# ON A WEAK FORM OF THE BLUMBERG PROPERTY

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In this paper we modify and extend the techniques used by Blumberg in his classic paper of 1922. In particular, we formulate a weakening of the Blumberg property and show that a wide class of not necessarily metrizable spaces has this property.

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## 1. Introduction

In 1922 Blumberg[1] showed that every separable completely metrizable space $X$ has the following surprising property:

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B: *for each real-valued function* $f : X \to \mathbb{R}$ *there is a dense subset* $D$ *of* $X$ *such that the restriction* $f|D$ *is continuous.*

Since then B has been called the *Blumberg property* and spaces having it have been called Blumberg spaces. The theory of Blumberg spaces is well-developed. For example, in 1960 Bradford and Goffman [2] showed that every Blumberg space is a Baire space and that the converse is true if the Baire space is metrizable. In 1977 Weiss [5] showed that there are compact Hausdorff spaces that are not Blumberg.

In this paper we consider spaces that need not be metrizable and develop a theory for them analogous to that developed in [1]. We define weak Blumberg spaces, show that every separable $T_1$ Baire space is weak Blumberg, and provide examples of non-metrizable weak Blumberg spaces.

## 2. The Weak Blumberg Property

Throughout let $(X, \tau)$ be a topological space (often denoted by $X$ alone), let $\mathbb{R}$ be the real numbers with the usual topology, let $\mathbb{Q}$ be the rational numbers and $\mathbb{N}$ be the natural numbers. For each $p \in X$, the set of open neighborhoods of $p$ will be denoted by $\mathcal{N}_p$.

**Definition 2.1.** Let $\mathfrak{R}$ be a binary relation between open sets and elements, i.e., $\mathfrak{R} \subseteq \tau \times X$. For $U$ a member of $\tau$ and $p$ an element of $X$, $U\mathfrak{R}p$ means that the open set $U$ has the relation $\mathfrak{R}$ to the element $p$, i.e., $(U, p) \in \mathfrak{R}$. We say that the relation $\mathfrak{R}$ is **closed** if for every subset $A$ of $X$ and every $U \in \tau$, $U\mathfrak{R}s$ for all $s \in A$ imply $U\mathfrak{R}p$ for all $p \in \overline{A}$.

**Definition 2.2.** A **partial neighborhood of** $p$, often denoted by $N_<$, is an open set with $p$ in its closure.

**Lemma 2.1.** *If a subset* $T$ *of* $X$ *has the property that each* $p \in T$ *has a partial neighborhood* $V_<^p$ *such that* $V_<^p \cap T = \varnothing$*, then* $T$ *is nowhere dense in* $X$*.*

**Proof.** Let $V = \bigcup\{V_<^p : p \in T\}$. Then for each $p \in T$ we have $p \in \overline{V_<^p} \subseteq \overline{V}$ which implies that $T \subseteq \overline{V}$. But $T \cap V = \varnothing$, so $T \subseteq \overline{V} \setminus V$, which is nowhere dense. □

**Lemma 2.2.** *For each* $f : X \longrightarrow \mathbb{R}$ *and each pair of real numbers* $r_1$ *and* $r_2$ *with* $r_1 < r_2$ *the relation* $\mathfrak{R}^f_{r_1 r_2} \subseteq \tau \times X$ *given by:*
$U\mathfrak{R}^f_{r_1 r_2}p$ *if and only if* $p \in \overline{U}$ *and there exists some* $q \in U$ *such that* $r_1 \leqslant f(q) < r_2$
*is a closed relation.*

**Proof.** Suppose that $A \subseteq X$ and $U \in \tau$ such that $U\Re^f_{r_1 r_2} s$ for all $s \in A$. Then for each such $s$ we have $s \in \overline{U}$ and there is some $q \in U$ such that $r_1 \leqslant f(q) < r_2$. Suppose that $p \in \overline{A}$. Since $A \subseteq \overline{U}$, we have $\overline{A} \subseteq \overline{U}$; thus $p \in \overline{U}$. Also $p \in \overline{A}$ implies that $A \neq \varnothing$. Hence there is some $q \in U$ with $r_1 \leqslant f(q) < r_2$. Consequently $U\Re^f_{r_1 r_2} p$. □

**Lemma 2.3.** *If $f : X \to \mathbb{R}$, then for every pair of real numbers $r_1, r_2$, the set of elements $p$ of $X$ that satisfy both*
*(a) $N\Re^f_{r_1 r_2} p$ for every open neighborhood $N$ of $p$ and*
*(b) $N_< \Re^f_{r_1 r_2} p$ is false for some partial neighborhood $N_<$ of $p$*
*constitute a nowhere dense set (denoted by $T^f_{r_1 r_2}$).*

**Proof.** let $T^f_{r_1 r_2}$ be the set of points satisfying both (a) and (b). If $p \in T^f_{r_1 r_2}$ then by (b) there is a partial neighborhood $N_<$ of $p$ such that $N_< \Re^f_{r_1 r_2} p$ does not hold. Suppose that
$(*)$ every neighborhood $W$ of $p$ contains some $q_W$ such that $N_< \Re^f_{r_1 r_2} q_W$.
Consider the set $A = \{q_W : W \in \mathcal{N}_p\}$. Then for each $q_W \in A$ we have $N_< \Re^f_{r_1 r_2} q_W$. By Lemma 2.2 $\Re^f_{r_1 r_2}$ is closed. Thus since $p \in \overline{A}$ we have $N_< \Re^f_{r_1 r_2} p$, which is a contradiction. So $(*)$ is false. Hence there exists some neighborhood $K$ of $p$ that contains no member that is related to $N_<$. So for each $q \in K \cap N_<$ we know that $N_< \Re^f_{r_1 r_2} q$ is false. Hence, because $N_<$ is an open neighborhood of each such $q$, condition (a) implies that no such $q$ can be in $T^f_{r_1 r_2}$. Thus $K \cap N_< \cap T^f_{r_1 r_2} = \varnothing$. Hence $K \cap N_<$ is a partial neighborhood of $p$ that misses $T^f_{r_1 r_2}$. From Lemma 2.1 it follows that $T^f_{r_1 r_2}$ must be nowhere dense. □

**Corollary 2.1.** *Under the conditions of Lemma 2.3, $T^f :\equiv \bigcup\{T^f_{r_1 r_2} : (r_1, r_2) \in \mathbb{Q} \times \mathbb{Q}\}$ is a subset of $X$ of the first category.*

**Definition 2.3.**
(a) A function $f : (X, \tau) \to \mathbb{R}$ is said to be **densely approached** at a point $p$ of $X$ if and only if for each $\epsilon > 0$ there exists an open neighborhood $N_\epsilon$ of $p$ such that the set of all elements $q$ of $N_\epsilon$ for which $|f(q) - f(p)| < \epsilon$ is dense in $N_\epsilon$.
(b) A subset of a topological space is called **residual** if and only if its complement is of the first category.

**Theorem 2.1.** *For each topological space $X$ a subset $R$ of $X$ is precisely the set of all elements of $X$ that are densely approached by some real-valued function $f : X \longrightarrow \mathbb{R}$ if and only if $R$ is a residual subset of $X$.*

**Proof.** Let $f : X \longrightarrow \mathbb{R}$. To show that $R = \{p \in X : f \text{ is densely approached at } p\}$ is residual, we will show that the complement $X \setminus R$ is a set of the first category. If $p \in X \setminus R$ then there is an $\epsilon > 0$ such that for each open neighborhood $N$ of $p$ the set $\{q \in N : |f(q) - f(p)| < \epsilon\}$ is not dense in $N$. Thus for each such neighborhood $N$ of $p$ we can choose an open set $U_N$ such that $U_N \cap N \neq \varnothing$ and $|f(q_N) - f(p)| \geqslant \epsilon$ for all $q_N \in U_N \cap N$. Let $U = \bigcup\{U_N \cap N : N \in \mathcal{N}_p\}$. Then $p \in \overline{U}$ and $|f(q) - f(p)| \geqslant \epsilon$ for all $q \in U$. Let $r_1,\ r_2 \in \mathbb{Q}$ such that $f(p) - \epsilon < r_1 < f(p) < r_2 < f(p) + \epsilon$. Then $U \mathfrak{R}^f_{r_1 r_2} p$ is false and $M \mathfrak{R}^f_{r_1 r_2} p$ for each $M \in \mathcal{N}_p$. Thus $p \in T^f_{r_1 r_2}$ (as defined in Lemma 2.3), so that $X \setminus R \subseteq T^f$(as defined in Corollary 2.1) which is a set of the first category. Hence $R$ is residual. To see the converse, if $R$ is residual, then $X \setminus R$ is of the first category. So there is a countable family $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$ of pairwise disjoint nowhere dense sets such that $X \setminus R = \bigcup \mathcal{G}$. Define $f : X \longrightarrow \mathbb{R}$ by

$$f(p) = \begin{cases} \frac{1}{n} & \text{if } p \in G_n, \\ 2 & \text{if } p \in R. \end{cases}$$

Then $f$ is densely approached at $p$ if and only if $p \in R$. □

**Definition 2.4.** A function $f : X \to \mathbb{R}$ is said to be **exhaustibly approached** at a point $p$ of $X$ if and only if an open neighborhood $N$ of $p$ and an $\epsilon > 0$ exist such that the set of elements $q$ of $N$ where $|f(q) - f(p)| < \epsilon$ is of the first category. If $f$ is not exhaustibly approached at $p$ then we say that $f$ is **inexhaustibly approached** at $p$.

**Theorem 2.2.** *For every separable space $X$ and every real-valued function $f : X \to \mathbb{R}$ the set consisting of those elements $x$ of $X$ at which $f$ is exhaustibly approached is of the first category.*

**Proof.** For each $r \in \mathbb{Q}$ and $n \in \mathbb{N}$ let $S_{rn} = (r - \frac{1}{n}, r + \frac{1}{n})$. If $f$ is exhaustibly approached at $q$, then there exist a positive number $\epsilon$ and an open neighborhood $N_q$ of $q$, such that $f^{-1}[(f(q) - \epsilon, f(q) + \epsilon)] \cap N_q$ is of the first category. Let's pick the set $S_{rn}$ such that $f(q) \in S_{rn} \subseteq (f(q) - \epsilon, f(q) + \epsilon)$. Then $f^{-1}[S_{rn}] \cap N_q$ is of the first category. Let

$$E_{rn} = \{x \in f^{-1}[S_{rn}] : \text{a neighborhood } N_x \text{ of } x \text{ exists such that } f^{-1}[S_{rn}] \cap N_x \text{ is of the first category}\}$$

Then $E = \bigcup\{E_{rn} : r \in \mathbb{Q}, n \in \mathbb{N}\}$ is the set of all elements at which $f$ is exhaustibly approached. Now, it is sufficient to prove that every $E_{rn}$ is of the first category since $E$ is a countable union of such sets.

Fix $E_{rn}$. $X$ is separable, i.e., $X$ has a countable dense subset, say $P = \{p_i : i \in \mathbb{N}\}$. For each $x$ in $E_{rn}$ pick and fix an open neighborhood $N_x$ of $x$ such that $f^{-1}[S_{rn}] \cap N_x$ is of the first category. $P$ is dense. So, for each such $N_x$, we can pick and fix an element $p_i$ of $P$ such that $p_i$ belongs to $N_x$. Let $\mathcal{E}_i = \{N_x : p_i \in N_x\}$. Take a member $M_i$ from each nonempty family $\mathcal{E}_i$. If $\mathcal{E}_i = \varnothing$, let $M_i = \varnothing$. Let $M = \bigcup\{M_i : i \in \mathbb{N}\}$. we claim that $E_{rn}\backslash M$ is nowhere dense. Suppose that the claim is not true. Then $\overline{E_{rn}\backslash M}$ contains a nonempty open subset $U$. Pick an element $y$ from the set $U \cap (E_{rn}\backslash M)$. Then $N_y \cap U$ is an open neighborhood of $y$. This implies that there exists an element $p_k$ of $P$ such that $p_k \in N_y \cap U$. Thus $p_k \in M_k \in \mathcal{E}_k$, so that $p_k$ belongs to $M$. Thus, $p_k$ belongs to $U \cap M$. Since $M$ is open, $X\backslash M$ is closed. $U \subseteq \overline{E_{rn}\backslash M} \subseteq \overline{X\backslash M} = X\backslash M$. Thus, $U \cap M$ is empty, which contradicts that $p_k$ belongs to $U \cap M$. Hence, the claim is true, so that $(E_{rn}\backslash M) \cup \{f^{-1}[S_{rn}] \cap M_i : i \in \mathbb{N}\}$ is of the first category. Also

$$\begin{aligned} E_{rn} &= (E_{rn} \cap M) \cup (E_{rn}\backslash M) \\ &\subseteq (f^{-1}[S_{rn}] \cap M) \cup (E_{rn}\backslash M) \\ &= \bigcup\{f^{-1}[S_{rn}] \cap M_i : i \in \mathbb{N}\} \cup (E_{rn}\backslash M). \end{aligned}$$

Thus $E_{rn}$ is of the first category. □

If $M$ is a subset of $X$, we shall use, in connection with approach, the expression "via $M$" to indicate that the range of $p$ is restricted to $M$.

**Definition 2.5.** For any set $M \subseteq X$ a function $f : X \to \mathbb{R}$ is said to be **exhaustibly approached at** $p$ **via** $M$ if and only if an open neighborhood $N$ of $p$ and a positive number $\epsilon$ exist such that the set of elements $q$ of $N \cap M$ where $|f(q) - f(p)| < \epsilon$ is of the first category; otherwise, $f$ is called **inexhaustibly approached at** $p$ **via** $M$.

We know that if $A$ is of the first category, then every subset of $A$ is also of the first category. Thus, if $f$ is exhaustibly approached at $p$, then $f$ is exhaustibly approached at $p$ via $M$ for each subset $M$ of $X$. On the other hand, for any $M \subseteq X$, if $f$ is inexhaustibly approached at $p$ via $M$, then $f$ is inexhaustibly approached at $p$.

**Definition 2.6.** A function $f$ is said to be **densely approached at** $p$ **via** $M$ if and only if for each $\epsilon > 0$ there exists an open neighborhood $N_\epsilon$ of $p$ such that the elements $q$ of $M \cap N_\epsilon$ for which $|f(p) - f(q)| < \epsilon$ form a dense subset of $M \cap N_\epsilon$.

**Lemma 2.4.** *If $M$ is a subset of $X$ such that $p$ is a limit point of $M$, the following statements are equivalent.*
*(1) $f : X \to \mathbb{R}$ is densely approached at $p$ via $M$.*
*(2) For every partial neighborhood $N_<$ of $p$ such that $N_< \cap M$ has $p$ as a limit point, $f[N_< \cap M]$ has $f(p)$ as a limit point.*

**Proof.** Assume that (1) holds, but (2) does not. Then there is some partial neighborhood $N_<$ of $p$ having $p$ as a limit point, but for which $f(p)$ is not a limit point of $f[N_< \cap M]$. Then there is some $\epsilon > 0$ and $U \in \mathcal{N}_p$ such that $U \cap N_< \cap M \neq \varnothing$ and $|f(p) - f(q)| \geq \epsilon$ for all $q \in U \cap N_< \cap M$ with $q \neq p$. By (1) there is some $N_\epsilon \in \mathcal{N}_p$ such that $\{q^* \in M \cap N_\epsilon : |f(q^*) - f(p)| < \epsilon\}$ is dense in $M \cap N_\epsilon$. Now $W = (U \cap M \cap N_<) \cap (M \cap N_\epsilon)$ is nonempty and open in $M \cap N_\epsilon$. Also $|f(r) - f(p)| \geq \epsilon$ for all $r \in W$ with $r \neq p$, which contradicts (1). Thus (1)⇒(2).

Now suppose that (1) does not hold; i.e., there is some $\epsilon > 0$ such that for each $N \in \mathcal{N}_p$ we can pick an open $U_N$ such that $U_N \cap M \cap N \neq \varnothing$ and $|f(p) - f(z)| \geq \epsilon$ for all $z \in U_N \cap M \cap N$. Let $U = \bigcup\{U_N : N \in \mathcal{N}_p\}$. Then $U$ is a partial neighborhood of $p$ and $|f(p) - f(x)| \geq \epsilon$ for each $x \in U \cap M$. Thus $f(p)$ is not a limit point of $f[U \cap M]$. Since $p$ is a limit point of $U \cap M$, we have that (2) does not hold. Thus (2)⇒(1). □

**Theorem 2.3.** *For every real-valued function $f$ defined on a separable $T_1$ space $X$, there exists a residual subset $S$ of $X$ such that if $p$ is an element of $S$ then the function $f$ is both inexhaustibly and densely approached at $p$ via $S$.*

**Proof.** Let $E_1$ be the set of elements at which $f$ is exhaustibly approached, and let $S_1 = X \backslash E_1$. By Theorem 2.2, we know that $E_1$ is of the first category. Then $S_1$ is residual and $f$ is inexhaustibly approached at the elements of $S_1$. That is, if $q$ is an element of $S_1$, then for each $\epsilon > 0$ and for each open neighborhood $N$ of $q$, there exists a subset $M$ of $N$ such that $M$ is of the second category and $|f(x) - f(q)| < \epsilon$ for all $x$ in $M$. $M \cap S_1 = M \backslash E_1$ is of the second category since $M \cap E_1$ is of the first category and $M$ is of the second category. Thus, $f$ is inexhaustibly approached at the elements of $S_1$ via $S_1$.

Case 1. Suppose that $f$ is bounded. Then there exists a real number $k$ such that $k > \sup f[X]$. Now define $g : X \to \mathbb{R}$ by:

$$g(x) = \begin{cases} f(x) & \text{if } x \in S_1. \\ k & \text{if } x \in E_1. \end{cases}$$

Then, by Theorem 2.1 the elements of $X$ at which $g$ is densely approached constitute a residual set, say, $S_g$. For every element $r$ in $E_1$ and for every $s$ in $S_1$, $g(r) - g(s) \geq k - \sup f[X] > 0$. Each of $g$ and $f$ is densely approached at the elements of $S_1 \cap S_g$ via $S_1$. Furthermore, if $K = X - S_g$, then $S_g \cap S_1 = X \backslash (K \cup E_1)$ is a residual set since $K$ and $E_1$ are each of the first category. It follows that the elements of $S_1$ at which $f$ is densely approached via $S_1$ constitute a residual set, say, $S$. Let $E_2 = S_1 \backslash S$. Then $X = E_1 \cup E_2 \cup S$. This implies that $E_2$ is of the first category. For every element $p$ of $S$, $f$ is inexhaustibly approached at $p$ via $S$ since $E_2$ is of the first category and $f$ is inexhaustibly approached at the elements of $S_1$ via $S_1$. Also, $f$ is densely approached at each such $p$ via $S_1$. For every partial neighborhood $N_<$ of $p$, if $p$ is a limit point of $N_< \cap S$ then $p$ is a limit point of $N_< \cap S_1$. By Lemma 2.4 this implies that $f(p)$ is a limit point of $f[N_< \cap S_1]$. That is, every open neighborhood of $f(p)$ intersects $f[N_< \cap S_1]$ in some element other than $f(p)$ itself. For each $\epsilon > 0$ and for each open neighborhood $N$ of $p$, there exists an element which is distinct from $p$, say, $p^*$ such that $p^*$ belongs to $N \cap (N_< \cap S_1)$ and $|f(p^*) - f(p)| < \epsilon/2$. It follows that $f$ is inexhaustibly approached at each such $p^*$ via $S$ since $p^*$ belongs to $S_1$ and $S_1 \backslash S = E_2$ is of the first category. $X$ is a $T_1$ space so $\{p\}$ is closed. It follows that $(N \cap N_<) \backslash \{p\}$ is an open neighborhood of $p^*$. Thus, a subset $M$ of $(N \cap N_< \cap S) \backslash \{p\}$ exists such that $M$ is of the second category and for every element $q^*$ of $M$, $|f(q^*) - f(p^*)| < \epsilon/2$. By the triangle inequality, we have that $|f(q^*) - f(p)| < \epsilon$ for all $q^*$ in $M$. We can pick an element $q$ from $M$ such that $q$ is distinct from $p$, $q \in N \cap N_< \cap S$ and $|f(q) - f(p)| < \epsilon$. This implies that $p$ is a limit point of $N_< \cap S$. Hence $f$ is densely approached at $p$ via $S$.

Case 2. Suppose that $f$ is unbounded. We define $\overline{f} : X \to \mathbb{R}$ by $\overline{f}(x) = f(x)/(1+|f(x)|)$. Then $\overline{f}$ is bounded and the properties of dense approach, exhaustible approach and inexhaustible approach of $f$ are preserved by $\overline{f}$. So we can proceed as in Case 1. □

**Corollary 2.2.** *For every real-valued function $f$ defined on a separable $T_1$ Baire space $X$, there exists a dense subset $S$ of $X$ such that if $p$ is an element of $S$ and $N_<$ is a partial neighborhood of $p$ then $f$ is inexhaustibly approached at $p$ via $N_< \cap S$.*

**Proof.** By Theorem 2.3, there exists a residual set $S$ of $X$ such that if $p$ is an element of $S$ then $f$ is densely approached at $p$ via $S$. Thus by Lemma 2.4 if $M_<$ is a partial neighborhood of $p$ such that $p$ is a limit point of $M_< \cap S$, then $f(p)$ is a limit point of $f[M_< \cap S]$. Given a partial neighborhood $N_<$ of $p$, $N_< \setminus \{p\}$ is open since $X$ is $T_1$. Since $X$ is a Baire space $S$ must be dense. So, $p$ is a limit point of $N_< \cap S$. It follows that $f(p)$ is a limit point of $f[N_< \cap S]$. So, for each open neighborhood $N$ of $p$ and for each $\epsilon > 0$, there exists an element $p^*$ which is distinct from $p$ such that $p^*$ belongs to $N \cap N_< \cap S$ and $|f(p^*) - f(p)| < \epsilon/2$. Now $p^*$ belongs to $S$, so that by Theorem 2.3, $f$ is inexhaustibly approached at $p^*$ via $S$. Since $N_< \cap N$ is an open neighborhood of $p$, there exists a subset $M$ of $N_< \cap N \cap S$ such that $M$ is of the second category and $|f(p^*) - f(q)| < \epsilon/2$ for all $q$ in $M$. Thus $|f(q) - f(p)| < \epsilon$ for all $q$ in $M$, so $f$ is inexhaustibly approached at $p$ via $N_< \cap S$. □

**Definition 2.7.** We say that $X$ is a **weak Blumberg space** or that $X$ **has the weak Blumberg property** if and only if for each real-valued function $f$ defined on $X$ there exists a dense subset $D$ of $X$ such that $f$ is densely approached at $p$ via $D$ for each $p \in D$.

**Theorem 2.4.** *The family of weak Blumberg spaces contains both the family of Blumberg spaces and the family of separable $T_1$ Baire spaces.*

**Proof.** The first containment is clear from the fact that every subset of a space is dense in itself. Also note that every residual subset of a Baire space is dense. Hence the second containment follows immediately from Theorem 2.3. □

**Example 2.1.** $[0,1]^{2^{\aleph_0}}$ is a non-metrizable weak Blumberg space.

**Proof.** $[0,1]^{2^{\aleph_0}}$ is a separable non-metrizable compact Hausdorff space, and every such space is $T_1$ and Baire. By Theorem 2.4 it is a weak Blumberg space. □

**Example 2.2.** The set of real numbers with the cofinite topology, $\mathbb{R}_{cf}$, is a non-metrizable weak Blumberg space that is not a Blumberg space.

**Proof.** $\mathbb{R}_{cf}$ is clearly $T_1$ and non-Hausdorff, thus non-metrizable. Since a subset of $\mathbb{R}_{cf}$ is dense if and only if it is infinite, it follows easily that $\mathbb{R}_{cf}$ is separable and Baire. By Theorem 2.4 it has the weak Blumberg property. Consider the identity function $f : \mathbb{R}_{cf} \longrightarrow \mathbb{R}$. If $f|D$ were continuous for any dense subset $D$ of $\mathbb{R}_{cf}$, then D would be an infinite Hausdorff subspace of $\mathbb{R}_{cf}$, which is impossible. Thus $\mathbb{R}_{cf}$ does not have the Blumberg property. □

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