Moments over Short Intervals

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Abstract. An asymptotic result for the kth moment ($k \le 9$) of the error term in the Dirichlet divisor problem over short intervals is obtained, which improves on an earlier result of Nowak.

1. Introduction. Define for $x > 0$,

$$\Delta(x) = \sum_{n \le x} d(n) - x(\log x + 2\gamma - 1)$$

where $d(n)$ denotes the number of positive divisors of n and γ is the Euler constant. This is the error term in the well-known Dirichlet divisor problem, which aims at the determination of the order of $\Delta(x)$. The conjecture that $\Delta(x) \ll x^{\beta+\epsilon}$ with $\beta = 1/4$ is very difficult and it remains unsolved as of to-day. The current best result is $\beta \le 131/416$, due to Huxley [3]. Throughout ϵ denotes any arbitrarily small positive constant which may differ at each occurrence.

Apart from the above conjecture, there are plenty of investigations focusing on various statistical properties of $\Delta(x)$, including the power moments and the probability distribution. For instance, the work of Voronoi [7] yields the asymptotic formula

$$\int_0^X \Delta(x)\,dx = \frac{1}{4}X + O(X^{3/4}), \tag{1.1}$$

and Cramér [1] obtained for the mean square of $\Delta(x)$ an asymptotic formula with explicit error term. For higher power moments, Ivić [4] used the large sieve inequality to derive the essentially best possible upper bounds

$$\int_0^X |\Delta(x)|^A\,dx \ll X^{1+A/4+\epsilon} \tag{1.2}$$

for $0 \le A \le 28/3$. On the other hand, asymptotic formulas for moments of $\Delta(x)$ higher than the second was first obtained by the second author [6], who established for $k = 3, 4$,

$$\int_0^X \Delta(x)^k\,dx = c_k X^{1+k/4} + F_k(X) \tag{1.3}$$

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with

$$F_k(X) \ll X^{1+k/4-\theta(k)}, \tag{1.4}$$

where c_k and $\theta(k)$ are explicit positive numbers. Here the case for $k = 3$ is particularly interesting and important, because it shows that the positive and negative parts of $\Delta(x)$ are not quite balanced, a fact which cannot be seen from (1.1) alone.

In another direction, Heath-Brown [2] initiated the study of the value distribution of $\Delta(x)$. He proved, among other things, that asymptotic formula for $\int_0^X \Delta(x)^k \, dx$ exist for all integers $k \in [0, A)$ whenever (1.2) holds. However the paper [2] does not contain any explicit estimate for the error term in the asymptotic formula. Recently Zhai [8, 9] proved (1.3) with $F_k(X)$ of the form (1.4) for $k = 5, \cdots, 9$, together with explicit form for the constant c_k.

To better understand the local behaviour of $\Delta(x)$, it is natural to look at the moments over short intervals around X instead of the full interval $[0, X]$. Nowak [5] considered this problem and his results, in summary, are as follows: let $\Lambda = \Lambda(X) = o(X)$. Then

$$\int_X^{X+\Lambda} \Delta(x)^k \, dx \sim (1 + \frac{k}{4}) c_k \Lambda X^{k/4} \quad (k = 2, 3, 4)$$

holds if $\lim_{X\to\infty} X^{1/2} \log^3 X/\Lambda = 0$ when $k = 2$, or for some constant $\delta > 0$, $\lim_{X\to\infty} X^{3/4+\delta}/\Lambda = 0$ when $k = 3, 4$.

Remark. It is sufficient to consider the situation $\Lambda = o(X)$ since otherwise it is contained in the asymptotics (1.3).

The main purpose of this paper is to prove the following theorems.

Theorem 1 *Suppose the estimate $\Delta(x) \ll x^{\beta+\epsilon}$ holds for a certain $\beta \geq \frac{1}{4}$. Let $k < min\{11, 8\beta/(4\beta-1)\}$ and let $\Lambda = \Lambda(X)$ be an increasing function of X such that $X \gg \Lambda(X) \gg X^\lambda$ for some constant λ, which satisfies*

$$\lambda > \frac{1}{2} + \max\{0, (k-2)(\beta - \frac{1}{4})\}. \tag{1.5}$$

Then the limit

$$\lim_{X\to\infty} \frac{1}{\Lambda X^{k/4}} \int_X^{X+\Lambda} |\Delta(x)|^k \, dx \tag{1.6}$$

exists. Furthermore, if $k(< min\{11, 8\beta/(4\beta-1)\})$ *is an odd integer and (1.5) holds, then*

$$\lim_{X\to\infty} \frac{1}{\Lambda X^{k/4}} \int_X^{X+\Lambda} \Delta(x)^k \, dx \tag{1.7}$$

also exists.

By using Huxley's estimate [3] that $\beta = 131/416$ is admissible, we deduce readily from Theorem 1 the following improvement on the aforementioned result of Nowak [5, Theorem 2].

Theorem 2 *The limit (1.6) exists for every* $k < 9.7\cdots$ *and the limit (1.7) exists for* $k = 1, 3, 5, 7, 9$*; provided* $\Lambda \gg X^{\lambda}$ *with*

$$\lambda > \frac{1}{2} + max\{0, (k-2)\frac{27}{416}\}.$$

In particular, we only need $\lambda > 0.5649\cdots$ *for* $k = 3$ *and* $\lambda > 0.6298\cdots$ *for* $k = 4$.

It is known that pointwise upper bound for $\Delta(x)$ can be derived from the power moments. Our next theorem shows that the conjectured upper bound: $\Delta(x) \ll x^{1/4+\epsilon}$ yields asymptotic results for all power moments over short intervals of the form $[X, X + X^{1/2+\epsilon}]$.

Theorem 3 *Suppose the conjecture that* $\Delta(x) \ll x^{1/4+\epsilon}$ *is true. Then for* $X \gg \Lambda \gg X^{\lambda}$ *where* $\lambda > 1/2$ *is any fixed constant, the limits (1.6) and (1.7) exist for all non-negative real* k *and all positive odd integers* k *respectively.*

To prove the theorems, we make use of a short interval version of the large value result and the distribution result of $\Delta(x)$. Our results suggest that this method is more effective than the direct treatment of the cross terms, as in the work of Nowak [5].

2. Preliminaries. Let $Y = Y(T)$ be a function of T such that $Y \to \infty$ as $T \to \infty$. Consider a function $F(t)$ which satisfies the following hypothesis.
(H_s): Let $a_1(t), a_2(t), \cdots$ be continuous real-valued functions of period 1, and suppose that there are non-zero constants $\gamma_1, \gamma_2, \cdots$ such that

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{Y} \int_T^{T+Y} \min\{1, |F(t) - \sum_{n\le N} a_n(\gamma_n t)|\} \, dt = 0.$$

(This is the short interval version of the Hypothesis (H) in [2].)

For any continuous function $f : \mathbb{R} \longrightarrow \mathbb{C}$, we define

$$\mathrm{m}_T(f) = \frac{1}{Y}\int_T^{T+Y} f(t)\,dt.$$

In particular, as $T \to \infty$,

$$\mathrm{m}_T(e(\gamma t)) \to \begin{cases} 1, & \gamma = 0, \\ 0, & \gamma \neq 0. \end{cases}$$

The results in [2, Section 2] are valid when the operator $m_T(\cdot)$ in [2] is replaced by our $\mathrm{m}_T(\cdot)$. Applying the argument in [2, Section 3] with Hypothesis (H_s) instead of (H), we obtain the following result which is analogous to [2, Theorem 6].

Theorem 4 *Suppose $F(t)$ satisfies Hypothesis* (H_s)*, and*

$$\int_T^{T+Y} |F(t)|^K\,dt \ll Y \tag{2.1}$$

for some positive K. Then all the limits

$$\lim_{T\to\infty} \frac{1}{Y}\int_T^{T+Y} |F(t)|^k\,dt, \tag{2.2}$$

for real $k \in [0, K)$, and

$$\lim_{T\to\infty} \frac{1}{Y}\int_T^{T+Y} F(t)^k\,dt, \tag{2.3}$$

for odd integers $k \in [0, K)$, exist.

Remark: Actually, replacing Hypothesis (H) by (H_s) we deduce, corresponding to Theorems 4 and 5 in [2] respectively, that

(i) $Y^{-1}\int_T^{T+Y} p(F(t))\,dt$ converges,

(ii) $Y^{-1}\mathrm{meas}\{t \in [T, T+Y] : F(t) \in I\} \to \int_I f(\alpha)\,d\alpha$ for any fixed interval I,

as $T \to \infty$. (Note that the density function $f(\alpha)$ depends only on the sequences $\{a_n(t)\}$ and $\{\gamma_n\}$, but not on the function Y.)

Next we derive a large value result for $\Delta(x)$ over a short interval along the line of argument in [4].

Lemma 5 *Let* $\sqrt{T} \ll \Lambda \ll T$ *and let* $T \le t_1 < t_2 < \cdots < t_R < T + \Lambda$ *such that* $|t_r - t_s| \ge V > T^{7/32+\epsilon}$ *for* $r \ne s$. *Suppose* $\Delta(t_r) \gg V$ *for all* $1 \le r \le R$. *Then*

$$R \ll T^{\epsilon}(TV^{-3} + \Lambda T^{11/4} V^{-12}).$$

This follows essentially the proof of [4, Theorem 1]. We carry out the same procedure of dividing the interval $[T,\, T+\Lambda]$ into subintervals of length T_0. Instead of [4, (2.2)], we have

$$R \ll R_0(1 + \frac{\Lambda}{T_0}).$$

The upper bound $R_0 \ll T^{1+\epsilon}V^{-3}$ in [4, (2.10)] still holds since the argument in [4, (2.3)-(2.10)] is only applied to each of the subintervals. The proof then finishes with the same choice of T_0 in [4].

Assume now that $\Delta(x) \ll x^{\beta+\epsilon}$ for a certain $\beta \ge \frac{1}{4}$. Take V to be a power of two such that

$$T^{1/4} \le 2^m = V \le T^{\beta+\epsilon}. \tag{2.4}$$

For each such V we divide the interval $[T,\, T+\Lambda]$ into subintervals of length V. As in [4, Section 3], we construct two systems of points, each of which satisfies

$$V \le |\Delta(t_r)| \le 2V \ , \quad |t_r - t_s| \ge V \ (r \ne s),$$

if the points in the system are labelled as $t_1, \cdots, t_R$, $R = R(V)$. This is obtained by picking the points alternatively from the odd-indexed and the even-indexed subintervals. This ensures the validity of the separation condition. Then, similar to [4, (3.2)-(3.3)] we find that

$$\begin{aligned} \int_T^{T+\Lambda} |\Delta(x)|^A \, dx &\ll \Lambda T^{A/4} + \sum_V V^{A+1} R(V) \\ &\ll \Lambda T^{A/4} + T^{\epsilon} \sum_V (TV^{A-2} + \Lambda T^{11/4} V^{A-11}), \end{aligned}$$

by Lemma 5. Invoking (2.4), we then obtain the following.

Theorem 6 *Suppose* $\Delta(x) \ll x^{\beta+\epsilon}$ *and* $\sqrt{X} \ll \Lambda \ll X$. *Then for any positive constant* A, *we have*

$$\int_X^{X+\Lambda} |\Delta(x)|^A \, dx \ll \Lambda X^{A/4+\epsilon} + X^{1/2+(A-2)(\beta-1/4)+A/4+\epsilon} + \Lambda X^{A/4+(A-11)(\beta-1/4)+\epsilon}.$$

Remark: The third term on the right-side is dominated by the first when $A \leq 11$.

3. Proofs of Theorems 1 and 3. The basis of the proofs of these two theorems is Theorem 4, in which $F(t) = \Delta(t^2)t^{-1/2}$.

Define $a_n(t)$ and γ_n as in [2, Section 5]. Then from [2, p.402] we have, for any $1 \leq N \leq \sqrt{T}$,

$$F(t) - \sum_{n\leq N} a_n(\gamma_n t) \ll \Big| \sum_{\substack{n\leq T^2\\ s(n)>N}} \frac{d(n)}{n^{3/4}} e(2t\sqrt{n}) \Big| + N^{1/2}T^{-1/2+\epsilon}$$

uniformly for $t \in [T, 2T]$, where $s(n)$ denotes the squarefree kernel of n. Integrating over $[T, T+Y]$, we have

$$\begin{aligned}\int_T^{T+Y} \Big| \sum_{\substack{n\leq T^2\\ s(n)>N}} \frac{d(n)}{n^{3/4}} e(2t\sqrt{n}) \Big|^2 dt \; &\ll \; Y \sum_{\substack{n\leq T^2\\ s(n)>N}} \frac{d(n)^2}{n^{3/2}} + \sum_{m\neq n\leq T^2} \frac{d(n)d(m)}{(mn)^{3/4}} \frac{1}{|\sqrt{m}-\sqrt{n}|} \\ &\ll \; YN^{\epsilon-1/2} + T^{\epsilon}.\end{aligned}$$

It follows that

$$\int_T^{T+Y} \Big| F(t) - \sum_{n\leq N} a_n(\gamma_n t) \Big|^2 dt \ll YN^{\epsilon-1/2} + T^{\epsilon} + YNT^{\epsilon-1} \tag{3.1}$$

and hence, by Hölder's inequality, Hypothesis (H_s) is valid whenever $Y \gg T^{\delta}$ for some $\delta > 0$.

Next, let L be any fixed positive integer. Analogous to [2, (5.4)], we take

$$N = Y^{2^{-(2L+2)}}. \tag{3.2}$$

Lemma 5 in [2] implies that for any positive integers $n_1, \cdots, n_{2L} \leq N^4$,

$$|\sqrt{n_1} \pm \cdots \pm \sqrt{n_{2L}}| \gg N^{-2^{2L+1}} = Y^{-1/2} \tag{3.3}$$

by (3.2), unless the product $n_1 \cdots n_{2L}$ is squarefull.

From [2, p.403], we get

$$\sum_{n\leq N} a_n(\gamma_n t) \;=\; \frac{1}{\pi\sqrt{2}} \sum_{\substack{n\leq N^4\\ s(n)\leq N}} \frac{d(n)}{n^{3/4}} \cos(4\pi t\sqrt{n} - \pi/4) + O(1).$$

Multiplying out and integrating term by term, we find that

$$(3.4)\qquad \int_T^{T+Y} \Big|\sum_{n\le N} a_n(\gamma_n t)\Big|^{2L} dt$$

$$\ll \int_T^{T+Y} \Big|\sum_{\substack{n\le N^4\\ s(n)\le N}} \frac{d(n)}{n^{3/4}} e(2t\sqrt{n})\Big|^{2L} dt + Y$$

$$\ll Y{\sum}' \frac{d(n_1)\cdots d(n_{2L})}{(n_1\cdots n_{2L})^{3/4}} + Y^{1/2} \sum_{n_1,\cdots,n_{2L}\le N^4} \frac{d(n_1)\cdots d(n_{2L})}{(n_1\cdots n_{2L})^{3/4}} + Y$$

by (3.3), where the summation $\sum'$ runs over all $n_1,\cdots,n_{2L}\le N^4$ such that the product $n_1\cdots n_{2L}$ is squarefull. The argument in [2, p.404] shows that

$${\sum}' \frac{d(n_1)\cdots d(n_{2L})}{(n_1\cdots n_{2L})^{3/4}} \ll \sum_{\substack{q=1\\ q \text{ squarefull}}}^{\infty} q^{-3/4+2\epsilon} \ll 1.$$

Moreover, the second sum $\sum_{n_1,\cdots,n_{2L}\le N^4}$ is clearly $\ll N^{2L(1+\epsilon)} \ll Y^{1/4}$, by (3.2). The integral in (3.4) is thus $\ll Y$. Hence, for any $k\ge 0$, by taking $L=\lceil k\rceil$ and using Hölder's inequality, we have

$$(3.5)\qquad \int_T^{T+Y} \Big|\sum_{n\le N} a_n(\gamma_n t)\Big|^{k} dt \ll_k Y \qquad (\text{any } k\ge 0).$$

(Note that N is a small positive power of Y, depending on k.)

Assume for the time being that for a certain positive K,

$$(3.6)\qquad \int_T^{T+Y} \big|F(t)\big|^K dt \ll Y^{1+\epsilon}.$$

Then, by (3.5)

$$\int_T^{T+Y} \Big|F(t) - \sum_{n\le N} a_n(\gamma t)\Big|^K dt$$

$$\ll \int_T^{T+Y} \big|F(t)\big|^K dt + \int_T^{T+Y} \Big|\sum_{n\le N} a_n(\gamma t)\Big|^K dt \ll Y^{1+\epsilon}.$$

By the same argument in the paragraph below [2, (5.7)], we then deduce from this and (3.1) that, for any $0\le k<K$,

$$(3.7)\qquad \int_T^{T+Y} \Big|F(t) - \sum_{n\le N} a_n(\gamma t)\Big|^k dt \ll Y.$$

(Recall that $Y \gg T^{\delta}$.) Hence, by (3.5) again,

$$\begin{aligned}(3.8)\qquad &\int_T^{T+Y} \big|F(t)\big|^k\,dt \\ \ll\ &\int_T^{T+Y} \big|F(t) - \sum_{n\le N} a_n(\gamma t)\big|^k\,dt + \int_T^{T+Y} \big|\sum_{n\le N} a_n(\gamma t)\big|^k\,dt \ll Y,\end{aligned}$$

for any $0 \le k < K$. In view of Theorem 4, the limits in (2.2) and (2.3) all exist. The passing from these limits to those for $\Delta(x)$ in (1.6) and (1.7) can be seen as follows. For $X \to \infty$, $X^{\lambda} \ll \Lambda = \Lambda(X) = o(X)$, set $T = \sqrt{X}$ and $Y = \sqrt{X+\Lambda} - \sqrt{X}$. Then

$$(3.9)\qquad Y \sim \frac{1}{2}\Lambda X^{-1/2} \gg \Lambda(T^2)T^{-1} \gg T^{2\lambda-1}.$$

Integrating by parts yields

$$\begin{aligned}&\int_X^{X+\Lambda} |\Delta(x)|^k\,dx = 2\int_T^{T+Y} t^{1+k/2}|F(t)|^k\,dt \\ =\ &2(T+Y)^{1+k/2}\int_T^{T+Y} |F(t)|^k\,dt - (2k+1)\int_T^{T+Y} t^{k/2}\int_T^t |F(u)|^k\,du\,dt.\end{aligned}$$

By (3.8) and (3.9), the second term is

$$\ll T^{k/2}\int_T^{T+Y}\int_T^t |F(u)|^k\,du\,dt \ll Y^2T^{k/2} \ll \Lambda^2 X^{k/4-1}.$$

As $2(T+Y)^{1+k/2} \sim \Lambda X^{k/4}Y^{-1}$ and $\Lambda = o(X)$, we deduce that

$$(3.10)\qquad \int_X^{X+\Lambda} |\Delta(x)|^k\,dx \sim \Lambda X^{k/4}Y^{-1}\int_T^{T+Y} |F(t)|^k\,dt.$$

The limit in (1.6) then follows from (2.2). The proof of (1.7) is the same.

It remains to determine the largest K for which (3.6) holds with the T and Y defined above. In view of (3.10) and Theorem 6, for $A = K \le 11$ we have

$$\int_T^{T+Y} |F(t)|^K\,dt \ll Y(X^{\epsilon} + \sqrt{X}\Lambda^{-1}X^{(K-2)(\beta-1/4)+\epsilon}),$$

which is $\ll Y^{1+\epsilon}$ provided $\frac{1}{2} - \lambda + (K-2)(\beta - 1/4) \le 0$. This is the condition (1.5). The fact that $\lambda \le 1$ gives rise to the requirement $K \le 8\beta/(4\beta-1)$. The proof of Theorem 1 is thus complete. To prove Theorem 3, we just have to note that, in this case (3.6) holds for all $K > 0$.

References

[1] H. Cramér, Über zwei Sätz von Herrn G.H. Hardy, Math. Zeit. 15 (1922), 200-210.

[2] D.R. Heath-Brown, The distribution and moments of the error term in the Dirichlet divisor problem, Acta Arith. 60 (1992), 389-415.

[3] M.N. Huxley, Exponential sums and lattice points III, Proc. London Math. Soc. 87 (2003), 591-609.

[4] A. Ivić, Large Values of the Error Term in the Divisor Problem Invent. math. 71 (1983), 513-520.

[5] W.G. Nowak, On the divisor problem: Moments of $\Delta(x)$ over short intervals, Acta Arith. 109 (2003), 329-341

[6] K.-M. Tsang, Higher power moments of $\Delta(x)$, $E(t)$ and $P(x)$, Proc. London Math. Soc. 65 (1992), 65-84.

[7] G.F. Voronoi, Sur une fonction transcendente et ses applications à la sommation de quelques séries, Ann. École Normale 21 (1904), 207-268 & 459-534.

[8] W. Zhai, On higher-power moments of $\Delta(x)$, Acta Arith. 112 (2004), 367-395.

[9] W. Zhai, On higher-power moments of $\Delta(x)$ (II), Acta Arith. 114 (2004), 35-54.

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