}$. Then there exist $\xi_2 \in \mathfrak{h}^{\sigma_z}$ and $u_2 \in \mathfrak{n}$ such that $y + \xi_1 + u_1 = \text{Ad}_{g_z}(\xi_2 + u_2)$. Write $g_z = n\hat{w}b$, where $n \in N, b \in B$, and $\hat{w}$ is a representative of $w$ in $K$. Write $\text{Ad}_{n^{-1}}(y + \xi_1 + u_1) = y + \xi_1 + u_1' \quad \text{and} \quad \text{Ad}_b(\xi_2 + u_2) = \xi_2 + u_2'$, where $u_1', u_2' \in \mathfrak{n}$. Then we have

$$y + \xi_1 + u_1' = \text{Ad}_b(\xi_2 + u_2').$$

Since $y + \xi_1, \text{Ad}_w\xi_2 \in \mathfrak{h}$ and $u_1', \text{Ad}_w u_2' \in \mathfrak{n} + \mathfrak{n}_-$, where $\mathfrak{n}_- = \theta(\mathfrak{n})$, we have $y + \xi_1 = \text{Ad}_w \xi_2 \in \mathfrak{h}(u^{w*}(z))^{\tau_v}$. Thus $y \in p(\mathfrak{h}(u^{w*}(z))^{\tau_v})$. Conversely, if $y \in \mathfrak{h}^{\tau_v}$ is such that $y + \xi_1 \in \mathfrak{h}(u^{w*}(z))^{\tau_v} = \text{Ad}_w \mathfrak{h}^{\sigma_z}$ for some $\xi_1 \in \mathfrak{h}^{-\tau_v}$, write $y + \xi_1 = \text{Ad}_w \xi_2$ for $\xi_2 \in \mathfrak{h}^{\sigma_z}$. Let $\text{Ad}_{n^{-1}} \xi_2 = \xi_2 + u_2$ for some $u_2 \in \mathfrak{n}$. We then have

$$\text{Ad}_n(y + \xi_1) = \text{Ad}_{n\hat{w}b}(\xi_2 + u_2) \in \mathcal{V}(I_x).$$

On the other hand, let $\text{Ad}_n(y + \xi_1) = y + \xi_1 + u_1$ with $u_1 \in \mathfrak{n}$. We see that $y = p(\text{Ad}_n(y + \xi_1))$ so $X_y(x) \in T_xS_x$. \qed
5. Invariant Poisson cohomology of open orbits.

Let \( O_z \) be a \( G_v \)-orbit in \( X \) equipped with the Poisson structure \( \Pi_v \). Then \((O_z, \Pi_v)\) is a Poisson homogeneous space for the Poisson Lie group \( G_v \). The \( G_v \)-invariant Poisson cohomology of \((O_z, \Pi_v)\), denoted by \( H^•_{\Pi_v, G_v}(O_z) \), is defined as the cohomology of the cochain complex \((\chi^•_{G_v}(O_z), \partial_{\Pi_v})\), where \( \chi^•(O_z)^{G_v} \) is the space of all \( G_v \)-invariant complex multi-vector fields on \( O_z \), \( d_{\Pi_v}(V) = [\Pi_v, V] \), and \([\cdot, \cdot]\) is the Schouten bracket of the multi-vector fields.

**Proposition 5.1.** When \( O_z \) is an open \( G_v \)-orbit in \( X \), the \( G_v \)-invariant Poisson cohomology \( H^•_{\Pi_v, G_v}(O_z) \) is isomorphic to the de Rham cohomology of \( X \).

**Proof.** As in the proof of Theorem 4.1, let \( x = g_zB \in X \) be an arbitrary point in \( O_z \), where \( g_z \in Z \) is in the coset \( z \), and let \( V(l_x) = \text{Ad}_{g_z}(\mathfrak{h}^x + \mathfrak{n}) \). Since \( O_z \) is open, the stabilizer subalgebra of \( g_v \) at \( x \) is \( g_v \cap V(l_x) = \text{Ad}_{g_z}(\mathfrak{h}^x) \). By [9, Theorem 7.5], the \( G_v \)-invariant Poisson cohomology \( H^•_{\Pi_v, G_v}(O_z) \) is isomorphic to the relative Lie algebra cohomology of the Lie algebra \( V(l_x) \otimes \mathbb{C} \) relative to the subalgebra \( (\text{Ad}_{g_z}(\mathfrak{h}^x)) \otimes \mathbb{C} \). Thus the \( G_v \)-invariant Poisson cohomology is isomorphic to the \( \mathfrak{h} \)-invariant part of the Lie algebra cohomology of the direct sum Lie algebra \( \mathfrak{n} \oplus \mathfrak{n} \) with coefficients in \( \mathbb{C} \), which by Kostant’s theorem [7], is isomorphic to the de Rham cohomology of \( X \). \( \square \)


We have constructed a Poisson structure \( \Pi_v \) on \( X \) for each Vogan diagram \( v \) for \( \mathfrak{g} \) (which is not necessarily normalized). In particular, each Bruhat cell \( C_w \) in \( X \) carries the Poisson structure \( \Pi_v \). It would be interesting to study connections between the Poisson structures for different \( v \). Especially interesting are the properties of \( \Pi_v \) that depend only on the inner class \( d \) of the real form \( \mathfrak{g}_v \). We also remark that the Poisson structure \( \Pi_v \) is defined on the whole variety \( \mathcal{L} \) of Lagrangian subalgebras of \( \mathfrak{g} \). We have only been looking at the restriction of \( \Pi_v \) to a particular \( G \)-orbit, namely the \( G \)-orbit through the Lagrangian subalgebra \( \mathfrak{t} + \mathfrak{n} \). There are many other interesting \( G \)-orbits in \( \mathcal{L} \), such as the \( G \)-orbit through a given real form of \( \mathfrak{g} \). It would be interesting to study the properties of the Poisson structure \( \Pi_v \) on these orbits as well as on their closures with respect to both the classical topology and the Zariski topology.

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REFERENCES


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