Some New Weighted Hardy-type Inequalities for Vector-Valued Functions

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Abstract

This is a continuation of an earlier work of Cheung-Pečarić. By using the C-technique developed by Cheung and Pečarić, some new and interesting Hardy-type inequalities involving vector-valued functions are established. These generalize and imporve some known results by Cheung, Cheung-Hanjš-Pečarić, Izumi-Izumi, and Pachpatte.

1. Introduction

One of the classical and important inequalities of G.H. Hardy is the following integral inequality [7, Theorem 330]:

If $p > 1$, $m \neq 1$, $f(x)$ is non-negative measurable on $(0, \infty)$, and

$$F(x) = \begin{cases} \int_0^x f(t)dt & \text{for } m > 1 \ , \\ \int_x^\infty f(t)dt & \text{for } m < 1 \ , \end{cases} \tag{A}$$

then

$$\int_0^\infty x^{-m} F(x)^p dx < \Big(\frac{p}{|m-1|}\Big)^p \int_0^\infty x^{-m+p} f(x)^p dx \tag{B}$$

unless $f \equiv 0$, where the constant on the right is best possible.

Because of its fundamental importance in the discipline, over the years much effort and time have been devoted to the improvement and generalizations of Hardy's inequality (B). These include, among others, the works by Cheung [1], Cheung-Hanjš-Pečarić [3], Isumi-Isumi [8], Levinson [9], Love [10], Pachpatte [14], and Pachpatte-Love [15]. Recently, Hanjš, Love and Pečarić [6] adopted a function more general than $F(x)$ in (A) and established some new and interesting Hardy-type integral inequalities. In this paper, using the C-technique developed by Cheung and Pečarić (see, e.g. [1-4]), by adopting also a function similar to that in [6] and using techniques parallel to those in Cheung-Pečarić [5], we obtain some new Hardy-type inequalities which generalize and imporve some existing results of Cheung [1], Cheung-Hanjš-Pečarić [3], Isumi-Isumi [8] and Pachpatte [14].

2. Main Results

We follow the notations used in [5], namely, $\mathbb{R}_+ = (0, \infty)$, $X \in \mathbb{R}_+$ a fixed number, $n \geq 1$ an integer, and i, j are indices running from 1 to n. Also, as all summations and products that will appear are taken over i, j from 1 to n, we shall drop the limits and denote these simply by $\sum_i$, $\sum_j$, $\prod_i$, $\prod_j$, etc.

Theorem 1. *Let $m > 1$, $p \geq 1$, and $q \geq 0$. Let $s(x)$, $w(x)$ and $z(x)$ be absolutely continuous and positive a.e. on $[0, X]$, with $z'(x)$ essentially bounded and positive. If $f(x)$ is nonnegative and integrable on $[0, X]$,*

$$F(x) := \frac{1}{s(x)} \int_{\frac{x}{2}}^x \frac{s(t)z'(t)}{z(t)} f(t)dt \quad \textit{for } 0 \leq x \leq X \ ,$$

and

$$1 + \frac{1}{m-1} \frac{z(x)}{z'(x)} \Big((p+q)\frac{s'(x)}{s(x)} - \frac{w'(x)}{w(x)} \Big) \geq \frac{1}{\alpha} > 0 \quad \textit{a.e.} \ , \tag{1}$$

then

$$\int_0^X w(x) \frac{z'(x)}{z(x)^m} F(x)^{p+q} dx \leq \Big[\frac{\alpha(p+q)}{m-1}\Big]^p \int_0^X w(x) \frac{z'(x)}{z(x)^m} F^q(x) \tilde{f}(x)^p dx \ , \tag{2}$$

where

$$\tilde{f}(x) := \frac{z(x)}{z'(x)s(x)}|\Delta(x)| \ ,$$
$$\Delta(x) := \frac{s(x)z'(x)}{z(x)}f(x) - \frac{s(\frac{x}{2})z'(\frac{x}{2})}{2z(\frac{x}{2})}f\Big(\frac{x}{2}\Big) \ . \tag{3}$$

Proof. (i) By arguments similar to those in the proof of (i) of Theorem 1 in [6], $F(x)$ is absolutely continuous. So the whole integrand in the left hand side of (2) is bounded, and the integral on the left hand side of (2) is convergent.

(ii) Again by arguments similar to those in the proof of (ii) of Theorem 1 in [6], the following integration by parts is valid:

$$\int_0^X w(x)F(x)^{p+q}(1-m)\frac{z'(x)}{z(x)^m}dx = \Big[w(x)F(x)^{p+q}z(x)^{1-m}\Big]_{x=0}^{x=X}$$
$$-\int_0^X z(x)^{1-m}\Big[w'(x)F(x)^{p+q} + w(x)(p+q)F(x)^{p+q-1}F'(x)\Big]dx \ ,$$

hence

$$\begin{aligned}
&(m-1)\int_0^X \frac{z'(x)}{z(x)^m}w(x)F^{p+q}(x)dx + z(X)^{1-m}w(X)F(X)^{p+q}\\
&=\int_0^X z(x)^{1-m}\Big[w'(x)F(x)^{p+q} + w(x)(p+q)F(x)^{p+q-1}F'(x)\Big]dx\\
&=\int_0^X z(x)^{1-m}\Big\{w'(x)F(x)^{p+q} + w(x)(p+q)F(x)^{p+q-1}\Big[-\frac{s'(x)}{s(x)}F(x)\\
&\quad + \frac{1}{s(x)}\Big(\frac{s(x)z'(x)}{z(x)}f(x) - \frac{s(\frac{x}{2})z'(\frac{x}{2})}{2z(\frac{x}{2})}f\big(\frac{x}{2}\big)\Big)\Big]\Big\}dx\\
&=\int_0^X \Big\{\frac{z'(x)}{z(x)^m}w(x)\Big[\frac{w'(x)}{w(x)}\ \frac{z(x)}{z'(x)} - (p+q)\frac{s'(x)}{s(x)}\ \frac{z(x)}{z'(x)}\Big]F(x)^{p+q}\\
&\quad + \frac{z(x)^{1-m}}{s(x)}w(x)(p+q)\Big(\frac{s(x)z'(x)}{z(x)}f(x) - \frac{s(\frac{x}{2})z'(\frac{x}{2})}{2z(\frac{x}{2})}f\big(\frac{x}{2}\big)\Big)F(x)^{p+q-1}\Big\}dx \ .
\end{aligned} \tag{4}$$

We note that

$$\frac{z(x)^{1-m}}{s(x)}w(x)\Big[\frac{s(x)z'(x)}{z(x)}f(x) - \frac{s(\frac{x}{2})z'(\frac{x}{2})}{2z(\frac{x}{2})}f\big(\frac{x}{2}\big)\Big]F(x)^{p+q-1}$$

is integrable. In fact, by the proof of (i) of Theorem 1 in [6], $\frac{s(x)z'(x)}{z(x)}f(x)$ is integrable, so the same is true for $\frac{s(\frac{x}{2})z'(\frac{x}{2})}{2z(\frac{x}{2})}f\big(\frac{x}{2}\big)$ and $\Delta(x)$, while the other factors in the item including $z(x)^{1-m}$ are absolutely continuous.

Now, by additivity and using condition (1), (4) can be restated as

$$\frac{1}{\alpha}\int_0^X \overline{w}(x)F(x)^{p+q}dx \le \frac{p+q}{m-1}\int_0^X \overline{w}(x)F(x)^{p+q-1}\tilde{f}(x)dx \ , \tag{5}$$

where $\overline{w}(x) = \frac{z'(x)}{z(x)^m}w(x)$ and $\tilde{f}(x)$ is defined as in (3).

From (5), we have

$$\frac{1}{\alpha}\int_0^X \overline{w}(x)F(x)^{p+q}dx \le \frac{p+q}{m-1}\int_0^X \Big(\overline{w}(x)^{1-\frac{1}{p}}F(x)^{p+q-1-\frac{q}{p}}\Big)\cdot\Big(\overline{w}(x)^{\frac{1}{p}}F(x)^{\frac{q}{p}}\tilde{f}(x)\Big)dx$$
$$\le \frac{p+q}{m-1}\Big(\int_0^X \overline{w}(x)F(x)^{p+q}dx\Big)^{1-\frac{1}{p}}\Big(\int_0^X \overline{w}(x)F^q(x)\tilde{f}(x)^p dx\Big)^{\frac{1}{p}}$$

by Hölder's inequality, and Theorem 1 follows. □

Theorem 2. *For any $i=1,\dots,n$, let $w, s_i, z_i : [0,X]\to\mathbb{R}_+$ be absolutely continuous with z_i' essentially bounded and positive a.e., and $k_i>1$, $p_i\ge q_i>0$, $r_i\ge 0$, $\alpha_i>0$ be real numbers such that $\sum_i q_i=1$ and*

$$1+\frac{1}{k_i-1}\,\frac{z_i(x)}{z_i'(x)}\left[\Big(\frac{p_i+r_i}{q_i}\Big)\frac{s_i'(x)}{s_i(x)}-\frac{w'(x)}{w(x)}\right]\ge\frac{1}{\alpha_i}>0 \quad a.e.\ .$$

If for any $i=1,\dots,n$, f_i is integrable and nonnegative and

$$F_i(x):=\frac{1}{s_i(x)}\int_{\frac{x}{2}}^x \frac{s_i(t)z_i'(t)}{z_i(t)}f_i(t)dt\ ,\quad 0\le x\le X\ ,$$

then

$$\int_0^X w(x)\prod_i\left[\Big(\frac{z_i'(x)}{z_i(x)^{k_i}}\Big)^{q_i}F_i(x)^{p_i+r_i}\right]dx$$
$$\le\Big(\prod_j C_j^{-p_j}\Big)\sum_i q_iC_i^{\frac{p_i}{q_i}}\left[\frac{\alpha_i(p_i+r_i)}{q_i(k_i-1)}\right]^{\frac{p_i}{q_i}}\int_0^X w(x)\frac{z_i'(x)}{z_i(x)^{k_i}}F_i(x)^{\frac{r_i}{q_i}}\tilde{f}_i(x)^{\frac{p_i}{q_i}}dx\ ,$$

where

$$\tilde{f}_i(x)=\frac{z_i(x)}{z_i'(x)s_i(x)}|\Delta_i(x)|$$
$$\Delta_i(x)=\frac{s_i(x)z_i'(x)}{z_i(x)}f_i(x)-\frac{s_i(\frac{x}{2})z_i'(\frac{x}{2})}{2z_i(\frac{x}{2})}f_i\Big(\frac{x}{2}\Big)\ .$$

Proof. By Theorem 1, we have

$$\int_0^X w(x)\frac{z_i'(x)}{z_i(x)^{k_i}}F(x)^{\frac{p_i+r_i}{q_i}}dx\le\left[\frac{\alpha_i(p_i+r_i)}{q_i(k_i-1)}\right]^{\frac{p_i}{q_i}}\int_0^X w(x)\frac{z_i'(x)}{z_i(x)^{k_i}}F_i(x)^{\frac{r_i}{q_i}}\tilde{f}_i^{\frac{p_i}{q_i}}(x)dx \qquad (6)$$

for all $i=1,\dots,n$. On the other hand, for any $C_i>0$, by the arithmetic-geometric inequality [7, 11-13], we have

$$w(x)\prod_i\left[\Big(\frac{z_i'(x)}{z_i(x)^{k_i}}\Big)^{q_i}F_i(x)^{p_i+r_i}\right]=w(x)\prod_i\left\{\left[C_i^{\frac{p_i}{q_i}}\frac{z_i'(x)}{z_i(x)^{k_i}}F_i(x)^{\frac{p_i+r_i}{q_i}}\right]^{q_i}C_i^{-p_i}\right\}$$
$$=\Big(\prod_j C_j^{-p_j}\Big)w(x)\prod_i\left[C_i^{\frac{p_i}{q_i}}\frac{z_i'(x)}{z_i(x)^{k_i}}F_i(x)^{\frac{p_i+r_i}{q_i}}\right]^{q_i}$$
$$\le\Big(\prod_j C_j^{-p_j}\Big)w(x)\sum_i q_iC_i^{\frac{p_i}{q_i}}\frac{z_i'(x)}{z_i(x)^{k_i}}F_i(x)^{\frac{p_i+r_i}{q_i}}\ .$$

Therefore, from (6) we obtain

$$\int_0^X w(x)\prod_i\left[\Big(\frac{z_i'(x)}{z_i(x)^{k_i}}\Big)^{q_i}F_i(x)^{p_i+r_i}\right]dx \le \Big(\prod_j C_j^{-p_j}\Big)\sum_i q_iC_i^{\frac{p_i}{q_i}}\int_0^X w(x)\frac{z_i'(x)}{z_i(x)^{k_i}}F_i(x)^{\frac{p_i+r_i}{q_i}}\,dx$$

$$\le\Big(\prod_j C_j^{-p_j}\Big)\sum_i(q_iC_i^{\frac{p_i}{q_i}})\Big[\frac{\alpha_i(p_i+r_i)}{q_i(k_i-1)}\Big]^{\frac{p_i}{q_i}}\int_0^X w(x)\frac{z_i'(x)}{z_i(x)^{k_i}}F_i(x)^{\frac{r_i}{q_i}}\tilde{f}_i(x)^{\frac{p_i}{q_i}}\,dx\ . \qquad \square$$

Corollary 1. *For any $i=1,\dots,n$, let $s_i:[0,X]\to\mathbb{R}_+$ be absolutely continuous and positive a.e., and $p_i\ge q_i>0$, $m_i>q_i$ be real numbers such that $\sum_i q_i=1$ and*

$$1+\frac{p_i}{m_i-q_i}\,\frac{xs_i'(x)}{s_i(x)}\ge\frac{1}{\alpha_i}>0 \quad a.e.$$

If for any $i=1,\dots,n$, f_i is integrable and nonnegative and

$$\tilde{F}_i(x):=\frac{1}{s_i(x)}\int_{\frac{x}{2}}^x\frac{s_i(t)}{t}f_i(t)dt\ ,\quad 0\le x\le X\ ,$$

then

$$\int_0^x x^{-\sum_i m_i}\prod_i\Big(\tilde{F}_i(x)^{p_i}\Big)dx \le\Big(\prod_j C_j^{-p_j}\Big)\sum_i q_iC_i^{\frac{p_i}{q_i}}\Big[\frac{p_i\alpha_i}{m_i-q_i}\Big]^{\frac{p_i}{q_i}}\int_0^X x^{-\frac{m_i}{q_i}}\tilde{f}_i^*(x)^{\frac{p_i}{q_i}}dx \tag{7}$$

where

$$\tilde{f}_i^*(x)=\frac{1}{s_i(x)}\Big|s_i(x)f_i(x)-s_i(\frac{x}{2})f_i(\frac{x}{2})\Big|\ .$$

Proof. This follows from Theorem 2 by setting $w(x)\equiv1$, $z_i(x)=x$, $k_i=\frac{m_i}{q_i}$, and $r_i=0$ for all i. $\square$

Remark 1. If we rename $f_i(x)$ as $g_i(x)f_i(x)^{\beta_i-\alpha_i}$ and $s_i(x)$ as $f_i(x)^{\alpha_i}$, (7) becomes

$$\int_0^X x^{-\sum_i m_i}\prod_i(\mu_i(x)^{p_i})dx \le\Big(\prod_j C_j^{-p_j}\Big)\sum_i q_iC_i^{\frac{p_i}{q_i}}\Big[\frac{p_i\alpha_i}{m_i-q_i}\Big]^{\frac{p_i}{q_i}}\int_0^X x^{-\frac{m_i}{q_i}}\left[\frac{1}{f_i(x)^{\alpha_i}}\Big|f_i^{\beta_i}(x)g_i(x)-f_i^{\beta_i}(\frac{x}{2})g_i(\frac{x}{2})\Big|\right]^{\frac{p_i}{q_i}}dx \tag{8}$$

which is exactly Theorem 2.9 in [1], where

$$\mu_i(x)=\frac{1}{f_i^{\alpha_i}(x)}\int_{\frac{x}{2}}^x\frac{f_i^{\beta_i}(t)g_i(t)}{t}\,dt\ ,\quad 0\le x\le X\ .$$

Furthermore, if we restrict to $\alpha_i=\beta_i=1$ for all i, (8) reduces to Theorem 3 in [3]. Note that inequality (8) also generalizes a result of Isumi-Isumi [8, Theorem 2], which only deals with the situation where $n=1$.

Now, we choose specific constants C_i, q_i etc. to derive two interesting Hardy-type inequalities from Theorem 2 in the following Corollaries.

Corollary 2. *Under the same conditions as in Theorem 2,*

$$\int_0^X w(x)\prod_i\left[\Big(\frac{z_i'(x)}{z_i(x)^{k_i}}\Big)^{q_i}F_i(x)^{p_i+r_i}\right]dx \le C\sum_i\int_0^X w(x)\frac{z_i'(x)}{z_i(x)^{k_i}}F_i(x)^{\frac{r_i}{q_i}}\tilde{f}_i(x)^{\frac{p_i}{q_i}}dx\ ,$$

where

$$C=\prod_j\left\{q_j^{q_j}\Big[\frac{\alpha_j(p_j+r_j)}{q_j(k_j-1)}\Big]^{p_j}\right\}\ ,$$

F_i and $\tilde{f}_i$ are defined as in Theorem 2.

Proof. It follows immediately from Theorem 2 by setting

$$C_i=q_i^{-\frac{q_i}{p_i}}\Big[\frac{\alpha_i(p_i+r_i)}{q_i(k_i-1)}\Big]^{-1}$$

for all $i=1,\dots,n$. □

Corollary 3. *For any $i=1,\dots,n$, let $w,s_i,z_i:[0,X]\to\mathbb{R}_+$ be absolutely continuous with z_i' essentially bounded and positive a.e. and $k_i>1$, $p_i\ge\frac{1}{n}$, $r_i\ge 0$ be real numbers such that*

$$1+\frac{1}{k_i-1}\,\frac{z_i(x)}{z_i'(x)}\Big[n(p_i+r_i)\frac{s_i'(x)}{s_i(x)}-\frac{w'(x)}{w(x)}\Big]\ge\frac{1}{\alpha_i}>0\quad a.e.$$

If for any $i=1,\dots,n$, f_i is integrable and nonnegative and

$$F_i(x):=\frac{1}{s_i(x)}\int_{\frac{x}{2}}^{x}\frac{s_i(t)z_i'(t)}{z_i(t)}f_i(t)dt\ ,\quad 0\le x\le X\ ,$$

then

$$\begin{aligned}&\int_0^X w(x)\prod_i\left[\Big(\frac{z_i'(x)}{z_i(x)^{k_i}}\Big)^{\frac{1}{n}}F_i(x)^{p_i+r_i}\right]dx\\ \le&\frac{1}{n}\Big(\prod_j C_j^{-p_j}\Big)\sum_i C_i^{np_i}\Big[\frac{\alpha_i(p_i+r_i)n}{k_i-1}\Big]^{np_i}\int_0^X w(x)\frac{z_i'(x)}{z_i(x)^{k_i}}F_i(x)^{nr_i}\tilde{f}_i(x)^{np_i}dx\end{aligned}\tag{9}$$

for any constants $C_i>0$, where F_i and $\tilde{f}_i$ are defined as in Theorem 2.

Proof. This follows immediately from Theorem 2 by setting $q_i=\frac{1}{n}$ for all i. □

Remark 2. If we choose $C_i=1$ for all i, (9) becomes

$$\begin{aligned}&\int_0^X w(x)\prod_i\left[\Big(\frac{z_i'(x)}{z_i(x)^{k_i}}\Big)^{\frac{1}{n}}F_i(x)^{p_i+r_i}\right]dx\\ \le&\frac{1}{n}\sum_i\Big[\frac{\alpha_i(p_i+r_i)n}{k_i-1}\Big]^{np_i}\int_0^X w(x)\frac{z_i'(x)}{z_i(x)^{k_i}}F_i(x)^{nr_i}\tilde{f}_i(x)^{np_i}dx\ .\end{aligned}$$

In particular, setting $z_i(x)=x$, $k_i=m$, $w\equiv 1$, and $r_i=0$, this reduces to an inequality obtained by Pachpatte in [13, Theorem 6]. Observe, though, that our assumption here are considerably milder. In [14] it was required that $p_i>1$ for all i, while here all we need is $p_i\ge\frac{1}{n}$ for all i.

Remark 3. In Theorems 1, 2 and Corollaries 1, 2 and 3, if the hypotheses on w, s_i, z_i and z_i' hold locally on $[0, \infty)$, the assertions are still true with X replaced by ∞ (but in this case the improper integrals concerned may not always be convergent).

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