On Holomorphic Immersions into Kähler Manifolds of Constant Holomorphic Sectional Curvature

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Abstract. We study holomorphic immersions $f : X \to M$ from a complex manifold X into a Kähler manifold of constant holomorphic sectional curvature M, i.e., a complex hyperbolic space form, a complex Euclidean space form, or the complex projective space equipped with the Fubini-Study metric. For X compact we show that the tangent sequence splits holomorphically if and only if f is a totally geodesic immersion. For X not necessarily compact we relate an intrinsic cohomological invariant $p(X)$ on X, viz., the invariant defined by Gunning measuring the obstruction to the existence of holomorphic projective connections, and an extrinsic cohomological invariant $\nu(f)$ measuring the obstruction to the holomorphic splitting of the tangent sequence. The two invariants $p(X)$ and $\nu(f)$ are related by a linear map on cohomology groups induced by the second fundamental form. In some cases, especially when X is a complex surface and M is of complex dimension 4, under the assumption that X admits a holomorphic projective connection we obtain a sufficient condition for the holomorphic splitting of the tangent sequence in terms of the second fundamental form.

The complex hyperbolic space is the complex unit ball B^n equipped with the canonical Kähler-Einstein metric. By a complex hyperbolic space form we mean B^n/Γ for some torsion-free discrete group Γ of holomorphic automorphisms. By a complex Euclidean space form we mean $\mathbb{C}^n/\Gamma$ for some torsion-free discrete group Γ of holomorphic rigid motions on the complex Euclidean space $\mathbb{C}^n$. The complex projective space $\mathbb{P}^n$ is always assumed to be equipped with the Fubini-Study metric.

Let M be a complex hyperbolic space form, a complex Euclidean space form, or the complex projective space. Such spaces M share the common geometric property of carrying complete Kähler metrics of constant holomorphic sectional curvature. On the other hand they carry holomorphic projective structures in the sense of Gunning [G]. Any compact Kähler-Einstein manifold admitting a holomorphic projective structure is either a complex hyperbolic space form, a complex Euclidean space form or the complex projective space, and it is not known whether these are the only examples which are compact and Kähler. Geometrically the salient feature for manifolds M with holomorphic projective structures is the existence of special local holomorphic curves corresponding under uniformization to open subsets of projective lines on the ambient projective space $\mathbb{P}^n$. This

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corresponds to a holomorphic foliation on the projectivized tangent bundle $\pi : \mathbb{P}T_M \to M$ by tautological liftings of holomorphic curves. In general the existence of such a holomorphic foliation is equivalent to the existence of a holomorphic projective connection in the sense of Gunning [G].

In this article we study holomorphic immersions from a complex manifold into a complex hyperbolic space form, a complex Euclidean space form or the complex projective space M. From the holomorphic projective structure on M, for the holomorphic immersion $f : X \to M$ we have the projective second fundamental form $\sigma : S^2T_X \to N_f$, where $N_f = f^*T_M/T_X$ is the normal bundle of the holomorphic immersion. σ is holomorphic, and it agrees with the second fundamental form of the holomorphic immersion $f : X \to M$ into the Kähler manifold M, with respect to a canonical Kähler-Einstein metric on M. Our study will focus on the properties of the holomorphic immersion that can be captured by the second fundamental form. Our first result concerns holomorphic immersions of compact complex manifolds into M. For its formulation, by the tangent sequence associated to a holomorphic immersion we mean the short exact sequence $0 \to T_X \to f^*T_M \to N_f \to 0$. We prove using methods of Complex Differential Geometry the following splitting criterion on the tangent sequence.

Theorem 1. *Let X be a compact complex manifold, M be a complex hyperbolic space form, a complex Euclidean space form or the complex projective space. Denote by $T = T_X$ the holomorphic tangent bundle of X, and by T_M the holomorphic tangent bundle of M. Let $f : X \to M$ be a holomorphic immersion, and denote by $N = N_f := f^*T_M/T$ its normal bundle. Then, the tangent sequence $0 \to T \to f^*T_M \to N \to 0$ for the holomorphic immersion splits holomorphically if and only if $f : X \to M$ is totally geodesic.*

Theorem 1 in the case where M is the complex projective space is a result of Van de Ven [VdV, 1958] (cf. also Mustată-Popa [MP, 1997]). Theorem 1 for complex Euclidean space forms M is a special case of results recently established by Jahnke [J,2004] on the splitting of the tangent sequence. All the known results are based on methods of Algebraic Geometry. Our proof makes use of harmonic forms, and relies on representing the obstruction class $\nu(f) \in H^1(X, T \otimes N^*)$ to the splitting of the tangent sequence by means of the second fundamental form σ.

Our second result is relevant to the general situation of a holomorphic immersion $f : X \to M$ of a (not necessarily compact) complex manifold X into a Kähler manifold M of constant holomorphic sectional curvature, relating two different cohomological classes associated to the complex manifold X resp. the holomorphic immersion $f : X \to M$. Associated to any complex manifold X, whose holomorphic tangent bundle we denote by T, there is a cohomology class $p(X) \in H^1(X, S^2T^* \otimes T)$ which is the obstruction to the existence of a holomorphic projective connection on X. For a holomorphic immersion $f : X \to M$ we relate the cohomology class $\nu(f) \in H^1(X, T \otimes N)$ to $p(X)$ and obtain

Theorem 2 (simplified form). *Let X be a complex manifold, and M be a complex hyperbolic space form, a complex Euclidean space form or the complex projective space. Denote by $T = T_X$ the holomorphic tangent bundle of X, and by T_M the holomorphic tangent bundle of M. Let $f : X \to M$ be a holomorphic immersion, $N := N_f = f^*T_M/T_X$ be its normal bundle, $\sigma \in \Gamma(X, S^2T^* \otimes N)$ be the second fundamental form, and $\nu(f) \in H^1(X, T \otimes N^*)$ be the obstruction to the holomorphic splitting of the tangent sequence $0 \to T \to f^*T_M/T \to N \to 0$. There is a canonically defined bundle homomorphism $\varphi : T \otimes N^* \to S^2T^* \otimes T$ completely determined by σ, such that X admits a holomorphic projective connection if and only if $\varphi_*(\nu(f)) = 0$ for the induced linear map $\varphi_* : H^1(X, T_X \otimes N^*) \to H^1(X, S^2T^* \otimes T)$ on first cohomology groups.*

Our third result is more special, pertaining to the case where $\dim(X) = 2$ and $\dim(M) = 4$, as follows.

Theorem 3. *In the notations of* Theorem 2 *suppose* $\dim(X) = 2$, $\dim(M) = 4$, *and X admits a holomorphic projective connection, i.e., $p(X) = 0$. Denote by $\pi : L \to \mathbb{P}T_X$ the tautological line bundle over the projectivized tangent bundle $\mathbb{P}T_X$, and by $s \in \Gamma(\mathbb{P}T_X, L^{-2} \otimes \pi^*N)$ the holomorphic section corresponding to $\sigma \in \Gamma(X, S^2T_X^* \otimes N)$. Suppose s is nowhere zero on $\mathbb{P}T_X$. Then, the tangent sequence splits holomorphically, i.e., $\nu(f) = 0$. When X is compact this can never happen.*

An analogue of Theorem 3 will also be formulated for $\dim(X) = n$ and $\dim(M) = \frac{(n+1)(n+2)}{2} - 2$, i.e., when $\dim(M)$ is one less than the dimension N in the Veronese embedding $v : \mathbb{P}^n \to \mathbb{P}^N$. The case where $\dim(X) = 2$ and $\dim(M) = 4$, as in Theorem 3, is particularly interesting and it reveals already the geometric features of the more general statement.

At the end of the article, we will discuss some problems in relation to holomorphic immersions between compact Kähler manifolds of constant holomorphic section curvatures, in the case of domain dimension 2 and target dimension 4, which was one of the original motivations for our study.

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§1 A splitting criterion for the tangent sequence

1.1 Holomorphicity of the second fundamental form

We start with the study of holomorphic immersions into Kähler manifolds of constant holomorphic curvature in the context of Kähler geometry. The following lemma concerns the second fundamental form, a fundamental geometric entity for our study, in terms of Kähler metrics. In §2 the second fundamental form will be related to holomorphic projective connections.

For a holomorphic immersion $f : X \to M$ between complex manifolds X resp. M, with holomorphic tangent bundles T resp. T_M, we write $N = N_f := f^*T_M/T$ for its normal bundle. We have the short exact sequence $0 \to T \to f^*T_M \to N \to 0$, called the tangent sequence (associated to the holomorphic immersion f). The obstruction to the holomorphic splitting of the tangent sequence is given by a cohomology class $\nu(f) \in H^1(X, T \otimes N^*)$, which is related to the second fundamental form σ. To start with we note the following well-known statement.

Lemma 1. *Let X be a compact complex manifold, M be a complex hyperbolic space form, a complex Euclidean space form or the complex projective space. Let $f : X \to M$ be a holomorphic immersion, and denote by σ the (1,0)-part of the second fundamental form of the holomorphic immersion f. Then, σ is holomorphic.*

Proof. By abuse of terminology in the sequel we will simply call σ the second fundamental form (of the immersion f). Denote by $g = 2\mathrm{Re}\big(\sum g_{i\bar{j}} dz^i \otimes \overline{dz^j}\big)$ a canonical Kähler-Einstein metric on M, of constant negative holomorphic sectional curvature when M is a complex hyperbolic space form, of zero curvature when M is a complex Euclidean space form, and of constant positive holomorphic sectional curvature when M is the complex projective space. We denote by ∇ covariant differentiation on the Kähler manifold (M, g) and by R its curvature tensor. Let ξ, μ, ν be holomorphic vector fields defined on some open subset U of X. Then,

$$\sigma(\mu, \nu) = \nabla_\mu \nu \bmod T_X. \tag{1}$$

$$\nabla_{\overline{\xi}}(\nabla_\mu \nu) = \nabla_\mu(\nabla_{\overline{\xi}}\nu) - \nabla_{[\mu,\overline{\xi}]}\nu - R(\mu,\overline{\xi})\nu. \tag{2}$$

Since ν is a holomorphic vector field we have $\nabla_{\overline{\xi}}\nu = 0$. Since μ and ξ are holomorphic vector fields we have $[\mu,\overline{\xi}] = 0$. For the curvature tensor R of (M,g), which is a complex hyperbolic space form resp. a complex Euclidean space form, resp. the complex projective space equipped with the Fubini-Study metric, denoting by $< \cdot, \cdot >$ the Hermitian inner product on T_M we have the curvature formula

$$R(\mu,\overline{\xi};\nu,\overline{\zeta}) = c\big(< \mu,\xi >< \nu,\zeta > + < \mu,\zeta >< \nu,\xi > \big) \tag{3}$$

for some constant c which is negative resp. zero resp. positive. Taking ζ to be orthogonal to T_X we see that $R(\mu,\overline{\xi};\nu,\overline{\zeta}) = 0$. It follows that in Eqn.(2) the curvature term is tangent to X. Projecting to the normal bundle we conclude that

$$\nabla_{\overline{\xi}}\big(\sigma(\mu,\nu)\big) = 0.$$

In other words, σ is holomorphic, $\sigma \in \Gamma(X, S^2T_X^* \otimes N)$, as desired. □

1.2 Proof of the splitting criterion using harmonic forms

Our first result is a splitting criterion for the tangent sequence of a holomorphic immersion of a compact complex manifold X into a Kähler manifold of constant holomorphic sectional curvature, which we recall here. The proof involves the use of the second fundamental form to represent the extension class of a short exact sequence of Hermitian holomorphic vector bundles, as in Griffiths [Gr].

Theorem 1. *Let X be a compact complex manifold, M be a complex hyperbolic space form, a complex Euclidean space form or the complex projective space. Denote by $T = T_X$ the holomorphic tangent bundle of X, and by T_M the holomorphic tangent bundle of M. Let $f : X \to M$ be a holomorphic immersion and denote by $N = N_f := f^*T_M/T$ its normal bundle. Then, the tangent sequence $0 \to T \to f^*T_M \to N \to 0$ for the holomorphic immersion splits holomorphically if and only if $f : X \to M$ is totally geodesic.*

Proof. We denote by g a canonical Kähler-Einstein metric on M and by h the Hermitian metric on the normal bundle N induced from g as a quotient metric. From the second fundamental form $\sigma \in \Gamma(X, S^2T^* \otimes N)$ taking conjugates we may inteprete $\overline{\sigma}$ as a $\overline{T^*} \otimes \overline{N}$-valued smooth (0,1)-form on X. Denote by η the $T \otimes N^*$-valued smooth (0,1)-form of X induced by $\overline{\sigma}$ by means of the canonical isomorphism $\overline{T_x^*} \cong T_x$ and $\overline{N_x} \cong N_x^*$ at each point $x \in X$ induced by the Hermitian metric g and the Hermitian metric h on N. Write m for the complex dimension of M and $n < m$ for the complex dimension on X. At $x \in X$ identify a sufficiently small coordinate neighborhood U of x on X with a locally closed complex submanifold of X by means of the holomorphic immersion f. Choose a system of local holomorphic coordinates $(z_1, \cdots, z_n)$ on U and extend the latter to a holomorphic coordinate system $(z_1, \cdots, z_m)$ of X at x. For $1 \leq \gamma \leq m-n$ we write e_γ for $\frac{\partial}{\partial z_\gamma}$ mod T_x. $\{e_1, \cdots, e_{m-n}\}$ constitutes a holomorphic basis of the normal bundle N over U. In terms of the chosen coordinates the Kähler metric g is represented on U by $(g_{i\bar{j}})_{1\leq i,j\leq m}$ and the Hermitian metric h on N is represented by $\left(h_{\alpha\bar{\gamma}}\right)_{1\leq\alpha,\gamma\leq m-n}$ on U. We have

$$\sigma = \sum_{i,k,\gamma} \sigma^\gamma_{ik} dz^i \otimes dz^k \otimes e_\gamma; \tag{1}$$

$$\eta = \sum_{i,k,\alpha} \eta^i_{\alpha\bar{k}} \frac{\partial}{\partial z_i} \otimes e^\alpha \otimes d\overline{z^k}, \quad \text{where} \tag{2}$$

$$\eta^i_{\alpha\bar{k}} = \sum_{j,\gamma} g^{i\bar{j}} h_{\alpha\bar{\gamma}} \overline{\sigma^\gamma_{jk}}. \tag{3}$$

Here $(g^{i\bar{j}})$ denotes the conjugate inverse of $(g_{i\bar{j}})$. We denote now by ∇ covariant differentiation on X of various vector bundles with respect to connections induced from (M, g). By the Codazzi equation σ satisfies

$$\nabla_{\frac{\partial}{\partial z_j}} \sigma^\gamma_{ik} = \nabla_{\frac{\partial}{\partial z_k}} \sigma^\gamma_{ij}, \tag{4}$$

which translates into

$$\nabla_{\frac{\partial}{\partial \bar{z}_j}} \eta^i_{\alpha\bar{k}} = \nabla_{\frac{\partial}{\partial \bar{z}_k}} \eta^i_{\alpha\bar{j}}, \tag{5}$$

Denote by $\left(\Gamma^\ell_{jk}\right)_{1\leq\ell,j,k\leq m}$ the Riemann-Christoffel symbols of the Kähler metric g for the (1,0)-part of the Riemannian connection ∇. Then, $\Gamma^\ell_{jk} = \Gamma^\ell_{kj}$ since ∇ is torsion-free. Taking conjugates $\Gamma^{\bar{\ell}}_{\overline{jk}} = \Gamma^{\bar{\ell}}_{\overline{kj}}$, and, observing that $T \otimes N^*$ is a holomorphic vector bundle hence parallel with respect to the (0,1)-part of covariant differentiation, we conclude from Eqn.(5) that

$$\frac{\partial}{\partial \overline{z_j}} \eta^i_{\alpha\bar{k}} = \frac{\partial}{\partial \overline{z_k}} \eta^i_{\alpha\bar{j}}. \tag{6}$$

In other words, η is a $\bar{\partial}$-closed $T \otimes N^*$-valued (0,1)-form, i.e., $\bar{\partial}\eta = 0$. We note that the latter statement is established for all Kähler submanifolds of Kähler manifolds. To prove Theorem 1 we have now to make use of the hypothesis that the ambient Kähler manifold (M, g) is a Kähler space form of constant holomorphic sectional curvature. By Lemma 1, the second fundamental form σ is holomorphic. This translates into the equation

$$\nabla_{\frac{\partial}{\partial z_j}}\eta^i_{\alpha\overline{k}} = 0 \tag{7}$$

for $1 \le i,j,k \le m$, $1 \le \alpha \le m-n$. Recall that for the compact Kähler manifold (X, f^*g) and for the holomorphic vector bundle $T \otimes N^*$ equipped with the Hermitian metric induced from g, the adjoint of the Cauchy-Riemann operator $\overline{\partial}$ on $T \otimes N$-valued (0,1)-forms is given by

$$\overline{\partial}^*\eta = \sum_{i,j,k,\alpha} g^{j\overline{k}}\nabla_{\frac{\partial}{\partial z_j}}\eta^i_{\alpha\overline{k}}\frac{\partial}{\partial z_i}\otimes e^\alpha. \tag{8}$$

In particular, it follows from Eqn.(7) that $\overline{\partial}^*\eta = 0$. Thus, η is both $\overline{\partial}$-closed and $\overline{\partial}^*$-closed. In other words, η is a harmonic form. It follows that the cohomology class $[\eta] \in H^1(X, T\otimes N^*)$ represented by η vanishes if and only if η vanishes. Since $[\eta]$ is precisely the obstruction to the splitting of the tangent sequence $0 \to T \to f^*T_M \to N \to 0$, we conclude that the tangent sequence splits if and only if $\eta \equiv 0$, i.e., if and only if $\sigma \equiv 0$. In other words, the tangent sequence splits holomorphically if and only if $f : X \to M$ is a totally-geodesic immersion. $\square$

§2 Holomorphic projective connections

2.1 Local expressions of holomorphic projective connections

For the description of holomorphic projective connections, we follow essentially Gunning [G]. A holomorphic projective connection Π on an n-dimensional complex manifold X, $n > 1$, consists of a covering $\mathcal{U} = \{U_\alpha\}$ of coordinate open sets, with holomorphic coordinates $(z_1^{(\alpha)}, \cdots, z_n^{(\alpha)})$, together with holomorphic functions $\left({}^\alpha\Phi^k_{ij}\right)_{1\le i,j,k\le n}$ on U_α symmetric in i,j satisfying the trace condition $\sum_k {}^\alpha\Phi^k_{ik} = 0$ for all i and satisfying furthermore on $U_{\alpha\beta} := U_\alpha \cap U_\beta$ the transformation rule (†)

$${}^\beta\Phi^\ell_{pq} = \sum_{i,j,k} {}^\alpha\Phi^k_{ij}\frac{\partial z_i^{(\alpha)}}{\partial z_p^{(\beta)}}\frac{\partial z_j^{(\alpha)}}{\partial z_q^{(\beta)}}\frac{\partial z_\ell^{(\beta)}}{\partial z_k^{(\alpha)}} + \left[\sum_\ell \frac{\partial z_\ell^{(\beta)}}{\partial z_k^{(\alpha)}}\frac{\partial^2 z_k^{(\alpha)}}{\partial z_p^{(\beta)}\partial z_q^{(\beta)}} - \delta^k_p\sigma_q^{(\alpha\beta)} - \delta^k_q\sigma_p^{(\alpha\beta)}\right],$$

where the expression in square brackets defines the Schwarzian derivative $S(f_{\alpha\beta})$ of the holomorphic transformation given by the change of variables $z^{(\alpha)} = f_{\alpha\beta}(z^{(\beta)})$, in which

$$\sigma_p^{(\alpha\beta)} = \frac{1}{n+1}\frac{\partial}{\partial z_p^{(\beta)}}\log J(f_{\alpha\beta}),$$

$J(f_{\alpha\beta}) = \det\left(\frac{\partial z_i^{(\alpha)}}{\partial z_p^{(\beta)}}\right)$ being the Jacobian determinant of the holomorphic change of variables $f_{\alpha\beta}$. The Schwarzian derivatives vanishes identically if and only if $f_{\alpha\beta}$ is a projective linear transformation.

Given a holomorphic projective connection Π on X with a given open covering $\mathcal{U} = \{U_\alpha\}$ and local expressions $\left({}^\alpha\Phi_{ij}^k\right)$ with respect to holomorphic coordinate systems $\left(z_1^{(\alpha)}, \cdots, z_n^{(\alpha)}\right)$ on $\mathcal{U}_\alpha$, the local expressions of Π with respect to another choice of holomorphic coordinate systems $\left(w_1^{(\alpha)}, \cdots, w_n^{(\alpha)}\right)$ on U_α can be obtained from the same transformation rule (†) in the above by considering $\left(z_1^{(\alpha)}, \cdots, z_n^{(\alpha)}\right)$ as a holomorphic change of variables from $\left(w_1^{(\alpha)}, \cdots, w_n^{(\alpha)}\right)$. From this, given two sets of data defining holomorphic projective connections Π and Π', we can introduce the notion of equivalence between them. Π and Π' are said to be equivalent if and only if there exists a common refinement $\mathcal{W} = \{W_\gamma\}$ of the respective open coverings $\mathcal{U}$ and $\mathcal{U}'$, such that for each W_γ the local expressions of Π and Π' with respect to a fixed choice of holomorphic coordinate system $\left(u_1^{(\gamma)}, \cdots, u_n^{(\gamma)}\right)$ agree with each other. In place of using local expressions and introducing equivalence, one can alternatively define holomorphic projective connections in an intrinsic manner in analogy to the axiomatic definition of affine connections, as done in Molzon-Mortensen [MM,§4], but we will not do so here.

2.2 Cohomological interpretation and tautological foliations

Holomorphic projective connections on a complex manifold X can be interpreted in cohomological terms as in Gunning [G,§6] (which deals with general pseudogroups), as follows. The Schwarzian derivatives $S(f_{\alpha\beta})$ constitutes a Čech 1-cocycle with values in $S^2T^* \otimes T$ with respect to the open covering $\mathcal{U} = \{U_\alpha\}$, where $T = T_X$ denotes the holomorphic tangent bundle on X. The transformation rule (†) on $\left({}^\alpha\Phi_{ij}^k\right)$ says that $\sum^\alpha \Phi_{ij}^k dz_i^{(\alpha)} \otimes dz_j^{(\alpha)} \otimes \frac{\partial}{\partial z_k}$ is a 0-cochain whose boundary is precisely $-S(f_{\alpha\beta}) = S(f_{\beta\alpha})$. We write $p(X) = \left[S(f_{\beta\alpha})\right]$, as an element of $H^1(X, S^2T^* \otimes T)$. $p(X)$ is the cohomological obstruction to the existence of holomorphic projective connections on X.

Let V be an n-dimensional complex vector space. Then, the natural action of $\mathrm{GL}(V)$ on V induces an action on $S^2V^* \otimes V$. Representing elements of the latter by $\left(A_{ij}^k\right)$ in terms of a basis, we have $A_{ij}^k = A_{ji}^k$. There is a $\mathrm{GL}(V)$-invariant subspace P consisting of those elements satisfying the trace condition $\sum_j A_{ij}^j = 0$. An element of P will be called a trace-free element. We have a decomposition $S^2V^* \otimes V = P \oplus Q$ as where both P and Q are irreducible $\mathrm{GL}(V)$-representation spaces (cf. Mok-Yeung [MY,§3]), and Q is isomorphic to V^*. An element of Q is called a trace element. Given $A \in S^2V^* \otimes V$, we have the decomposition

$A = A_P + A_Q$ with $A_P \in P$ and $A_Q \in Q$. We call $A_P \in P$ the trace-free part of A, and $A_Q \in Q$ the trace of A. The decomposition $A = P+Q$ induces on any complex manifold X a canonical direct sum decomposition $S^2T^* \otimes T = E \oplus F$, $F \cong T^*$, as holomorphic vector bundles. From the description of holomorphic projective connections we have $p(X) \in H^1(X,E)$. Note that $\mathrm{rank}(E) = \frac{n(n+2)(n-1)}{2}$. In particular, for $n = 2$, E is of rank 4.

We return to the local expressions of holomorphic projective connections and proceed to relate them to affine connections. Let $\left({}^\alpha\overline{\Gamma}^k_{ij}\right)$ be the Riemann-Christoffel symbols of any (smooth) affine connection on X. Then, we can define a torsion-free affine connection ∇ on X (cf. Molzon-Mortensen [MM,§4]) with Riemann-Christoffel symbols

$$ {}^\alpha\Gamma^k_{ij} = {}^\alpha\Phi^k_{ij} + \frac{1}{n+1}\sum_\ell \delta^k_i \; {}^\alpha\overline{\Gamma}^\ell_{\ell j} + \frac{1}{n+1}\sum_\ell \delta^k_j \; {}^\alpha\overline{\Gamma}^\ell_{i\ell} \qquad (\sharp) $$

We say that ∇ is an affine connection associated to Π. Two affine connections ∇ and ∇' on a complex manifold X are said to be projectively equivalent (cf. Molzon-Mortensen [MM, §4]) if and only if there exists a smooth (1,0)-form ω such that $\nabla_\xi\zeta - \nabla'_\xi\zeta = \omega(\xi)\zeta + \omega(\zeta)\xi$ for any smooth (1,0)-vector fields ξ and ζ on an open set of X. It is clear that any two affine connections ∇ defined as in ($\sharp$) are projectively equivalent. For any complex submanifold S of X, the second fundamental form of S in X is the same for two projectively equivalent affine connections. (Here and henceforth by complex submanifold $S \subset X$ we mean the images of an injective holomorphic immersion such that S is locally closed in X.) In particular, the class of complex geodesic submanifiolds S are the same. (A 1-dimensional complex geodesic submanifold will be called a complex geodesic.) We will say that a complex submanifold $S \subset X$ is geodesic with respect to the holomorphic projective connection Π to mean that it is geodesic with respect to any (torsion-free) affine connection ∇ associated to Π.

For an arbitrary affine connection one does not expect to find complex geodesic submanifolds, even locally. In the case of a holomorphic projective connection Π on X, given any coordinate open subset $U \subset X$, there is the flat affine connection on U; i.e., the Riemann-Christoffel symbols are $\overline{\Gamma}^k_{ij} = 0$ in the given holomorphic coordinate system; and hence we can associate a holomorphic torsion-free affine connection ∇_U to $\Pi|_U$. Since ∇_U is holomorphic, for every $x \in U$ and any non-zero (1,0)-vector $\eta \in T_xU$ there is a complex geodesic $\Lambda \subset U$ passing through x such that $T_x(\Lambda) = \mathbb{C}\eta$. Associated to X we have the projectivized holomorphic tangent bundle $\pi : \mathbb{P}T_X \to X$. Given any holomorphic curve C on X there is a tautological lifting of C to $\widehat{C} \subset \mathbb{P}T_X$ by lifting $x \in C$ to $[T_x(C)] \in \mathbb{P}T_x(X)$. Thus, given a holomorphic projective connection Π, by this tautological lifting we obtain a holomorphic foliation $\mathcal{F}$ on $\mathbb{P}T_X$ with 1-dimensional leaves, with the property that $d\pi(\mathcal{F}_{[\eta]}) = \mathbb{C}\eta$ for every non-zero $\eta \in T_x(X)$.

Suppose Π and Π' are equivalent holomorphic projective connections. Let ∇ resp. ∇' be an affine connection associated to Π resp. Π' according to ($\sharp$) with respect to the same background affine connection. By passing to some common refinement $\mathcal{W}$ of open coverings $\mathcal{U}$ resp. $\mathcal{U}'$ for Π resp. Π', it follows readily from the definition of equivalence of holomorphic projective connections that ∇ and ∇' are the same. In particular, complex geodesics and hence the foliations $\mathcal{F}$ on $\mathbb{P}T_X$ are the same for equivalent holomorphic projective connections. We have in fact

Proposition 1. *Let X be a complex manifold and $\pi : \mathbb{P}T_X \to X$ be its projectivized holomorphic tangent bundle. Then, there is a canonoical one-to-one correspondence between the set of equivalence classes of holomorphic projective connections on X and the set of holomorphic foliations $\mathcal{F}$ on $\mathbb{P}T_X$ by tautological liftings of holomorphic curves.*

Proof. In the sequel we will sometimes refer to holomorphic foliations $\mathcal{F}$ on $\mathbb{P}T_X$ as in Proposition 1 simply as tautological holomorphic foliations on $\mathbb{P}T_X$ (of leaf dimension 1). We have seen how an equivalence class of holomorphic projective connections on X defines a holomorphic foliation $\mathcal{F}$ on $\mathbb{P}T_X$ by tautological liftings of complex geodesics. Starting with a tautological holomorphic foliation on $\mathbb{P}T_X$, we are going to define an equivalence class of holomorphic projective connections. Let $U \subset X$ be an open subset and $\mathcal{F}^1$, $\mathcal{F}^2$ be holomorphic foliations on $\mathbb{P}T_U$ by tautological liftings of holomorphic curves. For any non-zero $\eta \in T_U$ we have $d\pi(\mathcal{F}^i_{[\eta]}) = \mathbb{C}\eta$, for $1 = 1, 2$. $\mathcal{F}^1$ can be compared to $\mathcal{F}^2$, as follows. Let $x \in U$ and η be a nonzero vector in $T_x(U)$. Then $d\pi(\mathcal{F}^1_{[\eta]}) = d\pi(\mathcal{F}^2_{[\eta]}) = \mathbb{C}\eta$. We can associate to $\eta \in T_x(U)$ two vectors $\tau_1 \in \mathcal{F}^1_{[\eta]}$ and $\tau_2 \in \mathcal{F}^2_{[\eta]}$, uniquely determined, such that $d\pi(\tau_1) = d\pi(\tau_2) = \eta$, so that $d\pi(\tau_1 - \tau_2) = 0$, and $v(\eta) := \tau_1 - \tau_2 \in T_{[\eta]}\big((\mathbb{P}T_x(X)\big)$ is a 'vertical' tangent vector with respect to the projection $\pi : \mathbb{P}T_X \to X$. Denoting by L the tautological line bundle over $\mathbb{P}T_X$, we have obtained by the assignment of $v(\eta)$ to η an element t in $\Gamma(\mathbb{P}T_U, L^* \otimes T_\pi)$, where T_π denotes the relative tangent bundle of $\pi : \mathbb{P}T_X \to X$. By Mok-Yeung [MY, §3] we have $\pi_*(L^* \otimes T_\pi) \subset \pi_*(L^*) \otimes \pi_*(T_\pi) \cong T^* \otimes \mathrm{End}_o(T)$, where $T = T_X$. Here $\pi_*(L^* \otimes T_\pi)_x$ corresponds precisely to $\big(S^2T^*_x \otimes T_x\big) \cap \big(T_x \otimes \mathrm{End}_o(T_x)\big) := E_x$. Thus, $t \in \Gamma(U, L^* \otimes T_\pi)$ corresponds to a holomorphic tensor $s \in \Gamma(U, E) \subset \Gamma(U, S^2T^* \otimes T)$. With respect to local holomorphic coordinates $(z_1, \cdots z_n)$ the holomorphic tensor s is represented by holomorphic functions $\big(\Phi^k_{ij}\big)_{1 \le i,j,k \le n}$ which are symmetric in i, j and which satisfy furthermore the trace condition $\sum_j \Phi^j_{ij} = 0$. We write $s := \delta(\mathcal{F}^1, \mathcal{F}^2)$, which is a difference between tautological foliations.

Fix now a covering $\mathcal{U} = \{U_\alpha\}$ of X by coordinate open sets U_α with holomorphic coordinates $\big(z_1^{(\alpha)}, \cdots, z_n^{(\alpha)}\big)$. The flat projective connection on each U_α with respect to $\big(z_i^{(\alpha)}\big)$ defines a holomorphic foliation $\mathcal{F}^\alpha$ on $\mathbb{P}T_{U_\alpha}$ by tautological liftings of open subsets of affine lines in the complex Euclidean space with coordinates $\big(z_i^{(\alpha)}\big)$. Define $s_\alpha := \delta(\mathcal{F}|_{U_\alpha}, \mathcal{F}^\alpha)$. In terms of the coordinates $\big(z_i^{(\alpha)}\big)$ we write $s_\alpha = \sum_{i,j,k} {}^\alpha\Phi^k_{ij} dz_i^{(\alpha)} \otimes dz_j^{(\alpha)} \otimes \frac{\partial}{\partial z_k^{(\alpha)}}$. We claim that $\big({}^\alpha\Phi^k_{ij}\big)_{1 \le i,j,k \le n}$ are the local expressions of a holomorphic projective connection Π on X. It suffices to check that the transformation rule (†) ${}^\beta\Phi^\ell_{pq} = \sum_{i,j,k} {}^\alpha\Phi^k_{ij} \frac{\partial z_i^{(\alpha)}}{\partial z_p^{(\beta)}} \frac{\partial z_j^{(\alpha)}}{\partial z_q^{(\beta)}} \frac{\partial z_\ell^{(\beta)}}{\partial z_k^{(\alpha)}} + S(f_{\alpha\beta})$ is satisfied. The first term on the right-hand side consists precisely of the local coefficients of the holomorphic tensor s_α in terms of the holomorphic coordinates $\big(z_i^{(\beta)}\big)$. From $s_\alpha := \delta(\mathcal{F}|_{U_\alpha}, \mathcal{F}^\alpha)$ it follows that on $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we have $s_\beta - s_\alpha = \delta(\mathcal{F}|_{U_\beta}, \mathcal{F}^\beta) - \delta(\mathcal{F}|_{U_\alpha}, \mathcal{F}^\alpha) = \delta(\mathcal{F}^\alpha|_{U_{\alpha\beta}}, \mathcal{F}^\beta|_{U_{\alpha\beta}})$, which in terms of the holomorphic coordinate system $\big(z_i^{(\beta)}\big)$ is nothing other than the Schwarzian derivative $S(f_{\alpha\beta})$. The construction of the holomorphic projective connection Π from the holomorphic foliation $\mathcal{F}$ depends on the choice of an atlas $\mathcal{U}$, but from its definition it follows readily that a change of atlas results in an equivalent holomorphic projective connection.

Starting with an equivalence class of holomorphic projective connections $\{\Pi\}$, giving rise to a tautological holomorphic foliation $\mathcal{F}$ on $\mathbb{P}T_X$, it follows from the second-order holomorphic differential equation satisfied by complex geodesics that from $s_\alpha := \delta(\mathcal{F}|_{U_\alpha}, \mathcal{F}^\alpha)$ we recuperate precisely $\{\Pi\}$. The same calculation implies that the process can be reversed, starting with a tautological holomorphic foliation of leaf dimension 1 on $\mathbb{P}T_X$, and we have established a canoncial one-to-one correspondence between equivalence classes of holomorphic projective connections on X and tautological holomorphic foliations of leaf dimension 1 on $\mathbb{P}T_X$. □

From now on we will not make any distinction between a holomorphic projective connection and an equivalence class of holomorphic projective connections.

Let M be a complex manifold equipped with a holomorphic projective connection Π, and $X \subset M$ be a complex submanifold. As implicit in the discussion preceding the statement of Proposition 1, where the notion of complex geodesics, the second fundamental form σ of X in M with respect to any affine connection ∇ associated to the projective connection Π by means of (♯) is independent of the choice of the background affine connection. We call σ the projective second fundamental form of $X \subset M$ with respect to Π. Since locally we can always choose the flat background affine connection it follows that the projective second fundamental form is *holomorphic*.

2.3 Holomorphic projective connections on Kähler manifolds of constant holomorphic sectional curvature

Consider now the situation where M is a complex hyperbolic space form, a complex Euclidean space form, or the complex projective space. M is equipped with a canonical Kähler metric g of constant negative resp. zero resp. positive holomorphic sectional curvature. The universal covering space of M is the complex unit ball B^n resp. the complex Euclidean space $\mathbb{C}^n$ resp. the complex projective space $\mathbb{P}^n$ (itself), equipped with the Bergman metric resp. the Euclidean metric resp. the Fubini-Study metric. We say that X is of noncompact resp. Euclidean resp. compact type. For $\mathbb{C}^n$ the family of affine lines leads to a tautological foliation $\mathcal{F}_o$ on the projective tangent bundle, and g is associated to the flat holomorphic projective connection. In the case of $\mathbb{P}^n$ the projective lines, which are closures of the affine lines in $\mathbb{C}^n \subset \mathbb{P}^n$, are totally-geodesic with the Fubini-Study metric g. In the case of $B^n \subset \mathbb{C}^n$, the intersections of affine lines with B^n give precisely the minimal disks which are totally-geodesic with respect to the Bergman metric g. As a consequence, the tautological foliation $\mathcal{F}$ on $\mathbb{P}T_{\mathbb{P}^n}$ defined by the tautological liftings of projective lines, which is invariant under the projective linear group $\mathrm{Aut}(\mathbb{P}^n) \cong \mathbb{P}\mathrm{GL}(n+1)$, restricts to tautological foliations on $\mathbb{C}^n$ resp. B^n, and they descend to quotients X of $\mathbb{C}^n$ resp. B^n by torsion-free discrete groups of holomorphic isometries of $\mathbb{C}^n$ resp. B^n, which are in particular projective linear transformations. The holomorphic projective connection on $\mathbb{P}^n$ corresponding to $\mathcal{F}$ by means of Proposition 1 will be called the canonical holomorphic projective connection. The same term will apply to holomorphic projective connections induced by the restriction of $\mathcal{F}$ to $\mathbb{C}^n$ and to B^n and to the tautological foliations induced on their quotient manifolds X as in the above. Relating the canonical holomorphic projective structures to the canonical Kähler metric g, we have the following result which in particular gives another proof of the holomorphicity of the second fundamental form σ defined in terms of Kähler geometry in (1.1), Lemma 1.

Lemma 2. *Let (M, g) be a complex hyperbolic space form, a complex Euclidean form, or the complex projective space equipped with the Fubini-Study metric. Then, the affine connection of the Kähler metric g is associated to the canonical holomorphic projective connection on M. As a consequence, given any complex submanifold $X \subset M$, the second fundamental form on X as a Kähler submanifold of*

(M, g) agrees with the projective second fundamental form of X in M with respect to the canonical holomorphic projective connection.

Proof. The complex geodesics on M with respect to the canonical holomorphic projective connection are also complex geodesics with respect to the canonical metric g. On a sufficiently small open subset $U \subset M$ choose holomorphic coordinates obtained by identifying M locally with open subset of the complex projective space by means of the universal covering map. In terms of these local holomorphic coordinates let $\left(\Gamma^k_{ij}\right)$ be the (1,0)-part of the Riemann-Christoffel symbols. Regard Γ^k_{ij} as components of a tensor Γ belonging to $S^2T^*_U \otimes T_U$, noting that this tensor depends on the choice of local coordinates. Then, from total geodesy of (open subsets of) affine lines we conclude that at every $x \in U$, $\beta^*(\Gamma(\alpha, \alpha)) = 0$ for every $\alpha \in T_x(U)$, and for every $\beta^* \in T^*_x$ such that $\beta^*(\alpha) = 0$. Recall the canonical decomposition $S^2T^*_U \otimes T_U = E \oplus F$ as explained in (2.2). Note that $\alpha \otimes \alpha \otimes \beta^*$ is trace-free, hence belonging to E_x. From the irreducibility of E_x under the natural action of $\mathrm{GL}(V)$ we conclude that Γ takes values in F. This means that there exist smooth functions v_j such that $\Gamma^k_{ij} = \delta^k_i v_j + \delta^k_j v_i$. By comparing with the equation ($\sharp$) relating holomorphic projective connections to Riemann-Christoffel symbols it follows that the affine connection of (M, g) is associated to the canonical holomorphic projective connection on M, as asserted. The last statement follows from our definition of the projective second fundamental form in the last paragraph of (2.2). $\square$

Using Proposition 1 in (2.2) it is possible to define the projective second fundamental form by means of the tautological foliation, without any reference to affine connections. Let M be a complex manifold equipped with a holomorphic projective connection, i.e., equivalently, a tautological foliation $\mathcal{F}$, by Proposition 1. Let $X \subset M$ be a complex submanifold, and denote by N the normal bundle of the embedding. Let $x \in X$ and $\alpha \in T_x(M)$ be a nonzero tangent vector. Let C_α be a local holomorphic curve on M passing through x such that $T_x(C_\alpha) = \mathbb{C}\alpha$ and such that the tautological lifting $\widehat{C}_\alpha$ to $\mathbb{P}T_M$ is a local leaf of the tautological foliation. Let now D_α be any local holomorphic curve on $X \subset M$ passing through x such that $T_x(D_\alpha) = \mathbb{C}\alpha$, and denote by $\widehat{D}_\alpha$ the tautological lifting of D_α. Denote by $\pi : \mathbb{P}T_M \to M$ the canonical projection, and by L the tautological line bundle over $\mathbb{P}T_M$. Let $\xi \in T_{[\alpha]}(\widehat{C}_{[\alpha]}) \subset T_{[\alpha]}(\mathbb{P}T_x(M))$ be such that $d\pi(\xi) = \alpha$. Let $\eta \in T_{[\alpha]}(\widehat{C}_{[\alpha]}) \subset T_{[\alpha]}(\mathbb{P}T_x(M)$ be such that $d\pi(\eta) = \alpha$. Then $d\pi(\xi - \eta) = 0$, so that $\xi - \eta \in T_{[\alpha]}\left(\mathbb{P}T_x(M)\right)$, which is canonically isomorphic to $L^*_{[\alpha]} \otimes T_x(M)/L_{[\alpha]}$. Composing with the canonical projection $\rho = id_{L_{[\alpha]}} \otimes \rho_o$, $\rho_o := T_x(M)/L_{[\alpha]} \to$

$T_x(M)/T_x(X) = \pi^* N_x$, the assignment of $[\alpha]$ to $\rho(\xi - \eta)$ is independent of the choice of $D_{[\alpha]}$. Furthermore this assignment is holomorpic in $[\alpha]$, leading therefore to a section $s \in \Gamma\big(\mathbb{P}T_X, L^{-2} \otimes \pi^* N\big)$, which is equivalent to a holomorphic section $\sigma \in \Gamma\big(X, S^2 T_X^* \otimes N\big)$. As can be easily verified, σ agrees with the projective second fundamental form of X in M with respect to the given holomorphic projective connection.

An n-dimensional complex manifold X is said to admit a holomorphic projective structure if its universal covering space $\widetilde{X}$ admits a local biholomorphism $\rho : \widetilde{X} \to \mathbb{P}^n$, which is called a developing map. The holomorphic projective connection on $\mathbb{P}^n$ then induces a holomorphic projective connection on X. A holomorphic projective connection on a complex manifold is said to be integrable if it is induced by a holomorphic projective structure, i.e., by a developing map into the complex projective space. Integrability of holomorphic projective connections do not concern us here. We note however that the canonical holomorphic projective connections on $\mathbb{P}^n$ and on compact complex hyperbolic space forms are the unique holomorphic projective connections on these manifolds (cf. Mok-Yeung [MY], for the latter case).

§3 Linking second fundamental forms to holomorphic projective connections

3.1 A homomorphism between first cohomology groups defined by the second fundamental form

We restate Theorem 2 in the following more precise form. For its formulation recall as in (2.1) that on a complex manifold X, writing $T = T_X$, we have a direct sum decomposition $S^2T^* \otimes T = E \oplus F$, where $F \cong T^*$ and E consists of elements (A^k_{ij}) satisfying the trace condition $\sum_j A^j_{ij} = 0$. Denote by $\pi_E : S^2T^* \otimes T \to E$ the canonical holomorphic bundle epimorphsm. We have

Theorem 2 (precise form). *Let X be a complex manifold, and M be a complex hyperbolic space form, a complex Euclidean space form or the complex projective space. Let $p(X) \in H^1(X, E)$ be the obstruction class to the existence of holomorphic projective connections on X. Denote by $T = T_X$ the holomorphic tangent bundle of X, and by T_M the holomorphic tangent bundle of M. Let $f : X \to M$ be a holomorphic immersion, and $N := N_f = f^*T_M/T$ be its normal bundle, $\sigma \in \Gamma(X, S^2T^* \otimes N)$ be the second fundamental form, and $\nu(f) \in H^1(X, T \otimes N^*)$ be the obstruction class to the holomorphic splitting of the tangent sequence $0 \to T \to f^*T_M/T \to N \to 0$. Let $\psi : T \otimes N^* \to S^2T^* \otimes T$*

be defined by $\big(\psi(\xi \otimes \theta^*)\big)(\lambda \otimes \mu) = \theta^*(\sigma(\lambda, \mu))\xi$ *for any* $x \in X$*;* $\xi, \lambda, \mu \in T_x$*;* $\theta^* \in N_x^*$*, and* $\varphi : T \otimes N^* \to E$ *be defined by* $\varphi = \pi_E \circ \psi$*. Then,* $\varphi_*\big(\nu(f)\big) = p(X)$ *for the induced linear map* $\varphi_* : H^1(X, T \otimes N^*) \to H^1(X, E)$ *on first cohomology groups. In particular,* X *admits a holomorphic projective connection if and only if* $\varphi_*(\nu(f)) = 0$*.*

Proof. By the proof of Theorem 1, $\nu(f)$ is represented by the $T \otimes N^*$-valued (0,1)-form η. Recall that

$$\eta = \sum_{i,j,k,\alpha,\gamma} g^{i\bar{j}} h_{\alpha\bar{\gamma}} \overline{\sigma_{jk}^{\gamma}} \frac{\partial}{\partial z_i} \otimes e^{\alpha} \otimes d\overline{z^k}. \tag{1}$$

In what follows the notation φ_*, etc. for a bundle homorphism φ will denote both the induced map on differential forms and an cohomology classes, as is clear from the context. For the homomorphism $\psi_* : H^1(X, T \otimes N^*) \to H^1(X, S^2T^* \otimes T)$ on first cohomology groups, the cohomology class $\psi_*(\nu(f))$ is represented by the $S^2T^* \otimes T$-valued (0,1)-form

$$\psi_*(\eta) := \sum_{i,j,k,p,q,\alpha,\gamma} g^{i\bar{j}} h_{\alpha\bar{\gamma}} \sigma_{pq}^{\alpha} \overline{\sigma_{jk}^{\gamma}} \frac{\partial}{\partial z_i} \otimes dz^p \otimes dz^q \otimes d\overline{z^k}. \tag{2}$$

Composing with the canonical projecting π_E of $S^2T^* \otimes T$ onto E we obtain

$$\varphi_*(\eta) := \sum_{i,k,p,q} A_{pq\bar{k}}^{i} \frac{\partial}{\partial z_i} \otimes dz^p \otimes dz^q \otimes d\overline{z^k};$$

$$A_{pq\bar{k}}^{i} = \sum_{j,\alpha,\gamma} \left(g^{i\bar{j}} h_{\alpha\bar{\gamma}} \sigma_{pq}^{\alpha} \overline{\sigma_{jk}^{\gamma}} - \frac{1}{n+1} \sum_{\ell} \left(\delta_p^i g^{\ell\bar{j}} h_{\alpha\bar{\gamma}} \sigma_{\ell q}^{\alpha} \overline{\sigma_{jk}^{\gamma}} + \delta_q^i g^{\ell\bar{j}} h_{\alpha\bar{\gamma}} \sigma_{p\ell}^{\alpha} \overline{\sigma_{jk}^{\gamma}} \right) \right). \tag{3}$$

To prove Theorem 2 assume first of all that X admits a holomorphic projective connection, i.e., $p(X) = 0$. We are going to prove in this case that $\varphi_*(\eta)$ is $\overline{\partial}$-exact. The main point is to make use of the holomorphic projective connections to lift the second fundamental form locally to a holomorphic section with values in $S^2T^* \otimes T_M$. The liftings will not be unique, but will be unique modulo trace elements of $S^2T^* \otimes T$. This will allow us to solve the equation $\overline{\partial}\tau = \varphi_*(\eta)$ by means of orthogonal projections.

Denote by $\mathcal{E}$ the tautological foliation on $\mathbb{P}T_M$ corresponding to the holomorphic projective structure on M, and by $\mathcal{F}$ the tautological foliation on $\mathbb{P}T_X$ corresponding to the given holomorphic projective connection on X. Let $x \in X$ and $\alpha \in T_x(X)$ be a nonzero tangent vector. Let C_α resp. D_α be local holomorphic curves on M resp. X passing through x and tangent to α such that their tautological liftings $\widehat{C}_\alpha$ to $\mathbb{P}T_M$ resp. $\widehat{D}_\alpha$ to $\mathbb{P}T_X$ are integral curves of $\mathcal{E}$ resp. $\mathcal{F}$. Denote by $\pi : \mathbb{P}T_M \to M$ the canonical projection, by L the tautological line bundle over $\mathbb{P}T_M$ and by T_π the relative tangent bundle of π. To simplify notations we will write as if X were embedded in M, so that $\mathbb{P}T_X \subset \mathbb{P}T_M$, $L\big|_{\mathbb{P}T_X}$ is the tautological line bundle over $\mathbb{P}T_X$, etc. Let $\xi \in T_{[\alpha]}(\widehat{C}_\alpha)$ and $\eta \in T_{[\alpha]}(\widehat{D}_\alpha)$ be such that $d\pi(\xi) = d\pi(\eta) = \alpha$. Then, $d\pi(\xi - \eta) = 0$, and the assignment of α to $\xi - \eta \in T_{\pi,[\alpha]} \subset T_{[\alpha]}\big(\mathbb{P}T_M\big)$ defines a holomorphic section $s^\flat$ in $\Gamma(\mathbb{P}T_X, L^* \otimes T_\pi)$. Note that here T_π is the relative tangent bundle of $\pi : \mathbb{P}T_M \to M$, restricted to $\mathbb{P}T_X$. From $T_\pi \cong L^* \otimes \pi^* T_M/L$ we have $s^\flat \in \Gamma(\mathbb{P}T_X, L^{-2} \otimes \pi^* T_M/L)$. On a Stein open neighborhood U of $x \in X$ consider the exact sequence $0 \to L^* \to L^{-2} \otimes \pi^* T_M \to L^{-2} \otimes \pi^* T_M/L \to 0$ over $\mathbb{P}T_U$. Since $H^1(\mathbb{P}T_x, L^*) = H^1\big(\mathbb{P}T_x, \mathcal{O}(1)\big) = 0$ we obtain by taking direct images a short exact sequence

$$0 \to \pi_* L^* \to \pi_*(L^{-2} \otimes \pi^* T_M) \to \pi_*(L^{-2} \otimes \pi^* T_M/L) \to 0 \qquad (\natural)$$

Since U is Stein, $H^1(U, \pi_* L^*) = 0$, and $s^\flat \in \Gamma(\mathbb{P}T_U, L^{-2} \otimes \pi^* T_M/L)$ lifts by ($\natural$) to some $\widetilde{s} \in \Gamma(\mathbb{P}T_U, L^{-2} \otimes \pi^* T_M)$, which corresponds to some $\widetilde{\sigma} \in \Gamma(U, S^2T^* \otimes T_M)$. Any two liftings $\widetilde{\sigma}$ of $\sigma \in \Gamma(U, S^2T^* \otimes N)$ obtained this way differ by a section $\zeta \in \Gamma(U, F)$, where $F \cong T^*$ is the vector bundle of trace elements in $S^2T^* \otimes T$.

Define now a smooth E-valued (0,1)-form τ from $\widetilde{\sigma}$, as follows. Let $\rho_o : T_M \to T$ be the orthogonal projection and $\rho : S^2T^* \otimes T_M \to S^2T^* \otimes T$ be given by $\rho = id_{S^2T^*} \otimes \rho_0$. Define τ to be the trace-free part of $\rho_* \widetilde{\sigma}$, so that τ is independent of the choice of lifting $\widetilde{\sigma}$ and is globally defined on X. We assert that $\overline{\partial}\tau = \varphi_*(\eta)$. In the following equations i, p, q range from 1 to n, while ℓ ranges from 1 to m. In terms of local holomorphic coordiantes write

$$\widetilde{\sigma} = \sum_{p,q,\ell} \widetilde{\sigma}^\ell_{pq} dz^p \otimes dz^q \otimes \frac{\partial}{\partial z_\ell}. \qquad (4)$$

Writing

$$\rho_o = \sum_{i,\ell} \rho^i_\ell dz^\ell \otimes \frac{\partial}{\partial z_i} \ , \qquad (5)$$

we have

$$\rho_* \widetilde{\sigma} = \sum_{i,p,q,\ell} \rho^i_\ell \widetilde{\sigma}^\ell_{pq} dz^p \otimes dz^q \otimes \frac{\partial}{\partial z_i} \qquad (6)$$

Now $\rho_o\big|_T$ is the identity map, so that $\overline{\partial}\rho_o$ as a $\mathrm{Hom}(T_M, T)$-valued (0,1)-form vanishes on T, inducing thus a $T \otimes N^*$-valued (0,1)-form. As is well-known, this is nothing other than η in the notation of Theorem 1 in (1.2). At the same time, when $\overline{\partial}\rho_o$ acts on $\widetilde{\sigma}$, only $\widetilde{\sigma}(\lambda,\mu)$ mod T matters, which gives precisely σ. Thus

$$\overline{\partial}(\rho_*\widetilde{\sigma}) = \sum_{i,k,p,q,\alpha} \eta^i_{\alpha\overline{k}}\sigma^\alpha_{pq}\frac{\partial}{\partial z_i}\otimes dz^p\otimes dz^q\otimes dz^{\overline{k}}; \tag{7}$$

where i,k,p,q range from 1 to n; and α ranges from 1 to $n-m$. By the definition of ψ the right-hand side of Eqn.(7) gives precisely $\psi_*(\eta)$, which is written out more explicitly in Eqn.(2). If we take the trace-free part of both sides, we get

$$\overline{\partial}\tau = \varphi_*(\eta), \tag{8}$$

where $\varphi_*(\eta)$ is written out more explicitly in Eqn.(3). This proves Theorem 3 assuming that X admits a holomorphic projective connection, i.e., $p(X)=0$.

In the general case we cover X by an atlas $\mathcal{U}=\{U_\alpha\}$ of Stein open sets U_α where each U_α is equipped with a holomorphic projective connection whose associated tautological foliation will be denoted by $\mathcal{F}^\alpha$. Over each U_α we have no difficulty lifting $\sigma\in\Gamma(U_\alpha,S^2T^*\otimes N)$ to $\widetilde{\sigma}^{(\alpha)}\in\Gamma(U_\alpha,S^2T^*\otimes T_M)$. From this we obtain smooth E-valued (0,1)-form $\tau^{(\alpha)}$ on U_α such that $\overline{\partial}\tau_\alpha=\varphi_*(\eta)$ on U_α. Define now $\lambda_{\alpha\beta}:=\tau^{(\alpha)}-\tau^{(\beta)}$. Then, $\overline{\partial}\lambda_{\alpha\beta}=0$ on the overlaps $U_{\alpha\beta}=U_\alpha\cap U_\beta$, so that $(\lambda_{\alpha\beta})$ defines a Čech 1-cocycle with values in E. But this 1-cocycle is obtained because we use possibly distinct background holomorphic projective connections on $U_{\alpha\beta}$, and $\lambda_{\alpha\beta}$ is nothing other than the difference $\delta(\mathcal{F}^\beta,\mathcal{F}^\alpha)$ of the tautological foliations. In other words, $(\lambda_{\alpha\beta})$ represents precisely $p(X)\in H^1(X,E)$. But by construction $(\lambda_{\alpha\beta})$ represents the same cohomology class as that of $\varphi_*(\eta)$, under the canonical isomorphism between Čech and Dolbeault-Grothendieck cohomologies. Thus, $\varphi_*\big(\nu(f)\big)=p(X)$, desired. $\square$.

3.2 Holomorphic splitting of the tangent sequence via the second fundamental form – the case of surfaces

Using Theorem 2 we proceed to examine the criterion for the holomorphic splitting of the tangent sequence in terms of holomorphic projective connections and in terms of properties of the second fundamental form. To start with we consider the case where the domain manifold is of dimension 2 and the target manifold is of dimension 4, as follows.

Theorem 3. *In the notations of* Theorem 2 *suppose* $\dim(X)=2$, $\dim(M)=4$, *and X admits a holomorphic projective connection, i.e., $p(X)=0$. Denote by $\pi: L\to\mathbb{P}T_X$ the tautological line bundle over the projectivized tangent bundle*

$\mathbb{P}T_X$, and by $s \in \Gamma\big(\mathbb{P}T_X, L^{-2} \otimes \pi^* N\big)$ the holomorphic section corresponding to $\sigma \in \Gamma\big(X, S^2T_X^* \otimes N\big)$. Suppose s is nowhere zero on $\mathbb{P}T_X$. Then, the tangent sequence splits holomorphically, i.e., $\nu(f) = 0$. When X is compact this can never happen.

For the proof of Theorem 3 we need first of all to examine the pointwise condition under which the induced bundle homomorphism φ as in the statement of Theorem 2 will be an ismorphism, in the special case of domain dimension 2 and target dimension 4. It is a completely algebraic statement, as follows.

Lemma 3. *Let V and W be 2-dimensional complex vector spaces, and write $S^2V^* \otimes V = P + Q$, where P consists of trace-free elements and Q consists of trace elements. Let $s \in S^2V^* \otimes W$, and $\psi_s : V \otimes W^* \to S^2V^* \otimes V$ be defined by $\psi_s(\xi \otimes \theta^*)(\lambda \otimes \mu) = \theta^*(s(\lambda,\mu))\xi$ for any $\xi, \lambda, \mu \in V; \theta^* \in W^*$. Let $\varphi_s : V \otimes W^* \to P$, $\dim(V \otimes W^*) = \dim(P) = 4$, be defined by $\varphi_s = \pi_P \circ \psi$, where $\pi_P : S^2V^* \otimes V \to P$ is the canonical projection. Then φ_s is an isomorphism if and only if for every nonzero $\alpha \in V$, we have $s(\alpha,\alpha) \neq 0$.*

Proof. Regard also $s \in S^2V^* \otimes W$ as an element of $\mathrm{Hom}(S^2V, W)$. We have at the same time a homomorphism $t \in \mathrm{Hom}(W^*, S^2V^*)$ defined by $t(\theta^*)(\lambda,\mu) = \theta^*(s(\lambda,\mu))$ for any $\theta^* \in W^*\lambda, \mu \in V$, which is the transpose of s. Thus, t is injective if and only if s is surjective. The homomorphism $\psi_s : V \otimes W^* \to S^2V^* \otimes V$ is nothing other than $t \otimes \mathrm{id}_V$. Representing elements of S^2V by 2-by-2 matrices any nonzero element is either of rank 1 or rank 2. Those of rank 1 correspond by projectivization to a smooth quadric $Z \subset \mathbb{P}(S^2V)$. If s fails to be surjective then $\mathrm{Ker}(s)$ is at least 2-dimensional, so that $\mathbb{P}\big(\mathrm{Ker}(s)\big) \cap Z \neq \emptyset$. Thus, when s is not surjective, on the one hand $s(\alpha,\alpha) = 0$ for some nonzero α, on the other hand ψ_s and hence φ_s fails to be injective. To prove Lemma 3 it suffices therefore to restrict to the case where s is surjective, in which case $\mathrm{Ker}(s)$ must be 1-dimensional. Write $\mathrm{Ker}(s) = \mathbb{C}\eta$. If $0 \neq \eta \in S^2V$ is of rank 2, then there exists linearly independent vectors $\alpha, \beta \in V$ such that $\eta = \alpha \otimes \beta + \beta \otimes \alpha$, so that $s(\alpha,\beta) = 0$. Thus, in terms of the basis α, β for V and any basis for W, $\big(s^1_{ij}\big)$ and $\big(s^2_{ij}\big)$ are simultaneously diagonal matrices such that $\big(s^1_{11}, s^1_{22}\big)$ and $\big(s^2_{11}, s^2_{22}\big)$ are linearly independent. (From this we also see that t and hence ψ_s is injective.) To show that φ_s is an isomorphism it remains to verify that $\big(\mathrm{Im}(t) \otimes V\big) \cap Q = 0$. Any element A in $\mathrm{Im}(t) \otimes V$ is of the form $\big(A^k_{ij}\big)$ with $A^k_{ij} = a^k_1 s^1_{ij} + a^k_2 s^2_{ij}$ for some constants $a^k_\ell; k, \ell = 1, 2$; which implies $A^1_{12} = A^2_{12} = 0$. From $A \in Q$ we have $A^1_{22} = A^2_{11} = 0$. When combined with $A \in \mathrm{Im}(t) \otimes V$ we have $A^1_{11} = 2A^2_{12} = 0, A^2_{22} = 2A^1_{21} = 0$, forcing $A = 0$. Thus, if $s(\alpha,\alpha) \neq 0$ for any nonzero α, we have proven that φ_s

is an isomorphism. If on the other hand $s(\alpha,\alpha) = 0$ for some nonzero α then choosing β independent of α and using α, β as an ordered basis of V we have $s^1_{11} = s^2_{11} = 0$. Since t is injective choosing a suitable basis of W we may assume $s^1_{12} = s^1_{21} = 1, s^1_{22} = 0; s^1_{12} = s^1_{21} = 0, s^1_{22} = 1$. Writing $\alpha = e_1, \beta = e_2$; and writing e^1, e^2 for a dual basis of e_1, e_2, we have $\mathrm{Im}(t) = \mathrm{Span}\big(e^1 \otimes e^2 + e^2 \otimes e^1, e^2 \otimes e^2\big)$. It follows that $\mathrm{Im}\psi_s \supset \mathrm{Span}\big(e^1 \otimes e^2 \otimes e_1 + e^2 \otimes e^1 \otimes e_1, e^2 \otimes e^2 \otimes e_2\big)$. In particular, $B = e^1 \otimes e^2 \otimes e_1 + e^2 \otimes e^1 \otimes e_1 + 2e^2 \otimes e^2 \otimes e_2$ lies on $\mathrm{Im}(\psi_s)$, where $B \in Q$. Thus, $\mathrm{Ker}(\varphi_s) \neq 0$. The proof of Lemma 3 is complete. $\square$

By means of Lemma 3 we have immediately

Proof of Theorem 3. In the notations of the statement of Theorem 3, and applying Lemma 3 to each point $x \in X$ with $V = T_x, W = N_x, s = \sigma(x)$, we obtain a holomorphic bundle isomorphism $\varphi : T \otimes N^* \cong E$. Thus, the induced map on first cohomology groups $\varphi_* : H^1(X, T \otimes N^*) \to H^1(X, E)$ is an isomorphism. If X admits a holomorphic projective connection, we have $p(X) = 0$. By Theorem 2 we have $\varphi_*(\nu(f)) = p(X)$, hence $\nu(f) = 0$. In other words, the tangent sequence $0 \to T \to T_M \to N \to 0$ splits holomorphically. In the case where X is compact, by Theorem 1 the tangent sequence splits holomorphically if and only if $f : X \to M$ is totally geodesic. In this case $\sigma \equiv 0$, which means that the hypothesis of Theorem 3 can never be satisfied in the case where X is compact. $\square$

Remarks

In the case of holomorphic immersions of compact complex surfaces X into 4-dimensional Kähler space forms of constant holomorphic sectional curvature satisfying some nondegeneracy assumptions (cf. §4, Questions 1 and 2), Theorem 3 says that outside of some nonempty algebraic curve determined by the holomorphic immersion, there exists a holomorphic splitting of the tangent sequence determined by the second fundamental form and the given holomorphic projective connection.

3.3 Holomorphic splitting of the tangent sequence via the second fundamental form – the higher-dimensional case

We examine now a generalization of Theorem 3 to the case of holomorphic immersions of n-dimensional complex manifolds into Kähler space forms of constant holomorphic sectional curvature. On an n-dimensional complex manifold X a symmetric 2-vector, i.e., an element of S^2T, where T denotes T_X, can be represented as a symmetric n-by-n matrix in terms of local coordinates. The determinant of the matrix representative gives a well-defined bundle homomorphism

$\delta : S^n(S^2T) \to K^{-2}$, where $K = K_X$ is the canonical line bundle. We will write $\det(\eta)$ for $\delta(\eta, \cdots, \eta)$. A symmetric 2-vector η will be said to be generic if and only if $\det(\eta) \neq 0$.

When X is the complex projective space $\mathbb{P}^n$, we have the Veronese embedding $v : \mathbb{P}^n \to \mathbb{P}^{\frac{(n+1)(n+2)}{2}-1}$, e.g., $v : \mathbb{P}^2 \to \mathbb{P}^5$ and $v : \mathbb{P}^3 \to \mathbb{P}^9$, for which the second fundamental form $\sigma : S^2T \to N$ is a holomorphic bundle isomorphism. In general, suppose $f : X = \mathbb{P}^n \to \mathbb{P}^{\frac{(n+1)(n+2)}{2}-1}$ is a holomorphic immersion such that the second fundamental form $\sigma : S^2T \to N$ is a bundle isomorphism, by computing the determinant of σ it follows readily that f is of degree 2, i.e., $f^*\mathcal{O}(1) \cong \mathcal{O}(2)$. Since $\dim\big(\Gamma(\mathbb{P}^n, \mathcal{O}(2)\big) = \frac{(n+1)(n+2)}{2}$ it follows that f is congruent to the Veronese embedding. On the other hand it is impossible for an n-dimensional compact complex hyperbolic space form X to be holomorpically immersed into a compact complex hyperbolic space form M of dimension $\frac{(n+1)(n+2)}{2} - 1$ in such a way that the second fundamental form $\sigma : S^2T \to N$ is a bundle isomorphism, as can be seen from a direct computation on the curvature of $\det(S^2T)$ and $\det(N)$. On the totally geodesic case $\det(S^2T)$ is already more negative than $\det(N)$, and that remains the case in general because of the curvature decreasing property on Hermitian holomorphic vector subbundles resulting from the Gauss equation. As a generalization to Theorem 3 we will examine holomorphic immersions where the target dimension is one short of the target dimension in the Veronese embedding. One motivation for this, as is already in the case of Theorem 3, is to examine the obstruction to the existence of holomorphic immersions between complex hyperbolic space form when the domain manifold is compact (cf. §4, Questions 1 and 2). As an algebraic preliminary we have the following analogue of Lemma 3.

Lemma 3'. *Let V be an n-dimensional complex vector spaces, and write $S^2V^* \otimes V = P + Q$, where P consists of trace-free elements and Q consists of trace elements. Let W be a complex vector space of dimension $\frac{(n+2)(n-1)}{2}$. Let $s \in S^2V^* \otimes W$ and $\psi_s : V \otimes W^* \to S^2V^* \otimes V$ be defined by $\psi_s(\xi \otimes \theta^*)(\lambda \otimes \mu) = \theta^*(s(\lambda, \mu))\xi$ for any $\xi, \lambda, \mu \in V; \theta^* \in W^*$; and $\varphi_s : V \otimes W^* \to P$, $\dim(V \otimes W^*) = \dim(P) = \frac{n(n+2)(n-1)}{2}$ be defined by $\varphi_s = \pi_P \circ \psi$, where $\pi_P : S^2V^* \otimes V \to P$ is the canonical projection. Then, φ_s is an isomorphism if and only if for every nonzero generic 2-vector $\eta \in S^2V$, we have $s(\eta) \neq 0$.*

Proof. The proof is a generalization of the special case of $n = 2$, i.e., Lemma 3. s and t carry similar meaning and interpretation as in the proof there. Representing elements of S^2V by 2-by-2 matrices any nonzero element is of rank $1, 2, \cdots$, or n. Those of rank less than n correspond by projectivization to a *hypersurface* $Z \subset \mathbb{P}(S^2V)$ defined by the vanishing of the determinant. If s fails to be surjective then $\mathrm{Ker}(s)$ is at least 2-dimensional, so that $\mathbb{P}\big(\mathrm{Ker}(s)\big) \cap Z \neq \emptyset$. Thus, as in the proof of Lemma 3 it suffices to restrict to the case where $s \in \mathrm{Hom}(S^2V, W^*)$ is surjective, equivalently where $t \in \mathrm{Hom}(W^*, S^2V^*)$ is injective.

If $0 \neq \eta \in S^2V$ is of rank n, then there exists a basis $e_1, \cdots, e_n$ of V; with dual basis $e^1, \cdots, e^n$; with respect to which $\eta = e_1 \otimes e_1 + \cdots + e_n \otimes e_n$. (Note that we are adopting a different choice of basis which is more convenient for n general.) Since t is injective and $\psi_s = t \otimes id_V$, $\mathrm{Im}(\psi_s)$ is of codimension n in $S^2V^* \otimes V$. To show that φ_s is an isomorphism it remains to verify that $\big(\mathrm{Im}(t) \otimes V\big) \cap Q = 0$. Choose any basis for W. Any element A in $\mathrm{Im}(t) \otimes V$ is of the form $\big(A^k_{ij}\big)$ with $A^k_{ij} = \sum_\ell a^k_\ell s^\ell_{ij}$ some constants a^k_ℓ; $1 \leq k \leq n$; $1 \leq \ell \leq \frac{(n+2)(n-1)}{2}$. Together with $s(e_1 \otimes e_1 + \cdots + e_n \otimes e_n) = 0$, this implies $A^k_{11} + \cdots + A^k_{nn} = 0$. From $A \in Q$ we have $A^k_{ij} = 0$ whenever $k \neq i, j$. It remains to verify that $A^i_{ii} = 0$ and $A^k_{ik} = 0$ for $1 \leq i, k \leq n$; $i \neq k$. It is enough to show $A^1_{11} = A^2_{11} = 0$. From $A \in \mathrm{Im}(t) \otimes V$ it follows that $A^1_{11} + \cdots + A^1_{nn} = 0$. From $A \in Q$ we have $A^1_{22} = \cdots = A^1_{nn} = 0$, which then forces $A^1_{11} = 0$. On the other hand from $A \in Q$ we have $A^1_{11} = 2A^2_{12}$, which forces also $A^2_{12} = 0$, as desired. Thus, if $\mathrm{Ker}(s)$ is generated by a 2-vector η of rank n we have proven that $\big(\mathrm{Im}(t) \otimes V\big) \cap Q = 0$, hence φ_s is an isomorphism.

If on the other hand $\mathrm{Ker}(s)$ is generated by some some nonzero η of rank $k < n$ we can choose a basis $e_1, \cdots, e_n$ of V; with dual basis $e^1, \cdots, e^n$; with respect to which $\eta = e_1 \otimes e_1 + \cdots + e_k \otimes e_k$. We have $\mathrm{Im}(t) \supset \mathrm{Span}\big(e^1 \otimes e^n + e^n \otimes e^1, \cdots, e^{n-1} \otimes e^n + e^n \otimes e^{n-1}, e^n \otimes e^n\big)$. It follows that $\mathrm{Im}\psi_s \supset \mathrm{Span}\big(e^1 \otimes e^n \otimes e_1 + e^n \otimes e^1 \otimes e_1, \cdots, e^{n-1} \otimes e^n \otimes e_{n-1} + e^n \otimes e^{n-1} \otimes e_{n-1}, e^n \otimes e^n \otimes e_n\big)$. In particular, $B = \big(e^1 \otimes e^n \otimes e_1 + e^n \otimes e^1 \otimes e_1\big) + \cdots + \big(e^{n-1} \otimes e^n \otimes e_{n-1} + e^n \otimes e^{n-1} \otimes e_{n-1}\big) + 2\big(e^n \otimes e^n \otimes e_n\big)$ lies on $\mathrm{Im}(\psi_s)$, where $B \in Q$. Thus, $\mathrm{Ker}(\varphi_s) \neq 0$. The proof of Lemma 3' is complete. $\square$

Using Lemma 3' in place of Lemma 3 by the same proof as that of Therorem 3 we have the following immediate generalization to the situataion of holomorphic immersions of n-dimensional complex manifolds.

Theorem 3'. *In the notations of* Theorem 2 *and* 3 *suppose* $\dim(X) = n$, $\dim(M) = \frac{(n+1)(n+2)}{2} - 2$, *and,* X *admits a holomorphic projective connection, i.e.,* $p(X) = 0$. *Suppose the second fundamental form* $\sigma : S^2T^* \to N$ *is a surjective bundle homomorphism, and the* 1*-dimensional kernel* $\mathrm{Ker}(\sigma)$ *is generated at each point by a generic symmetric* 2*-vector. Then, the tangent sequence splits holomorphically, i.e.,* $\nu(f) = 0$. *When* X *is compact this can never happen.*

§4 Some open problems

4.1 Holomorphic immersions on compact Kähler manifolds of constant holomorphic sectional curvature

One of our primary motivations is the study of holomorphic immersions between compact Kähler manifolds which are space forms of constant holomorphic sectional curvature. We note that up to a finite unramified covering a compact complex Euclidean space form is nothing other a compact complex torus. Because of the monotocity property on holomorphic sectional curvatures of complex submanifolds of Kähler manifolds, a number of cases are obviously ruled out by analogues of the Gauss-Bonnet Theorem. For instance, it is not possible to immerse a compact complex torus into a complex hyperbolic space form, and any holomorphic immersion from a compact complex torus to a complex Euclidean space form is necessarily totally-geodesic. On the more interesting side in the case of holomorphic immersions between complex projective spaces, Feder [F] proved that any holomorphic immersion from $\mathbb{P}^n$ into $\mathbb{P}^N$ is necessarily linear provided that $N \leq 2n-1$. He did this by using the vanishing of the top Chern class $c_n(\nu)$ for the normal bundle ν of the holomorphic immersion and by a computation of $c_n(\nu)$ using the topological splitting of the tangent sequence of the immersion. The same proof also shows that under the same dimension restriction an n-dimensional compact hyperbolic space form or compact complex torus cannot be holomorphically immersed into $\mathbb{P}^N$. The dual version of Feder's result for holomorphic immersions from an n-dimensional compact complex hyperbolic space form $X = B^n/\Gamma$ into a (not necessary compact) complex hyperbolic space form $Y = B^N/\Gamma'$ for $N \leq 2n-1$ was obtained by Cao-Mok [CM] using the Proportionality Principle, an adaptation of Feder's argument, and a study of holomorphic foliations associated to second fundamental forms. A simple computation of Chern classes also shows that under the same dimension restriction an n-dimensional compact hyperbolic space form cannot be holomorphically immersed into a compact complex torus. We can conclude that

Theorem 4. *Given $N \leq 2n-1$, an n-dimensional compact Kähler manifold X of constant holomorphic sectional curvature cannot be holomorphically immersed into a Kähler manifold M of constant holomorphic sectional curvature, except in the case of a totally-geodesic holomorphic immersion between Kähler space forms of the same type.*

4.2 Immersions of complex surfaces into 4-dimensional Kähler manifolds of constant holomorphic sectional curvature

The first interesting case not accessible by the methods described above is in the case when $n = 2$ and $N = 4$. In this case we have the following question

Question 1. *Let $\Sigma \subset \Omega$ be a germ of 2-dimensional complex submanifold of the 4-dimensional complex Euclidean space $\mathbb{C}^4$. Let T be the holomorphic tangent bundle T_Σ of Σ, N be the normal bundle on Σ of the germ of holomorphic embedding $\Sigma \subset \mathbb{C}^4$, and $\sigma \in \Gamma(\Sigma, S^2T^2 \otimes T)$ be the second fundamental form. Denote by $E \subset S^2T^* \otimes T$ the 4-dimensional holomorphic vector subbundle consisting of trace-free elements and $\varphi : T^* \otimes N \to E$ be the holomorphic bundle map defined in terms of the second fundamental form as in (2.3). Characterize those germs of holomorphic embedding $\Sigma \subset \mathbb{C}^4$ for which φ has a nontrivial kernel everywhere.*

Such germs of holomorphic embeddings $\Sigma \subset \mathbb{C}^4$ ought to be very special. Locally if we take Σ to be a complex surface in complex Euclidean space ruled by open subsets of complex affine lines which is not totally geodesic, then the Ricci form with respect to the restricted Euclidean metric may or may not have a nontrivial kernel (cf. Mok [M,§7]). In the case where the kernel is nontrivial, then the second fundamental form σ has 1-dimensional image at a general point. Otherwise, σ has two-dimensional images at a general point. In the latter case $\sigma(\alpha, \alpha) = 0$ for every (1,0) vector α tangent to leaves of the foliation (ruling) by open subsets of affine lines. If we take the ambient Euclidean space to be 4-dimensional this gives local examples of holomorphic embeddings $\Sigma \subset \mathbb{C}^4$ such that at every point the bundle homomorphism $\varphi : T \otimes N^* \to E$ fail to be an isomorphism.

When in place of a germ of holomorphic embedding we consider a global holomorphic immersion $f : X \to M$ from a compact 2-dimensional complex manifold X into a 4-dimensional complex manifold M which is a complex hyperbolic space form, a complex Euclidean space form or the projective space, further restrictions for global reasons should exclude most of those characterized by an answer to Question 1. Thus we have

Question 2. *In the global situation let $f : X \to M$ be a holomorphic immersion from a compact 2-dimensional complex manifold X into a 4-dimensional complex hyperbolic space form, complex Euclidean space form or complex projective space M. Is the induced holomorphic bundle map $\varphi : T \otimes N^* \to E$ necessarily a bundle isomorphism outide a complex-analytic curve on X, except in the case of a totally-geodesic holomorphic immersion between Kähler space forms of the same type?*

When M is of dimension 3 instead, by Theorem 4, the holomorphic immersion f is necessarily a totally-geodesic immersion between Kähler space forms of the same type. On the other hand, in the context of Question 2, and in the case when both X and M are complex projective spaces, there are certainly many holomorphic immersions and even holomorphic embeddings. They all arise in the following manner. Let W be a complex vector space and $X \subset \mathbb{P}(W)$ be a projective submanifold. Let $Z \subset W$ be a complex vector subspace such that $\mathbb{P}(Z) \cap X = \emptyset$. Consider the linear projection $p : \mathbb{P}(W) - \{[\eta]\} \to \mathbb{P}(W/\mathbb{C}\eta)$. Then, $f := p|_X : X \to \mathbb{P}(W/Z)$ is holomorphic. It is a holomorphic immersion whenever $\mathbb{P}(Z)$ is disjoint from the tangent variety $\mathrm{Tan}(X) \subset \mathbb{P}(W)$, which is the union of lines tangent to X at some point. Furthermore, f is a holomorphic embedding whenever $\mathbb{P}(Z)$ is disjoint from the secant variety $\mathrm{Sec}(X)$, i.e., the closure of the union of secant lines of X.

Let now $k \geq 2$ and $v_k : \mathbb{P}^2 \to \mathbb{P}(\Gamma(\mathbb{P}^2, \mathcal{O}(k))^*) \cong \mathbb{P}^{N(k)}$ be the k-th Veronese embedding, where $N(k) = \frac{3\cdot 4\cdots(k+2)}{k!} - 1$; $N(2) = 5, N(3) = 9$, etc. Let now $V \subset \Gamma(\mathbb{P}^2, \mathcal{O}(k))^* := W_k$, $\dim(W_k) = N(k) + 1$ be a linear subspace of dimension $N(k) - 4 \geq 1$ such that $\mathbb{P}(V) \cap v_k(\mathbb{P}^2) = \emptyset$. Composing v_k on the left with the linear projection $p : \mathbb{P}(W_k) - \mathbb{P}(V) \to \mathbb{P}(W_k/V) \cong \mathbb{P}^4$ we obtain a holomorphic mapping $f_k : X \to \mathbb{P}^4$. Let $S_k \subset \mathbb{P}(W_k)$ be the tangent variety of $v_k(\mathbb{P}^2)$, which is 4-dimensional. Since $\dim(\mathbb{P}(V)) = N(k) - 5$, for a generic choice of $V \subset W_k$, $S_k \cap \mathbb{P}(V) = \emptyset$. For such a choice of V the holomorphic mapping $f_k : \mathbb{P}^2 \to \mathbb{P}^4$ is a holomorphic immersion. A positive answer to Question 2 would say that φ is always a bundle isomorphism outside of an algebraic curve on $\mathbb{P}^2$. In place of arguing by explicit computation there ought to be geometric arguments which should allow us to answer Question 2 irrespective of the types of X and M.

4.3 The non-splitting locus

Let X be a compact 2-dimensional Kähler space form of constant holomorphic sectional curvature, and $f : X \to M$ be a holomorphic immersion into a 4-dimensional Kähler space form M of constant holomorphic sectional curvature. Suppose the induced holomorphic bundle map $\varphi : X \to N$ is a bundle isomorphism at a general point. Let $\Sigma_f \subset X$ be the complex-analytic curve over which φ fails to be a bundle isomorphism. We call $\Sigma_f \subset X$ the non-splitting locus of the holomorphic immersion $f : X \to M$. Obviously $\Sigma_f \subset X$ can be endowed the structure of a possibly reduced complex space as the zero divisor of the determinant of φ. We have a holomorphic splitting of the tangent sequence over $X - \Sigma_f$, and the holomorphic splitting can be determined in terms of the second fundamental form of the holomorphic immersion and the canonical holomorphic projective structure on X. It is tempting to believe that the answer to (4.2), Question 2 is positive, in which case one can always associate to the holomorphic immersion $f : X \to M$ its non-splitting locus $S \subset X$. In the case where X and M are complex projective spaces the non-splitting locus can be explicitly determined from the description of all holomorphic immersions $f : \mathbb{P}^2 \to \mathbb{P}^4$ as given in (4.2). We give here the description of the simplest case.

<u>The non-splitting locus arising from the Veronese embedding v_2</u>

Let V be an $(n+1)$-dimensional complex vector space. The Veronese embedding $v : \mathbb{P}(V) \to \mathbb{P}(S^2V)$, $v = v_2$, is defined by $v([\alpha]) = [\alpha \otimes \alpha]$. For $\eta \in S^2V$ we define its rank to be the rank of the symmetric matrix representing it (with respect to a choice of basis). In this very special case the tangent variety $\mathrm{Tan}(v(\mathbb{P}(V))$ agrees with the secant variety $\mathrm{Sec}\big(v(\mathbb{P}(V)\big)$, consisting precisely of projectivizations of symmetric 2-tensors of rank 2. Let $\eta \in S^2V$ be a nonzero symmetric 2-tensor of rank ≥ 3, then the projection $p : \mathbb{P}(S^2V) - \big\{[\eta]\big\} \to \mathbb{P}(S^2V/\mathbb{C}\eta)$ induces a holomorphic embedding $f_{[\eta]} := p \circ v : \mathbb{P}(V) \to \mathbb{P}(S^2V/\mathbb{C}\eta)$, where the target projective space is of complex dimension $\frac{(n+1)(n+2)}{2} - 2$, exactly the dimension in which Theorem 3' applies. Obviously the notion of the non-splitting locus generalizes to this context.

$\mathrm{GL}(V)$ acts naturally on $S^2(V)$. Under this action it acts transitively on the set of all elements (represented by symmetric matrices) of the same rank. Thus, up to congruence the holomorphic embedddings $f : \mathbb{P}(V) \to \mathbb{P}(S^2V/\mathbb{C}\eta)$ obtained from the Veronese embedding is determined by the rank of η.

We fix now η of rank k, $3 \leq k \leq n$, and let $f := f_{[\eta]}$. The seond fundamental form σ_f of f is obtained from the second fundemantal form σ_v of the Veronese embedding by projection. Note that σ_v is an isomorphism at every point. Write $X = f(\mathbb{P}(V)) \subset \mathbb{P}(S^2V/\mathbb{C}\eta)$. Let $x \in X, x = f([\alpha]) = p(v([\alpha]) = p([\alpha \otimes \alpha])$, and let $W_\alpha \subset V$ be a vector subspace complementary to $\mathbb{C}\alpha$. By Lemma 3 and Lemma 3', the holomorphic bundle homomorphism $\varphi : T \otimes N^* \to E$ over X associated to the holomorphic embedding f fails to be an isomorphism if and only if there exists some $\lambda \in \mathbb{C}$, $\beta \in V$, $\xi \in S^2W_\alpha$ of rank $< n$, $\xi \neq 0$, such that $\lambda\eta + \alpha\beta = \xi$, where we write $\alpha\beta$ for $\frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$. Since the equation is never satisfied for $\lambda = 0$, we may assume $\lambda = 1$.

In what follows we specialize to the case where $n = 2$. In this case η is always of rank 3 and all holomorphic embeddings $f_{[\eta]}$ are congruent. The equation which determines the non-splitting locus of f amounts to finding all nonzero $\alpha \in V$ such that ($\flat$) $\eta + \alpha\beta = \gamma^2$ for some $\gamma \in W_\alpha$. Fix a basis e^1, e^2, e^3 of $V \cong \mathbb{C}^3$. Without loss of generality we may take $\eta = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 = e_1^2 + e_2^2 + e_3^2$. Let $\alpha = ae_1 + be_2 + ce_3$. Suppose $a \neq 0$, in which case we may assume $a = 1$. Choose $W_\alpha = \mathrm{Span}(e_2, e_3)$. Write $e_1 = \alpha - be_2 - ce_3$. Then, $\eta = (\alpha - be_2 - ce_3)^2 + e_2^2 + e_3^2 =$

$\big(\alpha^2 - 2b\alpha e_2 - 2c\alpha e_3\big) + \big((1+b^2)e_2^2 + 2bce_2e_3 + (1+c^2)e_3^2\big)$. Thus ($\flat$) is satisfied for $\alpha = e_1+be_2+ce_3$ if and only if the symmetric 2-tensor $(1+b^2)e_2^2+2bce_2e_3+(1+c^2)e_3^2$ is of rank 1. This is the case if and only if $(1+b^2)(1+c^2) = (bc)^2$, i.e., $1+b^2+c^2 = 0$. Homogenizing we see that for $\alpha = ae_1 + be_2 + ce_3$, $x = f([\alpha]) = p(v([\alpha]))$ lies on the non-splitting if and only if $a^2 + b^2 + c^2 = 0$. Thus, the non-splitting locus Σ_f of the holomorphic embedding $f : \mathbb{P}^2 \to \mathbb{P}^4$ is precisely a smooth quadric curve.

In invariant terms the non-splitting locus Σ_f can be described as follows. Let V be a 3-dimensional complex vector space and $\eta \in S^2(V)$ be a symmetric two-tensor of rank 3. It determines a homogeneous quadratic polynomial Q_η on the dual vector space V^*. The zero locus of Q_η then corresponds to a smooth quadric curve Γ_f on $\mathbb{P}(V^*)$. $\Sigma_f \subset \mathbb{P}(V)$ is then the dual variety of Γ_f when we identify $\mathbb{P}(V)$ canonically as the dual projective space of $\mathbb{P}(V^*) \cong \mathbb{P}^2$. It is straightforward to check that this description for Σ_f is valid in the case of $\eta = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$ described above. Since any symmetric 2-tensor of rank 3 can expressed in this form with respect to some choice of coordinates on $V \cong \mathcal{C}^3$, the description of Σ_f in projective-geometric terms as given holds true in general.

4.4 Rank-2 vector bundles associated to some holomorphic immersions

In relation to holomorphic immersions between Kähler manifolds of constant holomorphic sectional curvature where the domain manifold is compact and of dimension 2, the most remarkable example is the discovery by Mumford of an embedding of some Abelian surface X into $\mathbb{P}^4$ as a surface of degree 10. Since the determinant bundle of the normal bundle extends to $\mathbb{P}^4$, this leads to the existence of a rank-2 holomorphic vectort bundle on $\mathbb{P}^4$, which is indecomposable since X is not a globally a complete intersection. (See Okonek-Schneider-Spindler [OSS] for the background on vector bundles on $\mathbb{P}^n$.) This rank-2 indecomposable bundle was constructed explicitly in Horrocks-Mumford [HM,1973], where it is shown that the zero locus of a general section of F is nonsingular and isomorphic to an Abelian surface. Furthermore, it was proven that up to congruence this accounts for all holomorphic embeddings of Abelian surfaces into $\mathbb{P}^4$. Concerning the Horrocks-Mumford bundle we have the following natural question in relation to the study of holomorphic immersions in this article.

Question 3. *Let $f : X \to \mathbb{P}^4$ be a holomorphic embedding of an Abelian surface into the 4-dimensional complex projective space. Is the associated bundle homomorphism $\varphi : T \otimes N^* \to E$ an isomorphism at a general point? If this is the case, describe the non-splitting locus Σ_f of f in terms of the geometry of the Horrocks-Mumford bundle.*

There has so far not been any nontrivial example of holomorphic immersions of 2-dimensional compact complex hyperbolic space forms into a 4-dimensional complex hyperbolic space form. In this regard we have the following question in part motivated by the case of holomorphic embeddings of Abelian surfaces in $\mathbb{P}^4$.

Question 4. *Let $f : X \to M$ be a (hypothetical) holomorphic immersion of a compact 2-dimensional complex hyperbolic space form X into a 4-dimensional complex hyperbolic space form M which is not totally geodesic. Is the normal bundle of X in M necessarily indecomposable?*

It is tempting to believe that the answer to Question 4 is positive. An affirmative answer will introduce a new element into the study of obstructions to the existence of such holomorphic immersions, viz., indecomposable rank-2 holomorphic vector bundles on 2-dimensional compact complex hyperbolic space forms.

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