Periodic Solutions for p-Laplacian Liénard Equation with a Deviating Argument

Wing-Sum Cheung*

Department of Mathematics, The University of Hong Kong

Pokfulam, Hong Kong

wscheung@hkucc.hku.hk

and

Jingli Ren†

Department of Mathematics, Zhengzhou University

Zhengzhou, Henan 450052, P.R. China

ren_jingli@yahoo.com

Abstract—By employing Mawhin's continuation theorem, the existence of periodic solutions of the p-Laplacian Liénard equation with a deviating argument

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(x(t-\tau(t))) = e(t)$$

under various assumptions are obtained.

Keywords—periodic solution, Mawhin's continuation theorem, deviating argument.

1. INTRODUCTION

As it is well known, the existence of periodic solutions for Duffing equation and Liénard equation was extensively studied (see [1, 4-6, 9] and the references therein). In recent years, the existence of $T-$periodic solutions to several types of second order scalar differential equations with deviating arguments were studied in [10, 12-14]. For example, In [10], Huang and Xiang studied the following type of Duffing equation with a single constant deviating argument

$$x''(t) + g(x(t-\tau)) = p(t). \tag{1.1}$$

*Correspondence author. Research is partially supported by the Research Grants Council of the Hong Kong SAR, China (Project No. HKU7130/00P)

†Research is partially supported by the National Natural Science Foundation, China (Project No. 10371006)

Under a one-sided boundedness condition imposed on $g(x)$ such as

$$|g(x)| < R_0 \quad \text{for } |x| > M, \tag{1.2}$$

and a signal condition $xg(x) > 0$ for $|x| > M$, where $M > 0, R_0 > 0$ are constants, the authors obtained a periodic solution for Eq.(1.1). In [14], Ma, Wang and Yu studied delay Duffing equations of the type

$$x''(t) + m^2 x(t) + g(x(t-\tau)) = p(t). \tag{1.3}$$

They established several criteria to guarantee the existence of periodic solutions of Eq.(1.3) by assuming

$$M = \sup_{x\in R} |g(x)| < \infty. \tag{1.4}$$

In [12], Lu and Ge discussed the existence of periodic solutions for the second order differential equation with multiple deviating arguments

$$x''(t) + f(x(t))x'(t) + \sum_{j=1}^{n} \beta_j(t) g(x(t-\gamma_j(t))) = p(t).$$

In their work, some linear growth condition imposed on $g(x)$ such as

$$\lim_{|x|\to+\infty} \frac{|g(x)|}{|x|} = r \in [0, +\infty) \tag{1.5}$$

was required.

In this paper, we study the existence of periodic solutions for a p-Laplacian Liénard equation with a deviating argument

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(x(t-\tau(t))) = e(t), \tag{1.6}$$

where $p > 1$ is a constant, $\varphi_p : \mathbb{R} \to \mathbb{R}, \varphi_p(u) = |u|^{p-2}u$, $f, g, e, \tau \in C(\mathbb{R}, \mathbb{R})$, e, τ are periodic with period T, and $\int_0^T e(s)ds = 0$. By using the time maps and the phase plane analysis, some researchers discussed the existence of periodic solutions to Eq.(1.6) for $\tau(t) \equiv 0$ in [2, 3, 7, 11, 15]. But the corresponding problem of Eq.(1.6) for $p \neq 2$ and $\tau(t) \not\equiv 0$, as far as we know, has not been studied. The purpose of this paper is to establish some criteria to guarantee the existence of T-periodic solutions to Eq.(1.6). The methods used to estimate *a priori* bound of periodic solutions are different from the corresponding ones in [10, 12-14]. Furthermore, the significance of this paper is that Theorem 3.1 does not require any one-sided growth condition on g but instead, it is related to the deviating argument $\tau(t)$, and the one-sided growth condition imposed on $g(x)$ in Theorem 3.2 is weaker than the corresponding ones, namely, (1.2), (1.4) and (1.5) in [10], [14] and [12].

2. MAIN LEMMAS

First, we recall Mawhin's continuation theorem which our study is based upon.

Let X and Y be real Banach Spaces and let $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $Im\ L$ is closed in Y and $\dim Ker\ L = \dim(Y/Im\ L) < +\infty$. Consider the supplementary subspaces X_1 and Y_1 such that $X = Ker\ L \oplus X_1$ and $Y = Im\ L \oplus Y_1$ and let $P : X \to Ker\ L$ and $Q : Y \to Y_1$ be the natural projections. Clearly, $Ker\ L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Denote by K the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \phi$. A map $N : \overline{\Omega} \to Y$ is said to be $L-$ compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact.

LEMMA 2.1[8] Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is $L-$ compact on $\overline{\Omega}$. If

(1)$Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;

(2)$Nx \notin Im\ L, \forall x \in \partial\Omega \cap Ker\ L$; and

(3)$deg\{JQN, \Omega \cap Ker\ L, 0\} \neq 0$, where $J : Im\ Q \to Ker\ L$ is an isomorphism,

then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \bigcap D(L)$.

The following Lemma is important for us to estimate *a priori* bound of periodic solutions in Section 3.

LEMMA 2.2[12] Let $0 \leq \alpha \leq T$ be a constant, $s \in C(\mathbb{R}, \mathbb{R})$ be periodic with period T, and $\max\limits_{t\in[0,T]} |s(t)| \leq \alpha$. Then for any $u \in C^1(\mathbb{R}, \mathbb{R})$ which is periodic with period T, we have

$$\int_0^T |u(t) - u(t - s(t))|^2 dt \leq 2\alpha^2 \int_0^T |u'(t)|^2 dt.$$

In order to use Mawhin's continuation theorem to study the existence of $T-$periodic solutions for Eq.(1.6), we rewrite Eq.(1.6) in the following form

$$\begin{cases} x_1'(t) &= \varphi_q(x_2(t)) = |x_2(t)|^{q-2}x_2(t) \\ x_2'(t) &= -g(x_1(t - \tau(t))) - f(x_1(t))\varphi_q(x_2(t)) + e(t), \end{cases} \tag{2.1}$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a $T-$periodic solution to Eqs.(2.1), then $x_1(t)$ must be a $T-$periodic solution to Eq.(1.6). Thus, the problem of finding a $T-$periodic solution for Eq. (1.6) reduces to finding one for Eq. (2.1).

Now, we set $C_T = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \ \phi(t + T) \equiv \phi(t)\}$ with norm $|\phi|_0 = \max\limits_{t\in[0,T]} |\phi(t)|$, $X = Y = \{x = (x_1(\cdot), x_2(\cdot)) \in C(\mathbb{R}, \mathbb{R}^2) : \ x(t) \equiv x(t + T)\}$ with norm $||x|| = \max\{|x_1|_0, |x_2|_0\}$. Clearly, X and Y are Banach spaces. Meanwhile, let

$$L : D(L) \subset X \to Y, \quad Lx = x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$N : X \to Y, \quad Nx = \begin{pmatrix} \varphi_q(x_2) \\ -g(x_1(t-\tau(t))) - f(x_1(t))\varphi_q(x_2(t)) + e(t). \end{pmatrix}$$

It is easy to see that $Ker\ L = \mathbb{R}^2$, $Im\ L = \{y \in Y : \int_0^T y(s)ds = 0\}$. So L is a Fredholm operator with index zero. Let $P : X \to Ker\ L$ and $Q : Y \to Im\ Q \subset \mathbb{R}^2$ be defined by

$$Px = \frac{1}{T}\int_0^T x(s)ds; \quad Qy = \frac{1}{T}\int_0^T y(s)ds,$$

and let K denote the inverse of $L|_{KerP\cap D(L)}$. Obviously, $Ker\ L = Im\ Q = \mathbb{R}^2$ and

$$[Ky](t) = \int_0^T G(t,s)y(s)ds \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \dfrac{s}{T}, & 0 \le s < t \le T. \\ \dfrac{s-T}{T}, & 0 \le t \le s \le T. \end{cases}$$

From (2.2), one can easily see that N is $L-$compact on $\overline{\Omega}$, where Ω is an open, bounded subset of X.

For the sake of convenience, we list the following assumptions which will be used for us to study the existence of $T-$periodic solutions to Eq.(1.6) in Section 3.

$[H_1]$ There exists an integer m such that $\delta := |\tau(t) - mT|_0 \le T$.

$[H_2]$ There is a constant $d > 0$ such that $ug(u)$ does not change sign for $|u| > d$.

$[H_3]$ There is a constant $l > 0$ such that

$$|g(u_1) - g(u_2)| \le l|u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}.$$

$[H_4]$ There is a constant $r \in [0, +\infty)$ such that $\lim\limits_{u\to-\infty} \frac{|g(u)|}{|u|^{p-1}} = r$.

3. MAIN RESULTS

THEOREM 3.1 If $[H_1] - [H_3]$ hold, and there is a constant $\sigma > 0$ such that $|f(s)| \ge \sigma$ for all $s \in \mathbb{R}$, then Eq.(1.6) has at least one $T-$periodic solution if $2^{1/2}l\delta < \sigma$.

Proof: Consider the following operator equation

$$Lx = \lambda Nx, \lambda \in (0,1). \tag{3.1}$$

Let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in (0,1)\}$. If $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \Omega_1$, then from (3.1)we have

$$\begin{cases} x_1'(t) &= \lambda\varphi_q(x_2(t)) = \lambda|x_2(t)|^{q-2}x_2(t) \\ x_2'(t) &= -\lambda g(x_1(t-\tau(t))) - \lambda f(x_1(t))\varphi_q(x_2(t)) + \lambda e(t). \end{cases} \tag{3.2}$$

From the first equation of (3.2), we have $x_2(t) = \varphi_p(\frac{1}{\lambda}x_1'(t))$, which together with the second equation of (3.2) yields

$$\Big[\varphi_p\Big(\frac{1}{\lambda}x_1'(t)\Big)\Big]' + f(x_1(t))x_1'(t) + \lambda g(x_1(t-\tau(t))) = \lambda e(t),$$

i.e.,

$$\big[\varphi_p\big(x_1'(t)\big)\big]' + \lambda^{p-1} f(x_1(t))x_1'(t) + \lambda^p g(x_1(t-\tau(t))) = \lambda^p e(t), \tag{3.3}$$

Integrating both sides of (3.3) over $[0,T]$, we get

$$\int_0^T g(x_1(t-\tau(t)))dt = 0. \tag{3.4}$$

So there is a constant $\xi \in [0,T]$ such that

$$g(x_1(\xi-\tau(\xi))) = 0.$$

From assumption $[H_2]$, we see $|x_1(\xi-\tau(\xi))| \le d$. Write $\xi - \tau(\xi) = kT + t_0$, where $k \in \mathbb{Z}$ and $t_0 \in [0,T)$. Thus $|x_1(t_0)| = |x_1(\xi-\tau(\xi))| \le d$, which implies

$$|x_1|_0 \le d + \int_0^T |x_1'(s)|ds. \tag{3.5}$$

On the other hand, multiplying both sides of Eq.(3.3) by $x_1'(t)$ and integrating over $[0,T]$, we have

$$\begin{aligned} &\int_0^T [\varphi_p(x_1'(t))]'x_1'(t)dt + \lambda^{p-1}\int_0^T f(x_1(t))[x_1'(t)]^2 dt + \lambda^p \int_0^T g(x_1(t-\tau(t)))x_1'(t)dt \\ = \;& \lambda^p \int_0^T e(t)x_1'(t)dt. \end{aligned} \tag{3.6}$$

If we write $w(t) = \varphi_p(x_1'(t))$, then

$$\int_0^T [\varphi_p(x_1'(t))]'x_1'(t)dt = \int_0^T \varphi_q(w(t))dw(t) = 0,$$

which together with (3.6) yields

$$\Big|\int_0^T f(x_1(t))[x_1'(t)]^2 dt\Big| < \Big|\int_0^T g(x_1(t-\tau(t)))x_1'(t)dt\Big| + \Big|\int_0^T e(t)x_1'(t)dt\Big|. \tag{3.7}$$

Furthermore, from condition $|f(s)| \ge \sigma$ for all $s \in \mathbb{R}$ and $|f| \ge \sigma$ everywhere, we see that

$$\sigma \int_0^T |x_1'(t)|^2 dt \le \int_0^T |f(x_1(t))||x_1'(t)|^2 dt = \Big|\int_0^T f(x_1(t))[x_1'(t)]^2 dt\Big|.$$

So we have from (3.7) that

$$\sigma \int_0^T |x_1'(t)|^2 dt \le \Big|\int_0^T g(x_1(t-\tau(t)))x_1'(t)dt\Big| + \Big|\int_0^T e(t)x_1'(t)dt\Big|. \tag{3.8}$$

In view of $\int_0^T g(x_1(t))x'(t)dt = 0$, we find from (3.8) that

$$
\begin{aligned}
& \sigma \int_0^T |x_1'(t)|^2 dt \\
\leq\ & \Big| \int_0^T [g(x_1(t)) - g(x_1(t-\tau(t)))] x_1'(t) dt \Big| + \Big| \int_0^T e(t) x_1'(t) dt \Big| \\
\leq\ & \int_0^T |g(x_1(t)) - g(x_1(t-\tau(t)))||x_1'(t)| dt + \int_0^T |e(t)||x_1'(t)| dt \\
\leq\ & l \int_0^T |x_1'(t)||x_1(t) - x_1(t-\tau(t))| dt + \int_0^T |e(t)||x_1'(t)| dt \\
\leq\ & l(\int_0^T |x_1'(t)|^2 dt)^{1/2} (\int_0^T \big|x_1(t) - x_1(t-\tau(t))\big|^2 dt)^{1/2} + \\
& (\int_0^T |e(s)|^2 ds)^{1/2} (\int_0^T |x_1'(t)|^2 dt)^{1/2} \\
=\ & l(\int_0^T |x_1'(t)|^2 dt)^{1/2} (\int_0^T \big|x_1(t) - x_1(t-\tau(t)+mT)\big|^2 dt)^{1/2} + \\
& (\int_0^T |e(s)|^2 ds)^{1/2} (\int_0^T |x_1'(t)|^2 dt)^{1/2}.
\end{aligned}
\tag{3.9}
$$

By Lemma 2.2, we have

$$
\Big(\int_0^T \big| x_1(t) - x_1(t-\tau(t)+mT) \big|^2 dt \Big)^{1/2} \leq 2^{1/2} \delta \Big(\int_0^T |x_1'(t)|^2 dt \Big)^{1/2}.
$$

Substituting this into (3.9), we obtain

$$
\begin{aligned}
& \sigma \int_0^T |x_1'(t)|^2 dt \\
\leq\ & 2^{1/2} l \delta \int_0^T |x_1'(t)|^2 dt + \Big(\int_0^T |e(s)|^2 ds \Big)^{1/2} \Big(\int_0^T |x_1'(t)|^2 dt \Big)^{1/2},
\end{aligned}
$$

and so

$$
\Big(\int_0^T |x_1'(t)|^2 dt \Big)^{1/2} \leq \frac{(\int_0^T |e(s)|^2 ds)^{1/2}}{\sigma - 2^{1/2} l \delta} =: R_0.
$$

It follows from (3.5) that

$$
|x_1|_0 \leq d + T^{1/2} R_0 =: M_1. \tag{3.10}
$$

By the first equation of (3.2), we have

$$
\int_0^T |x_2(s)|^{q-2} x_2(s) ds = 0,
$$

which implies that there is a constant $t_2 \in [0, T]$ such that $x_2(t_2) = 0$. So

$$
|x_2|_0 \leq \int_0^T |x_2'(s)| ds.
$$

Then by the second equation of (3.2), we obtain

$$
\begin{aligned}
\int_0^T |x_2'(s)| ds \ &\leq\ \lambda g_{M_1} T + \lambda \int_0^T |f(x_1(s))||x_2(s)|^{q-1} ds + \lambda |e|_1 \\
&=\ \lambda g_{M_1} T + \int_0^T |f(x_1(s))||x_1'(s)| ds + \lambda |e|_1 \\
&\leq\ g_{M_1} T + |f|_0 T^{1/2} R_0 + |e|_1,
\end{aligned}
$$

where $g_{M_1} = \max\limits_{|s|\le M_1} |g(s)|$ and $|e|_1 = \int_0^T |e(s)|ds$. So we have

$$|x_2|_0 \le g_{M_1}T + |f|_0 T^{1/2} R_0 + |e|_1 =: M_2. \tag{3.11}$$

Let $\Omega_2 = \{x \in Ker\ L : Nx \in Im\ L\}$. If $x \in \Omega_2$, then $x \in Ker\ L$ and $QNx = 0$. By the assumption on e, we see that

$$\begin{cases} |x_2|^{q-2}x_2 = 0, \\ g(x_1) = 0. \end{cases}$$

So

$$|x_1| \le d \le M_1, \quad x_2 = 0 \le M_2.$$

Let $\Omega = \{x = (x_1, x_2)^\top \in X : |x_1|_0 < N_1, |x_2|_0 < N_2\}$, where N_1 and N_2 are constants with $N_1 > M_1$, $N_2 > M_2$ and $(N_2)^q > dg_d$, where $g_d = \max\limits_{|u|\le d} |g(u)|$. Then $\overline{\Omega}_1 \subset \Omega$, $\overline{\Omega}_2 \subset \Omega$. From (3.10), (3.11) and the above, it is easy to see that conditions (1) and (2) of Lemma 2.1 are satisfied.

Next, we claim that condition (3) of Lemma 2.1 is also satisfied. For this, define the isomorphism $J : Im\ Q \to Ker\ L$ by

$$J(x_1, x_2) = \begin{cases} (x_2, x_1), & \text{if } \ ug(u) < 0 \text{ for } |u| > d, \\ (-x_2, x_1), & \text{if } \ ug(u) > 0 \text{ for } |u| > d, \end{cases}$$

and let $H(v, \mu) := \mu v + \frac{1-\mu}{T} JQNv, \quad (v, \mu) \in \Omega \times [0, 1]$. By simple calculation, we obtain, for $(x, \mu) \in \partial(\Omega \cap KerL) \times [0, 1]$,

$$x^\top H(x, \mu) = \begin{cases} \mu(x_1^2 + x_2^2) + \frac{1-\mu}{T}(-x_1 g(x_1) + |x_2|^q) > 0, & \text{if } \ ug(u) < 0 \text{ for } |u| > d, \\ \mu(x_1^2 + x_2^2) + \frac{1-\mu}{T}(x_1 g(x_1) + |x_2|^q) > 0, & \text{if } \ ug(u) > 0 \text{ for } |u| > d. \end{cases}$$

Hence

$$\begin{aligned} & deg\{JQN, \Omega \cap KerL, 0\} = deg\{H(x, 0), \Omega \cap KerL, 0\} \\ = \ & deg\{H(x, 1), \Omega \cap KerL, 0\} = deg\{I, \Omega \cap KerL, 0\} \\ \ne \ & 0, \end{aligned}$$

and so condition (3) of Lemma 2.1 is satisfied.

Therefore, by Lemma 2.1, we conclude that equation

$$Lx = Nx$$

has a solution $x(t) = (x_1(t), x_2(t))^\top$ on $\overline{\Omega}$, i.e., Eq.(1.6) has a $T-$periodic solution $x_1(t)$ with $|x_1|_0 \le M_1$. This completes the proof of Theorem 3.1.

Observe that if $\tau(t) \equiv kT$, where k is an integer, then we may take $m = k$ and $\delta = 0$, and so in this case assumption $[H_3]$ in Theorem 3.1 can be removed. In fact, from (3.8) we see that

$$\begin{aligned}
&\sigma\int_0^T |x_1'(t)|^2dt\\
\le\ & \Big|\int_0^T g(x_1(t-\tau(t)))x_1'(t)dt\Big| + \Big|\int_0^T e(t)x_1'(t)dt\Big|\\
=\ & \Big|\int_0^T g(x_1(t-kT))x_1'(t)dt\Big| + \Big|\int_0^T e(t)x_1'(t)dt\Big|\\
=\ & \Big|\int_0^T g(x_1(t))x_1'(t)dt\Big| + \Big|\int_0^T e(t)x_1'(t)dt\Big|\\
=\ & \Big|\int_0^T e(t)x_1'(t)dt\Big|\\
\le\ & \Big(\int_0^T |e(s)|^2ds\Big)^{1/2}\Big(\int_0^T |x_1'(s)|^2ds\Big)^{1/2},
\end{aligned}$$

which implies

$$\Big(\int_0^T |x_1'(s)|^2ds\Big)^{1/2} \le \frac{(\int_0^T |e(s)|^2ds)^{1/2}}{\sigma}.$$

So

$$|x_1|_0 \le d + T^{1/2}\frac{(\int_0^T |e(s)|^2ds)^{1/2}}{\sigma}.$$

Therefore, we have

COROLLARY 3.1 Suppose $[H_2]$ holds and there is a constant $\sigma > 0$ such that $|f(s)| \ge \sigma$ for all $s \in \mathbb{R}$. Furthermore, assume $\tau(t) \equiv kT$, where k is an integer. Then Eq.(1.6) has at least one $T-$periodic solution.

THEOREM 3.2 If $[H_2]$ and $[H_4]$ hold, then Eq.(1.6) has at least one $T-$periodic solution if $2rT^p < 1$.

PROOF: Let Ω_1 be defined as in Theorem 3.1. If $x(t) = \begin{pmatrix} x_1(t)\\ x_2(t)\end{pmatrix} \in \Omega_1$, then from the proof of Theorem 3.1 we see that

$$[\varphi_p(x_1'(t))]' + \lambda^{p-1}f(x_1(t))x_1'(t) + \lambda^p g(x_1(t-\tau(t))) = \lambda^p e(t), \tag{3.12}$$

$$\int_0^T g(x_1(t-\tau(t)))dt = 0. \tag{3.13}$$

and

$$|x_1|_0 \le d + \int_0^T |x_1'(s)|ds. \tag{3.14}$$

Multiplying both sides of Eq.(3.12) by $x_1(t)$ and integrating over $[0,T]$, we have

$$\begin{aligned}
&\textstyle\int_0^T \left[\varphi_p(x_1'(t))\right]'x_1(t)dt + \lambda^{p-1}\int_0^T x_1(t)f(x_1(t))x_1'(t)dt + \lambda^p\int_0^T g\big(x_1(t-\tau(t))\big)x_1(t)dt\\
=\ & \textstyle\lambda^p\int_0^T x_1(t)e(t)dt.
\end{aligned} \tag{3.15}$$

In view of $\int_0^T x_1(t)f(x_1(t))x_1'(t)dt = 0$ and $\int_0^T[\varphi_p(x_1'(t))]'x_1(t)dt = \int_0^T |x_1'(t)|^pdt$, it follows from (3.15) that

$$\begin{aligned}
\textstyle\int_0^T |x_1'(t)|^pdt\ &<\ \textstyle\int_0^T \big|g\big(x_1(t-\tau(t))\big)\big||x_1(t)|dt + \int_0^T |x_1(t)||e(t)|dt\\
&\le\ \textstyle|x_1|_0\int_0^T \big|g\big(x_1(t-\tau(t))\big)\big|dt + |x_1|_0|e|_1,
\end{aligned} \tag{3.16}$$

where $|e|_1=\int_0^T|e(s)|ds$. From $2rT^p<1$, we easily see that there is a constant $\varepsilon>0$ (independent of λ) such that

$$2T^p(r+\varepsilon)<1. \tag{3.17}$$

For such a constant $\varepsilon>0$, we have from assumption $[H_4]$ that there is a constant $\rho>d$ (independent of λ) such that

$$|g(u)|\le(r+\varepsilon)|u|^{p-1},\ \text{ for } u<-\rho. \tag{3.18}$$

Let $E_1=\{t\in[0,T]:|x_1(t-\tau(t))|\le\rho\}$, $E_2=\{t\in[0,T]:x_1(t-\tau(t))>\rho\}$ and $E_3=\{t\in[0,T]:x_1(t-\tau(t))<-\rho\}$. From (3.13), we have

$$\Big(\int_{E_1}+\int_{E_2}+\int_{E_3}\Big)g\big(x_1(t-\tau(t))\big)dt=0.$$

It follows from assumption $[H_2]$ that

$$\int_{E_2}\big|g\big(x_1(t-\tau(t))\big)\big|dt=\Big|\int_{E_2}g\big(x_1(t-\tau(t))\big)dt\Big|\le\int_{E_1}\big|g\big(x_1(t-\tau(t))\big)\big|dt+\int_{E_3}\big|g\big(x_1(t-\tau(t))\big)\big|dt,$$

and then by (3.18), we have

$$\begin{aligned}&\int_0^T\big|g\big(x_1(t-\tau(t))\big)\big|dt\\ =&\ \Big(\int_{E_1}+\int_{E_2}+\int_{E_3}\Big)\big|g\big(x_1(t-\tau(t))\big)\big|dt\\ \le&\ 2\int_{E_1}\big|g\big(x_1(t-\tau(t))\big)\big|dt+2\int_{E_3}\big|g\big(x_1(t-\tau(t))\big)\big|dt\\ \le&\ 2g_\rho T+2(r+\varepsilon)T|x_1|_0^{p-1},\end{aligned} \tag{3.19}$$

where $g_\rho=\max\limits_{|s|\le\rho}|g(s)|$. Substituting (3.19) and (3.14) into (3.16), we get

$$\begin{aligned}&\int_0^T|x_1'(t)|^pdt\\ \le&\ 2(r+\varepsilon)T|x_1|_0^p+[2g_\rho T+|e|_1]|x_1|_0\\ \le&\ 2(r+\varepsilon)T\Big(d+\int_0^T|x_1'(s)|ds\Big)^p+\big(2g_\rho T+|e|_1\big)\int_0^T|x_1'(s)|ds+\big(2g_\rho T+|e|_1\big)d.\end{aligned} \tag{3.20}$$

We claim that there exists a constant $M_1>0$ such that

$$|x_1|_0\le M_1. \tag{3.21}$$

Case 1. If $\int_0^T|x_1'(s)|ds=0$, then by (3.14), $|x_1|_0\le d$.
Case 2. If $\int_0^T|x_1'(s)|ds>0$, then

$$\Big[d+\int_0^T|x_1'(s)|ds\Big]^p=\Big(\int_0^T|x_1'(s)|ds\Big)^p\Big[1+\frac{d}{\int_0^T|x_1'(s)|ds}\Big]^p. \tag{3.22}$$

From elementary analysis, there is a constant $h>0$ (independent of λ) such that

$$(1+x)^p<1+(1+p)x\quad\forall x\in(0,h]. \tag{3.23}$$

If $\frac{d}{\int_0^T |x_1'(s)|ds} \geq h$, then

$$\int_0^T |x_1'(s)|ds \leq d/h,$$

which implies that

$$|x_1|_0 \leq d + d/h. \tag{3.24}$$

If $\frac{d}{\int_0^T |x_1'(s)|ds} < h$, then from (3.22) we have

$$\begin{aligned}
& \Big[d + \int_0^T |x_1'(s)|ds\Big]^p \\
\leq\ & \Big(\int_0^T |x_1'(s)|ds\Big)^p \Big[1 + \frac{(p+1)d}{\int_0^T |x_1'(s)|ds}\Big] \\
=\ & \Big(\int_0^T |x_1'(s)|ds\Big)^p + (p+1)d\Big(\int_0^T |x_1'(s)|ds\Big)^{p-1} \\
\leq\ & T^{p/q}\int_0^T |x_1'(s)|^p ds + (p+1)dT^{(p-1)/q}\Big(\int_0^T |x_1'(s)|^p ds\Big)^{1/q}.
\end{aligned}$$

Substituting this into (3.20), we obtain

$$\begin{aligned}
& \int_0^T |x_1'(t)|^p dt \\
\leq\ & 2(r+\varepsilon)T^{1+p/q}\int_0^T |x_1'(s)|^p ds + 2(r+\varepsilon)(p+1)dT^{(p+q-1)/q}\Big(\int_0^T |x_1'(s)|^p ds\Big)^{1/q} + \\
& \big[2g_\rho T + |e|_1\big]T^{1/q}\Big(\int_0^T |x_1'(s)|^p ds\Big)^{1/p} + \big[2g_\rho T + |e|_1\big]d \\
=\ & 2(r+\varepsilon)T^{p}\int_0^T |x_1'(s)|^p ds + 2(r+\varepsilon)(p+1)dT^{(p+q-1)/q}\Big(\int_0^T |x_1'(s)|^p ds\Big)^{1/q} + \\
& \big[2g_\rho T + |e|_1\big]T^{1/q}\Big(\int_0^T |x_1'(s)|^p ds\Big)^{1/p} + \big[2g_\rho T + |e|_1\big]d.
\end{aligned}$$

In view of $1/q < 1, 1/p < 1$ and $2(r+\varepsilon)T^p < 1$, it follows that there is a constant $M_0 > 0$ (independent of λ) such that $\int_0^T |x_1'(t)|^p dt \leq M_0$, which together with (3.14) yields that

$$|x_1|_0 \leq d + T^{1/q}(M_0)^{1/p}. \tag{3.25}$$

This proves the claim and the rest of the proof of the theorem is identical to that of Theorem 3.1.

REMARK 3.1 From the proof of Theorem 3.2, it is easy to see that if assumption $[H_4]$ is replaced by

$[H_4']$ $\lim\limits_{u\to+\infty} \frac{|g(u)|}{|u^{p-1}|} = r \in [0, +\infty)$,

the conclusion of Theorem 3.2 is still true.

REMARK 3.2 The one-sided linear growth condition $[H_4]$ imposed on $g(x)$ in Theorem 3.2 is weaker than the corresponding ones in [12] and [14].

Example 3.1. Let us consider the following equation

$$x''(t) + f(x(t))x'(t) + g(x(t - \sin t)) = \cos t, \tag{3.26}$$

where $f(x)$ is an arbitrary continuous function,

$$g(x) = \begin{cases} \frac{x}{8\pi^2+1}, & x < 0 \\ x\exp x^3, & x \geq 0. \end{cases}$$

Corresponding to Eq.(1.6), we have $p = 2$, $T = 2\pi$ and $r = \lim_{x\to-\infty} \frac{|g(x)|}{|x|^{p-1}} = \frac{1}{8\pi^2+1}$. Since $2T^p r = \frac{8\pi^2}{1+8\pi^2} < 1$, by Theorem 3.2, we conclude that Eq.(3.26) has at least one $2\pi-$periodic solution.

Remark 3.3: In example 3.1, one can easily see that $g(x)$ does not satisfy conditions (1.2), (1.4) or (1.5). So the above result cannot be obtained by [10, 12, 14].

Example 3.2. Let us consider the following equation

$$[\varphi_p(x'(t))]' + (1 + x^2(t))x'(t) + x(t - \theta\cos t) = \sin t, \tag{3.27}$$

where $p > 1$ is an arbitrary constant, and $\theta \in (0, 1)$ is a parameter. Corresponding to Eq.(1.6), we have $T = 2\pi$, $f(x) = 1 + x^2$, $g(x) = x$ and $\tau(t) = \theta\cos t$. So we can choose $\sigma = 1, \delta = \theta, d = L = 1$ such that assumptions $[H_1] - [H_3]$ hold. If $\theta \in (0, \sqrt{2}/2)$, then $\sqrt{2}L\delta < \sigma$. By using Theorem 3.1, Eq.(1.6) has a $2\pi-$periodic solution.

REFERENCES

[1] A. Capietto, J. Mawhin and F. Zanolin, A continuation approach to superlinear periodic boundary value problems, *J. Differential equations*, **88**(1990), 347-395.

[2] M.A. del Pino, M. Elgueta and R.F. Manásevich, A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t, u) = 0, u(0) = u(T) = 0, p > 1$, *J. Differential Equations*, **80**(1989), 1-13.

[3] M.A. del Pino and R.F. Manásevich, Multiple solutions for the p-Laplacian under global nonresonance, *Proc. Amer. Math. Soc.*, **112**(1991), 131-138.

[4] M.A. del Pino, R.F. Manásevich, and A. Murua, Existence and multiple of solutions with prescribed period for a second order quasilinear ODE, *Nonlinear Analysis, TMA*, **18**(1992), 79-92.

[5] T. Din, R. Iannacci, and F. Zanolin, Existence and multiplicity results for periodic solutions of semilinear Duffing equations, *J. Differential Equations*, **105**(1993), 364-409.

[6] T. Ding, and F. Zanoli, Time-maps for the solvability of perturbed nonlinear Duffing equations, *Nonlinear Analysis, TMA*, **17**(1991), 635-653.

[7] C. Fabry and D. Fayyad, Periodic solutions of second order differential equations with a p-Laplacian and asymmetric nonlinearities, *Rend. Ist. Univ. Trieste*, **24**(1992), 207-227.

[8] R.E. Gaines and J.L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Springer-Verlag, Berlin, 1977.

[9] W.G. Ge, On the existence of harmonic solution of Liénard systems, *Nonlinear Analysis, TMA*, **16**, No.2(1991), 183-190.

[10] X. Huang and Z. Xiang, On the existence of 2π-periodic solutions of Duffing type equation $x''(t) + g(x(t - \tau)) = p(t)$, *Chinese Science Bulletin*, **39**, No.1(1994), 201-203 [in Chinese].

[11] B. Liu, Multiplicity results for periodic solutions of a second order quasilinear ODE with asymmetric nonlinearities, *Nonlinear Analysis, TMA*, **24**(1992), 139-160.

[12] S.P. Lu and W.G. Ge, Periodic solutions for a kind of second order differential equations with multiple deviating arguments, *Applied Mathematics and Computation*, **146**, No.1(2003), 195-209.

[13] S.W. Ma, Z.C. Wang and J.S. Yu, An abstract theorem at resonance and its applications, *J. Differential Equations*, **145**(1998), 274-294.

[14] S.W. Ma, Z.C. Wang and J.S. Yu, Coincidence degree and periodic solutions of Duffing equations, *Nonlinear Analysis, TMA*, **34**(1998), 443-460.

[15] R.F. Manásevich and J. Mawhin, Periodic solutions for nonlinear systems with p-Laplacian like operarors, *J. Differential Equations*, **145**(1998), 367-393.