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Twin positive solutions for quasi-linear multi-point boundary value problems

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Abstract

In this paper, we investigate two classes of quasi-linear multi-point boundary value problems with sign-changing nonlinearity. By applications of fixed point index theory, sufficient conditions for the existence of twin positive solutions are established.

Keywords: Multi-point boundary value problems; p-Laplacian; Fixed point index theory

1. Introduction

In recent years, the existence of positive solutions for multi-point (including three-point) boundary value problems (BVP) has attracted much attention, see [4,7–9] and the references cited therein. Most results so far have been obtained mainly by using the fixed-point theorem in cones, such as Kransnosel'skii fixed-point theorem [5], Leggett–Williams' theorem [6], Avery and Henderson's theorem [1], and so on. In order to apply the concavity of solutions in the proofs, all the existing works were done under the assumption that the nonlinear term

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is nonnegative. For example, in [7], Liu studied the three-point boundary value problem

$$y''(t) + a(t)f(y(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$y'(0) = 0, \quad y(1) = \beta y(\eta), \tag{1.2}$$

where $0 < \eta < 1$, $0 < \beta < 1$. By applying Kransnosel'skii fixed-point theorem, Liu obtained some existence results for positive solutions of the BVP (1.1)–(1.2) under the condition that both f and a are in $C([0, \infty), [0, \infty))$.

Motivated by this, we consider a quasi-linear differential equation with sign-changing nonlinearity

$$(\Phi_p(u'))' + h(t)f(t, u) = 0, \quad 0 < t < 1, \tag{1.3}$$

where $h : [0, 1] \to \mathbb{R}^+$ and $f : [0, 1] \times [0, \infty) \to \mathbb{R}$ are continuous, and

$$\Phi_p(u) := |u|^{p-2}u, \quad p > 1,$$

is a one-dimensional p-Laplacian. Note that the nonlinear term $f(t, u)$ here is allowed to change sign. Observe also that if q is the conjugate exponent of p, i.e., if $1/q + 1/p = 1$, then $\Phi_q = (\Phi_p)^{-1}$. We investigate Eq. (1.3) subject to one of the following multi-point boundary conditions:

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \tag{1.4}$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) = 0. \tag{1.5}$$

Here $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\alpha_i \geqslant 0$ with $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $m \geqslant 3$. It is obvious that BVP (1.3)–(1.4) can be regarded as a generalization of BVP (1.1)–(1.2). We shall apply fixed point index theory to obtain sufficient conditions for the existence of twin positive solutions for the prescribed problems (1.3)–(1.4) and (1.3)–(1.5) by constructing available operators. In doing so the usual restriction $f \geqslant 0$ is removed.

2. Main results

Before giving the main theorems, we need a couple of preliminary results. The first one is a well-known result of the fixed point index theory which our main results are based on.

Lemma 2.1 (*Deimling [2], Guo and Lakshmikantham [3] and Krasnoselskii [5]*). *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|x\| = r\}$.*

(i) *If $\|Tx\| \geqslant \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

(ii) *If* $\|Tx\| \leqslant \|x\|$ *for* $x \in \partial K_r$, *then*

$$i(T, K_r, K) = 1.$$

Lemma 2.2. *Let* $X = C[0,1]$ *with the sup norm* $\|u\| := \sup_{0\leqslant t\leqslant 1}|u(t)|$, *and* $K = \{x \in X : x(t) \geqslant 0\}$. *Suppose* $T : X \to X$ *is completely continuous. Define* $\Theta : TX \to K$ *by* $(\Theta y)(t) = \max\{y(t), 0\}$ *for* $y \in TX$, $t \in [0,1]$. *Then* $\Theta \circ T : X \to K$ *is also a completely continuous operator.*

Proof. The complete continuity of T implies that T is continuous and relatively compact. Therefore, given any bounded $D \subset X$ and any $\varepsilon > 0$, there are y_i, $i = 1, \ldots, m$, such that

$$TD \subset \bigcup_{i=1}^{m} B(y_i, \varepsilon),$$

where $B(y_i, \varepsilon) = \{x \in X : \|x - y_i\| < \varepsilon\}$. Hence for any $\overline{y} \in (\Theta \circ T)(D)$, there is $y \in TD$ such that $\overline{y}(t) = \max\{y(t), 0\}$. We choose an $i \in \{1, \ldots, m\}$ such that $\max_{0\leqslant t\leqslant 1}|y(t) - y_i(t)| < \varepsilon$. The fact

$$\max_{0\leqslant t\leqslant 1}|\overline{y}(t) - \overline{y}_i(t)| \leqslant \max_{0\leqslant t\leqslant 1}|y(t) - y_i(t)| < \varepsilon$$

implies that $\overline{y} \in B(\overline{y}_i, \varepsilon)$. So $(\Theta \circ T)(D)$ is relatively compact.

On the other hand, for arbitrary $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|Ty - Tx\| < \varepsilon \quad \text{whenever} \quad \|y - x\| < \delta$$

and hence

$$\begin{aligned}
\|(\Theta \circ T)y - (\Theta \circ T)x\| &= \max_{0\leqslant t\leqslant 1}\|(\overline{Ty})(t) - (\overline{Tx})(t)\| \\
&\leqslant \max_{0\leqslant t\leqslant 1}\|(Ty)(t) - (Tx)(t)\| \\
&= \|Ty - Tx\| \\
&< \varepsilon
\end{aligned}$$

whenever $\|y - x\| < \delta$. Therefore, $\Theta \circ T$ is continuous on X and hence $\Theta \circ T$ is completely continuous. □

Below we discuss BVP (1.3)–(1.4) and BVP (1.3)–(1.5), separately.

2.1. Positive solution to BVP (1.3)–(1.4)

Lemma 2.3. *Suppose* $\sum_{i=1}^{m-2}\alpha_i \neq 1$ *and* $y(\cdot) \in C[0,1]$. *Then the problem*

$$(\Phi_p(u'))' + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i) \tag{2.2}$$

has a unique solution

$$u(t) = \int_t^1 \Phi_q \left(\int_0^s y(r)\,\mathrm{d}r \right) \mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \Phi_q(\int_0^s y(r)\,\mathrm{d}r)\,\mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}. \tag{2.3}$$

Proof. Integrate (2.1) from 0 to t, we get

$$\Phi_p(u'(t)) = -\int_0^t y(r)\,\mathrm{d}r,$$

i.e.,

$$u'(t) = -\Phi_q \left(\int_0^t y(r)\,\mathrm{d}r \right).$$

Integrating this from t to 1 yields

$$u(t) = u(1) + \int_t^1 \Phi_q \left(\int_0^s y(r)\,\mathrm{d}r \right) \mathrm{d}s.$$

From the boundary condition (2.2), we have

$$u(1) = \frac{\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \Phi_q(\int_0^s y(r)\,\mathrm{d}r)\,\mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}.$$

This proves Lemma 2.3. □

Now let $X = C[0,1]$, $K = \{u \in X : u(t) \geqslant 0\ \forall t \in [0,1]\}$, $K' = \{u \in X : u$ is nonnegative, concave, and nonincreasing$\}$. Equip X with the sup norm $\|u\| := \sup_{0 \leqslant t \leqslant 1} |u(t)|$. Clearly, $K, K' \subseteq X$ are cones with $K' \subseteq K$. Define operators A, T and T' as follows: for any $u \in K$, define

$$(Au)(t) := \int_t^1 \Phi_q \left(\int_0^s h(r) f(r, u(r))\,\mathrm{d}r \right) \mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \Phi_q(\int_0^s h(r) f(r, u(r))\,\mathrm{d}r)\,\mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}, \quad t \in [0,1].$$

By Lemma 2.3, for any $u \in K$, Au satisfies BVP (1.3)–(1.4). Furthermore, BVP (1.3)–(1.4) is equivalent to the fixed point problem of the operator A. Next, for any $u \in K$, define

$$(Tu)(t) := \left[\int_t^1 \Phi_q \left(\int_0^s h(r) f(r, u(r))\,\mathrm{d}r \right) \mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \Phi_q(\int_0^s h(r) f(r, u(r))\,\mathrm{d}r)\,\mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i} \right]^+, \quad t \in [0,1],$$

where $g^+ := \max\{g, 0\}$. Obviously $T = \Theta \circ A$, where Θ is as defined in Lemma 2.2. Finally, for any $u \in K'$, define

$$(T'u)(t) := \int_t^1 \Phi_q\left(\int_0^s h(r) f^+(r, u(r))\,\mathrm{d}r\right)\mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \Phi_q(\int_0^s h(r) f^+(r, u(r))\,\mathrm{d}r)\,\mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}, \quad t \in [0, 1].$$

Lemma 2.4. *If $u \in K'$ satisfies BVP* (1.3)–(1.4), *then*

$$\min_{0 \leqslant t \leqslant 1} u(t) \geqslant \frac{\sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i} \|u\|.$$

Proof. For $u \in K'$, it is easy to see that $u(0) = \|u\|$, $u(1) = \min_{0 \leqslant t \leqslant 1} u(t)$. By the concavity of $u(t)$, we have

$$\frac{u(1) - u(\xi_i)}{1 - \xi_i} \leqslant \frac{u(1) - u(0)}{1}, \quad i = 1, \ldots, m - 2,$$

i.e.,

$$u(\xi_i) - \xi_i u(1) \geqslant (1 - \xi_i) u(0), \quad i = 1, \ldots, m - 2.$$

Then

$$\sum_{i=1}^{m-2} \alpha_i u(\xi_i) - \sum_{i=1}^{m-2} \alpha_i \xi_i u(1) \geqslant \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i) u(0).$$

By (1.4), $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i)$. Hence

$$u(1) \geqslant \frac{\sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i} u(0).$$

This proves Lemma 2.4. □

Next, define

$$\delta = \frac{\sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} \alpha_i \xi_i}$$

and

$$M = \int_0^1 \Phi_q\left(\int_0^s h(r)\,\mathrm{d}r\right)\mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^1 \Phi_q(\int_0^s h(r)\,\mathrm{d}r)\,\mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}.$$

We have

Theorem 2.1. *Suppose* $f(t,0)\geqslant 0$ *for* $t\in[0,1]$ *and* $h(t)f(t,0)\not\equiv 0$ *on any subinterval of* $[0,1]$. *If there exist nonnegative numbers a, b and d such that* $0<(1/\delta)d<a<\delta b<b$ *and f satisfies the following conditions*:

(H1) $f(t,u)\geqslant 0$ *for* $(t,u)\in[0,1]\times[d,b]$;
(H2) $f(t,u)<\Phi_p(a/M)$ *for* $(t,u)\in[0,1]\times[0,a]$;
(H3) $f(t,u)>\Phi_p(b/M)$ *for* $(t,u)\in[0,1]\times[\delta b,b]$,

then BVP (1.3)–(1.4) *has two positive solutions* u_1 *and* u_2 *with*

$$0<\|u_1\|<a<\|u_2\|<b.$$

Proof. First of all, from the definitions of T and T', it is clear that $T(K)\subset K$ and $T'(K')\subset K'$. Moreover, by the continuity of f, it is easy to see that $A:K\to X$ and $T':K'\to K'$ are completely continuous. So by Lemma 2.2, $\Theta\circ A:K\to K$ and $T:K\to K$ are completely continuous.

Now we show that T has a fixed point $u_1\in K$ with $0<\|u_1\|<a$. For simplicity, for any $r>0$, we write $K_r:=\{u\in K:\|u\|<r\}$. Observe that $\partial K_r=\{u\in K:\|u\|=r\}$. Hence for any $u\in\partial K_a$, we have $\|u\|=a$ and $0\leqslant u(t)\leqslant a$ for all $t\in[0,1]$. So it follows from (H2) that

$$\begin{aligned}
\|Tu\| &= \max_{0\leqslant t\leqslant 1}\Bigg[\int_t^1\Phi_q\left(\int_0^s h(r)f(r,u(r))\,\mathrm{d}r\right)\mathrm{d}s\\
&\qquad+\frac{\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^1\Phi_q(\int_0^s h(r)f(r,u(r))\,\mathrm{d}r)\,\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\Bigg]^+\\
&=\max_{0\leqslant t\leqslant 1}\max\Bigg\{\int_t^1\Phi_q\left(\int_0^s h(r)f(r,u(r))\,\mathrm{d}r\right)\mathrm{d}s\\
&\qquad+\frac{\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^1\Phi_q(\int_0^s h(r)f(r,u(r))\,\mathrm{d}r)\,\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i},0\Bigg\}\\
&<\frac{a}{M}\max_{0\leqslant t\leqslant 1}\left(\int_t^1\Phi_q\left(\int_0^s h(r)\,\mathrm{d}r\right)\mathrm{d}s+\frac{\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^1\Phi_q\left(\int_0^s h(r)\,\mathrm{d}r\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\right)\\
&=\frac{a}{M}\left(\int_0^1\Phi_q\left(\int_0^s h(r)\,\mathrm{d}r\right)\mathrm{d}s+\frac{\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^1\Phi_q\left(\int_0^s h(r)\,\mathrm{d}r\right)\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\right)\\
&=a.
\end{aligned}$$

It follows from Lemma 2.1 that

$$i(T, K_a, K) = 1$$

and hence T has a fixed point u_1 in K_a. We claim that u_1 is also a fixed point of A in K_a. Suppose not, then there exists $t_0 \in [0, 1]$ such that

$$(Au_1)(t_0) \neq u_1(t_0) = (Tu_1)(t_0) = \max\{(Au_1)(t_0), 0\}$$

and so this forces

$$(Au_1)(t_0) < 0 = u_1(t_0).$$

Let $I = I(t_0)$ be the largest interval in $\mathbb{R}$ that is open in [0, 1] and contains t_0 such that $(Au_1)(t) < 0$ for all $t \in I$. Note that

$$u_1(t) = Tu_1(t) = \max\{(Au_1)(t), 0\} = 0 \quad \text{for all } t \in I.$$

Let $\bar{I}$= the closure of I in [0, 1]. Obviously $\bar{I} \neq [0, 1]$, for otherwise we would have

$$u_1(t) = 0 \quad \text{for all } t \in [0, 1],$$

which contradicts to the assumption that $h(t)f(t, 0) \not\equiv 0$ in any subinterval of [0, 1]. So we should have either $1 \notin \bar{I}$ or $0 \notin \bar{I}$.

Case 1: $1 \notin \bar{I}$. We have $u_1(t) = 0$ for all $t \in \bar{I}$. It is easy to check that

$$(\Phi_p[(Au_1)'](t))' = -h(t)f(t, 0) \leqslant 0 \quad \text{for all } t \in \bar{I}.$$

In particular, this implies that $(Au_1)'(t)$ is decreasing on $\bar{I}$. On the other hand, if $\beta < 1$ denotes the right-hand end-point of I, since $(Au_1)(t) < 0$ for all $t \in I$ and $(Au_1)(\beta) = 0$, we have $(Au_1)'(\beta) \geqslant 0$. Hence $(Au_1)(t) < 0$ and is bounded away from 0 everywhere in I. This forces $t_0 = 0$ and $(Au_1)'(t) > 0$ for all $t \in I = [0, \beta)$, which contradicts with the first condition of (1.4).

Case 2: $0 \notin \bar{I}$. Similar to Case 1, it can be shown that $1 \in I$ and $(Au_1)'(t) < 0$ for all $t \in I$. Hence $(Au_1)(t) < 0$ is strictly decreasing on I. For any $j \in \{1, \ldots, m-2\}$, if $(Au_1)(\xi_j) \geqslant 0$, then since $(Au_1)(1) < 0$, we have $(Au_1)(\xi_j) > (Au_1)(1)$. On the other hand, if $(Au_1)(\xi_j) < 0$, we must have $\xi_j \in I$. For if not, denote by $I(\xi_j)$ the largest interval in $\mathbb{R}$ that is open in [0, 1] and contains ξ_j such that $(Au_1)(t) < 0$ for all $t \in I(\xi_j)$. Since I must be disjoint from $I(\xi_j)$ and $1 \in I$, we have $1 \notin \bar{I}(\xi_j)$, which is impossible as shown in Case 1. So in this situation we also have $(Au_1)(\xi_j) > (Au_1)(1)$. Thus $(Au_1)(\xi_i) > (Au_1)(1)$ for all $i \in \{1, \ldots, m-2\}$ and so

$$(Au_1)(1) = \sum_{i=1}^{m-2} \alpha_i (Au_1)(\xi_i) \geqslant \sum_{i=1}^{m-2} \alpha_i (Au_1)(1) > (Au_1)(1),$$

which is also impossible. Combining, we conclude that u_1 is also a fixed point of A and, therefore, is a positive solution of BVP (1.3)–(1.4).

Next we show the existence of another fixed point of A. For $u \in \partial K'_a$, similar to above, we have from (H2) that

$$\begin{aligned}\|T'u\| &= (T'u)(0)\\ &= \int_0^1 \Phi_q\left(\int_0^s h(r) f^+(r, u(r))\,\mathrm{d}r\right)\mathrm{d}s\\ &\quad + \frac{\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^1 \Phi_q(\int_0^s h(r) f^+(r, u(r))\,\mathrm{d}r)\,\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\\ &< \frac{a}{M}\left[\int_0^1 \Phi_q\left(\int_0^s h(r)\,\mathrm{d}r\right)\mathrm{d}s + \frac{\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^1 \Phi_q(\int_0^s h(r)\,\mathrm{d}r)\,\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\right]\\ &= a.\end{aligned}$$

For $u \in \partial K'_b$, we have $\|u(t)\| = b$. For $0 \leqslant t \leqslant 1$, in view of Lemma 2.4, we have $\delta b \leqslant \min_{0\leqslant s\leqslant 1} u(s) \leqslant u(t) \leqslant b$. Hence $u(t) \in [\delta b, b]$ for $t \in [0, 1]$. By (H3), we get

$$\begin{aligned}\|T'u\| &= \int_0^1 \Phi_q\left(\int_0^s h(r) f^+(r, u(r))\,\mathrm{d}r\right)\mathrm{d}s\\ &\quad + \frac{\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^1 \Phi_q(\int_0^s h(r) f^+(r, u(r))\,\mathrm{d}r)\,\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\\ &> \frac{b}{M}\left[\int_0^1 \Phi_q\left(\int_0^s h(r)\,\mathrm{d}r\right)\mathrm{d}s + \frac{\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^1 \Phi_q(\int_0^s h(r)\,\mathrm{d}r)\,\mathrm{d}s}{1-\sum_{i=1}^{m-2}\alpha_i}\right]\\ &= b.\end{aligned}$$

It follows from Lemma 2.1 that

$$i(T', K'_a, K') = 1, \quad i(T', K'_b, K') = 0.$$

Thus $i(T', K'_b \backslash \bar{K}'_a, K') = -1$ and T' has a fixed point u_2 in $K'_b \backslash \bar{K}'_a$.

We claim that $Ax = T'x$ for $x \in K'_b \backslash \bar{K}'_a \cap \{u : T'u = u\}$. In fact, for $u_2 \in K'_b \backslash \bar{K}'_a \cap \{u : T'u = u\}$, it is clear that $u_2(0) = \|u_2\| > a$ and so by Lemma 2.4,

$$u_2(1) = \min_{0\leqslant t\leqslant 1} u_2(t) \geqslant \delta u_2(0) > \delta a > d.$$

Thus for $t \in [0, 1]$, $d \leqslant u_2(t) \leqslant b$. From (H1), we know that $f^+(t, u_2(t)) = f(t, u_2(t))$. This implies that $Au_2 = T'u_2$ for $u_2 \in K'_b \backslash \bar{K}'_a \cap \{u : T'u = u\}$. Hence u_2 is a fixed point of A in K', which is also a positive solution of (1.3)–(1.4). □

2.2. Positive solution to BVP (1.3)–(1.5)

Lemma 2.5. *Suppose* $\sum_{i=1}^{m-2} \alpha_i \neq 1$ *and* $y(\cdot) \in C[0, 1]$. *Then the problem*

$$(\Phi_p(u'))' + y(t) = 0, \quad 0 < t < 1, \tag{2.4}$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(1) = 0 \tag{2.5}$$

has a unique solution

$$u(t) = \int_0^t \Phi_q \left(\int_s^1 y(r)\,\mathrm{d}r \right) \mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \Phi_q(\int_s^1 y(r)\,\mathrm{d}r)\,\mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}. \tag{2.6}$$

Proof. Integrate (2.4) from t to 1, we get

$$\Phi_p(u'(t)) = \int_t^1 y(r)\,\mathrm{d}r,$$

i.e.,

$$u'(t) = \Phi_q \left(\int_t^1 y(r)\,\mathrm{d}r \right).$$

Integrating this from 0 to t yields

$$u(t) = u(0) + \int_0^t \Phi_q \left(\int_s^1 y(r)\,\mathrm{d}r \right) \mathrm{d}s.$$

From the boundary condition (2.5), we have

$$u(0) = \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \Phi_q(\int_s^1 y(r)\,\mathrm{d}r)\,\mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}.$$

This proves Lemma 2.5. □

Now let X, K, K', $\|\cdot\|$, K_r, and ∂K_r be as defined in Section 2.1. For any $u \in K$, we define

$$(\tilde{A}u)(t) := \int_0^t \Phi_q \left(\int_s^1 h(r) f(r, u(r))\,\mathrm{d}r \right) \mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \Phi_q(\int_s^1 h(r) f(r, u(r))\,\mathrm{d}r)\,\mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}, \quad t \in [0, 1].$$

By Lemma 2.5, BVP (1.3)–(1.5) is equivalent to the fixed point problem of the operator $\tilde{A}$. For any $u \in K$ we also define

$$(\tilde{T}u)(t) := \left[\int_0^t \Phi_q \left(\int_s^1 h(r) f(r, u(r))\, \mathrm{d}r \right) \mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \Phi_q (\int_s^1 h(r) f(r, u(r))\, \mathrm{d}r)\, \mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i} \right]^+, \quad t \in [0, 1],$$

where $g^+ = \max\{g, 0\}$. Obviously $\tilde{T} = \Theta \circ \tilde{A}$, where Θ is defined as in Lemma 2.2. For any $u \in K'$, define

$$(\tilde{T}'u)(t) := \int_0^t \Phi_q \left(\int_s^1 h(r) f^+(r, u(r))\, \mathrm{d}r \right) \mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \Phi_q (\int_s^1 h(r) f^+(r, u(r))\, \mathrm{d}r)\, \mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}, \quad t \in [0, 1].$$

Lemma 2.6. *If $u \in K'$ satisfies BVP* (1.3)–(1.5), *then*

$$\min_{0 \leqslant t \leqslant 1} u(t) \geqslant \frac{\sum_{i=1}^{m-2} \alpha_i \xi_i}{1 - \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)} \|u\|.$$

Proof. For $u \in K'$, it is easy to see that $u(1) = \|u\|$, $u(0) = \min_{0 \leqslant t \leqslant 1} u(t)$. By the concavity of $u(t)$, we have

$$\frac{u(\xi_i) - u(0)}{\xi_i} \geqslant \frac{u(1) - u(0)}{1}, \quad i = 1, \ldots, m-2,$$

i.e.,

$$u(\xi_i) - u(0) + \xi_i u(0) \geqslant \xi_i u(1), \quad i = 1, \ldots, m-2.$$

By (1.5), $u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i)$. Hence we have

$$\min_{0 \leqslant t \leqslant 1} u(t) \geqslant \frac{\sum_{i=1}^{m-2} \alpha_i \xi_i}{1 - \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)} \|u\|.$$

This proves Lemma 2.6. □

As in Section 2.1, define

$$\tilde{\delta} := \frac{\sum_{i=1}^{m-2} \alpha_i \xi_i}{1 - \sum_{i=1}^{m-2} \alpha_i (1 - \xi_i)},$$

$$\tilde{M} := \int_0^1 \Phi_q \left(\int_s^1 h(r)\, \mathrm{d}r \right) \mathrm{d}s + \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \Phi_q (\int_s^1 h(r)\, \mathrm{d}r)\, \mathrm{d}s}{1 - \sum_{i=1}^{m-2} \alpha_i}.$$

Analogous to Theorem 2.1, by using Lemmas 2.1, 2.2, 2.5, and 2.6, it is not hard to show that BVP (1.3)–(1.5) has at least two positive solutions:

Theorem 2.2. *Suppose* $f(t,0)\geqslant 0$ *for* $t\in[0,1]$ *and* $h(t)f(t,0)\not\equiv 0$ *on any subinterval of* $[0,1]$. *If there exist nonnegative numbers a, b and d such that* $0<(1/\tilde{\delta})d<a<\tilde{\delta}b<b$ *and f satisfies the following conditions*:

$(\mathrm{H}1)'$ $f(t,u)\geqslant 0$ *for* $(t,u)\in[0,1]\times[d,b]$;
$(\mathrm{H}2)'$ $f(t,u)<\Phi_p(a/\tilde{M})$ *for* $(t,u)\in[0,1]\times[0,a]$;
$(\mathrm{H}3)'$ $f(t,u)>\Phi_p(b/\tilde{M})$ *for* $(t,u)\in[0,1]\times[\tilde{\delta}b,b]$,

then BVP (1.3)–(1.5) *has at least two positive solutions* u_1 *and* u_2 *with*

$$0<\|u_1\|<a<\|u_2\|<b.$$

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