Some Discrete Nonlinear Inequalities and Applications To Boundary Value Problems for Difference Equations

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Abstract.

In this paper, we establish some new discrete Gronwall-Bellman-Ou-Iang type inequalities over 2-dimensional lattices. These on the one hand generalize some existing results in the literature and on the other hand provide a handy tool for the study of qualitative properties of solutions of difference equations. We illustrate this by applying these new results to certain boundary value problem for difference equations.

1. Introduction

It is well-recognized that integral inequalities in general provide a very useful and handy device for the study of qualitative as well as quantitative properties of solutions of differential equations. Among varies types of integral inequalities, the Gronwall-Bellman type (see, e.g. [3-8, 10-13, 16-18]) is particularly useful in that they provide explicit bounds for the unknown functions. A specific branch of this type of integral inequalities is originated by Ou-Iang. In [14], in order to study the boundedness behavior of the solutions of certain 2nd order differential equations, Ou-Iang established the following integral inequality which is now known as Ou-Iang's inequality in the literature.

Theorem (Ou-Iang [14]). *If u and f are non-negative functions on $[0,\infty)$ satisfying*

$$u^2(x) \le k^2 + 2\int_0^x f(s)u(s)ds\ , \quad x \in [0,\infty)\ ,$$

for some constant $k \ge 0$, then

$$u(x) \le k + \int_0^x f(s)ds\ , \quad x \in [0,\infty)\ .$$

While Ou-Iang's inequality is having a neat form and is interesting in its own right as an integral inequality, its importance lies equally heavily on its many beautiful applications in differential equations (see, e.g., [2,3,9,12,13]). Over the years, many generalizations of Ou-Iang's inequality to various situations have been established. Among these, the discretization is an interesting direction because one naturally expects that discrete versions of these inequalities should play an important role in the study of difference equations, just as the continuous versions are playing a fundamental role in the study of differential equations. Results in this direction can be found in e.g., [1,18,19] and the references cited there.

The purpose of this paper is to establish some new discrete Gronwall-Bellman-Ou-Iang-type inequalities with explicit bounds which on the one hand generalize some existing results and on the other hand give a handy tool for the study of qualitative properties of solutions of difference equations. We illustrate this by applying these new inequalities to study the boundedness, uniqueness, and continuous dependence of the solutions of a boundary value problem for difference equations.

2. Discrete Gronwall-Bellman-Ou-Iang-type inequalities

Throughout this paper, $I := [m_0, M) \cap \mathbb{Z}$ and $J := [n_0, N) \cap \mathbb{Z}$ are two fixed lattices of integral points in $\mathbb{R}$, where $m_0, n_0 \in \mathbb{Z}$, $M, N \in \mathbb{Z} \cup \{\infty\}$. Let $\Omega := I \times J \subset \mathbb{Z}^2$, $\mathbb{R}_+ := [0, \infty)$, and for any $(s,t) \in \Omega$, the sub-lattice $[m_0, s] \times [n_0, t] \cap \Omega$ of Ω will be denoted as $\Omega_{(s,t)}$.

If U is a lattice in $\mathbb{Z}$ (resp. $\mathbb{Z}^2$), the collection of all $\mathbb{R}$-valued functions on U is denoted by $\mathcal{F}(U)$, and that of all $\mathbb{R}_+$-valued functions by $\mathcal{F}_+(U)$. For the sake of convenience, we extend the domain of definition of each function in $\mathcal{F}(U)$ and $\mathcal{F}_+(U)$ trivially to the ambient space $\mathbb{Z}$ (resp., $\mathbb{Z}^2$). So for example, a function in $\mathcal{F}(U)$ is regarded as a function defined on $\mathbb{Z}$ (resp., $\mathbb{Z}^2$) with support in U. As usual, the collection of all continuous functions of a topological space X into a topological space Y will be denoted by $C(X, Y)$.

If U is a lattice in $\mathbb{Z}$, the difference operator Δ on $f \in \mathcal{F}(\mathbb{Z})$ or $\mathcal{F}_+(\mathbb{Z})$ is defined as

$$\Delta f(n) := f(n+1) - f(n) \ , \quad n \in U \ ,$$

and if V is a lattice in $\mathbb{Z}^2$, the partial difference operators Δ_1 and Δ_2 on $u \in \mathcal{F}(\mathbb{Z}^2)$ or $\mathcal{F}_+(\mathbb{Z}^2)$ are defined as

$$\Delta_1 u(m,n) := u(m+1,n) - u(m,n) \ , \quad (m,n) \in V \ ,$$

$$\Delta_2 u(m,n) := u(m,n+1) - u(m,n) \ , \quad (m,n) \in V \ .$$

Theorem 2.1. *Suppose $u \in \mathcal{F}_+(\Omega)$. If $c \geq 0$ is a constant and $b \in \mathcal{F}_+(\Omega)$, $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ are functions satisfying*

(i) *w is non-decreasing with $w(r) > 0$ for $r > 0$; and*

(ii) *for any $(m,n) \in \Omega$,*

$$u(m,n) \le c + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) w\big(u(s,t)\big) \ , \tag{1}$$

then

$$u(m,n) \le \Phi^{-1}\big[\Phi(c) + B(m,n)\big] \tag{2}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$, where

$$\begin{aligned} B(m,n) &:= \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) \ , \\ \Phi(r) &:= \int_1^r \frac{ds}{w(s)} \ , \quad r > 0 \ , \\ \Phi(0) &:= \lim_{r \to 0^+} \Phi(r) \ , \end{aligned}$$

Φ^{-1} is the inverse of Φ, and $(m_1,n_1) \in \Omega$ is chosen such that $\Phi(c) + B(m,n) \in Dom(\Phi^{-1})$ for all $(m,n) \in \Omega_{(m_1,n_1)}$.

Proof. It suffices to consider the case $c > 0$, for then the case $c = 0$ can be arrived at by continuity argument. Denote by $p(m,n)$ the right hand side of (1). Then $p > 0$, $u \le p$ on Ω, and p is non-decreasing in each variable. Hence for any $(m,n) \in \Omega$,

$$\begin{aligned} \Delta_1 p(m,n) &= p(m+1,n) - p(m,n) \\ &= \sum_{t=n_0}^{n-1} b(m,t) w\big(u(m,t)\big) \\ &\le \sum_{t=n_0}^{n-1} b(m,t) w\big(p(m,t)\big) \\ &\le w\big(p(m,n-1)\big) \sum_{t=n_0}^{n-1} b(m,t) \ . \end{aligned}$$

Therefore, by the Mean-Value Theorem for integrals, for each $(m,n) \in \Omega$, there exists

$p(m,n) \le \xi \le p(m+1,n)$ such that

$$\begin{aligned}\Delta_1(\Phi\circ p)(m,n) &= \Phi\big(p(m+1,n)\big) - \Phi\big(p(m,n)\big)\\ &= \int_{p(m,n)}^{p(m+1,n)} \frac{ds}{w(s)}\\ &= \frac{1}{w(\xi)}\Delta_1 p(m,n)\ .\end{aligned}$$

Since w is non-decreasing, $w(\xi) \ge w\big(p(m,n)\big)$ and so

$$\begin{aligned}\Delta_1(\Phi\circ p)(m,n) &\le \frac{1}{w\big(p(m,n)\big)}\Delta_1 p(m,n)\\ &\le \frac{w\big(p(m,n-1)\big)}{w\big(p(m,n)\big)}\sum_{t=n_0}^{n-1} b(m,t)\\ &\le \sum_{t=n_0}^{n-1} b(m,t)\end{aligned}$$

for all $(m,n)\in\Omega$. Therefore,

$$\sum_{s=m_0}^{m-1} \Delta_1(\Phi\circ p)(s,n) \le \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t) = B(m,n)\ .$$

On the other hand, it is elementary to check that

$$\sum_{s=m_0}^{m-1} \Delta_1(\Phi\circ p)(s,n) = \Phi\circ p(m,n) - \Phi\circ p(m_0,n)\ ,$$

thus

$$\begin{aligned}\Phi\circ p(m,n) &\le \Phi\circ p(m_0,n) + B(m,n)\\ &= \Phi(c) + B(m,n)\ .\end{aligned}$$

Since Φ^{-1} is increasing on Dom Φ^{-1}, this yields

$$p(m,n) \le \Phi^{-1}\big[\Phi(c) + B(m,n)\big]$$

for all $(m,n)\in\Omega_{(m_1,n_1)}$. ■

Remark. In many cases the non-decreasng function w satisfies $\int_1^\infty \frac{ds}{w(s)} = \infty$. For example, $w =$ constant > 0, $w(s) = s$, $w(s) = \sqrt{s}$, etc., are such functions. In such cases $\Phi(\infty) = \infty$ and so we may take $m_1 = M$, $n_1 = N$. In particular, inequality (2) holds for all $(m,n)\in\Omega$.

Theorem 2.2. *Suppose* $u \in \mathcal{F}_+(\Omega)$. *If* $k \geq 0$ *is a constant and* $a, b \in \mathcal{F}_+(\Omega)$, $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ *are functions satisfying*

(i) w *is non-decreasing with* $w(r) > 0$ *for* $r > 0$; *and*

(ii) *for any* $(m, n) \in \Omega$,

$$u^2(m,n) \leq k^2 + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)u(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)u(s,t)w\big(u(s,t)\big) \ , \tag{3}$$

then

$$u(m,n) \leq \Phi^{-1}\Big[\Phi\big(k + A(m,n)\big) + B(m,n)\Big] \tag{4}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$, *where*

$$A(m,n) := \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t) \ ,$$
$$B(m,n) := \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) \ ,$$

Φ *is defined as in Theorem 2.1, and* $(m_1, n_1) \in \Omega$ *is chosen such that* $\Phi\big(k + A(m,n)\big) + B(m,n) \in \mathit{Dom}\ \Phi^{-1}$ *for all* $(m,n) \in \Omega_{(m_1,n_1)}$.

Proof. Similar to the proof of Theorem 2.1, it suffices to consider the case $k > 0$. Denote by $q(s,t)$ the right hand side of (3). Then $q > 0$, $u \leq \sqrt{q}$ on Ω, and q is non-decreasing in each variable. Hence for any $(m,n) \in \Omega$,

$$\begin{aligned}
\Delta_1 q(m,n) &= q(m+1,n) - q(m,n) \\
&= \sum_{t=n_0}^{n-1} a(m,t)u(m,t) + \sum_{t=n_0}^{n-1} b(m,t)u(m,t)w\big(u(m,t)\big) \\
&\leq \sum_{t=n_0}^{n-1} a(m,t)\sqrt{q(m,t)} + \sum_{t=n_0}^{n-1} b(m,t)\sqrt{q(m,t)}w\Big(\sqrt{q(m,t)}\Big) \\
&\leq \sqrt{q(m,n-1)}\left[\sum_{t=n_0}^{n-1} a(m,t) + \sum_{t=n_0}^{n-1} b(m,t)w\Big(\sqrt{q(m,t)}\Big)\right] ,
\end{aligned}$$

or

$$\frac{\Delta_1 q(m,n)}{\sqrt{q(m,n-1)}} \leq \sum_{t=n_0}^{n-1} a(m,t) + \sum_{t=n_0}^{n-1} b(m,t)w\Big(\sqrt{q(m,t)}\Big) \ .$$

Therefore, for any $(m,n)\in\Omega$,

$$\sum_{s=m_0}^{m-1}\frac{\Delta_1 q(s,n)}{\sqrt{q(s,n-1)}}\le\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}a(s,t)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t)w\Big(\sqrt{q(s,t)}\Big)$$
$$=A(m,n)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t)w\Big(\sqrt{q(s,t)}\Big)\ .$$

On the other hand, by the non-decreasing property of q in each variable, it is easy to check that

$$\begin{aligned}
&\sum_{s=m_0}^{m-1}\frac{\Delta_1 q(s,n)}{\sqrt{q(s,n-1)}}\\
=&\frac{q(m,n)}{\sqrt{q(m-1,n-1)}}-\frac{q(m-1,n)}{\sqrt{q(m-1,n-1)}}+\frac{q(m-1,n)}{\sqrt{q(m-2,n-1)}}-\frac{q(m-2,n)}{\sqrt{q(m-2,n-1)}}+\cdots\\
&\quad+\frac{q(m_0+1,n)}{\sqrt{q(m_0,n-1)}}-\frac{q(m_0,n)}{\sqrt{q(m_0,n-1)}}\\
=&\frac{q(m,n)}{\sqrt{q(m-1,n-1)}}+\sum_{s=1}^{m-m_0-1}q(m-s,n)\Big[\frac{1}{\sqrt{q(m-s-1,n-1)}}-\frac{1}{\sqrt{q(m-s,n-1)}}\Big]\\
&\quad-\frac{q(m_0,n)}{\sqrt{q(m_0,n-1)}}\\
\ge&\frac{q(m,n)}{\sqrt{q(m,n)}}-\frac{q(m_0,n)}{\sqrt{q(m_0,n-1)}}\\
=&\sqrt{q(m,n)}-k
\end{aligned}$$

for all $(m,n)\in\Omega$. Hence we have

$$\sqrt{q(m,n)}\le k+A(m,n)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t)w\Big(\sqrt{q(s,t)}\Big)$$

for all $(m,n)\in\Omega$. In particular, since A is non-decreasing in each variable, for any fixed $(\overline{m},\overline{n})\in\Omega_{(m_1,n_1)}$,

$$\sqrt{q(m,n)}\le\big(k+A(\overline{m},\overline{n})\big)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t)w\big(\sqrt{q(s,t)}\big)$$

for all $(m,n)\in\Omega_{(\overline{m},\overline{n})}$. Now by applying Theorem 2.1 to the function $\sqrt{q(m,n)}$, we have

$$u(m,n)\le\sqrt{q(m,n)}\le\Phi^{-1}\Big[\Phi\big(k+A(\overline{m},\overline{n})\big)+B(m,n)\Big]$$

for all $(m,n) \in \Omega_{(\overline{m},\overline{n})}$. In particular, this gives

$$u(\overline{m},\overline{n}) \leq \Phi^{-1}\Big[\Phi\big(k + A(\overline{m},\overline{n})\big) + B(\overline{m},\overline{n})\Big] \ .$$

Since $(\overline{m},\overline{n}) \in \Omega_{(m_1,n_1)}$ is arbitrary, this concludes the proof of the theorem. ∎

Remark. Similar to the previous remark, in many cases $\Phi(\infty) = \infty$ and so in these situations, inequality (4) holds for all $(m,n) \in \Omega$.

In case Ω degenerates into a 1-dimensional lattice, Theorem 2.2 takes the following simpler form which is equivalent to a result of Pachpatte in [18].

Corollary 2.3. *Suppose $u \in \mathcal{F}_+(I)$. If $k \geq 0$ is a constant and $a, b \in \mathcal{F}_+(I)$, $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ are functions satisfying*

(i) *w is non-decreasing with $w(r) > 0$ for $r > 0$; and*

(ii) *for any $m \in I$,*

$$u^2(m) \leq k^2 + \sum_{s=m_0}^{m-1} a(s)u(s) + \sum_{s=m_0}^{m-1} b(s)u(s)w\big(u(s)\big) \ ,$$

then

$$u(m) \leq \Phi^{-1}\left[\Phi\Big(k + \sum_{s=m_0}^{m-1} a(s)\Big) + \sum_{s=m_0}^{m-1} b(s)\right]$$

for all $m \in [m_0, m_1] \cap I$, where Φ is defined in Theorem 2.1, and $m_1 \in I$ is chosen such that $\Phi\Big(k + \sum_{s=m_0}^{m-1} a(s)\Big) \in Dom\ \Phi^{-1}$ for all $m \in [m_0, m_1] \cap I$.

Proof. It follows immediately from Theorem 2.2 by setting $\Omega = I \times \{n_0\}$ for some $n_0 \in \mathbb{Z}$, and extending the functions $a(s)$, $b(s)$, $u(s)$ to $a(s, n_0)$, $b(s, n_0)$ and $u(s, n_0)$ respectively in the obvious way. ∎

A useful special case of Theorem 2.2 is the following

Corollary 2.4. *Suppose $u \in \mathcal{F}_+(\Omega)$. If $k \geq 0$ is a constant and $a, b \in \mathcal{F}_+(\Omega)$ are functions such that for any $(m,n) \in \Omega$,*

$$u^2(m,n) \leq k^2 + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u^2(s,t) \ ,$$

then

$$u(m,n) \le \big[k + A(m,n)\big] \exp B(m,n)$$

for all $(m,n) \in \Omega$, where $A(m,n)$ and $B(m,n)$ are as defined in Theorem 2.2.

Proof. Assume first that $k > 0$. Let w be the identity mapping of $\mathbb{R}_+$ onto itself. Then all conditions of Theorem 2.2 are satisfied. Note that in this cases $\Phi = \ln$ and so $\Phi^{-1} = \exp$. In particular, Φ^{-1} is defined everywhere on $\mathbb{R}$. By Theorem 2.2, we have

$$u(m,n) \le \exp\Big[\ln\big(k + A(m,n)\big) + B(m,n)\Big] = \big[k + A(m,n)\big] \exp B(m,n)$$

for all $(m,n) \in \Omega$. Finally, as this is true for all $k > 0$, by continuity, this should also hold for the case $k = 0$. ■

In case Ω degenerates into a 1-dimensional lattice, Corollary 2.4 takes the following simpler form which is equivalent to another result of Pachpatte in [18].

Corollary 2.5. *Suppose $u \in \mathcal{F}_+(I)$. If $k \ge 0$ is a constant and $a, b \in \mathcal{F}_+(I)$ are functions such that for any $m \in I$,*

$$u^2(m) \le k^2 + \sum_{s=m_0}^{m-1} a(s)u(s) + \sum_{s=m_0}^{m-1} b(s)u^2(s) \ ,$$

then

$$u(m) \le \Big[k + \sum_{s=m_0}^{m-1} a(s)\Big] \prod_{s=m_0}^{m-1} \exp b(s)$$

for all $m \in I$.

Proof. Analogous to that of Corollary 2.3 and apply Corollary 2.4. ■

Another special situation of Corollary 2.4 is the following 2-dimensional discrete version of Ou-Iang's inequality.

Corollary 2.6. *Suppose $u \in \mathcal{F}_+(\Omega)$. If $k \ge 0$ is a constant and $b \in \mathcal{F}_+(\Omega)$ is a function such that for any $(m,n) \in \Omega$,*

$$u^2(m,n) \le k^2 + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)u^2(s,t) \ ,$$

then

$$u(m,n) \leq k \exp B(m,n)$$

for all $(m,n) \in \Omega$, *where* $B(m,n)$ *is as defined in Theorem 2.2.*

Proof. This follows immediately from Corollary 2.4 by setting $a \equiv 0$. ■

In case Ω degenerates into a 1-dimensional lattice, Corollary 2.6 takes the following simpler form which is the 1-dimensional discrete analogue of Ou-Iang's inequality.

Corollary 2.7. *Suppose* $u \in \mathcal{F}_+(I)$. *If* $k \geq 0$ *is a constant and* $b \in \mathcal{F}_+(I)$ *is a function such that for any* $m \in I$,

$$u^2(m) \leq k^2 + \sum_{s=m_0}^{m-1} b(s)u^2(s) \ ,$$

then

$$u(m) \leq k \prod_{s=m_0}^{m-1} \exp b(s)$$

for all $m \in I$.

Proof. It follows from Corollary 2.5 by setting $a \equiv 0$, or by immitating the Proof of Corollary 2.3 and applying Corollary 2.6. ■

Remark. It is evident that the results above can be generalized to obtain explicit bounds for functions satisfying certain discrete sum inequalities involving more retarded arguments. It is also clear that these results can be extended to functions on higher dimensional lattices in the obvious way. As details of these are rather algorithmic, they will not be carried out here.

3. Applications to boundary value problems

In this section, we shall illustrate how the results obtained in §2 can be applied to study the boundedness, uniqueness, and continuous dependence of the solutions of certain boundary value problems for difference equations involving 2 independent variables. We consider the following:

Boundary Value Problem (BVP):

$$\Delta_{12} z^2(m,n) = f\big(m,n,z(m,n)\big)$$

satisfying

$$z(m,n_0) = p(m)\ ,\ z(m_0,n) = q(n)\ ,\ p(m_0) = q(n_0) = 0\ ,$$

where $f \in \mathcal{F}(\Omega \times \mathbb{R})$, $p \in \mathcal{F}(I)$, and $q \in \mathcal{F}(J)$ are given.

Our first result deals with the boundedness of solutions.

Theorem 3.1. *Consider (BVP). If*

$$\big|f(m,n,v)\big| \le b(m,n)|v|^2 \tag{5}$$

and

$$p^2(m) + q^2(n) \le k^2 \tag{6}$$

for some $k \ge 0$*, where* $b \in \mathcal{F}_+(\Omega)$*, then all solutions of (BVP) satisfy*

$$|z(m,n)| \le k \exp B(m,n)\ ,\quad (m,n) \in \Omega\ ,$$

where $B(m,n)$ *is defined as in Theorem 2.1. In particular, if* $B(m,n)$ *is bounded on* Ω*, then every solution of (BVP) is bounded on* Ω*.*

Proof. Observe first that $z = z(m,n)$ solves (BVP) if and only if it satisfies the sum-difference equation

$$z^2(m,n) = p^2(m) + q^2(n) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} f\big(s,t,z(s,t)\big)\ . \tag{7}$$

Hence by (5) and (6),

$$z^2(m,n) \le k^2 + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) z^2(s,t)$$

for all $(m,n) \in \Omega$. An application of Corollary 2.6 to the function $\big|z(m,n)\big|$ gives the assertion immediately. ■

The next result is about uniqueness.

Theorem 3.2. *Consider (BVP). If*

$$\left|f(m,n,v_1)-f(m,n,v_2)\right| \le b(m,n)\left|v_1^2-v_2^2\right|$$

for some $b \in \mathcal{F}_+(\Omega)$, then (BVP) has at most one solution on Ω.

Proof. Let $z(m,n)$ and $\overline{z}(m,n)$ be two solutions of (BVP) on Ω. By (7), we have

$$\begin{aligned}\left|z^2(m,n)-\overline{z}^2(m,n)\right| &\le \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|f\big(s,t,z(s,t)\big)-f\big(s,t,\overline{z}(s,t)\big)\right| \\ &\le \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\left|z^2(s,t)-\overline{z}^2(s,t)\right| .\end{aligned}$$

An application of Corollary 2.6 to the function $\sqrt{|z^2(s,t)-\overline{z}^2(s,t)|}$ shows that

$$\sqrt{|z^2(s,t)-\overline{z}^2(s,t)|} \le 0 \quad \text{for all } (s,t)\in\Omega .$$

Hence $z=\overline{z}$ on Ω. ■

Finally, we investigate the continuous dependence of the solutions of (BVP) on the function f and the boundary data p and q. For this we consider the following variation of (BVP):

($\overline{\text{BVP}}$):

$$\Delta_{12} z^2(m,n) = \overline{f}\big(m,n,z(m,n)\big)$$

with

$$z(m,n_0)=\overline{p}(m)\ ,\ z(m_0,n)=\overline{q}(n)\ ,\ \overline{p}(m_0)=\overline{q}(n_0)=0\ ,$$

where $\overline{f}\in\mathcal{F}(\Omega\times\mathbb{R})$, $\overline{p}\in\mathcal{F}(I)$, and $\overline{q}\in\mathcal{F}(J)$ are given.

Theorem 3.3. *Consider (BVP) and ($\overline{BVP}$). If*

(i) $|f(m,n,v_1)-f(m,n,v_2)| \le b(m,n)|v_1^2-v_2^2|$ *for some* $b\in\mathcal{F}_+(\Omega)$*;*

(ii) $\left|\big(p^2(m)-\overline{p}^2(m)\big)+\big(q^2(n)-\overline{q}^2(n)\big)\right| \le \frac{\varepsilon}{2}$*; and*

(iii) *for all solutions $\overline{z}(m,n)$ of ($\overline{BVP}$),*

$$\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|f\big(s,t,\overline{z}(s,t)\big)-\overline{f}\big(s,t,\overline{z}(s,t)\big)\right| \le \frac{\varepsilon}{2}$$

for all $(m,n)\in\Omega$*,* $v_1, v_2 \in \mathbb{R}$*, then*

$$\left|z^2(m,n)-\overline{z}^2(m,n)\right| \le \varepsilon \exp\left(2B(m,n)\right)\ ,$$

where $B(m,n)$ *is as defined in Theorem 2.1. Hence* z^2 *depends continuously on* f, p*, and* q*. In particular, if* z *does not change sign, it depends continuously on* f*,* p *and* q*.*

Proof. Let $z(m,n)$ and $\overline{z}(m,n)$ be solutions of (BVP) and ($\overline{\text{BVP}}$), respectively. Then z satisfies (7) and $\overline{z}$ satisfies the corresponding equation

$$\overline{z}^2(m,n)=\overline{p}^2(m)+\overline{q}^2(n)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\overline{f}\left(s,t,\overline{z}(s,t)\right)\ .$$

Hence

$$\begin{aligned}
&\left|z^2(m,n)-\overline{z}^2(m,n)\right| \\
&\le \left|\left(p^2(m)-\overline{p}^2(m)\right)+\left(q^2(n)-\overline{q}^2(n)\right)\right|+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|f\left(s,t,z(s,t)\right)-\overline{f}\left(s,t,\overline{z}(s,t)\right)\right| \\
&\le \frac{\varepsilon}{2}+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|f\left(s,t,z(s,t)\right)-f\left(s,t,\overline{z}(s,t)\right)\right|+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|f\left(s,t,\overline{z}(s,t)\right)-\overline{f}\left(s,t,\overline{z}(s,t)\right)\right| \\
&\le \varepsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t)\left|z^2(s,t)-\overline{z}^2(s,t)\right|
\end{aligned}$$

by assumptions (i), (ii) and (iii). Now by applying Corollary 2.6 to the function $\sqrt{|z^2(m,n)-\overline{z}^2(m,n)|}$, we have

$$\sqrt{|z^2(m,n)-\overline{z}^2(m,n)|}\le\sqrt{\varepsilon}\exp B(m,n)$$

for all $(m,n)\in\Omega$, or

$$\left|z^2(m,n)-\overline{z}^2(m,n)\right| \le \varepsilon \exp\left(2B(m,n)\right)\ .$$

Now when restricted to any compact sub-lattice, $B(m,n)$ is bounded, so

$$\left|z^2(m,n)-\overline{z}^2(m,n)\right| \le \varepsilon\cdot K$$

for some $K > 0$ for all (m, n) in this compact sub-lattice. Hence z^2 depends continuously on f, p and q. ■

Remark. The boundary value problem (BVP) is clearly not the only problem for which the boundedness, uniqueness, and continuous dependence of its solutions can be studied by using the results in §2. For example, one can arrive at similar results (with much more complicated computations) for the following variation of the (BVP):

$$\Delta_{12} z^2(m,n) = f\Big(m, n, z(m,n)\ ,\ z(m,n) \cdot w\big(|z(m,n)|\big)\Big)$$

with

$$z(m, n_0) = p(m)\ ,\ z(m_0, n) = q(n)\ ,\ p(m_0) = q(n_0) = 0\ ,$$

where $f \in \mathcal{F}(\Omega \times \mathbb{R}^2)$, $p \in \mathcal{F}(I)$, $q \in \mathcal{F}(J)$, and $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ are given.

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