Hamilton Paths in Toroidal Graphs

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Abstract

Tutte has shown that every 4-connected planar graph contains a Hamilton cycle. Grünbaum and Nash-Williams independently conjectured that the same is true for toroidal graphs. In this paper, we prove that every 4-connected toroidal graph contains a Hamilton path.

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1 Introduction and notation

In 1931, Whitney [9] proved that every 4-connected triangulation of the sphere contains a Hamilton cycle (and hence, is 4-face-colorable). Tutte [8] generalized this result to 4-connected planar graphs. Extending the technique of Tutte, Thomassen [7] proved that in any 4-connected planar graph there is a Hamilton path between any given pair of distinct vertices. Grünbaum [3] conjectured that every 4-connected graph embeddable in the projective plane contains a Hamilton cycle. This conjecture was proved by Thomas and Yu [5]. For graphs embeddable in the torus (*toroidal graphs*, for short), Grünbaum [3] and Nash-Williams [4] independently made the following

(1.1) **Conjecture**. *Every 4-connected toroidal graph contains a Hamilton cycle.*

Conjecture (1.1) is established in [1] for 6-connected toroidal graphs. Brunet and Richter [2] proved (1.1) for 5-connected triangulations of the torus. Later, Thomas and Yu [6] proved (1.1) for all 5-connected toroidal graphs. In this paper, we offer further evidence to (1.1) by proving the following

(1.2) **Theorem**. *Every 4-connected toroidal graph contains a Hamilton path.*

Only simple graphs will be considered. Let G be a graph. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. If e is an edge of G with ends u and v, then we also denote e by uv. For $X \subseteq E(G)$ or $X \subseteq V(G)$, $G - X$ denotes the graph obtained from G by deleting X and (if $X \subseteq V(G)$) by deleting all edges of G incident with vertices in X. When $X = \{x\}$, we write $G - x$ instead of $G - \{x\}$. Let S be a set of 2-element subsets of $V(G)$; then $G+S$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup S$. If $S = \{\{x_i, y_i\} : i = 1, \dots, k\}$ then we sometimes use $G + \{x_i y_i : i = 1, \dots, k\}$ instead of $G + S$, and if $S = \{\{x, y\}\}$ we use $G + xy$ instead of $G + S$.

Let G and H be subgraphs of a graph; then $G \cap H$ (respectively, $G \cup H$) denotes the *intersection* (respectively, *union*) of G and H. We write $G - H$ instead of $G - V(G \cap H)$. We use $P \subseteq G$ to mean that P is a subgraph of G. For $S \subseteq V(G)$, we also view S as the subgraph of G with vertex set S and no edges. Hence $P \cup S$ makes sense for $P \subseteq G$ and $S \subseteq V(G)$. A *block* in a graph is a maximal 2-connected subgraph or is induced by a cut edge of the graph.

A graph G is *embedded* in a surface Σ if it is drawn in Σ with no pair of edges crossing. The *faces* of G are the connected components (in topological sense) of $\Sigma - G$. The boundary of a face is called a *facial walk*. The *face width* or *representativity* of G in Σ is defined to be the minimum number $|\gamma \cap G|$ taken over all non-null homotopic simple closed curves γ in Σ. When Σ is the plane (or equivalently, the sphere), G is a *plane graph*. For convenience, an embedding of a graph G in the plane is called a *plane representation* of G. The boundary of the infinite face of a plane graph G is called the *outer walk* of G, or *outer cycle* if it is a cycle.

For a path P and two vertices $x, y \in V(P)$, we use xPy to denote the subpath of P with ends x and y. For a cycle C and distinct vertices x, y on C, an *xy-segment* of C is a path in C with ends x and y. If C is a cycle in a graph embedded in an orientable surface

Σ such that C bounds a closed disc in Σ, then we can speak of clockwise and counter clockwise orders along C. Given two vertices x and y on a cycle C bounding a closed disc, let $xCy = \{x\}$ if $x = y$, and otherwise, let xCy denote the xy-segment of C which is clockwise from x to y.

Let G be a graph and $P \subseteq G$. A *P-bridge* of G is a subgraph of G which either (1) is induced by an edge of $G - E(P)$ with both ends on P or (2) is induced by the edges in a component of $G - V(P)$ and all edges from that component to P. For a P-bridge B of G, the vertices of $B \cap P$ are the *attachments* of B (on P). We say that P is a *Tutte subgraph* of G if every P-bridge of G has at most three attachments on P. For $C \subseteq G$, P is a *C-Tutte subgraph* of G if P is a Tutte subgraph of G and every P-bridge of G containing an edge of C has at most two attachments on P. A *Tutte path* (respectively, *Tutte cycle*) in a graph is a path (respectively, cycle) which is a Tutte subgraph.

Note that if P is a Tutte path in a 4-connected graph and $|V(P)| \geq 4$, then P is in fact a Hamilton path. Hence, in order to prove (1.2), it suffices to find Tutte paths in 2-connected toroidal graphs. This will be done by applying induction on the face-width of a graph embedded in the torus. In Section 2, we state and prove a few lemmas about Tutte paths in plane graphs. In Section 3, we prove a result which will be used to treat the induction basis: the face width of a graph in the torus is at most two. We then complete the proof of (1.2) in Section 4.

For convenience, we use $A := B$ to rename B as A.

2 C-flaps and Tutte subgraphs

We begin this section with several known results on Tutte paths in plane graphs. The first result is the main theorem in [7], where a P-bridge is called a "P-component".

(2.1) **Lemma**. *Let G be a 2-connected plane graph with a facial cycle C. Assume that $x \in V(C)$, $e \in E(C)$, and $y \in V(G - x)$. Then G contains a C-Tutte path P between x and y such that $e \in E(P)$.*

Lemma (2.1) can easily be generalized as follows. Let G be a connected plane graph with a facial walk C. Assume that $x \in V(C)$, $e \in E(C)$, $y \in V(G - x)$, and G has a path between x and y and containing e. Then G contains a C-Tutte path P between x and y such that $e \in E(P)$. Hence, when we apply Lemma (2.1), we actually apply this general version.

In order to state the next result, we need the following concept first introduced in [5]. Let C be a cycle or a path in a graph G. A *C-flap* in G is either the null graph or an $\{a, b, c\}$-bridge H of G such that

(i) $a, b \in V(C) \cap V(H)$, $a \neq b$, and $c \in V(H) - V(C)$;

(ii) H contains an ab-segment S of C; and

(iii) H has a plane representation with S and c on its outer walk.

When a C-flap is an $\{a,b,c\}$-bridge H, we say that a,b,c are its attachments and define $I(H) = V(H) - \{a,b,c\}$. When a C-flap is null, we say that a,b,c are its attachments if $a = b = c \in V(C)$ and define $I(H) = \emptyset$.

See Figure 1 for an illustration. Note that we do not specify the order of a,b on C, and therefore we do not need the condition in [5] that H contains the clockwise segment of C between a and b.

Lemmas (2.2) and (2.3) below are the first half and second half, respectively, of (2.5) in [5], where a C-Tutte path is called an $E(C)$-snake. Lemma (2.2) is illustrated in Figure 1.

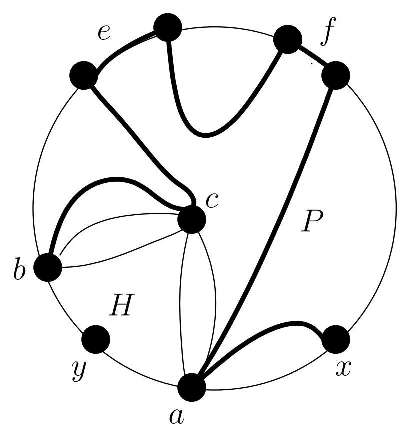

Figure 1: C-flap and Lemma (2.2)

(2.2) **Lemma**. *Let G be a 2-connected plane graph with outer cycle C. Let $x,y \in V(C)$ be distinct, let $e,f \in E(C)$, and assume that x,y,e,f occur on C in this clockwise order. Then there exist a C-flap H in G with attachments a,b,c ($a = b = c = y$ if H is null) and a $(C - I(H))$-Tutte path P between b and x in $G - I(H)$ such that x,a,y,b,e,f occur on C in this clockwise order, $y \in (V(H) - \{a\}) \cup \{b\}$, $\{e,f\} \subseteq E(P)$, and $a,c \in V(P)$.*

(2.3) **Lemma**. *Let G be a 2-connected plane graph with outer cycle C. Let $x,y \in V(C)$ be distinct, let $e,f \in E(C)$, and assume that x,y,e,f occur on C in this clockwise order. Then there exists a yCx-Tutte path P between x and y in G such that $\{e,f\} \subseteq E(P)$.*

Note that the above three lemmas hold when e or f or both are vertices of C. This can be seen by choosing edges in $E(C)$ incident with these vertices. Hence, when these lemmas are applied, we will allow e or f or both to be vertices.

The next lemma shows that if in (2.3) we do not insist that y be an end of P, then we can require P be a C-Tutte path.

(2.4) **Lemma**. *Let G be a 2-connected plane graph with outer cycle C. Let $x,y \in V(C)$ be distinct, let $e,f \in E(C)$, and assume that x,y,e,f occur on C in this clockwise order. Then there exists a C-Tutte path Q from x in G such that $y \in V(Q)$ and $\{e,f\} \subseteq E(Q)$.*

Proof. By (2.2), there exist a C-flap H in G with attachments a,b,c ($a = b = c = y$

if H is null) and a $(C - I(H))$-Tutte path P between b and x in $G - I(H)$ such that x, a, y, b, e, f occur on C in this clockwise order, $y \in (V(H) - \{a\}) \cup \{b\}$, $\{e, f\} \subseteq E(P)$, and $a, c \in V(P)$. See Figure 1.

If H is null, then $Q := P$ gives the desired path. So assume that H is non-null. Since G is 2-connected and $c \notin V(C)$, $H' := H + \{ac, bc\}$ is 2-connected. Without loss of generality, assume that ac, bc are added so that H' is a plane graph with outer cycle $C' := (aCb \cup \{c\}) + \{ac, bc\}$. By (2.3) (with H', C', b, c, ac, y as G, C, x, y, e, f, respectively), there exists a $cC'b$-Tutte path P' between b and c in H' such that $y \in V(P')$ and $ac \in E(P')$. Clearly, $bc \notin E(P')$.

Let $Q := (P' - \{a, c\}) \cup P$. Then every Q-bridge of G is one of the following: a P-bridge of $G - I(H)$, or a P'-bridge of H', or a subgraph of G induced by an edge of P' incident with a. Hence, Q is a C-Tutte path from x in G such that $y \in V(Q)$ and $\{e, f\} \subseteq E(Q)$. ∎

Again, when Lemma (2.4) is applied, e or f or both may be vertices. Next result is a technical lemma, and it will be used many times in later proofs. In order to cover all situations when this lemma is applied, we need to state it in a fairly general setting. See Figure 2 for an illustration.

(2.5) **Lemma**. *Let K be a connected graph, let Q be a path in K with ends p and q, let L be a subgraph of $K - Q$, let Q' be a cycle in L, and let $u \in V(Q')$. Suppose the following three conditions are satisfied.*

(1) If B is a $(L \cup Q)$-bridge of G, then $|V(B \cap L)| \leq 1$ and $V(B \cap L) \subseteq V(Q')$.

(2) Let H be the union of Q, Q', and those $(L \cup Q)$-bridges of G with an attachment on Q; then H has a plane representation such that Q and u are on a facial walk and Q' is its outer cycle.

(3) L contains a Q'-Tutte subgraph T such that (i) $u \in V(T)$ and $|V(Q') \cap V(T)| \geq 2$, and (ii) every T-bridge X of L containing an edge of Q' has a plane representation such that $X \cap Q'$ is a path on its outer walk.

Then $K - T$ contains a path S between p and q such that $S \cup T$ is a Q-Tutte subgraph of K, and every T-bridge of L containing no edge of Q' is also an $(S \cup T)$-bridge of K.

Remark. In most cases when (2.5) is applied, K is a plane graph, L is a block of $K - Q$, Q and u are on a facial walk of K, and Q' is a facial cycle of L which bounds the face of L containing Q. Hence, conditions (1) and (2) of (2.5) hold. Moreover, condition (3) holds if L has a Q'-Tutte subgraph T such that $u \in V(T)$ and $|V(T) \cap V(Q')| \geq 2$.

Proof. Let W denote the set of attachments on Q' of $(L \cup Q)$-bridges of K. Note that for each $w \in W$, either $w \in V(T)$ or there is a T-bridge X of L such that $w \in V(X - T)$. For $w, w' \in W$, we define $w \sim w'$ if $w = w'$ or there is a T-bridge X of L such that $\{w, w'\} \subseteq V(X - T)$. Clearly, $\sim$ is an equivalence relation on W. Let $W_1, W_2, \ldots, W_m$ denote the equivalence classes of W with respect to $\sim$. Then for $i \in \{1, \ldots, m\}$, either $|W_i| = 1$ and $W_i \subseteq V(T)$ (in this case, $B_i := W_i \subseteq V(Q')$) or $W_i \subseteq V(B_i - T)$ for some T-bridge B_i of L. Since T is a Q'-Tutte subgraph of L and $W \subseteq V(Q')$ (by (1)), $|V(B_i \cap T)| \leq 2$. Hence, by (i) of (3), $V(B_i \cap T) \subseteq V(Q')$

For each $i \in \{1, \ldots, m\}$, let $s_i, t_i \in V(Q)$ such that (I) p, s_i, t_i, q occur on Q in this order, (II) there are $w_s, w_t \in W_i$ such that $\{s_i, w_s\}$ is contained in a $(L \cup Q)$-bridge of K and $\{t_i, w_t\}$ is contained in a $(L \cup Q)$-bridge of K, and (III) subject to (I) and (II), s_iQt_i is maximal. By (2), s_iQt_i, $i = 1, \ldots, m$, are edge disjoint. We may therefore assume that $p, s_1, t_1, s_2, t_2, \ldots, s_m, t_m, q$ occur on Q in this order. Let $t_0 := p$ and $s_{m+1} := q$.

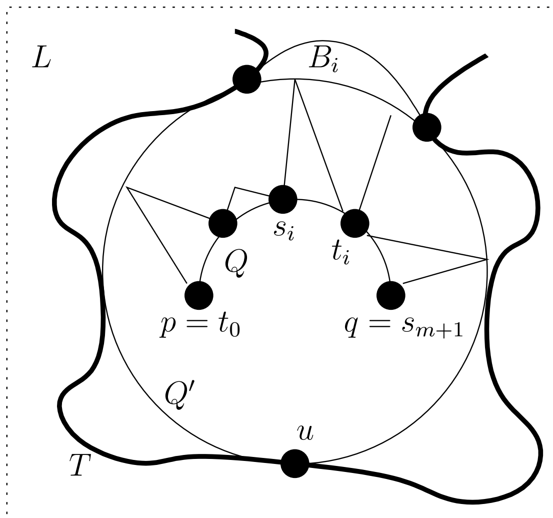

Figure 2: Illustration of Lemma (2.5)

For each $i \in \{0, \ldots, m\}$, let T_i denote the union of t_iQs_{i+1} and those $(L \cup Q)$-bridges of K whose attachments are all contained in $V(t_iQs_{i+1})$. For each $i \in \{1, \ldots, m\}$, let U_i denote the union of s_iQt_i, B_i, and those $(L \cup Q)$-bridges of K whose attachments are all contained in $V(s_iQt_i) \cup W_i$.

(a) By the definition of s_iQt_i, we conclude that for $i \leq j$, $U_i \cap T_j$ (and for $i < j$, $(U_i - T) \cap (U_j - T)$) is one of the following: $\emptyset$, or $\{t_i\}$, or the union of those $(L \cup Q)$-bridges of K with t_i as their only attachment on $L \cup Q$. Similarly, for $i < j$, $T_i \cap T_j$ (and also $T_i \cap U_j$) is one of the following: $\emptyset$, or $\{s_{i+1}\}$, or the union of those $(L \cup Q)$-bridges of K with s_{i+1} as their only attachment on $L \cup Q$.

(b) We claim that for each $i \in \{0, \ldots, m\}$, T_i contains a t_iQs_{i+1}-Tutte path R_i between t_i and s_{i+1}.

If $|V(t_iQs_{i+1})| \leq 2$, then $R_i := t_iQs_{i+1}$ gives the desired path for (b). Now assume that $|V(t_iQs_{i+1})| \geq 3$. By (2), T_i has a plane representation such that t_iQs_{i+1} is on its outer walk. Let C_i denote the outer walk of T_i, and choose an edge e from $E(t_iQs_{i+1})$. By applying (2.1) (with T_i, C_i, t_i, s_{i+1} as G, C, x, y, respectively), T_i has a C_i-Tutte path R_i between t_i and s_{i+1} such that $e \in E(R_i)$. Clearly, R_i is a t_iQs_{i+1}-Tutte path in T_i.

(c) We claim that for each $i \in \{1, \ldots, m\}$, $U_i - T$ contains a path S_i between s_i and t_i

such that $S_i \cup (U_i \cap T)$ is an s_iQt_i-Tutte subgraph of U_i.

Note that for all $i \in \{1, \ldots, m\}$, $|V(U_i \cap T)| = |V(B_i \cap T)| \leq 2$. By (ii) of (3), B_i has a plane representation such that $B_i \cap Q'$ is a path on its outer walk. Hence, by (2), U_i has a plane representation such that s_iQt_i and $B_i \cap T$ are on its outer walk. We will work on such a plane representation of U_i.

If $s_i = t_i$, then let $S_i := s_iQt_i$, and clearly, $S_i \cup (U_i \cap T)$ is an s_iQt_i-Tutte subgraph of U_i (because $|V(U_i \cap T)| \leq 2$). So assume that $s_i \neq t_i$. We distinguish two cases.

First assume that $W_i \subseteq V(T)$. Then $|W_i| = 1$. So let w be the only vertex in W_i. Without loss of generality, we can assume that $(s_iQt_i \cup \{w\}) + t_iw$ is contained in the outer walk D_i of $U_i + t_iw$. By (2.1) (with $U_i + t_iw, D_i, s_i, w, t_iw$ as G, C, x, y, e, respectively), $U_i + t_iw$ contains a D_i-Tutte path S_i' between s_i and w such that $t_iw \in E(S_i')$. Let $S_i := S_i' - w$. Then $S_i \subseteq U_i - T$, and it is easy to see that $S_i \cup (U_i \cap T) = S_i \cup \{w\}$ is an s_iQt_i-Tutte subgraph of U_i.

Now assume that $W_i \not\subseteq V(T)$. Then $B_i \neq W_i$ and B_i is a T-bridge of L containing an edge of Q'. Hence, $V(B_i \cap T)$ consists of two vertices w and w'. Assume that w, w', t_i, s_i occur on the outer walk of U_i in this clockwise order. Note that $(s_iQt_i \cup \{w, w'\}) + \{ws_i, t_iw'\}$ is contained in a cycle of $U_i + \{ws_i, t_iw'\}$, and so, let U_i' denote the block in $U_i + \{ws_i, t_iw'\}$ containing one such cycle. Without loss of generality, we may assume that $(s_iQt_i \cup \{w, w'\}) + \{ws_i, t_iw'\}$ is contained in the outer cycle D_i' of U_i'. By (2.3) (with $U_i', D_i', w, w', t_iw', ws_i$ as G, C, x, y, e, f, respectively), U_i' contains a $w'D_i'w$-Tutte path S_i' between w and w' such that $\{ws_i, t_iw'\} \subseteq E(S_i')$. Clearly, S_i' is also an s_iQt_i-Tutte path in $U_i + \{ws_i, t_iw'\}$. Let $S_i := S_i' - \{w, w'\}$. Then $S_i \subseteq U_i - T$, and it is easy to see that $S_i \cup (U_i \cap T) = S_i \cup \{w, w'\}$ is an s_iQt_i-Tutte subgraph of U_i.

By (a), (b) and (c), $S := (\bigcup_{i=0}^{m} R_i) \cup (\bigcup_{i=1}^{m} S_i)$ is a path between p and q in $K - T$. It is easy to see that every $(S \cup T)$-bridge of K is one of the following: a T-bridge of L not contained in any U_i, or a R_i-bridge of T_i, or an $(S_i \cup (U_i \cap T))$-bridge of U_i. Thus, $S \cup T$ is a Q-Tutte subgraph of K, and every T-bridge of L containing no edge of Q' is also an $(S \cup T)$-bridge of K. ∎

3 Planar graphs

In this section, we prove Theorem (3.3) which will be used to take care of the base case in the inductive proof of Theorem (1.2). First, we need two lemmas about Tutte paths in planar graphs, and for the sake of induction we will prove them simultaneously.

(3.1) **Lemma**. *Let G be a 2-connected plane graph with outer cycle C and a facial cycle D, let $y \in V(C)$, and let $x \in V(D)$. Then G contains a $(C \cup D)$-Tutte path P from y such that $x \in V(P)$ and no P-bridge of G contains vertices of both $C - P$ and $D - P$.*

A *separation* (G_1, G_2) in a graph G is a pair of edge disjoint subgraphs of G such that $G = G_1 \cup G_2$ and $E(G_1) \neq \emptyset \neq E(G_2)$. A separation (G_1, G_2) in G is a *k-separation* if $|V(G_1 \cap G_2)| = k$.

(3.2) **Lemma**. *Let G be a 2-connected plane graph with outer cycle C and another facial cycle D. Let $y \in V(C)$, $x \in V(D)$, and $e \in E(C)$. Assume that there do not exist distinct vertices $p, q \in V(C)$ and a 2-separation (G'_1, G'_2) in G such that $V(G'_1) \cap V(G'_2) = \{p, q\}$, p, y, e, q occur on C in the clockwise order listed, $pCq \subseteq G'_1$, and $qCp \cup D \subseteq G'_2$. Then one of the following holds:*

- *(A) there exist a C-flap H in G with attachments a, b, c ($a = b = c = y$ if H is null) and a $((C - I(H)) \cup D)$-Tutte path P from b in $G - I(H)$ such that $D \subseteq G - I(H)$, e, b, y, a occur on C in this clockwise order, $y \in (V(H) - \{a\}) \cup \{b\}$, $e \in E(P)$, $x, a, c \in V(P)$, and no P-bridge of G contains vertices of both $C - P$ and $D - P$; or*
- *(B) there exist $a, b \in V(C) \cap V(D)$, a separation (H, H^*) in G with $V(H) \cap V(H^*) = \{a, b\}$, and a $(C \cup D)$-Tutte path P from b in G such that e, b, y, a occur on C in this clockwise order, $bCa \cup bDa \subseteq H$, $aCb \cup aDb \subseteq H^*$, $P \subseteq H^*$, $y \in (V(H) - \{a\}) \cup \{b\}$, $a, x \in V(P)$, $e \in E(P)$, and no P-bridge of G distinct from H contains vertices of both $C - P$ and $D - P$.*

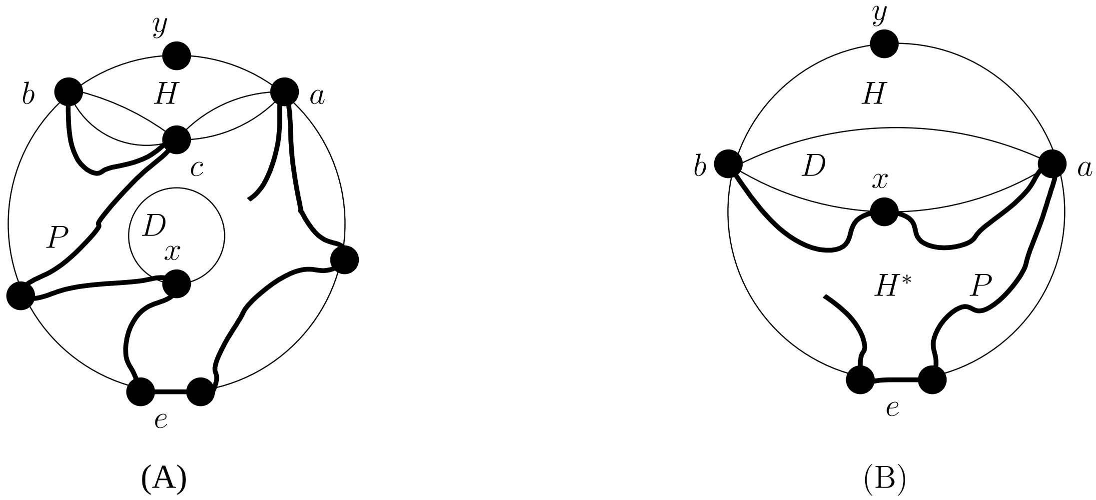

Figure 3: Illustration of Lemma (3.2)

Proof of (3.1) and (3.2). We proceed by induction on $|V(G)|$. We may assume that $C \neq D$; for otherwise, choose $f \in E(C) - \{e\}$ such that f is incident with y, then $P := C - f$ satisfies both the conclusion of (3.1) and (A) of (3.2) with H null. Thus $|V(G)| \geq 4$. Because we are proving (3.1) and (3.2) simultaneously, we can inductively assume that both (3.1) and (3.2) hold for all graphs on strictly less than $|V(G)|$ vertices. Let us remark that e is not defined in (3.1); hence in the proof of (3.1) we are free to specify e when inductively applying (3.2).

Let G_2 be a block of $G - y$ chosen as follows: for (3.1), G_2 is a block of $G - y$ containing an edge of $C - y$; for (3.2), if e is not incident with y let G_2 be the unique block of $G - y$ containing e, and otherwise let G_2 be the unique block of $G - y$ containing the unique edge of C adjacent to e but not incident with y. If G_2 has only one edge let $C_2 = G_2$; otherwise let C_2 be the outer cycle of G_2. It follows that every $(G_2 \cup \{y\})$-bridge of G has exactly one attachment in G_2, and this attachment belongs to C_2. Let $v_1, v_2, \ldots, v_k$ be those attachments of $(G_2 \cup \{y\})$-bridges of G on C_2 listed in clockwise order such that $v_k C v_1 = v_k C_2 v_1$. For $i = 1, 2, \ldots, k$, let J_i be the union of those $(G_2 \cup \{y\})$-bridges of G containing v_i, and let D_i denote the outer walk of J_i.

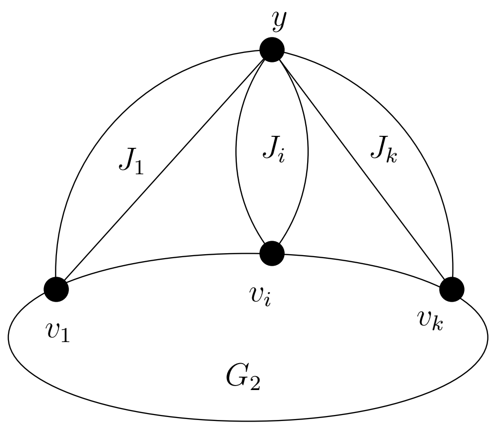

Figure 4: G_2 and $J_1, \ldots, J_k$

Since $C \neq D$, we have the following.

(a) Either there is some $i \in \{2, \ldots, k\}$ such that D is the union of $v_{i-1}C_2v_i$, a subpath of J_{i-1}, and a subpath of J_i; or there is some $i \in \{1, 2, \ldots, k\}$ such that D is a subgraph of J_i; or D is a subgraph of G_2.

Let us make three observations for the proof of (3.2) before we handle these cases separately.

(b) If e is incident with y, and if we denote the other end of e by z, then we may assume that y, z, e occur on C in the clockwise order listed.

To prove (b) assume that y, e, z occur on C in this clockwise order. Let $p := y$, $q := z$, let G_1' be the subgraph of G induced by e, and let $G_2' := G - e$. It follows from the hypothesis of (3.2) that D is not a subgraph of G_2', and hence $e \in E(D)$. Applying (2.1) to the graph G_2' (with $G_2', zCy \cup zDy, y, z, x$ as G, C, y, x, e, respectively), we find a $(zCy \cup zDy)$-Tutte path P' between y and z in G_2' such that $x \in V(P')$. Hence, $T := P' + e$ is a $(C \cup D)$-Tutte cycle in G with $e \in E(T)$ and $x \in V(T)$. By planarity, no T-bridge of G contains edges of both C and D. Let P be the path obtained from T by deleting the unique edge f of $T - e$ incident with y. Note that any P-bridge of G either is induced by f or is a T-bridge of G. Hence, no P-bridge of G contains vertices of both $C - P$ and $D - P$, and so, P satisfies (A) of (3.2) with H null. Thus we may assume that (b) holds.

We deduce from (b) that

(c) $e \in E(J_1) \cup E(G_2)$.

Moreover, the choice of G_2 implies that

(d) if $e \notin E(G_2)$, then e has ends y and v_1.

Now we distinguish three cases according to (a).

Case 1. There exists some $i \in \{2, \ldots, k\}$ such that D is the union of $v_{i-1}C_2v_i$, a subpath Q_{i-1} of J_{i-1}, and a subpath Q_i of J_i.

In this case, $y \in V(D)$, Q_{i-1} has ends v_{i-1} and y, and Q_i has ends v_i and y.

Suppose $i = 2$ and assume both $e \in E(J_1)$ and $x \in V(J_1)$. In this case, we only need to prove (3.2) (because for (3.1), we specify below so that e is in G_2). Since $e \in E(J_1)$ and by (d), $E(v_1Cy) = \{e\}$. Let $H^* := J_1$, $H := G - (V(J_1) - \{y, v_1\})$, and let $a := v_1$ and $b := y$. Then (H, H^*) is a separation in G with $V(H) \cap V(H^*) = \{a, b\}$, $bCa \cup bDa \subseteq H$, $aCb \cup aDb \subseteq H^*$, and $y \in (V(H) - \{a\}) \cup \{b\}$. If $|V(J_1)| = 2$ then $x = v_1$ and $P := J_1$ gives a D_1-Tutte path from $b = y$ in H^* such that $a, x \in V(P)$ and $e \in E(P)$, and we have (B) of (3.2). Now assume that J_1 is 2-connected. By (2.1) (with H^* as G), we find a D_1-Tutte path P between $b = y$ and x in H^* such that $e \in E(P)$. Therefore, we have (B) of (3.2).

So assume that either $i \neq 2$ or one of $\{e, x\}$ does not belong to J_1.

Let G' be obtained from G by deleting $V(J_j) - \{v_j, y\}$ for all $j = i, i+1, \ldots, k$. Then $C' := v_kCy \cup Q_{i-1} \cup v_{i-1}C_2v_k$ is the outer walk of G'. The following will serve as a proof of both (3.1) and (3.2) with the proviso that when inductively applying (3.2) in the proof of (3.1) the edge e is chosen to be the unique edge of $C \cap G'$ incident with v_k.

Next, we find a $(C' \cup D_i)$-Tutte path P^* from y in $G' \cup J_i$ such that $x, v_i \in V(P^*)$ and $e \in E(P^*)$. Assume first that $x \in V(G')$. If G' is 2-connected, then P^* can be found by applying (2.4) to G' (with $G', C', y, x, v_i, e, P^*$ as G, C, x, y, e, f, P, respectively). So assume that G' is not 2-connected. Then $i = 2$, and so, either $e \notin E(J_1)$ or $x \notin V(J_1)$. By (2.1) (with J_1, D_1, y, v_1, e or x as G, C, y, x, e, respectively), we find a D_1-Tutte path P_1 between y and v_1 in J_1 such that $e \in E(P_1)$ if $e \in E(J_1)$ and $x \in V(P_1)$ if $x \in V(J_1)$. If $x \notin V(G_2)$ then by (2.1) (with G_2, C_2, v_1, v_2, e as G, C, x, y, e, respectively), and if $x \in V(G_2)$ then by (2.4) (with G_2, C_2, v_1, x, v_2, e as G, C, x, y, e, f, respectively), we find a C_2-Tutte path P_2 from v_1 in G_2 such that $v_2 \in V(P_2)$, $e \in E(P_2)$, and $x \in V(P_2)$ if $x \in V(G_2)$. It is easy to check that $P^* := P_1 \cup P_2$ is the desired path. Now we may assume that $x \notin V(G')$. Thus $x \in V(J_i) - \{y, v_i\}$. By (2.1) (with G', C', y, v_i, e as G, C, x, y, e, respectively), we find a C'-Tutte path P' in G' with ends y and v_i such that $e \in E(P')$. Again by (2.1) (with J_i, D_i, v_i, y, x as G, C, x, y, e, respectively), we find a D_i-Tutte path P'' with ends y and v_i such that $x \in V(P'')$. It is easy to check that $P^* := P' \cup (P'' - y)$ gives the desired path. This completes the construction of the path P^*.

Suppose $v_k \in V(P^*)$. If there exists no P^*-bridge of G containing edges of both C and D, then $P := P^*$ satisfies both the conclusion of (3.1) and (A) of (3.2) with H null. So assume that there exists a P^*-bridge of G containing edges of both C and D, then this bridge is J_k and $i = k$. In this case, y is an end of P^* and $x \notin V(J_k) - \{y, v_k\}$. Therefore for (3.2), the path $P := P^*$, graphs $H := J_k$ and $H^* = G - (V(H) - \{y, v_k\})$, and vertices $b := y, a := v_k$ satisfy (B). To prove (3.1), we apply (2.1) (with G', C', y, v_k, x as G, C, x, y, e, respectively) to find a C'-Tutte path Q between y and v_k in G' such that $x \in V(Q)$. We also apply (2.1) to J_k (with J_k, D_k, y, v_k as G, C, x, y, respectively) to find a D_k-Tutte path R between y and v_k. Then it is easy to see that $P := Q \cup (R - y)$ satisfies the conclusion of (3.1).

Now assume that $v_k \notin V(P^*)$. Then $i \neq k$ (since $v_i \in V(P^*)$) and e is not incident with v_k, and so, we only need to prove (3.2) (because for (3.1), e is incident with v_k). Let H be the P^*-bridge of G containing v_k, and let a, b, c be its attachments labeled so that $b = y$, $a \in V(C)$ and $c \in V(C_2) - V(C)$. Since $i \neq k$, $D \subseteq G - I(H)$. It follows from the construction of P^* (in particular, $v_i \in V(P^*)$) that P^* is a $((C - I(H) \cup D)$-Tutte path in $G - I(H)$ and no P^*-bridge of G contains edges of both C and D. Hence, $P := P^*, H, a, b, c$ satisfy (A) of (3.2).

Case 2. There is some $i \in \{1, 2, \ldots, k\}$ such that D is a subgraph of J_i.

First, we dispose of the case $i = 1$. To this end assume that $i = 1$, let $p := y$, $q := v_1$, let K be obtained from G by deleting vertices and edges of J_1 except p and q, and let us consider the separation (K, J_1) in G. To prove (3.1), let $J_1' := J_1 + yv_1$ and $C' := v_1Cy + yv_1$ (so that C' is the outer cycle of J_1'). Inductively applying (3.1) to J_1' (with J_1', C', x, y as G, C, x, y, respectively) we find a $(C' \cup D)$-Tutte path P' from y in J_1' such that $x \in V(P')$ and no P'-bridge of J_1' contains vertices of both $C' - P'$ and $D - P'$. If $yv_1 \notin E(P')$, then $P := P'$ satisfies the conclusion of (3.1) (and K is a P-bridge of G with two attachments and contains no vertex of D). If $yv_1 \in E(P')$, then we apply (2.1) to K (with K, y, v_1, v_k as G, x, y, e, respectively) to find a yCv_1-Tutte path P'' between y and v_1 such that $v_k \in V(P'')$, and clearly, $P := (P' - yv_1) \cup P''$ satisfies the conclusion of (3.1). Now let us prove (3.2). The hypothesis of (3.2) implies that $e \in E(J_1)$. By (d), the edge e has ends y and v_1, and so, J_1 is 2-connected. By inductively applying (3.2) to J_1 (with J_1, D_1, y, x as G, C, y, x, respectively), we find a_1, b_1, c_1, H_1, P (as a, b, c, H, P, respectively) satisfying (A) of (3.2) or we find $a_1, b_1, (H_1, H_1^*), P$, (as $a, b, (H, H^*), P$, respectively) satisfying (B) of (3.2). Notice that K becomes a P-bridge of G with attachments y and v_1. Also note that if the graph H_1 obtained by induction is non-null then H_1 contains no vertex of C and y is an attachment of H_1. Hence, (A) of (3.2) holds with H null.

So we may assume that $i > 1$.

Let G' be obtained from G by deleting $V(J_j) - \{v_j, y\}$ for all $j = i, i+1, \ldots, k$. Note that $G' + yv_i$ is 2-connected, and let $C' := (v_iC_2v_k \cup v_kCy) + yv_i$ (so that C' is the outer cycle of $G' + yv_i$). By (2.3) (with $G' + yv_i, C', y, v_i, v_k, e$ as G, C, x, y, e, f, respectively), there exists a C'-Tutte path P' between y and v_i in G' such that $e \in E(P')$ and $v_k \in V(P')$. Let $G_3 = J_i + yv_i$ and let e_3 be the edge of G_3 with ends y and v_i. If $i = k$, then let C_3 be the cycle of G_3 consisting of the edge e_3 and the path yCv_k. If $i < k$ let C_3 be an arbitrary facial cycle of G_3 that includes the edge e_3. In either case we may assume that a plane representation of G_3 is chosen so that C_3 is its outer cycle and y, v_i, e_3 appear on C_3 in the clockwise order listed.

To prove (3.1), we inductively apply (3.1) to G_3 (with G_3, C_3, y, x as G, C, y, x, respectively), and we find a $(C_3 \cup D)$-Tutte path P_3 from y in G_3 such that $x \in V(P_3)$ and no P_3-bridge of G_3 contains vertices of both $C_3 - P_3$ and $D - P_3$. If $e_3 \notin E(P_3)$, then $P := P_3$ satisfies the conclusion of (3.1) (and $G' \cup J_{i+1} \cup \ldots \cup J_k$ is contained in a P_3-bridge of G

with two attachments). If $e \in E(P_3)$, then $P := P' \cup (P_3 - y)$ satisfies the conclusion of (3.1).

To prove (3.2), we wish to inductively apply (3.2) to G_3 (with G_3, C_3, y, x, e_3 as G, C, y, x, e, respectively). Since e_3 is incident with y and because y, v_i, e_3 occur on C_3 in this clockwise order, we see that the hypothesis of (3.2) is satisfied. Hence, there are two possibilities. (A) There exist a C_3-flap H_3 in G_3 with attachments a_3, b_3, c_3 ($a_3 = b_3 = c_3$ if H_3 is null) and a $((C_3 - I(H_3)) \cup D)$-Tutte path P_3 from b_3 in $G_3 - I(H_3)$ such that $D \subseteq G - I(H_3)$, e_3, b_3, y, a_3 occur on C_3 in this clockwise order, $y \in (V(H_3) - \{a_3\}) \cup \{b_3\}$, $e_3 \in E(P_3)$, $x, a_3, c_3 \in V(P_3)$, no P_3-bridge of G_3 contains vertices of both $C_3 - P_3$ and $D - P_3$. (B) There exist $a_3, b_3 \in V(C_3) \cap V(D)$, a separation (H_3, H_3^*) in G_3 with $V(H_3) \cap V(H_3^*) = \{a_3, b_3\}$, and a $(C_3 \cup D)$-Tutte path P_3 from b_3 in G_3 such that e_3, b_3, y, a_3 occur on C_3 in this clockwise order, $b_3C_3a_3 \cup b_3Da_3 \subseteq H_3$, $a_3C_3b_3 \cup a_3Db_3 \subseteq H_3^*$, $P_3 \subseteq H_3^*$, $y \in (V(H_3) - \{a_3\}) \cup \{b_3\}$, $e_3 \in E(P_3)$, $a_3, x \in V(P_3)$, and no P_3-bridge of G_3 distinct from H_3 contains vertices of both $C_3 - P_3$ and $D - P_3$. Since e_3 is incident with y we see that $b_3 = y$. If $i \neq k$, then $P := P' \cup (P_3 - y)$ satisfies (A) of (3.2) with H null. If $i = k$, then $P := P' \cup (P_3 - y)$, H_3 or $(H_3, H_3^* \cup G')$ (as H or (H, H^*)), and the vertices a_3, b_3, c_3 (as a, b, c) satisfy (A) or (B) of (3.2).

Case 3. D is a subgraph of G_2.

We will construct the desired path P as the union of two paths P_1, P_2 in graphs G_1 (to be defined), G_2, respectively. This will be done simultaneously for (3.1) and (3.2), with the understanding that when inductively applying (3.2) to G_2 in the proof of (3.1) we let e be the unique edge of $C_2 \cap C$ incident with v_k.

Since $D \subset G_2$ and G_2 is a block, G_2 is 2-connected. To construct P_2 we wish to inductively apply (3.2) to G_2, C_2, D, v_1, x, e (as G, C, D, y, x, e, respectively) if $e \in E(G_2)$, or we wish to inductively apply (3.1) to G_2, C_2, D, v_1, x (as G, C, D, y, x, respectively) if $e \notin E(G_2)$. In order to apply (3.2) to G_2, we must verify the absence in G_2 of vertices p, q and a 2-separation as in the hypothesis of (3.2). Suppose for a contradiction that there exist distinct vertices $p, q \in V(C_2)$ and a 2-separation (G_2'', G_2') in G_2 such that $V(G_2'') \cap V(G_2') = \{p, q\}$, p, v_1, e, q occur on C_2 in the clockwise order listed, $pC_2q \subseteq G_2''$, and $qC_2p \cup D \subseteq G_2'$. Since v_1 and e both belong to $pC_2q \cap C$, we deduce that $p, q \in V(C)$. Let $G_1' := G_2'' \cup J_1 \cup J_2 \cup \ldots \cup J_k$; then (G_1', G_2') is a 2-separation in G that contradicts the hypothesis of (3.2). This verifies the absence of vertices p, q and a corresponding 2-separation in G_2, and hence we may inductively apply (3.2) to G_2, C_2, D, v_1, x, e when $e \in E(G_2)$.

First, we construct P_2. If $e \notin E(G_2)$ (this only applies to the proof of (3.2)), then by inductively applying (3.1) to G_2, we find a $(C_2 \cup D)$-Tutte path P_2 from v_1 in G_2 such that $x \in V(P_2)$ and no P_2-bridge of G_2 contains vertices of both $C_2 - P$ and $D - P$, and we define H_2 to be null. Now assume $e \in E(G_2)$. By inductively applying (3.2) to G_2, we have two possibilities. (A) There exist a C_2-flap H_2 in G_2 with attachments a_2, b_2, c_2 ($a_2 = b_2 = c_2 = v_1$ if H_2 is null), and a $((C_2 - I(H_2)) \cup D)$-Tutte path P_2 from b_2 in $G - I(H_2)$ such that $D \subseteq G_2 - I(H_2)$, e, b_2, v_1, a_2 occur on C_2 in this clockwise order, $v_1 \in (V(H_2) - \{a_2\}) \cup \{b_2\}$, $e \in E(P_2)$, $x, a_2, c_2 \in V(P_2)$, and no P_2-bridge of G_2 contains vertices of both $C_2 - P_2$ and $D - P_2$. (B) There exist $a_2, b_2 \in V(C_2) \cap V(D)$, a separation

(H_2, H_2^*) in G_2 with $V(H_2) \cap V(H_2^*) = \{a_2, b_2\}$, and a $(C_2 \cup D)$-Tutte path P_2 from b_2 in G_2 such that e, b_2, v_1, a_2 occur on C_2 in this clockwise order, $b_2C_2a_2 \cup b_2Da_2 \subseteq H_2$, $a_2C_2b_2 \cup a_2Db_2 \subseteq H_2^*$, $P_2 \subseteq H_2^*$, $v_1 \in (V(H_2) - \{a_2\}) \cup \{b_2\}$, $a_2, x \in V(P_3)$, $e_2 \in E(P_2)$, and no P_2-bridge of G_2 distinct from H_2 contains vertices of both $C_2 - P_2$ and $D - P_2$.

We now define the graph G_1 and construct the path P_1 in G_1. Assume first that H_2 is null. Let $G_1 := J_1$, and apply (2.1) to J_1 to find a D_1-Tutte path P_1 between y and v_1 in G_1 such that $e \in E(P_1)$ if $e \in E(J_1)$. Now assume that H_2 is non-null. Then $e \in E(G_2)$. Since e, b_2, v_1, a_2 occur on C_2 in the clockwise order listed, it follows that $b_2 \in V(C)$. We may assume that $a_2 \in V(c_1C_2v_k)$; for otherwise, we only need to prove (3.2) (because $e \in E(P)$ is incident with v_k), and the vertices $a2, b_2, c_2$ (as a, b, c) or a_2, b_2 (as a, b), the graph $H_2 \cup J_1 \cup \ldots \cup J_k$ (as H), and the path P_2 (as P) satisfy (A) or (B) of (3.2). Let G_1 be the union of H_2 and those J_i with $v_i \in (V(H_2) - \{a_2\}) \cup \{b_2\}$. Assume first that we have (A) above for P_2. Note that $G_1' := G_1 + \{a_2c_2, c_2b_2, ya_2\}$ is 2-connected, and we may assume that G_1' has a plane representation such that a_2c_2, b_2c_2, ya_2 and $C \cap G_1$ are on its outer cycle C_1'. By applying (2.3) to the graph G_1' we find a C_1'-Tutte path P_1' between y and a_2 such that $a_2c_2, c_2b_2 \in E(P_1')$, and let $P_1 := P_1' - \{a_2, c_2\}$. Now assume that we have (B) above for P_2. Then we apply (2.1) to the graph $G_1'' := G_1 + a_2y$ to get a $((C \cup D) \cap G_1'')$-Tutte path P_1' between a_2 and b_2 such that $a_2y \in E(P_1')$, and let $P_1 = P_1' - a_2$.

Let $P := P_1 \cup P_2$. If $v_k \in V(P)$ we let $a = b = c = y$ and let H be null. If $v_k \notin V(P)$, then v_k belongs to a P_2-bridge H' of G_2 with two attachments. Let a, c be the attachments of H' such that $c \in V(C_2) - V(C)$ and $a \in V(C) \cap V(C_2)$, let $b = y$, and let H be the union of H' and those J_i with $v_i \in V(H') - \{c\}$. Then H is a C-flap with attachments a, b, c, and P is a path from b in $G - I(H)$ such that $a, c \in V(P)$ and $e \in E(P)$. Note when $v_k \notin V(P)$, $e \notin E(G_2)$ and, therefore, we only need to show (3.2).

To prove that P satisfies the conclusions of (3.1) and (3.2), we first notice that P is a $((C - I(H)) \cup D)$-Tutte path in $G_1 \cup G_2 = G_1 \cup (G_2 - I(H_2))$, and by planarity, no P-bridge of $G_1 \cup G_2$ contains vertices of both $C - P$ and $D - P$. Let J be a P-bridge of G distinct from H. Then J is either a P-bridge of $G_1 \cup G_2$, or $J = J_i$ for some i (in which case J has at most two attachments), or J is the union of a P_2-bridge J' of $G_2 - I(H_2)$ and some of the J_i's (in which case, J' includes an edge of C_2, and hence has exactly two attachments on P_2, and so we deduce that J has exactly three attachments on P). Hence, P is a $((C - I(H)) \cup D)$-Tutte path in $G - I(H)$. Note that $D \subseteq G - I(H)$, and so, we have (A) of (3.2). For (3.1), because e is incident with v_k, we have $v_k \in V(P_2)$. Therefore H is null, and hence P is a $(C \cup D)$-Tutte path in G satisfying (3.1). ∎

Now we show that (3.2) also holds when the vertex x is replaced by an edge f not incident with e. See Figure 3 for an illustration (with f replacing x).

(3.3) **Theorem**. *Let G be a 2-connected plane graph with outer cycle C and another facial cycle D. Let $y \in V(C)$, $f \in E(D)$, and $e \in E(C)$. Assume that $\{e, f\}$ is a matching in G, and assume that there do not exist distinct vertices $p, q \in V(C)$ and a 2-separation (G_1', G_2') in G such that $V(G_1') \cap V(G_2') = \{p, q\}$, p, y, e, q occur on C in the clockwise order listed, $pCq \subseteq G_1'$, and $qCp \cup D \subseteq G_2'$. Then one of the following holds:*

(A) *there exist a C-flap H in G with attachments a, b, c ($a = b = c = y$ if H is null) and a $((C - I(H)) \cup D)$-Tutte path P from b in $G - I(H)$ such that $D \subseteq G - I(H)$, e, b, y, a occur on C in this clockwise order, $y \in (V(H) - \{a\}) \cup \{b\}$, $e, f \in E(P)$, $a, c \in V(P)$, and no P-bridge of G contains vertices of both $C - P$ and $D - P$; or*

(B) *there exist $a, b \in V(C) \cap V(D)$, a separation (H, H^*) in G with $V(H) \cap V(H^*) = \{a, b\}$, and a $(C \cup D)$-Tutte path P from b in G such that e, b, y, a occur on C in this clockwise order, $bCa \cup bDa \subseteq H$, $aCb \cup aDb \subseteq H^*$, $P \subseteq H^*$, $y \in (V(H) - \{a\}) \cup \{b\}$, $a \in V(P)$, $e, f \in E(P)$, and no P-bridge of G distinct from H contains vertices of both $C - P$ and $D - P$.*

Proof. Let G', D' denote the graphs obtained from G, D, respectively, by subdividing the edge f with a vertex x. It is clear that G', C, D', y, x, e (as G, C, D, y, x, e, respectively) satisfy the hypothesis of (3.2). By applying (3.2) to G', C, D', y, x, e, we find P' (as P in (3.2)) and H (respectively, (H, H^*)) satisfying (A) (respectively, (B)) of (3.2).

First assume that x is not an end of P'. Then it is easy to check that $P := (P' - x) + f$ and H (respectively, $(H, (H^* - x) + f)$) satisfy (A) (respectively, (B)) of (3.3).

So assume that x is an end of P'. Let x_1, x_2 denote the neighbors of x, and assume that $xx_1 \in E(P')$. If $x_2 \notin V(P')$ then x_2 is contained in some P'-bridge of G' with two attachments (one of these is x), and so, it is easy to check that $P := ((P' - x) \cup \{x_2\}) + f$ and H (respectively, $(H, (H^* - x) + f)$) satisfy (A) (respectively, (B)) of (3.3). Therefore, we may assume that $x_2 \in V(P')$. Choose the edge g of P' incident with x_2 such that $P := ((P' - x) + f) - g$ is a path. Because $\{e, f\}$ is a matching in G, we have $g \neq e$. Now P and H (respectively, $(H, (H^* - x) + f)$) satisfy (A) (respectively, (B)) of (3.3). ∎

4 Toroidal graphs

Let G be a graph embedded in the torus and let R be a face of G. We define the *R-width* of G to be the minimum number $|\gamma \cap G|$ taken over all non-null homotopic simple closed curves γ in the torus passing through R. Note that we can homotopically shift curves in the torus so that curves that we will deal with meet G only at vertices.

To prove (1.2), we will choose a face R of G and apply induction on the R-width of G. Lemma (4.1) below deals with the base case. For the sake of induction, we introduce (4,C)-connected graphs. Let G be a connected graph and C be a subgraph of G; then we say that G is (4,C)-connected if, for any $T \subseteq V(G)$ such that $|T| \leq 3$ and $G - T$ is not connected, every component of $G - T$ contains a vertex of C.

(4.1) **Lemma**. *Let G be a 2-connected graph embedded in the torus with face width at least 2, let R be a face of G bounded by a cycle C in G, and let $y \in V(C)$. Assume that the R-width of G is 2 and G is (4,C)-connected. Then there exist a C-flap H in G with attachments a, b, c ($a = b = c = y$ if H is null) and a $(C - I(H))$-Tutte path P from b in $G - I(H)$ such that b, y, a occur on C in this clockwise order, $y \in (V(H) - \{a\}) \cup \{b\}$, $a, c \in V(P)$, and every P-bridge B of G containing an edge of C has a plane representation with $B \cap (C \cup P)$ on its outer walk. Moreover, $|V(P) \cap V(C)| \geq 2$ and $|V(P)| \geq 4$.*

Proof. Let $u, v \in V(C)$ be distinct such that there is a non-null homotopic simple closed curve γ passing through R and meeting G only in u and v. We may choose the triple (γ, u, v) such that $y \in V(uCv) - \{v\}$, and subject to this, uCy is minimal.

We cut the torus open along γ and, as a result, we obtain a plane graph G' and vertices u', v', u'', v'' of G' such that (by choosing appropriate notation) (1) u' and v' belong to the outer walk C' of G', and $E(uCv)$ induces a path F' on C' in the clockwise order from u' to v', (2) u'' and v'' belong to a facial walk C'' of G', and $E(vCu)$ induces a path F'' on C'' in the clockwise order from u'' to v'', and (3) identifying u' with u'' as u and identifying v' and v'' as v, we obtain G from G'.

Clearly, $u', v' \notin V(C'')$ and $u'', v'' \notin V(C')$. For otherwise, there is a non-null homotopic simple closed curve in the torus intersecting G only at u or v. This contradicts the assumption that the face-width of G is at least 2.

Note that if $y \neq u$ then $y \in V(F') - \{u', v'\}$. Let $y := u'$ if $y = u$. Also note that if $|V(F')| \geq 3$ then $F' + u'v'$ is a cycle, and if $|V(F')| = 2$ then $F' + u'v' = F'$ and $G' + u'v' = G'$. So when $|V(F')| \geq 3$, we draw $u'v'$ in the infinite face of G' so that $F' + u'v'$ is the outer cycle of $G' + u'v'$. Because the face width of G is 2, $F' \cap F'' = \emptyset$ and $(G' + u'v') - F''$ has a cycle containing $F' + u'v'$. Let L denote the block of $(G' + u'v') - F''$ containing one such cycle, and let D' denote the outer cycle of L. Then $D' = F' + u'v'$ if $|V(F')| \geq 3$, and otherwise, $F' \subseteq D'$. Let D be the cycle bounding the face of L which contains F'' (as a subset of the plane). Let $u^*, v^* \in V(F'')$ with $u^* F'' v^*$ maximal such that u'', u^*, v^*, v'' occur on F'' in this order and u^* and v^* are attachments of some $(L \cup F'')$-bridges of $G' + u'v'$ which also have attachments on L. Let $x, z \in V(D)$ such that $\{x, u^*\}$ is contained in a $(L \cup F'')$-bridge of $G' + u'v'$, $\{z, v^*\}$ is contained in a $(L \cup F'')$-bridge of $G' + u'v'$, and subject to these conditions, zDx is minimal. Then $zDx - \{z, x\}$ contains no attachment of any $(L \cup F'')$-bridge of $G' + u'v'$. See Figure 5. Possibly, $z = x$.

Next we define a graph M and a subgraph P_M of M, according to whether or not we have $x = z$. First, assume $x \neq z$. We let $u_1 = u''$ and $v_1 = v''$, let $V(M) = \{x, u'', v''\}$ and $E(M) = \emptyset$, and let $P_M = \emptyset$. Now assume $x = z$. Let $u_1, v_1 \in V(F'')$ such that u'', u_1, v_1, v'' occur on F'' in order, $\{x, u_1\}$ is contained in a $(L \cup F'')$-bridge of $G' + u'v'$, $\{x, v_1\}$ is contained in a $(L \cup F'')$-bridge of $G' + u'v'$, and subject to these conditions, $u_1 F'' v_1$ is minimal. Let M denote the union of $u_1 F'' u'' \cup v'' F'' v_1$ and those $(L \cup F'')$-bridges of $G' + u'v'$ whose attachments are all contained in $V(u_1 F'' u'') \cup V(v'' F'' v_1) \cup \{x\}$. By applying (2.3) to $M + \{u_1 x, v_1 x\}$ we find a $(u'' F'' u_1 \cup v'' F'' v_1)$-Tutte path P'_M from u'' to v'' and through $u_1 x$ and $v_1 x$. Let $P_M = P'_M - x$, which consists of disjoint paths from u'', v'' to u_1, v_1, respectively.

We divide the remainder of the proof into two cases.

Case 1. $x \notin \{u', v'\}$, and both edges of D incident with x are incident with the edge $u'v'$.

In this case, we must have $x = z$, for otherwise, the edge of zDx incident with x is not incident with u' or v' (because $zDx \subseteq C''$ and $u', v' \notin V(C'')$). Also the two edges of D incident with x must be xu' and xv'.

Therefore $u_1 \neq u''$ and $v_1 \neq v''$, for otherwise, there is a non-null homotopic simple closed curve in the torus intersecting G only at u or v, contradicting the assumption that the face width of G is at least 2. Moreover, because G is $(4, C)$-connected, $\{u', v', x\}$ induces a facial triangle in $G' + u'v'$. Hence, x has exactly two neighbors in L, namely u' and v'.

Let L' denote the plane graph obtained from $G' + u'v'$ by deleting $M - \{u_1, v_1\}$, adding edges u_1u' and v_1v', and deleting $u'v'$ when $|V(F')| > 2$, such that $Q := (u_1F''v_1 \cup F') + \{u_1u', v_1v'\}$ is the outer cycle of L' on which $v', v'v_1, u_1u', u', y$ occur in clockwise order.

By applying the mirror image version of (2.2) (with $L', Q, v', y, u'u_1, v'v_1$ as G, C, x, y, e, f, respectively), there exist a C-flap H in L' with attachments a, b, c ($a = b = c = y$ if H is null) and a $(Q - I(H))$-Tutte path P' from b to v' in $L' - I(H)$ such that u', b, y, a, v' occur on Q (and hence, on D') in clockwise order, $y \in (V(H) - \{a\}) \cup \{b\}$, $u_1u', v_1v' \in E(P')$, and $a, c \in V(P')$. (Note that H is null when $|V(F')| = 2$.) By planarity, no P'-bridge of L' contains vertices of both $D' - P'$ and $D - P'$.

Let P denote the path in G induced by $(E(P') - \{u'u_1, v'v_1\}) \cup E(P_M) \cup \{xv'\}$. Then P is a path from y to x, $u, v, x, u_1, v_1 \in V(P)$, and b, y, a occur on C in clockwise order. Hence, $|V(P) \cap V(C)| \geq 2$ and $|V(P)| \geq 4$ (because $x \notin \{u', v'\}$, $u'' \neq u_1$ and $v'' \neq v_1$). Clearly, H is a C-flap in G. Moreover, every P-bridge of G is either a P'-bridge of L', or a P'_M-bridge of M, or a bridge induced by the edge $u'x$. Hence, P is a $(C - I(H))$-Tutte path from b in $G - I(H)$ such that $a, c \in V(P)$. It is also clear that every P-bridge of G containing an edge of C is either a P'_M-bridge of M containing an edge of $M \cap F''$, or a P'-bridge of L' containing an edge of F' or $u_1F''v_1$ but not both. Hence, every P-bridge B of G containing an edge of C has a plane representation with $B \cap (C \cup P)$ on its outer walk. So P gives the desired path.

Case 2. Either $x \in \{u', v'\}$ or there is an edge $f \in E(D)$ incident with x such that $\{f, u'v'\}$ is a matching.

When $x \in \{u', v'\}$, we pick an arbitrary vertex $x' \in V(D) - \{u', v'\}$. Next, we show that we may apply (3.2) to $L, D', D, y, x', u'v'$ (as G, C, D, y, x, e, respectively) and, in the case when the edge f exists, apply (3.3) to $L, D', D, y, f, u'v'$ (as G, C, D, y, f, e, respectively). When $|V(F')| = 2$, we apply a mirror image version of (3.2) or (3.3), and the hypothesis of (3.2) or (3.3) holds because $y = u'$. When $|V(F')| \geq 3$, we claim that there do not exist vertices $p, q \in V(D')$ and a 2-separation (G'_1, G'_2) in L such that $V(G'_1) \cap V(G'_2) = \{p, q\}$, $p, y, u'v', q$ occur on D' in this clockwise order, $pD'q \subseteq G'_1$, and $qD'p \cup D \subseteq G'_2$. This is obvious if $u' = y$. So assume that $u' \neq y$. Suppose the above p, q, and (G'_1, G'_2) do exist. Then we can choose a non-null homotopic simple closed curve γ_1 in the torus such that γ_1 meets G only in p and q and passes through R. Because $y \in pCq - q$ and because pCy is properly contained in uCy, (γ_1, p, q) contradicts the choice of (γ, u, v). So we may apply (3.2) to $L, D', D, y, x', u'v'$ or apply (3.3) to $L, D', D, y, f, u'v'$.

By (3.2) (when $x \in \{u', v'\}$) and (3.3) (when f exists), there are two possibilities, and we treat them in two separate cases.

Subcase 2.1. There exist a C-flap H in L with attachments a, b, c ($a = b = c = y$ if H is null) and a $((D' - I(H)) \cup D)$-Tutte path P' from b in $L - I(H)$ such that $D \subseteq L - I(H)$, $u'v', b, y, a$ occur on D' in this clockwise order (or when $|V(F')| = 2$, $u'v', a, y = b$ occur on D' in this clockwise order), $y \in (V(H) - \{a\}) \cup \{b\}$, $x' \in V(P')$ when $x \in \{u', v'\}$, $f \in E(P')$ when the edge f exists, $u'v' \in E(P')$, $a, c \in V(P')$, and no P'-bridge of L

contains vertices of both $D' - P'$ and $D - P'$.

Note that when $|V(F')| = 2$, H must be null. For otherwise, $H - \{a, b, c\}$ is a component of $G - \{a, b, c\}$ containing no vertex of C, contradicting the assumption that G is (4,C)-connected. See Figure 5.

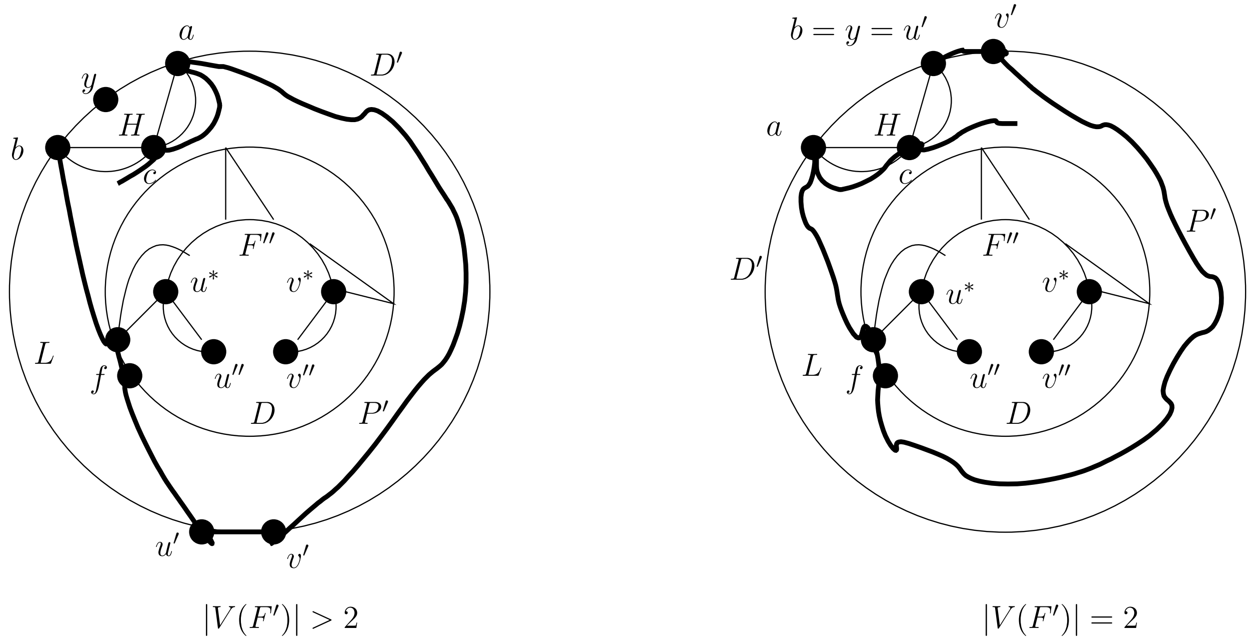

Figure 5: Subcase 2.1

Next we apply (2.5) to find a path P_1 from u_1 to v_1. For this purpose, we view $(G' + u'v') - (M - \{u_1, v_1, x\}), L, u_1, v_1, x, u_1F''v_1, D, P'$ as K, L, p, q, u, Q, Q', T in (2.5), respectively. It is straightforward to verify that the conditions of (2.5) are satisfied. (In particular, $|V(P') \cap V(D)| \geq 2$ because $f \in E(D) \cap E(P')$ or because $x, x' \in V(P') \cap V(D)$.) By (2.5), there is a path P_1 (as S in (2.5)) between u_1 and v_1 in $((G' + u'v') - (M - \{u_1, v_1, x\})) - P'$ such that $P' \cup P_1$ is an $u_1F''v_1$-Tutte subgraph of $(G' + u'v') - (M - \{u_1, v_1, x\})$ and every P'-bridge of L containing no edge of D is also a $(P' \cup P_1)$-bridge of $(G' + u'v') - (M - \{u_1, v_1, x\})$. By planarity, no $(P' \cup P_1)$-bridge of $(G' + u'v') - (M - \{u_1, v_1, x\})$ contains edges of both F' and $u_1F''v_1$. Hence, $P' \cup P_1 \cup P_M$ is a $((D' - I(H)) \cup F'')$-Tutte subgraph of $G' + u'v'$.

Clearly, H is a C-flap in G, and $E((P' - u'v') \cup P_1 \cup P_M)$ induces a $(C - I(H))$-Tutte path P from b in $G - I(H)$ such that $a, c \in V(P)$. It is also clear that every P-bridge of G containing an edge of C is a $(P' \cup P_1 \cup P_M)$-bridge of $G' + u'v'$ containing an edge of F' or F'' but not both. Hence, every P-bridge B of G containing an edge of C has a plane representation with $B \cap (C \cup P)$ on its outer walk. Because $u, v \in V(P) \cap V(C)$, $|V(P) \cap V(C)| \geq 2$.

We conclude this case by showing $|V(P)| \geq 4$. This is obvious when the edge f exists, because $f \in E(P \cap D)$ and $\{f, u'v'\}$ is a matching. Now assume $x \in \{u', v'\}$. Then $|V(P)| \geq 3$ (because $u, v, x' \in V(P)$). Suppose $|V(P)| = 3$. Then $V(P_M \cup P_1) = 2$. This implies $|V(F'')| = 2$ and, therefore, since G is (4,C)-connected, $\{x, u'', v''\}$ is not a cut in

G. Hence $V(M) = \{x, u'', v''\}$ and $E(M) = \{xu'', xv''\}$. This shows $x \in C''$, contradicting the fact that $u', v' \notin C''$. So we have $|V(P)| \geq 4$.

Subcase 2.2. There exist $a', b' \in V(D') \cap V(D)$, a separation (H', H^*) in L with $V(H') \cap V(H^*) = \{a', b'\}$, and a $(D' \cup D)$-Tutte path P' from b' in L such that $u'v', b', y, a'$ ($u'v', a', y, b'$ when $|V(F')| = 2$) occur on D' in this clockwise order, $b'D'a' \cup b'Da' \subseteq H'$ (or $a'D'b' \cup a'Db' \subseteq H'$ when $|V(F')| = 2$), $a'D'b' \cup a'Db' \subseteq H^*$ (or $b'D'a' \cup b'Da' \subseteq H^*$ when $|V(F')| = 2$), $P' \subseteq H^*$, $y \in (V(H') - \{a'\}) \cup \{b'\}$ ($y = b'$ when $|V(F')| = 2$), $x' \in V(P')$ when $x \in \{u', v'\}$, $f \in E(P')$ when the edge f exists, $u'v' \in E(P')$, $a' \in V(P')$, and no P'-bridge of L other than H' contains vertices of both $D' - P'$ and $D - P'$.

Note that there are vertices of $u_1F''v_1$ which are co-facial with a' or b'. Let $s, t \in V(u_1F''v_1)$ such that (a) u'', u_1, s, t, v_1, v'' occur on F'' in this order, (b) s is co-facial with b' (or a' when $|V(F')| = 2$) and t is co-facial with a' (or b' when $|V(F')| = 2$), and (c) subject to (a) and (b), $sF''t$ is maximal. Let J denote the union of H', $sF''t$, and those $(L \cup F'')$-bridges of $G' + u'v'$ whose attachments are all contained in $V(sF''t) \cup V(H')$.

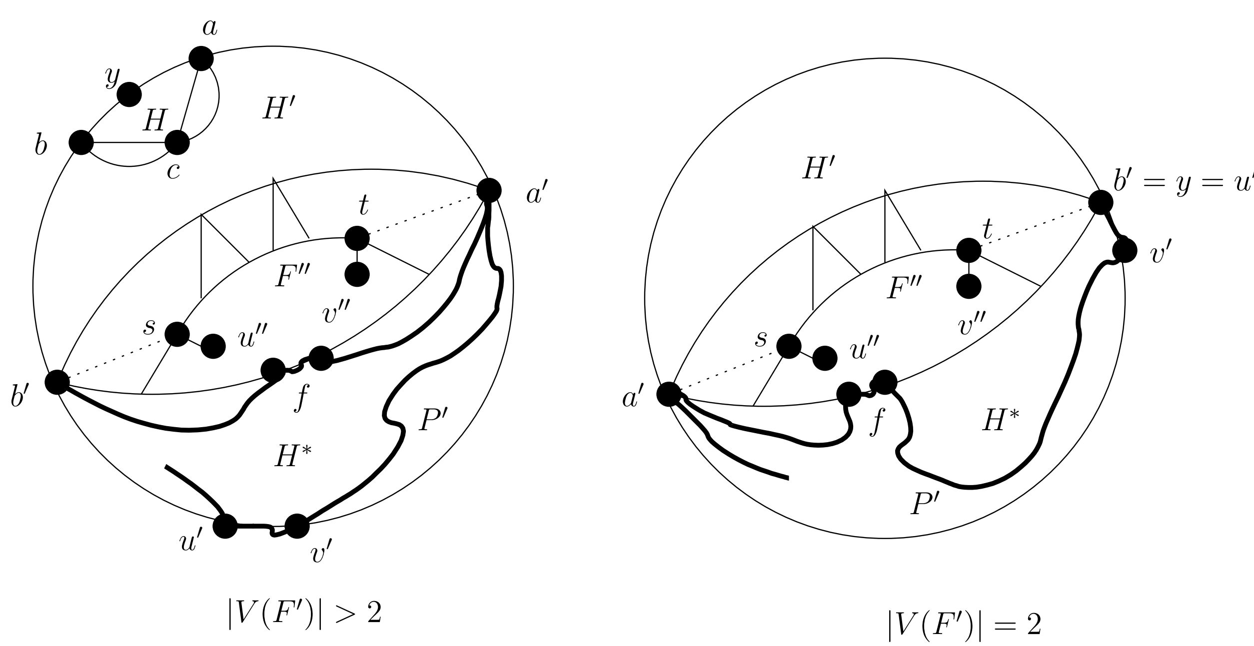

Figure 6: Subcase 2.2

Since G is 2-connected, $J^* := J + \{a't, sb'\}$ (or $J^* := J + \{a's, b't\}$ when $|V(F')| = 2$) is 2-connected. Without loss of generality, assume that $C^* := (b'D'a' \cup sF''t) + \{a't, sb'\}$ (or $C^* := (a'D'b' \cup sF''t) + \{a's, tb'\}$ when $|V(F')| = 2$) is the outer cycle of J^*. By (2.2), with $J^*, C^*, a', y, sb', a't$ (or $a', y, b't, a's$ when $|V(F')| = 2$) as G, C, x, y, e, f, respectively, there exist a C^*-flap H with attachments a, b, c ($a = b = c = y$ if H is null) and a $(C^* - I(H))$-Tutte path P^* between b and a' in $J^* - I(H)$ such that sb', b, y, a, a' (or $a, y = b, b't, sa'$ when $|V(F')| = 2$) occur on C^* in this clockwise order, $y \in (V(H) - \{a\}) \cup \{b\}$, $a, c \in V(P^*)$, and $\{a't, sb'\} \subseteq E(P^*)$ (or $\{a's, b't\} \subseteq E(P^*)$ when $|V(F')| = 2$). Note when $|V(F')| = 2$,

$y = b$ and H is null because G is (4,C)-connected. So H is also a C-flap in G. By planarity, no P^*-bridge of J^* contains edges of both D' and F''.

Next we apply (2.5) to find a path P_1 from u_1 to s and a path P_2 from t to v_1. Note that, since $x \in V(D) \cap V(P')$, x divides the attachments on $D \cap H^*$ of $(L \cup F'')$-bridges so that the P'-bridges of H^* used in constructing P_1 are different from those used in constructing P_2.

Let L_1 denote the union of L, $u_1F''s$, and those $(L \cup F'')$-bridges of G whose attachments are all contained in $V(H^*) \cup V(u_1F''s)$. Then L_1 is connected because $x \in V(H^*)$. We view $L_1, L, u_1, s, x, u_1F''s, D, P'$ as K, L, p, q, u, Q, Q', T in (2.5), respectively. It is straightforward to verify that the conditions of (2.5) hold. (Recall that $|V(P') \cap V(D)| \geq 2$ because $f \in E(D \cap P')$.) By (2.5), $L_1 - P'$ has a path P_1 (as S in (2.5)) between u_1 and s such that $P_1 \cup P'$ is a $u_1F''s$-Tutte subgraph of L_1 and every P'-bridge of L containing no edge of D is also a $(P_1 \cup P')$-bridge of L_1. By planarity, no $(P_1 \cup P')$-bridge of L_1 contains edges of both $u_1F''s$ and D'. Hence, $P_1 \cup P'$ is a $(D' \cup u_1F''s)$-Tutte subgraph of L_1.

Let L_2 denote the union of L, edge ta' (or tb' when $|V(F')| = 2$), $tF''v_1$, and those $(L \cup F'')$-bridges of G whose attachments are all contained in $V(H^*) \cup V(tF''v_1)$. Note that L_2 is connected because of the edge ta' (or tb' when $|V(F')| = 2$). We view $L_2, L, t, v_1, x, tF''v_1, D, P'$ as K, L, p, q, u, Q, Q', T in (2.5), respectively. By (2.5), $L_2 - P'$ has a path P_2 between t and v_1 such that $P_2 \cup P'$ is a $tF''v_1$-Tutte subgraph of L_2 and every P'-bridge of L containing no edge of D is also a $(P_2 \cup P')$-bridge of L_2. By planarity, no $(P_2 \cup P')$-bridge of L_2 contains edges of both $tF''v_1$ and D'. Hence, $P_2 \cup P'$ is a $(D' \cup tF''v_1)$-Tutte subgraph of L_2.

Now $E(P' \cup P^* \cup P_1 \cup P_2 \cup P_M) - \{u'v', a't, sb'\}$ (or $E(P' \cup P^* \cup P_1 \cup P_2 \cup P_M) - \{u'v', a's, b't\}$ when $|V(F')| = 2$) induces a $(C - I(H))$-Tutte path P from b in $G - I(H)$ such that b, y, a occur on C in this clockwise order, $y \in (V(H) - \{a\}) \cup \{b\}$, and $a, c \in V(P)$. It is clear that every P-bridge of G is one of the following: a $(P' \cup P_1)$-bridge of L_1, or $(P' \cup P_2)$-bridge of L_2, or a P^*-bridge of J^*, or a P'_M-bridge of M. Hence, every P-bridge B of G containing an edge of C has a plane representation with $B \cap (C \cup P)$ on its outer walk. Because $u, v \in V(P) \cap V(C)$, $|V(P) \cap V(C)| \geq 2$.

By exactly the same argument as in the end of Subcase 2.1, we can show $|V(P)| \geq 4$. ∎

Before we proceed to the general case, let us prove the following lemma which will be used to extend Tutte paths through C-flaps. See Figure 7 for an illustration.

(4.2) **Lemma**. *Let G be a 2-connected plane graph with outer cycle C, let $s, t, b, c, a \in V(C)$ be distinct such that $tCs = tbcas$, and let $y \in V(sCt - s)$. Suppose that $G - ((sCt - s) \cup \{a, c\})$ contains a path from b to s. Then there exist an sCt-flap H' in G with attachments a', b', c' ($a' = b' = c' = y$ if H' is null) and disjoint paths S and T in $(G - \{a, c\}) - I(H')$ such that s, a', y, b', t occur on C in this clockwise order, $y \in (V(H') - \{a'\}) \cup \{b'\}$, S is from s to b, T is from t to b', $a', c' \in V(S \cup T)$, and $S \cup T \cup \{a, c\}$ is a $(C - I(H'))$-Tutte subgraph of G.*

Proof. Because $G - ((sCt - s) \cup \{a, c\})$ contains a path between b and s, sCb is contained in a cycle in $G - \{a, c\}$. Let G' denote the block of $G - \{a, c\}$ containing sCb, and let C' denote the outer cycle of G'. Thus $sC'b = sCb$. Note that every $(G' \cup \{a, c\})$-bridge of G

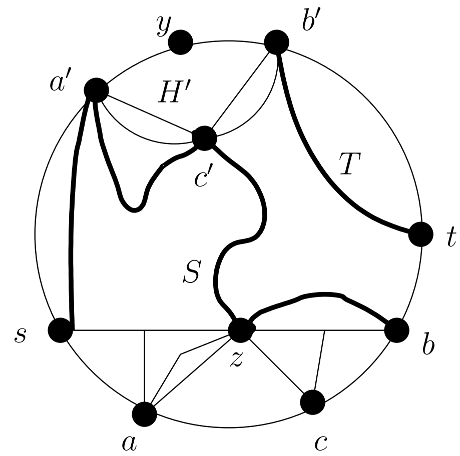

Figure 7: An illustration of Lemma (4.2)

has at most one attachment on G' (because G' is a block of G) and at least one attachment in $\{a, c\}$ (because G is 2-connected).

By planarity of G, there exists $z \in V(bC's)$ such that, for each $(G' \cup \{a, c\})$-bridge B of G, $B \cap C' \subseteq zC's$ if $a \in V(B)$, and $B \cap C' \subseteq bC'z$ if $c \in V(B)$. See Figure 7.

By (2.2) (with G', C', s, y, tb, z as G, C, x, y, e, f, respectively), there exist a C'-flap H' with attachments a', b', c' ($a' = b' = c' = y$ if H' is null) and a $(C' - I(H'))$-Tutte path P between b' and s in $G' - I(H')$ such that s, a', y, b', tb, z occur on C' in this clockwise order, $y \in (V(H') - \{a'\}) \cup \{b'\}$, $tb \in E(P)$, and $a', c', z \in V(P)$. Let S and T denote the disjoint paths in $P - tb$ such that S is between s and b and T is between t and b'.

Note that $a', c' \in V(S \cup T)$ and H' is an sCt-flap in G. Also note that every $(S \cup T \cup \{a, c\})$-bridge of G other than those contained in H' is either a P-bridge of $G' - I(H')$, or a $(G' \cup \{a, c\})$-bridge of G, or induced by the edge tb. Because $tCs = tbcas$, no $(G' \cup \{a, c\})$-bridge of G with three attachments contains an edge of C. Hence, $S \cup T \cup \{a, c\}$ is a $(C - I(H'))$-Tutte subgraph of G. ∎

We now prove (1.2) when the R-width is even for some face R.

(4.3) **Lemma**. *Let G be a 2-connected graph embedded in the torus with face width at least 2. Let R be a face of G, let C be the subgraph of G consisting of vertices and edges of G incident with R, and let $y \in V(C)$. Assume that the R-width of G is a positive even integer, and G is (4,C)-connected. Then there exist a C-flap H' in G with attachments a', b', c' ($a' = b' = c' = y$ if H' is null) and a $(C - I(H'))$-Tutte path P from b' in $G - I(H')$ such that b', y, a' occur on C in this clockwise order, $y \in (V(H') - \{a'\}) \cup \{b'\}$, $a', c' \in V(P)$, and every P-bridge B of G containing an edge of C has a plane representation with $B \cap (C \cup P)$ on its outer walk. Moreover, $|V(P) \cap V(C)| \geq 2$ and $|V(P)| \geq 4$.*

Proof. Because the face width of G is at least 2, C is a cycle in G. By (4.1), we may assume that the R-width of G is at least 4. Hence, $G - C$ contains a cycle which bounds an open disc in the plane containing R. Let L be the block of $G-C$ containing one such cycle. Then L has a face R' which contains R (as subsets of the torus). Since L is 2-connected and the R'-width of L is at least 2, R' is bounded by a cycle, say D. Since G has face width at least 2 and because D bounds a disc in the torus, L has face width at least 2. Since G is (4,C)-connected, L is (4,D)-connected.

Clearly, every $(L \cup C)$-bridge of G has at most one attachment on L and all these attachments are contained in $V(D)$. Let L^* be the union of C, D, and all $(L \cup C)$-bridges of G. Then L^* is contained in the closed disc in the torus bounded by D. Hence, we view L^* as a plane graph such that D is its outer cycle and C is a facial cycle. See Figure 8.

Let $v_1, \ldots, v_n$ be the attachments on D of $(L \cup C)$-bridges of G, and occur on D in this clockwise order. Let $p_i, q_i \in V(C)$ such that q_iCp_i is maximal subject to the following conditions: (a) $\{p_i, v_i\}$ is contained in a $(L \cup C)$-bridge of G, (b) $\{q_i, v_i\}$ is contained in a $(L \cup C)$-bridge of G, and (c) no $(L \cup C)$-bridge of G with an attachment in $V(D)-\{v_i\}$ has an attachment in $V(q_iCp_i)-\{p_i, q_i\}$. Since G is (4,C)-connected, p_i and q_i are well defined and there is some $p_j \neq p_i$ (because otherwise $G - \{v_i, p_i\}$ has a component containing no vertex of C, contradicting $(4, C)$-connectivity of G). For $i = 1, \ldots, n$, let J_i denote the union of q_iCp_i and those $(L \cup C)$-bridges of G whose attachments are all contained in $V(q_iCp_i) \cup \{v_i\}$. Note that $y \in V(p_kCp_{k+1}) - \{p_{k+1}\}$ for some $k \in \{1, \ldots, n\}$, where $p_{n+1} = p_1$, $q_{n+1} = q_1$ and $v_{n+1} = v_1$. In particular, $p_k \neq p_{k+1}$.

Because the R'-width of L is both even and less than the R-width of G, by induction hypothesis (with L, D, R', v_{k+1} as G, C, R, y, respectively), there exist a D-flap H with attachments a, b, c ($a = b = c = y$ if H is null) and a $(D - I(H))$-Tutte path T in $L - I(H)$ from b such that b, v_{k+1}, a occur on D in this clockwise order, $v_{k+1} \in (V(H) - \{a\}) \cup \{b\}$, $a, c \in V(T)$, and every T-bridge B of L containing an edge of D has a plane representation with $B \cap (D \cup P)$ on its outer walk. Note that $|V(T)| \geq 4$ and $|V(T) \cap V(D)| \geq 2$ by (4.1) or by induction hypothesis when the R'-width of L is at least 4.

We distinguish two cases.

Case 1. H is null.

In this case, T is a D-Tutte path from v_{k+1} in L.

Let J denote the union of p_kCp_{k+1} and those $(L \cup C)$-bridges of G whose attachments are all contained in $V(p_kCp_{k+1}) \cup \{v_{k+1}\}$. Since G is 2-connected, $J + p_kv_{k+1}$ is also 2-connected. We can view $J + p_kv_{k+1}$ as a 2-connected plane graph such that p_kCp_{k+1} and p_kv_{k+1} are contained in its outer cycle C_k. By (2.1) (with $J + p_kv_{k+1}, C_k, p_{k+1}, y, p_kv_{k+1}$ as G, C, x, y, e, respectively), $J + p_kv_{k+1}$ has a C_k-Tutte path R between p_{k+1} and y such that $p_kv_{k+1} \in E(R)$.

To find a path S from p_{k+1} to p_k, we let $K = G - (V(J) - \{p_k, p_{k+1}, v_{k+1}\})$. We view $p_{k+1}Cp_k, D, v_{k+1}$ as Q, Q', u in (2.5), respectively. It is straightforward to check that the conditions of (2.5) are satisfied (using the plane representation of L^*). By (2.5), there is a path S in $K - T$ between p_k and p_{k+1} such that $S \cup T$ is a $p_{k+1}Cp_k$-Tutte subgraph of K and every T-bridge of L containing no edge of D is also an $(S \cup T)$-bridge of K. By the plane representation of L^*, every $(S \cup T)$-bridge B of K containing an edge of C has a plane representation with $B \cap (S \cup T \cup C)$ on its outer walk.

Let $P := S \cup T \cup (R - p_kv_{k+1})$. Then every P-bridge of G is either an $(S \cup T)$-bridge

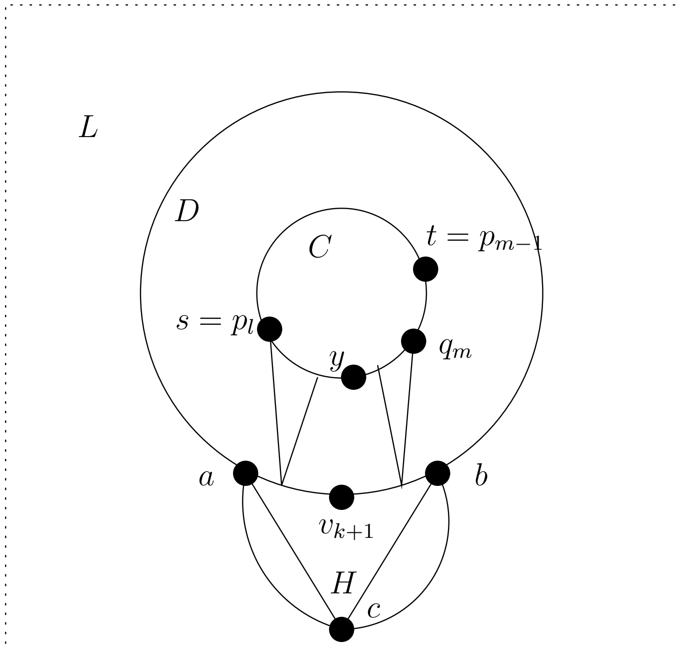

Figure 8: The graph G

of K or a R-bridge of $J + p_k v_{k+1}$. Hence P is a C-Tutte path from y in G such that every P-bridge B of G containing an edge of C has a plane representation with $B \cap (C \cup P)$ on its outer walk. Clearly, $|V(P) \cap V(C)| \geq 2$ (because $p_k, p_{k+1} \in V(P) \cap V(C)$) and $V(P) \not\subseteq V(C)$ and $|V(P)| \geq 4$ (because $|V(T)| \geq 4$).

Case 2. H is non-null.

Let $v_l, v_m \in I(H) \cap V(D)$ such that b, v_m, v_{k+1}, v_l, a occur on D in this clockwise order and such that $v_m D v_l$ is maximal. See Figure 8. Note that if $p_{m-1} = p_l$ then we do not have $p_{m-1} = p_m = \ldots = p_{k+1} = \ldots = p_l$ because $p_{k+1} \neq p_k$. Let J denote the union of $p_{m-1} C p_l$ (when $p_{m-1} \neq p_l$) or C (when $p_{m-1} = p_l$), H, and those $(L \cup C)$-bridges of G whose attachments are all contained in $V(p_{m-1} C p_l) \cup I(H)$ (when $p_{m-1} \neq p_l$) or $V(C) \cup I(H)$ (when $p_{m-1} = p_l$), where $p_0 = p_n$.

Next we define J'. If $p_{m-1} \neq p_l$, then let $J' = J$ (and so, J' has a plane representation with $p_{m-1} C p_l$ and $\{a, b, c\}$ on its outer walk), and let $t := p_{m-1}$ and $s := p_l$. See Figure 8. Now assume that $p_{m-1} = p_l$. Then $p_k \neq p_l$ or $p_{k+1} \neq p_l$ (because $p_k \neq p_{k+1}$. Since G is $(4, C)$-connected, every $(L \cup C)$-bridge of G with an attachment in $D - I(H)$ is induced by a single edge and has $p_{m-1} = p_l$ as an attachment. Also $v_m \neq v_l$; for otherwise, $G - \{v_l, p_l\}$ has a component containing no vertex of C, contradicting $(4, C)$-connectivity of G. Let J' be the plane graph obtained from J by splitting the vertex $p_{m-1} = p_l$ to s and t in a natural way such that C becomes a path and the neighbors of $p_{m-1} = p_l$ in J contained in J_l (respectively, not contained in J_l) become the neighbors of s (respectively, t) and such that J' has a plane representation with $E(C)$ and $\{a, b, c\}$ on its outer walk. See Figure 9.

Let $J'' := J' + \{tb, bc, ca, as\}$, and let C'' be the cycle of J'' induced by $E(C) \cup$

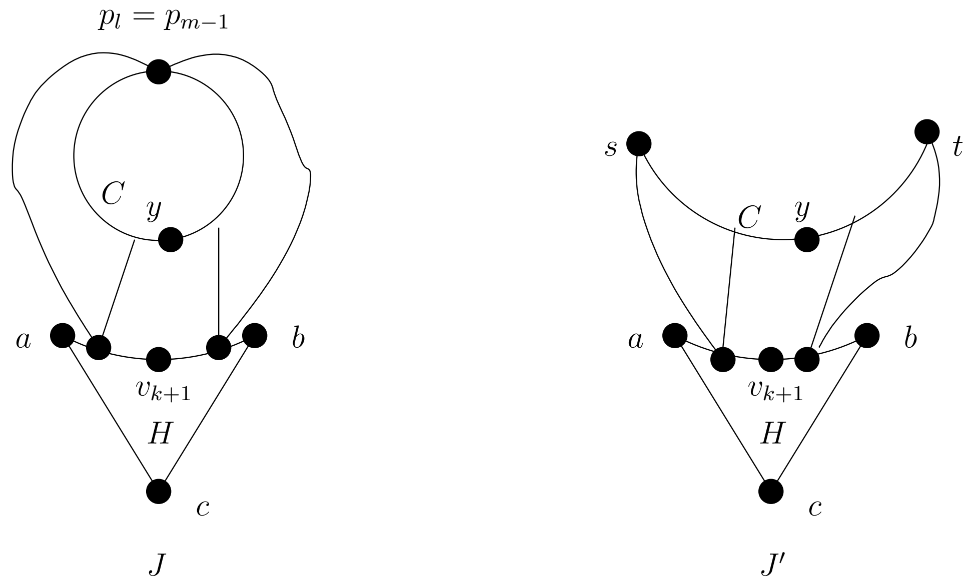

Figure 9: The graph J' when $p_{m-1} = p_l$

$\{tb, bc, ca, as\}$. We can view J'' as a 2-connected plane graph with outer cycle C'' such that $tC''s = tbcas$. It is easy to see that $y \in V(sC''t - s)$ (since $y \in p_kCp_{k+1} - p_{k+1}$) and $J'' - ((sC''t - s) \cup \{a, c\})$ has a path from b to s.

By (4.2) (with J'', C'' as G, C, respectively), there exist an $sC''t$-flap H' in J'' with attachments a', b', c' ($a' = b' = c' = y$ if H' is null) and disjoint paths S', T' in $(J'' - \{a, c\}) - I(H')$ such that s, a', y, b', t occur on C'' in this clockwise order, $y \in (V(H') - \{a'\}) \cup \{b'\}$, S' is between s and b, T' is between b' and t, $a', c' \in V(S' \cup T')$, and $S' \cup T' \cup \{a, c\}$ is an $(C'' - I(H'))$-Tutte subgraph of J''. Note that t, b', y, a', s occur on C in this clockwise order,

If $p_{m-1} = p_l$, then identifying t and s to $p_{m-1} = p_l$ in $T \cup S' \cup T'$ gives the desired path P in G. Note that $|V(P)| \geq 4$ because $|V(T)| \geq 4$. Moreover, $|V(P) \cap V(C)| \geq 2$, for otherwise, C is contained in a P-bridge of G whose attachemnts on P is a cut of size at most 3 in G (because $|V(T)| \geq 4$) showing that G is not $(4, C)$-connected, a contradiction.

So assume that $p_{m-1} \neq p_l$. Let $K = G - (V(J) - (V(H) \cup \{s, t\}))$. We view s, t, sCt, D, a as p, q, Q, Q', u in (2.5), respectively. It is easy to verify that the conditions of (2.5) are satisfied (using the plane representation of L^*). By (2.5), there is a path S in $K - T$ between s and t such that $S \cup T$ is an sCt-Tutte subgraph of K and every T-bridge of L containing no edge of D is also an $(S \cup T)$-bridge of K. Note that each $(S \cup T)$-bridge B of K containing an edge of C has a plane representation with $B \cap (S \cup T \cup C)$ on its outer walk. Because $|V(T)| \geq 4$, $|V(P)| \geq 4$. Since $p_{m-1}, p_l \in V(P)$, we have $|V(P) \cap V(C)| \geq 2$. Hence, $P := S \cup T \cup S' \cup T'$ gives the desired path in G. ∎

Proof of (1.2). Let G be a 4-connected graph embedded in the torus, and let ρ be the face width of G. If $\rho = 0$, then G is a 4-connected planar graph, and so, has a Hamilton cycle by a theorem of Tutte [8]. So assume that $\rho \geq 1$.

Suppose $\rho = 1$. Then there is a non-null homotopic simple closed curve γ meeting G only in one vertex, say u. Cutting the torus open along γ, we obtain a plane graph G' with two vertices u', u'' such that G can be obtained from G' by identifying u' and u'' as u. Since G is 4-connected, the blocks of G' can be labeled as $B_1, \ldots, B_n$ such that $B_i \cap B_{i+1}$ consists of a single vertex v_i, $B_i \cap B_j = \emptyset$ if $j > i+1$, and $v_0 := u' \in V(B_1) - \{v_1\}$ and $v_n := u'' \in V(B_n) - \{v_{n-1}\}$. Because G is 4-connected, $n = 1$ or $n = 2$, and if $n = 2$ then exactly one of B_1 or B_2 is induced by an edge. Without loss of generality, assume that B_1 is 2-connected and v_0 is contained in the outer cycle C_1 of B_1. We may assume $v_1 \notin V(C_1)$; as otherwise, G is planar and G has a Hamilton cycle by a theorem of Tutte. By (2.1) (with B_1, C_1, v_0, v_1 as G, C, x, y, respectively), B_1 has a C_1-Tutte path P_1 between v_0 and v_1. Since G is 4-connected and P_1 is a C_1-Tutte path in B_1, $V(C_1) \subseteq V(P_1)$. Hence $|V(P_1)| \geq 4$ (because $v_1 \notin V(C_1)$). Let $P' := P_1$ if $n = 1$, and otherwise let $P' := P_1 \cup B_2$. Then P' is a Tutte path in G' between u' and u''. Because G is 4-connected and $|V(P')| \geq |V(P_1)| \geq 4$, P' is a Hamilton path in G' between u' and u''. Clearly, $E(P')$ induces a Hamilton cycle T in G. Hence, G has a Hamilton path.

Therefore we may assume that $\rho \geq 2$. Let γ be a non-null homotopic simple closed curve in the torus that meets G exactly ρ times (only at vertices).

Case 1. ρ is even.

Let R be a face of G that γ passes through, and let C be the cycle of G consisting of vertices and edges of G incident with R.

Assume $\rho = 2$. Then the R-width of G is 2. By (4.1), G has a Tutte path P from some $b \in V(C)$ such that $|V(P)| \geq 4$. Since G is 4-connected, P is a Hamilton path in G.

Now assume $\rho \geq 4$. By (4.3), G has a Tutte path P from some $b' \in V(C)$ such that $|V(P)| \geq 4$. Since G is 4-connected and $|V(P)| \geq 4$, P is a Hamilton path in G.

Case 2. ρ is odd.

Let $u \in \gamma \cap V(G)$. Then $G - u$ has face width $\rho - 1 \geq 2$. Let R be the face of $G - u$ which contains u (as a subset of the torus). Let C denote the cycle in $G - u$ consisting of vertices and edges incident with R, and choose a vertex $y \in V(C)$ such that $yu \in E(G)$. Then $G - u$ is (4,C)-connected. Applying (4.1) or (4.3) to $G - u, R, y$, we can show that there is a C-flap H' in $G - u$ with attachments a', b', c' ($a' = b' = c' = y$ if H' is null) and a $(C - I(H'))$-Tutte path P' from b' in $(G - u) - I(H')$ such that b', y, a' occur on C in this clockwise order, $y \in (V(H') - \{a'\}) \cup \{b'\}$, $a', c' \in V(P')$, and $|V(P')| \geq 4$.

If H' is null, then P' is a C-Tutte path in $G - u$ from y. In this case, $P := (P' \cup \{u\}) + yu$ is a Tutte path in G. Because G is 4-connected and $|V(P)| \geq 4$, P is a Hamilton path in G.

Therefore we may assume that H' is non-null. Let H^* denote the union of H', u, and all edges of G with both ends in $I(H') \cup \{u\}$. Because G is 4-connected, H^* is connected. In fact, $H^* + \{a'u, b'c'\}$ is 2-connected. Assume without loss of generality that $a'u$ and $b'c'$ are contained in the outer cycle C^* of $H^* + \{a'u, b'c'\}$. By applying (2.3) (with $H^* + \{a'u, b'c'\}, C^*, c', a', a'u, b'c'$ as G, C, x, y, e, f, respectively), $H^* + \{a'u, b'c'\}$ contains a Tutte path P^* between a' and c' such that $\{a'u, b'c'\} \subseteq E(P^*)$. Let $P := P' \cup (P^* - \{a', c'\})$.

Then every P-bridge of G is either a P'-bridge of $G - u$, or a P^*-bridge of $H^* + \{a'u, b'c'\}$. Hence, P is a Tutte path of G. Because G is 4-connected and $|V(P)| \geq 4$, P is a Hamilton path in G. ∎

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