QUASI-ISOMETRY OF METRICS ON TEICHMÜLLER SPACES

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Abstract *The purpose of this paper is to provide a coherent study of the quasi-isometries among various invariant metrics on the Teichmüller space of a finite hyperbolic Riemann surface. In particular, we prove the quasi-isometry of the Teichmüller metric, the Kobayashi metric, the Caratheodory metric, the Bergman metric, the Kähler-Einstein metric with a fixed constant negative scalar curvature and the metrics constructed by McMullen.*

§1 Introduction

In this paper, we produce a simple proof of the quasi-isometry of the following metrics on a Teichmüller space of a compact Riemann surface of genus g with n punctures. The metrics include Teichmüller metric g_T, the Kobayashi metric g_K, the Caratheodory metric g_C, the Bergman metric g_B, the Kähler-Einstein metric g_{KE} of constant negative scalar curvature $-2(3g-2+n)$, and the metrics g_M constructed by McMullen. This is done through Bers Embedding, Schwarz Lemma and L^2-estimates with the construction of appropriate plurisubharmonic functions.

Theorem 1. *Let $\mathcal{T}_{g,n}$ be the a Teichmüller space of a compact hyperbolic Riemann surface of genus g with n punctures. Then $g_T, g_K, g_C, g_B, g_{KE}$ and g_M are all quasi-isometric on $\mathcal{T}_g$.*

Remarks
1. In [Ya], page 141, Yau proposed the problem of comparing g_B, g_K, g_C and g_{KE}. Theorem 1 provides a satisfactory answer to the problem.
2. In contrast, the Weil-Petersson metric g_{WP} is incomplete following from a well-known result of Wolpert (cf. [W]). From Theorem 1 and the results of [Mc], it is strictly smaller up to a positive constant to any of the above metrics.

Let us first recall some well-known results together with some recent developments. It is a well-known result of Royden that $g_T = g_K$ (cf. [G]). McMullen proved in [Mc] that g_M and g_T are quasi-isometric. It follows from the result of Look [L] and later on by Hahn [Ha] that $g_B \geq g_C$ (cf. [Ye]). Very recently, there appear on the preprint archive the results of [LSY] on the quasi-isometry of g_{KE} and g_M on the moduli space of compact Riemann surface $\mathcal{M}_g$ with no punctures ($n = 0$), and the results of Chen [C] on the quasi-isometry of g_M and g_B.

Our method is simple and direct. For the classical invariant metrics g_K, g_C, g_B and g_{KE}, the quasi-isometry follows essentially from Bers Embedding Theorem and Schwarz type lemma. The quasi-isometry involving g_M requires estimates of short geodesics on the moduli space, similar to the discussions in [Mc] and [Ye]. There was already a proof in [Mc] for this latter fact. To give a coherent treatment of the results, we give two alternate simpler proofs for results involving g_M. One of the

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proofs follows from a general criterion for the quasi-isometry of the Bergman metric with a complete Kähler metric on a complex manifold (Theorem 3), which has independent interest. This aspect of the paper can be considered as a continuation of the study in [Ye] and also in [C] for the geometric consequences of L^2-estimates via construction of appropriate plurisubharmonic functions in terms of geometric length functions. Relating to the theme of this paper, the construction of bounded plurisubharmonic exhaustion functions in [Ye] enables one to construct lots of holomorphic forms of top degree, which leads to $g_B \geq g_C$ from the observation of Look [L] and Hahn [Ha] in terms of peak functions.

As a notation, we use g^M to denote a metric on a manifold M, where the superscript would be dropped when there is no danger of confusion. Quasi-isometry between two metrics g_1 and g_2 is denoted by $g_1 \sim g_2$, in the sense that $\frac{1}{c} g_1(v, \bar{v}) \leqslant g_2(v, \bar{v}) \leqslant c g_1(v, \bar{v})$ for some positive constant $c > 0$ and all holomorphic tangent vector v on the manifold.

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§2 Metrics on Teichmüller space

In this section, we recall briefly different notions of metrics on $\mathcal{T}_{g,n}$. First of all, we mention that we denote by $|f(x)|$ the pointwise norm and $||f||$ the L^2-norm of a function over an appropriate space.

The notion of Kobayashi pseudometric g_K and Caratheodory pseudometric g_C are well-known for complex manifolds. For a unit tangent vector $v \in T_x M$ on a complex manifold M, the respective length functions are

$$\sqrt{g_K(x,v)} = \inf\{\frac{1}{r} | \exists f : B_r^1 \to M \text{ holomorphic}, f(0) = x, f'(0) = v\}.$$
$$\sqrt{g_C(x,v)} = \sup\{\frac{1}{r} | \exists h : M \to B_r^1 \text{ holomorphic}, h(x) = 0, |dh(v)| = 1\},$$

where we use $B_r^n = B_r^n(0)$ to denote a ball of radius r centered at 0 in $\mathbb{C}^n$. From Bers Embedding Theorem (cf [G]), we know that $\mathcal{T}_{g,n}$ can be realized as a bounded pseudoconvex domain in $\mathbb{C}^N$, $N = 3g - 3 + n$. Hence both g_K and g_C are non-degenerate Finsler metrics.

Consider now Kähler-Einstein metric of constant negative scalar curvature. We normalize the curvature so that g_{KE} satisfies $Ric(g_{KE}) = -2(3g - 2 + n)$, where ω_{KE} is the Kähler form associated to g_{KE}. We choose the normalization so that it agrees with the one for the hyperbolic metric on $B_{\mathbb{C}}^N$ of constant holomorphic sectional curvature -4. It is well-known from the work of Cheng-Yau [CY] and Mok-Yau [MY] on bounded psudoconvex domains that a complete Kähler-Einstein metric of negative Ricci curvature exists on a bounded domain in $\mathbb{C}^N$, hence on $\mathcal{T}_g$ from Bers Embedding Theorem again.

The Bergman pseudometric g_B on a general complex manifold M of complex dimension N is a Kähler pseudometric with local potential given by the coefficients of the Bergman kernel $K(x,x)$. It is clearly non-degenerate for $\mathcal{T}_g$. Since our proof depends on the estimates of g_B, we recall the following interpretation of g_B.

Let f be a L^2-holomorphic N-form on M, where $\dim_{\mathbb{C}} M = N$. In terms of local coordinates $(z_1, \cdots, z_N)$ on a coordinate chart U, let $e_{K_M} = dz^1 \wedge \cdots \wedge dz^N$ be a local basis of the canonical line bundle K_M on U. We can write f as $f_U e_{K_M}$ on U. Let $f_i, i \in N$ be an orthonormal basis of L^2-sections in $H^0_{(2)}(M, K_M)$. Note that from conformality, the choice is independent of the metric on M. The Bergman kernel is given by $K(x,x) = \sum_i f_i \wedge \overline{f_i}$. Let $k_U(x,x) = \sum_i f_{U,i} \wedge \overline{f_{U,i}}$ be the coefficient of $K(x,x)$ in terms of the local coordinates. The Bergman metric is given by a Kähler form

$$\omega_B = \sqrt{-1}\partial\bar{\partial} \log k_U(x,x) = \sqrt{-1}\frac{1}{k_U(x,x)^2}\sum_{i<j}(f_i \partial f_j - f_j \partial f_i) \wedge \overline{(f_i \partial f_j - f_j \partial f_i)},$$

which is clearly independent of the choice of a basis and U. As the Bergman kernel is independent of basis, for each fixed point $x \in M$,

$$k_U(x,x) = \sup_{f \in H^0_{(2)}(M,K_M), \|f\|=1} |f_U(x)|^2.$$

We may assume that $\sup_{f \in H^0_{(2)}(M,K_M)} |f_U(x)|$ is realized by $f_x \in H^0_{(2)}(M,K)$ with $\|f_x\| = 1$ so that $k_U(x,x) = |f_{x,U}(x)|^2$. Using the fact that the Bergman kernel is independent of the choice of a basis again and letting $V \in T_x M$,

$$\omega_B(V, \bar{V}) = \frac{1}{|f_{x,U}(x)|^2} \sup_{f \in H^0_{(2)}(M,K_M), \|f\|=1, f(x)=0} |V(f_U)|^2.$$

Consider in particular $V = \frac{\partial}{\partial z^i}$. We may also assume that the supremum for $|\frac{\partial}{\partial z^i}(f_U)|^2$ among all $f \in H^0_{(2)}(M, K_M), \|f\| = 1, f(x) = 0$ is achieved by $g_{i,x} \in H^0_{(2)}(M, K_M)$ of L^2-norm 1. Hence $\sup_{f \in H^0_{(2)}(M,K_M), \|f\|=1} |\frac{\partial}{\partial z^i} f_U|^2 = |\frac{\partial}{\partial z^i} g_{i,x,U}(x)|^2$. To simplify our notation, we may simply write

$$\omega_B(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^i}}) = \frac{1}{|f_x(x)|^2} \sup_{f \in H^0_{(2)}(M,K_M), \|f\|=1, f(0)=0} |\frac{\partial}{\partial z^i}(f)|^2 = \frac{|\frac{\partial}{\partial z^i} g_{i,x}(x)|^2}{|f_x(x)|^2},$$

since the expression is clearly independent of the choice of U and metric on e_U.

We now consider more specifically Teichmüller spaces which are also complex manifolds. We refer the readers to [G], [N] or [W] for standard facts on Teichmüller spaces including the notions of the Teichmüller metric g_T and the Weil-Petersson metric g_{WP} respectively. It is well-known that g_T is complete but g_{WP} is incomplete.

Recently, McMullen [Mc] constructed complete metrics with bounded sectional curvature in his proof of the Kähler hyperbolicity of Teichmüller space. Up to quasi-isometry the metric is of the form

$$g_{WP} + C \sum_{\gamma, \ell_\gamma < \epsilon} \sqrt{-1}\frac{\partial \ell_\gamma}{\ell_\gamma} \wedge \overline{\frac{\partial \ell_\gamma}{\ell_\gamma}}$$

for all $\epsilon > 0$ sufficiently small and an appropriate positive constant C, where the summation is taken over all primitive closed geodesics of length less than ϵ. Let us fix a pair of constants ϵ and C and denote by g_M such a metric. g_M can be considered either as a metric on $\mathcal{T}_g$ or on $\mathcal{M}_g$.

§3 Equivalence of invariant metrics

Theorem 2. *Let $\mathcal{T}_{g,n}$ be the Teichmüller space of a compact hyperbolic Riemann surface of genus g with n punctures. Then g_K, g_C, g_B and g_{KE} of constant Ricci curvature $-2(3g-2+n)$ are quasi-isometric on M with estimates given by*

$$\begin{aligned} g_K \leqslant\ & g_C && \leqslant 9 g_K \\ g_K \leqslant\ & g_B && \leqslant 16^2\pi^2(6^{2N}) g_K \\ \frac{1}{9N} g_K \leqslant\ & g_{KE} && \leqslant \frac{9^{2N-1}N^{N-1}}{4^N} g_K \end{aligned}$$

For the proof of the Theorem, let us recall the following well-known result (cf. [G], page 101 and [Mc], page 336)).

Proposition 1. *(Bers Embedding Theorem). Let S be a Riemann surface of genus g representing a point $x \in \mathcal{T}_{g,n}$. There exists an embedding $\Phi : \mathcal{T}_S \to \mathbb{C}^N$, so that $B^N_{\frac{1}{2}} \subset \mathcal{T}_S \subset B^N_{\frac{3}{2}}$, where $\mathbb{C}^N$ is identified with the space of holomorphic quadratic differentials based at S equipped with L^∞ norm, and $\Phi(x) = 0$.*

Since we are going to give some explicit estimates, we recall here several versions of Schwarz Lemma to be used.

Proposition 2. *Let $f : M \to N$ be a holomorphic mapping of rank N from a complete Kähler manifold (M, g) of complex dimension N to another Kähler manifold (N, h) of the same dimension.*
(a). (Mok-Yau [MY]). Suppose that the scalar curvature of M is bounded from below by $-a_1$ and the Ricci form of N is negative and satisfies $|Ric(h)|^n \geq b_1 \det(h)$. Then $\det(f^*h) \leqslant \frac{a_1^N}{N^N b_1} \det g$.
(b). (Royden [Ro]) Suppose that the Ricci curvature of M is bounded from below by $-a_2$ and the holomorphic sectional curvature of N is bounded from above by $-b_2$. Then $f^*h \leqslant \frac{2N}{N+1}\frac{a_2}{b_2} g$.
(c). (Chen-Cheng-Lu [CCL]) Syppose that the sectional curvature of M is bounded from below and in particular that the holomorphic sectional curvature of M is bounded from below by $-a_3$. Suppose also that holomorphic sectional curvature of N is bounded from above by $-b_3$. Then $f^*h \leqslant \frac{a_3}{b_3} g$.

Proof of Theorem 2
(1). $g_C \sim g_K$: It follows from Ahlfors Schwarz Lemma that $g_C \leqslant g_K$. For the other direction, Let $v \in T_x M$ identified with a vector in $\mathbb{C}^N$ by the Bers Embedding Φ. Then from definition of g_K and g_C, we conclude that $\sqrt{g_K(x,v)} \leqslant 2$ and $\sqrt{g_C(x,v)} \geq \frac{2}{3}$. Hence $g_K \leqslant 9 g_C$.
(2). $g_B \sim g_K$: We recall from section 2 that

$$\omega_B\Big(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^i}}\Big) = \frac{|\frac{\partial}{\partial z^i} g_{i,x}(x)|^2}{|f_x(x)|^2},$$

where f_x is a function with $||f_x|| = 1$ realizing the supremum of $|f(x)|$ among L^2-holomorphic functions $f \in H^0_{(2)}(\mathcal{T}), ||f|| = 1$ on $\mathcal{T}$, and $g_{i,x}$ is a holomorphic function realizing supremum of $|\frac{\partial}{\partial z^i}(f)|^2$ among all $f \in H^0_{(2)}(\mathcal{T}), ||f|| = 1, f(x) = 0$.

From Proposition 1 again, we may regard $B^N_{\frac{1}{2}}(x) \subset \mathcal{T} \subset B^{N(x)}_{\frac{3}{2}}$, where $B^N_r(x)$ denotes a complex ball of radius r centered at x identified with 0 in $\mathbb{C}^N$. Let vol_o denotes the Euclidean volume on $\mathbb{C}^N$. Clearly from Mean Value Inequality

$$(f_x(x))^2 \leqslant \frac{\int_{B^N_{\frac{1}{2}}(x)} |f_x|^2}{\mathrm{vol}_o(B^N_{\frac{1}{2}}(x))} \leqslant \frac{\int_{\mathcal{T}} |f_x|^2}{\mathrm{vol}_o(B^N_{\frac{1}{2}}(x))} = \frac{1}{(\frac{1}{2})^{2N}\mathrm{vol}_o(B^N_1)}.$$

The constant function $h_1(x) = 1$ satisfies $h_1(1) = 1$ and

$$||h_1||^2 = \mathrm{vol}_o(\mathcal{T}) \leqslant \mathrm{vol}_o(B^N_{\frac{3}{2}}) = (\frac{3}{2})^{2N}\mathrm{vol}_o(B^N_1)$$

with respect to the Bers Embedding. Hence $|f_x(x)| \geq \frac{1}{[(\frac{3}{2})^{2N}\mathrm{vol}_o(B^N_1)]^{\frac{1}{2}}}$. We conclude that

$$[\frac{2^{2N}}{(\mathrm{vol}_o(B^N_1)}]^{\frac{1}{2}} \geq |f_x(x)| \geq [\frac{2^{2N}}{(3)^{2N}\mathrm{vol}_o(B^N_1)}]^{\frac{1}{2}}.$$

Let V_i be the complex line generated by $\frac{\partial}{\partial z^i}$ in $\mathbb{C}^N$. Then from Generalized Cauchy Inequality and Mean Value Inequality,

$$\begin{aligned}
|\frac{\partial}{\partial z^i} g_{i,x}(x)| &\leqslant \frac{\int_{\partial(B^N_{\frac{1}{4}}(x))\cap V_i} |g_{i,x}(y)|dy}{(\frac{1}{4})^2} \leqslant \frac{[\int_{\partial(B^N_{\frac{1}{4}}(x))\cap V_i} |g_{i,x}(y)|^2 dy]^{\frac{1}{2}}[2\pi\frac{1}{4}]^{\frac{1}{2}}}{(\frac{1}{4})^2} \\
&\leqslant \frac{[\int_{\partial(B^N_{\frac{1}{4}}(x))\cap V_i} dy \int_{B^N_{\frac{1}{4}}(y)} |g_{i,x}(w)|^2 d\mathrm{vol}_o(w)]^{\frac{1}{2}}[\frac{\pi}{2}]^{\frac{1}{2}}}{(\frac{1}{4})^2[\mathrm{vol}_o(B^N_{\frac{1}{4}})]^{\frac{1}{2}}} \\
&\leqslant \frac{[\int_{\partial(B^N_{\frac{1}{4}}(x))\cap V_i} dy \int_{\mathcal{T}} |g_{i,x}(w)|^2 d\mathrm{vol}_o(w)]^{\frac{1}{2}}[\frac{\pi}{2}]^{\frac{1}{2}}}{(\frac{1}{4})^2[\mathrm{vol}_o(B^N_{\frac{1}{4}})]^{\frac{1}{2}}} \leqslant \frac{\frac{\pi}{2}}{(\frac{1}{4})^{2+N}[\mathrm{vol}_o(B^N_1)]^{\frac{1}{2}}}.
\end{aligned}$$

On the other hand the function $h_{i,x} = z_i$ satisfies $\frac{\partial}{\partial z^i} h_{i,x} = 1$ and $h_{i,x}(0) = 0$. As $\int_{B^N_1} |z_i|^2 = \frac{1}{N+1}\mathrm{vol}(B^N_1)$, we know that

$$||h_{i,x}||^2 = \int_{B^N_{\frac{3}{2}}} |z_i|^2 = (\frac{3}{2})^{2N+1}\int_{B^N_1} |z_i|^2 \leqslant \frac{1}{N+1}(\frac{3}{2})^{2N+1}\mathrm{vol}_o(B^N_1).$$

Hence the function $k_{i,x} := \frac{h_{i,x}}{||h_{i,x}||}$ satisfies $|\frac{\partial}{\partial z^i} k_{i,x}| = \frac{N+1}{[(\frac{3}{2})^{2N+1}\mathrm{vol}_o(B^N_1)]^{\frac{1}{2}}}$, $k_{i,x}(0) = 0$ and $||k_{i,x}||^2 = 1$. We conclude as before that

$$\frac{\frac{\pi}{2}}{(\frac{1}{4})^{N+2}[\mathrm{vol}_o(B^N_1)]^{\frac{1}{2}}} \geq |\frac{\partial}{\partial z^i} g_{i,x}(x)| \geq \frac{\sqrt{N+1}}{(\frac{3}{2})^{N+\frac{1}{2}}[\mathrm{vol}_o(B^N_1)]^{\frac{1}{2}}}.$$

Combining the above estimates for $f_x(x)$ and $g_{i,x}(x)$, we arrive at

$$8\pi 6^N \geq \sqrt{g_B(x, \frac{\partial}{\partial z^i})} \geq \sqrt{\frac{2N+2}{3}}\frac{1}{3^N}.$$

Since $\frac{1}{2} \leqslant \sqrt{g_K(x, \frac{\partial}{\partial z^i})} \leqslant \frac{3}{2}$, we conclude that

$$16\pi(6^N)\sqrt{g_K(x, \frac{\partial}{\partial z^i})} \geq \sqrt{g_B(x, \frac{\partial}{\partial z^i})} \geq \frac{2}{3^{N+1}}\sqrt{\frac{2N+2}{3}}\sqrt{g_K(x, \frac{\partial}{\partial z^i})}.$$

Clearly, the same estimates hold for any holomorphic tangent vector v. We can however have a better lower bound. We present the lower bound estimate above because a corresponding technique would be used in the proof Theorem 3 in the next section. In fact, from the estimates of Hahn [H] and Look [L] mentioned earlier, we conclude that $g_B \geq g_C \geq g_K$.
(3). $g_{KE} \sim g_K$: Consider the Poincare metric on a ball of radius r in $\mathbb{C}^N$. Since the metric is invariant under the automorphism group and the metric is homogeneous, it suffices for us to understand the metric at the origin 0. We normalized the Poincare metric so that the potential is $\log(r^2 - |z|^2)$ and the metric and curvature tensors at the origin are obtained by taking the Taylor expansion of the above function and evaluating the respective derivatives at the origin. Let $g_{KE}^{B_r}$ denote the resulting metric, which is a Kähler-Einstein metric of Ricci curvature $-\frac{2N+2}{\frac{1}{2}^2}$ with constant holomorphic sectional curvature $-4 < 0$. $g_{KE}^{B_r}(v,\overline{v}) = \frac{1}{r^2} = g_K^{B_r}(x,v)$. The explicit metric implies that each holomorphic tangent vector $v \in T_0 B_r$ spans a totally geodesic disk with constant holomorphic sectional curvature -4. It follows from definition that on B_r, $\sqrt{g_K^{B_r}(0,v)} \leqslant \frac{1}{r}$ and $\sqrt{g_C^{B_r}(0,v)} \geq \frac{1}{r}$. $g_K^{B_r} \geq g_C^{B_r}$ from Ahlfors Schwarz Lemma. Hence $g_K^{B_r}(0,v) = g_C^{B_r}(0,v) = [\frac{1}{r}]^2 = g_{KE}^{B_r}(v,\overline{v})$. From Proposition 3, $B_{\frac{1}{2}}^N \subset \mathcal{T}_S \subset B_{\frac{3}{2}}^N$. Applying Schwarz Lemma of Mok-Yau [MY] to the first inclusion with respect to the Kähler-Einstein metrics $g_{KE}^{B_{\frac{1}{2}}^N}$ and $g_{KE}^{\mathcal{T}}$ on $B_{\frac{1}{2}}^N$ and $\mathcal{T}$ of Ricci curvature $-\frac{2N+2}{(\frac{1}{2})^2}$ and $-(2N+2)$ respectively, we get

$$\begin{aligned} \mathrm{vol}(g_{KE}^{\mathcal{T}}) &\leqslant (\frac{1}{4})^N \mathrm{vol}(g_{KE}^{B_{\frac{1}{2}}^N}) = (\frac{1}{4})^N \mathrm{vol}(g_K^{B_{\frac{1}{2}}^N}) \\ &= (\frac{1}{4})^N 3^{2N} \mathrm{vol}(g_K^{B_{\frac{3}{2}}^N}) \leqslant (\frac{9}{4})^N \mathrm{vol}(g_K^{\mathcal{T}}) \end{aligned}$$

Applying Schwarz Lemma of [R] to $g_{KE}^{\mathcal{T}}$ which has constant Ricci curvature $-(2N+2)$ and $g_{KE}^{B_{\frac{3}{2}}^N}$ which has constant holomorphic sectional curvature -4, we conclude that

$$g_{KE}^{\mathcal{T}} \geq \frac{1}{N} g_{KE}^{B_{\frac{3}{2}}^N} = \frac{1}{N} g_K^{B_{\frac{3}{2}}^N} \geq \frac{1}{9N} g_K^{B_{\frac{1}{2}}^N} \geq \frac{1}{9N} g_K^{\mathcal{T}}.$$

Let $\mu_i > 0, i = 1, \ldots, N$ be the eigenvalues of $g_{KE}^{\mathcal{T}}$ with respect to $g_K^{\mathcal{T}}$. We conclude from the second estimate that $\mu_i \geq \frac{1}{9N}$ for all i, and from the first statement that $\prod_{i=1}^N \mu_i \leqslant (\frac{9}{4})^N$. It follows that $(9N)^{N-1}(\frac{9}{4})^N \geq \mu_i \geq \frac{1}{9N}$. Hence

$$(\frac{9^{2N-1}}{4^N})^{2N-1} N^{N-1} g_K \geq g_{KE} \geq \frac{1}{9N} g_K.$$

This concludes the proof of Theorem 2

It is already proved by McMullen in [Mc] that g_M is quasi-isometric to g_K, but the proof is rather long and complicated. To make the arguments in this article more consistent and self-content, we would give two proofs different from the one in [Mc]. The first proof is more in line with the arguments in Theorem 2 and the second one follows from some general results on the quasi-isometry of Bergman metric and the underlying Kähler metric of a complete Kähler manifold treated in the next section.

Proposition 3. *g_M is quasi-isometric to any of the metrics in Theorem 2.*

Proof Recall that up to quasi-isometry, the metric g_M is of the form

$$g_{WP} + C \sum_{\gamma, \ell_\gamma < \epsilon} \sqrt{-1} \frac{\partial \ell_\gamma}{\ell_\gamma} \wedge \overline{\frac{\partial \ell_\gamma}{\ell_\gamma}}$$

for some $\epsilon > 0$ and $C > 0$. We note the following three facts which have already been used in [Mc], (a) the sectional curvature of g_M is bounded, (b) $g_{WP} \leqslant cg_K$, and (c)$||d \log \ell_\gamma||_{g_K} \leqslant 2$. (a) follows essentially from Bers Embedding and Mean Value Inequality (cf. page 343-344 of [Mc]). For (b), we note that for a holomorphic quadratic differential ϕ on Riemann surface representing a cotangent vector in $T^*\mathcal{T}_S$,

$$||\phi||_T = \int_S |\phi| \leqslant [\int_S \frac{|\phi|^2}{g_S^2} g_S]^{\frac{1}{2}} (\mathrm{vol}(S, g_S))^{\frac{1}{2}} = c||\phi||_{WP},$$

where g_S is the hyperbolic metric on S and c is a topological number. Hence by duality, $g_{WP} \leqslant cg_T = cg_K$ on tangent vectors to the moduli, following the results of Royden. (c) follows from the computations of Theorem 3.1 of Wolpert [W] (cf. [M, page 341).

Utilizing Bers Embedding and (a), we apply the Schwarz Lemma of [CCL] to $(\mathcal{T}, g_M)$ and $(B^N_{\frac{3}{2}}(0), g_K^{B^N_{\frac{3}{2}}(0)})$ to conclude that

$$g_M \geq c g_K^{B^N_{\frac{3}{2}}(0)} \sim c g_K^{B^N_{\frac{1}{2}}(0)} \geq c g_K^{\mathcal{T}}.$$

On the other hand, (b) and (c) implies that $g_M \leqslant c' g_T = c' g_K$ following the results of Royden as mentioned before (cf. [G]). Hence $g_M \sim g_K$.

In section 5, we would give another proof of the proposition, using a rather different general approach.

Proof of Theorem 1 Theorem 1 now follows from Theorem 2 and Proposition 3, and the results of Royden mentioned before.

§4 L^2-estimates and Bergman metric

The following result is a direct consequence of L^2-estimates (cf. [Ho], [GW]).

Theorem 3. *Let (M, g) be a complete Kähler manifold with Kähler form ω. Assume that there exist positive constants r, δ, c, c_1, c_2 independent of x satisfying the following conditions.*
(i)There exist a coordinate chart (U_x, λ_x) of x such that $\lambda_x(U_x) = B_r^N(0)$, the Euclidean ball of radius r in $\mathbb{C}^N$, $\lambda_x(x) = 0$, $||\frac{\partial}{\partial z^i}||_g(0) = 1$ and $||\frac{\partial}{\partial z^i}||_g \geq \delta$ under the identification by the biholomorphic map λ_x.
(ii) There exists a strictly plurisubharmonic function ψ_x satisfying $\psi_x < c_1$ on M, and $\sqrt{-1}\partial\overline{\partial}\psi_x \geq c\omega$, $\psi_x > c_2$ on U_x.
Then the Bergman metric g_B on M and g are quasi-isometric.

Proof The idea of proof is similar to the proof of the part of Theorem 1 involving g_B, except that it is formulated in a more general setting for which L^2-estimate is needed. We use the notation in the last section and let f_x be the function in $H^0_{(2)}(M, K_M), ||f|| = 1$ realizing $\sup_{f \in H^0_{(2)}(M, K_M), ||f||=1} |f(x)|$. Let g_o be the Euclidean metric on $U_x \equiv B_r^N(0)$.

To prove the statement that $g_B \leqslant c_1 g$, it suffices to show that

$$\frac{1}{|f_x(x)|^2} \sup_{f\in H^0_{(2)}(M,K_M),\|f\|=1,f(x)=0} |\frac{\partial}{\partial z^i} f(x)|^2 \leqslant c_2$$

for each z_i. Let $x \in M$. Consider the neighborhood $U_x \cong B_r^N(0)$ of x satisfying condition (i). Let $\mathbb{C}z_i$ be the complex line through x generated by $\frac{\partial}{\partial z^i}$. It follows easily from Generalized Cauchy Integral Formula and Mean Value Theorem that

$$\begin{aligned} |\frac{\partial}{\partial z^i} g_{i,x}(x)|^2| &\leqslant c_3 \int_{\partial B^N_{\frac{r}{2}}(x)\cap \mathbb{C}z_i} |g_{i,x}(x)|^2| \\ &\leqslant c_4 \int_{B^N_r} |g_{i,x}(y)|^2| \\ &\leqslant c_4 \|g_{i,x}\|^2 = c_4. \end{aligned}$$

We now resort to standard L^2-estimates to show that $|f_x(x)|^2 \geq c_5$ for some constant $c_5 > 0$ independent of c_5. We use $B^N_{r,g}$ to denote a geodesic ball of radius r with respect to the metric g. Condition (i) also implies $g \geq \delta^2 g_o$ and hence that $B^N_{\delta r,g} \subset B^N_{r,g_o}(0) = B^N_r(0) = B^N_r(x) \cong U_x$, where we identify x with 0 by the coordinate map λ_x. Letting $|z|^2 = \sum |z_i|^2$ in terms Euclidean coordinates on B^N_{r,g_o}, then $|\nabla |z|^2|_g \leqslant c$ for some constant c on $B_r(x)$ independent of x on M. Let χ_x be a cut-off bumping function supported on $B_{\delta r,g}(x)$ and identically one on $B_{\frac{\delta r}{2},g}(x)$. Let $\eta_x = (\log \sum_{i=1}^N |z_i|^2)\chi_x$. Let $\rho = C\psi_x + N\eta_x$ with constant C large enough such that

$$\sqrt{-1}\partial\bar\partial\rho(y) \geq c_6(y)\omega(y)$$

for some positive function $c_6(y) > 0$ on M. Furthermore, from our hypothesis, there exists a constant $c_7 > 0$ such that $c_6(y) \geq c_7$ on U_x. Let $h = \bar\partial[\chi e_K]$. Standard L^2-estimates allow us to solve for $\partial f = h$ with

$$\begin{aligned} \int_M |f|^2 e^{-\rho} dV_g &\leqslant \int_M \frac{1}{c_6} |h|^2 e^{-\rho} dV_g \\ &= \int_{B^N_{\delta r,g}(x)} \frac{1}{c_6} |h|^2 e^{-\rho} dV_g \\ &\leqslant \frac{1}{c_7} \int_{B^N_{\delta r,g}(x)} |h|^2 e^{-\rho} dV_g \\ &< A, \end{aligned}$$

where A is a positive constant independent of x. Here we use assumptions in (i) and (ii). Let $f_1 = f - \chi e_K$. Since $-\rho$ is bounded from below by a constant on M by construction, $\|f_1\|^2 \leqslant c_8$ independent of x, and $|f_1(x)| = 1$. It follows that $f_2 = \frac{1}{\|f_1\|} f_1$ satisfies $\|f_2\|^2 = 1$ and $|f_2(x)| \geq c_9$ independent of $x \in M$. Hence $|f_x(x)| \geq c_9$ and we conclude that $g_B(x) \leqslant c_2 g(x)$.

The proof for $g_B \geq c_1 g$ is similar. Note that $|f_x(x)|^2 \leqslant c_{10} \|f_x\|^2 = c_{10}$ for some positive constant c_{10} by Cauchy Integral Formula or Mean Value Inequality. By letting $\rho = 2C\psi + (N+1)\eta_x$ and $h = \bar\partial[z^i \chi e_K]$, we can solve $\bar\partial f = g$ similar to the above discussion by L^2-estimates with $\int_M |f|^2 e^{-\rho} \leqslant \frac{1}{c_{11}} \int_{B^N_{\delta r,g}(x))} |h|^2 e^{-\rho} < B < \infty$, where B is a constant independent of x. Let $f_2 = f - z^i \chi e_K$. Clearly, $\|f_2\|^2 \leqslant c_{12}$ independent of x, and $|\frac{\partial}{\partial z^i} f_2(x)| = 1$. It follows that $f_3 = \frac{1}{\|f_2\|} f_2$ satisfies $\|f_2\|^2 = 1$

and $|\frac{\partial}{\partial z^i} f_2(x)| \geq c_{13} = \frac{1}{\sqrt{c_{12}}}$ independent of $x \in M$. Hence $g_B \geq \frac{1}{c_{13}} g$, the metric associated to ω. This concludes the proof of Theorem 3.

§5 Plurisubharmonic function for the equivalence of g_B and g_M.

In this section, we construct a special plurisubharmonic function and use it to apply Theorem 3 to prove the equivalence of g_B and g_M, providing a conceptually different proof of Proposition 3. Here is one of the key steps for our construction, which involves structure of the moduli space or Teichmüller space. We denote by Δ and Δ^* the unit disk and the punctured unit disk respectively.

Lemma 1. *There exist a finite covering $\mathcal{M}'_{g,n}$ corresponding to a torsion free subgroup of the mapping class group, a positive number ϵ and a relatively compact subset V of $\mathcal{M}'_{g,n}$ satisfying the following properties. $\mathcal{M}'_{g,n} - V$ is covered by a finite number of open sets $U_i, i = 1, \ldots, l$ each of form $(\Delta^*)^k \times \Delta^{N-k}$ for some $k > 0$. Denote by $A_i = A_i(S_x)$ the finite set of primitive closed geodesics on Riemann surface S_x represented by $x \in U_i$ consisting of geodesics on S_x which contracts to a node on the limiting noded Riemann surface corresponding to the Deligne-Mumford compactification. Then a primitive closed geodesic loop γ of length less than ϵ exists on a Riemann surface S_x for $x \in \mathcal{M}_{g,n}$ if and only if $x \in U_i$ and $\gamma \in A_i$, for some $l \geq i \geq 1$.*

Proof Note that the mapping class group acts as a group of biholomorphisms and isometries with respect to the metrics considered in Theorem 1. The moduli space $\mathcal{M}_{g,n}$ is the quotient of the Teichmüller space by the mapping class group. Consider first the case of $n = 0$. We recall that a finite covering $\mathcal{M}'_g$ of the moduli space of curves $\mathcal{M}_g$ corresponding to a subgroup of the mapping class group with finite index admits a compactification, Deligne-Mumford compactification [DM], in which the boundary components $D_i, i = 1 \cdots, k$, consist of noded Riemann surfaces obtained by contracting a finite number of geodesics to nodes on S. Suppose that a boundary component D_i is obtained by contracting $\gamma_{i1}, \cdots \gamma_{in_i}$ from a Riemann surface in $\mathcal{M}'_g$. Let $A = \cup_{i=1}^k \cup_{j=1}^{n_j} \gamma_{ij}$. D_i can be broken into subsets E_j in terms of strata which are mutually disjoint, in the forms of $D_i - \cup_{j \neq i} D_j, D_i \cap D_j - \cup_{k \neq i,j} D_i \cap D_j \cap D_k$, and in general $\cap_{l=1}^a D_{i_l} - \cup_{t \neq i_1, \ldots, i_l} [\cap_{l=1}^a D_{i_l}] \cap D_t$. For a generic point on $\cap_{l=1}^a D_{i_l} - \cup_{t \neq i_1, \ldots, i_l} [\cap_{l=1}^a D_{i_l}] \cap D_t$, there exists a neighbourhood of the form $(\Delta^*)^a \times \Delta^{N-a}$.

Suppose that we have a sequence of primitive short geodesics which are simple loops σ_i on Riemann surfaces S_i lying on $\mathcal{M}'_S$ such that their lengths satisfy $\lim_{i \to \infty} \ell_{\sigma_i} = 0$. We may assume that S_i approaches to a point S_∞ on some strata E_j as above. Since noded surfaces represented by E_j are obtained by contracting a finite number of primitive geodesic cycles $\gamma_{ji}^{S'}, i = 1, \ldots, n_i$ on all Riemann surfaces S' diffeomorphic to S on $\mathcal{M}'_S$, and contracting no other geodesic cycles, we conclude that there exists γ_{ji} such that the hyperbolic distance on Riemann surfaces S_k, $d(\sigma_k, \gamma_{ji}^{S_k}) \to 0$ as $k \to \infty$. However, by Collar Theorem of Keens (cf. [Ra]), we know that there exists $\tau_o > 0$ such that the collar of any geodesic of length at most τ_o has width greater than any preassigned constant, we may assume that a τ_o neighborhood of any γ_{ji} does not intersect with any other geodesic of length at most τ_o which is not a multiple of γ_{ji}. In particular, the contradiction establishes $\delta_j > 0$ such that any primitive geodesic cycle of length less than δ_j has to be one of the γ_{ji}. Since the cardinality of the set of D_i and hence the set of E_j is finite, and the complement of any neighborhood of the union of D_j is relatively compact on $\mathcal{M}_S$, there obviously exists a constant $\delta > 0$, such that the conclusion of the lemma is satisfied for $\mathcal{M}'_g$.

In the general case of $n > 0$, there is a holomorphic projection given by the map forgetting the punctures. The compactification in this case has been discussed by Knudsen-Mumford [KM] and Knudsen [K]. The boundary of $\mathcal{M}'_{g,n}$, some finite unramified covering of $\mathcal{M}_{g,n}$ with finite index as in the case of $n = 0$, is obtained again by contracting some primitive geodesic cycles on a Riemann surface S to nodes and the same argument applies. This concludes the proof of Lemma 1.

Recall that in [Ye], we use the sum $L_B = \sum_{\gamma \in B} \ell_\gamma$ for a set B of geodesics which are filling, in the sense that the complements of these geodesics in the Riemann surface are either 2-cells or cylinders, to construct bounded plurisubharmonic exhaustion on $\mathcal{T}_{g,n}$. Such a set of geodesics is fixed within the free homotopy class of geodesics on the Riemann surfaces represented by $\mathcal{T}_S$. The mapping class group however conjugates geodesic cycles with respect to a base Riemann surface S. Hence a short geodesic on a Riemann surface under the action of an element in the mapping class group maps to another closed geodesic which may be quite long on the same Riemann surface and vice versa. In this case, the Riemann surface is fixed but the marking is changed. There is in general no way of fixing a finite set B of homotopy classes of closed geodesics to take care of all the geodesics of length less than ϵ.

The way that we handle the problem is to consider for each point $x \in \mathcal{T}_{g,n}$, a connected component of some fixed open set of $\mathcal{M}'_{g,n}$ within which the action of the mapping class group would not conjugate those short geodesics relevant in the definition of the metric constructed by McMullen. The relevant fixed open sets are actually neighbourhoods of points around the Deligne-Mumford compactification of $\mathcal{M}'_{g,n}$ so that the monodromy around the compactifying divisor would not affect a simple geodesic loop contracting to a point on the divisor.

Hence in this paper we use appropriate product of the length of a set of geodesics which is not fixed within the free homotopy class of geodesics on a Riemann surface represented by $x \in \mathcal{T}_g$. However, the image of the set of geodesics by the uniformization map $\pi : \mathcal{T}_{g,n} \to \mathcal{M}'_{g,n}$, on the family of Riemann surfaces regarded as points on the moduli space $\mathcal{M}'_{g,n}$, has only a finite number of choices. This is one of the main differences between the approach here and the one in [C].

Here are the details. Let $\pi : \mathcal{T}_{g,n} \to \mathcal{M}'_{g,n}$ be the universal covering map. For $x \in \pi^{-1}(\cup_{i=1}^{l} U_i)$, choose an i such that $x \in \pi^{-1}(U_i)$. Let $U_{i,x}$ be the connected component of $\pi^{-1}(U_i)$ containing x. By adding non-empty intersections of those U_i to the set of open sets U_i if necessary, we may assume that there exists a ball $B_r(x)$ of radius r uniformly bounded from below on $\mathcal{T}_{g,n}$ such that $B_r(x) \subset U_{i,x}$, here r is measured with respect to the the Kobayashi metric. This can be seen easily by considering the lifting of a finite number of open sets of the form $(\Delta_r^*)^a \times \Delta_r^{N-a} \subset U_i$, where $(\Delta_r^*)^a$ consists of directions transversal to D in U_i, r is sufficiently small, and the union of these finite number of open sets covers $\cup_i U_i$. A Riemann surface S_y represented by $y \in U_{i,x}$ is biholomorphic to the Riemann surface represented by $\pi(y) \in \mathcal{M}'_{g,n}$. For $y \in U_{i,x}$, we use $A_{i,x} = A_{i,x}(y)$ to denote the set of closed geodesics on S_y corresponding to the set of geodesics A on the Riemann surface $\pi(y)$ on $\mathcal{M}'_{g,n}$. Note that the lifting of the contracting geodesic loops γ to $U_{i,x}$ is well-defined since there is no monodromy around D for the contracting geodesic loops.

Definition 1. *For a point $x \in \mathcal{T}_{g,n}$, choose an open set $U_{i,x}$ as above containing $B_r(x)$. Define for $y \in U_{i,x}$ $P_{A,x}(y) = \prod_{\gamma \in A_{i,x}(y)} \ell_\gamma$.*

Lemma 2. *(a). Let $\alpha = \frac{1}{4N_x}$, where N_x is the cardinality of $A_{i,x}$. Then on $U_{i,x}$,*

$$\sqrt{-1}\partial\overline{\partial}(-P_{A,x}^{-\alpha}) \geq \frac{\alpha}{4} P_{A,x}^{-\alpha} \sum_{\gamma, \ell_\gamma < \epsilon} \sqrt{-1}\frac{\partial \ell_\gamma}{\ell_\gamma} \wedge \overline{\frac{\partial \ell_\gamma}{\ell_\gamma}},$$

where the sum is taken over all primitive closed geodesics of length less than ϵ.
(b). Let y be a point on $\mathcal{M}'_{g,n}$ such that $d_K(x,y) \leqslant r$. Then $\frac{1}{c} \leqslant \frac{P_{A,x}(x)}{P_{A,x}(y)} \leqslant c$ for some constant $c > 0$ depending only on r but independent of x, y.

Proof The lemma follows essentially from [Ye] and computations of Wolpert and McMullen, as was used in [C]. For (a), we recall the following estimates of [Ye] based on the result of [W]. Let γ be a geodesic on S. Then on the Teichmüller space $\mathcal{T}_S$, $\sqrt{-1}\ell_\gamma \partial\overline{\partial}\ell_\gamma \geq 2\sqrt{-1}\partial\ell_\gamma \wedge \overline{\partial}\ell_\gamma$. We conclude that

$$\begin{aligned}
& \sqrt{-1}\partial\overline{\partial}[-P_{A,x}^{-\alpha}] \\
= \ & \sqrt{-1}\alpha\partial[(\sum_{\gamma \in A_{i,x}} \frac{\overline{\partial}\ell_\gamma}{\ell_\gamma})P_{A,x}^{-\alpha}] \\
= \ & \sqrt{-1}\alpha P_{A,x}^{-\alpha}\{\sum_{\gamma \in A_{i,x}} \frac{\sqrt{-1}}{\ell_\gamma^2}[\ell_\gamma\partial\overline{\partial}\ell_\gamma - \partial\ell_\gamma \wedge \overline{\partial}\ell_\gamma]\} \\
& -\alpha^2 P_{A,x}^{-\alpha}\{\sum_{\gamma \in A_{i,x}} \sqrt{-1}\frac{\partial\ell_\gamma}{\ell_\gamma} \wedge \sum_{\delta \in A_{i,x}} \overline{\frac{\partial\ell_\delta}{\ell_\delta}}\} \\
\geq \ & \sqrt{-1}\alpha(\frac{1}{2} - N_x\alpha)P_{A,x}^{-\alpha} \sum_{\gamma \in A_{i,x}} \sqrt{-1}\frac{\partial\ell_\gamma}{\ell_\gamma} \wedge \overline{\frac{\partial\ell_\gamma}{\ell_\gamma}},
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality.

Choose now $\alpha = \frac{1}{4N_x}$. Note that

$$\sum_{\gamma \in A_{i,x}(y)} \sqrt{-1}\frac{\partial\ell_\gamma}{\ell_\gamma} \wedge \overline{\frac{\partial\ell_\gamma}{\ell_\gamma}} = \sum_{\gamma \subset S_{\pi(x)}(y), \ell_\gamma < \epsilon} \sqrt{-1}\frac{\partial\ell_\gamma}{\ell_\gamma} \wedge \overline{\frac{\partial\ell_\gamma}{\ell_\gamma}}$$

from our choice of A_i and $A_{i,x}$. (a) follows from these estimates.

For (b), we use the fact that

$$\begin{aligned}
|\log \frac{P_{A,x}(x)}{P_{A,x}(y)}| & \leqslant \int_x^y \sum_{\gamma \in A} ||d\log\ell_\gamma||_{g_K} ds_{g_K} \\
& \leqslant \sum_{\gamma \in A} ||d\log\ell_\gamma||_{g_K} d_K(x,y) \\
& \leqslant c_c d_K(x,y),
\end{aligned}$$

since $||d\log\ell_\gamma||_{g_K} \leqslant 2$ from the estimates of [W] and [Mc].

We now give a second proof of the equivalence of g_M and the classical metrics stated in Theorem 2.

Proposition 4. $g_B \sim g_M$.

Proof We are going to apply Theorem 3 to conclude the proof. Recall from the proof of Lemma 1 that $\mathcal{M}'_{g,n}$ has a covering given by $U_i, i = 1, \ldots, l$ and V, which is a relatively compact set on $\mathcal{M}'_{g,n}$. Let $\widetilde{V} = \pi^{-1}(V)$ and $\widetilde{U} = \pi^{-1}(\cup_{i=1}^{l} U_i - V)$. $\widetilde{V}$ and $\widetilde{U}$ gives rise to a covering of $\mathcal{T}_{g,n}$.

Let us first consider a point $x \in \widetilde{U}$. We recall from our construction that there exists $x \in B_r(x) \subset U_{i,x}$ for some i. Since the curvature of g_M is bounded, the Schwarz Lemma of [CCL] or [Ro] as stated in Proposition 2 when applied to Bers Embedding implies that $g_M \geq c g_{KE}^{B^N_{\frac{1}{2}}} = g_K^{B^N_{\frac{1}{2}}}$. As in the proof of Theorem 2, $g_K^{\mathcal{T}} \sim g_K^{B_r^N}$ Clearly condition (i) of Theorem 3 is satisfied.

For conditions in (ii), we define $\psi_x(y) = d_{x,WP}(y)^2 - (\frac{P_{A,x}(x)}{P_{A,x}(y)})^\alpha$, where $d_{x,WP}(y)$ is the distance from y to x in terms of the Weil-Petersson metric g_{WP}. Note that the distance $d_{x,WP}$ is also utilized in [C]. According to [W], the Weil-Petersson metric on $\mathcal{T}_S$ has negative sectional curvature and the exponential map is a diffeomorphism from a domain in $T(\mathcal{T}_S)$ to $\mathcal{T}_S$. Moreover, standard comparison theorem (cf. [GW]) comparing $\mathcal{T}_S$ with Euclidean flat space implies that

$$\sqrt{-1}\partial\overline{\partial} d^2_{x,WP} \geq g_{WP}.$$

From Lemma 2(a),

$$\sqrt{-1}\partial\overline{\partial}[-(\frac{P_{A,x}(x)}{P_{A,x}(y)})^\alpha] \geq c_1[\frac{P_{A,x}(x)}{P_{A,y}(x)}]^\alpha \sum_{\gamma \in A} \sqrt{-1}\frac{\partial \ell_\gamma}{\ell_\gamma} \wedge \frac{\overline{\partial}\ell_\gamma}{\ell_\gamma},$$

where the differentiation is taken with respect to y. It follows that $\sqrt{-1}\partial\overline{\partial}\psi_x > 0$ on M. From Lemma 2(b), $\frac{1}{c} \leqslant \frac{P_{A,x}(x)}{P_{A,x}(y)} \leqslant c$ for $y \in B_r^N$. Hence

$$\sqrt{-1}\partial\overline{\partial}\psi_x(y) \geq c_2\omega_M(y)$$

for $y \in B_r^N$. Furthermore, $d_{x,WP}(y) \leqslant c_2 d_{x,M}(y)$ for some constant c_2 in terms of the distance $d_{x,M}$ measured with respect to g_M constructed by McMullen. Hence $d_{x,WP}(y) < c_3$ for some constant c_3 independent of x on $B_r^N(x)$. From Lemma 2b, this implies that $\psi_x \geq c_4$ for some constant c_4 on $B_r^N(0)$. Note also that $\psi_x(y)$ is bounded from above since $d_{x,WP}(p)$ is a bounded function for $y \in \mathcal{T}_{g,n}$ according to [Ma]. Hence condition (ii) of Theorem 3 is also satisfied for $x \in \widetilde{U}$.

Now consider $x \in \widetilde{V}$. Condition (i) is clearly satisfied since V is a relatively compact set of $\mathcal{M}'_{g,n}$ and we may just pull-back neighbourhoods from $\mathcal{M}'_{g,n}$. For (ii), we simply choose $\psi_x(y) = d^2_{x,WP}$. Note that the second term involving geodesic length functions in the metric constructed by McMullen is trivial by our choice of V according to Lemma 1. As in the calculation above, we clearly have $\sqrt{-1}\partial\overline{\partial}\psi_x(y) \geq c_2\omega_M(y)$ on $\widetilde{V}$ and $\sqrt{-1}\partial\overline{\partial}\psi_x(y) \geq 0$ on $\mathcal{T}_{g,n}$. Hence condition (ii) is satisfied for $x \in \widetilde{V}$ as well.

Proposition 3 now follows from Theorem 3.

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