Omega result for the mean square of the Riemann zeta function

Yuk-Kam Lau and Kai-Man Tsang

Abstract A recent method of Soundararajan enables one to obtain improved Ω-result for finite series of the form $\sum_n f(n)\cos(2\pi\lambda_n x+\beta)$ where $0\le\lambda_1\le\lambda_2\le\ldots$ and β are real numbers and the coefficients $f(n)$ are all non-negative. In this paper, Soundararajan's method is adapted to obtain improved Ω-result for $E(t)$, the remainder term in the mean-square formula for the Riemann zeta-function on the critical line. The Atkinson series for $E(t)$ is of the above type, but with an oscillating factor $(-1)^n$ attached to each of its terms.

2000 Mathematics Subject Classification: 11M06.

Key words and phrases: mean-square formula, Atkinson's formula, omega-results.

1 Introduction

Let $d(n)$ denote the divisor function. The error terms

$$\Delta(x):=\sum_{n\le x}d(n)-x\log x+(2\gamma-1)x,\quad (x\ge 0)$$

for the divisor sum, and

$$E(t):=\int_0^t|\zeta(\frac{1}{2}+iu)|^2du-t\log\frac{t}{2\pi}-(2\gamma-1)t,\quad (t>0)$$

for the mean square of the Riemann zeta function are two central objects in analytic number theory. Accounts of the research on these two error terms can be found in [7], [13], for example. In his fundamental paper [1], Atkinson showed that $E(t)$ and $\Delta(x)$ are indeed closely related. Corresponding to the well-known truncated Voronoi series for $\Delta(x)$ (see [13], eq.(12.4.4))

$$\Delta(x)=\frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n\le N}d(n)n^{-\frac{3}{4}}\cos(4\pi\sqrt{nx}-\frac{\pi}{4})+O(x^{\frac{1}{2}+\varepsilon}N^{-\frac{1}{2}})\quad\text{for}\quad 1\le N\ll x,\tag{1.1}$$

Atkinson found in [2] a similar series representation for $E(t)$. The terms in the series for $(2\pi)^{-1}E(2\pi x)$ for $n\ll x^{1/3}$ are, up to first order approximations, just those in (1.1) but with the extra oscillating factor $(-1)^n$. Atkinson's formula is instrumental in many of the subsequent studies on $E(t)$.

A central problem in this area is the widely held conjecture that for any $\varepsilon>0$, both $\Delta(x)\ll x^{1/4+\varepsilon}$ and $E(t)\ll t^{1/4+\varepsilon}$ hold. The best upper bounds we have for $\Delta(x)$ and $E(t)$ to this date are still quite far from the conjecture. On the other hand the current best omega results for $\Delta(x)$ and $E(t)$ are just marginally smaller than the conjectured bounds. Therefore any improvement on the omega results is of significance.

The main purpose of this paper is to improve on the omega result for $E(t)$. Investigation on the omega results for $\Delta(x)$ and $E(t)$ have made gradual progress over the past few decades. In 2003, Soundararajan [12] sharpened the previous best omega result for $\Delta(x)$ (and also the error term in the circle problem) obtained in [4] by introducing a clever trick to the method of Hafner. His method works for series of the form $\sum_n f(n)\cos(2\pi\lambda_n x+\beta)$, where $0\le\lambda_1\le\lambda_2\le\cdots$ and β are real numbers and the coefficients $f(n)$ are non-negative. For applications of this method of Soundararajan to other related problems, see [3], [10]. However, Soundararajan's method cannot be applied to $E(t)$ to yield a similar improvement, since the Atkinson series for $E(t)$ has the additional oscillating factor $(-1)^n$ attached to each of its terms, its coefficients are not of a constant sign. In this paper, we shall show that, via an indirect argument, Soundararajan's method can indeed be adapted to the case of $E(t)$. Hence we obtain the following.

Theorem 1.1. *We have*

$$E(t) = \Omega\bigg((t\log t)^{\frac{1}{4}}(\log_2 t)^{\frac{3}{4}(2^{4/3}-1)}(\log_3 t)^{-\frac{5}{8}}\bigg). \tag{1.2}$$

Here $\log_j$ *denotes the* j*-th iterated logarithm, so that* $\log_2 = \log\log,\ \log_3 = \log\log\log$ *and so on.*

The omega result obtained here is of the same order of magnitude as that for $\Delta(x)$ in [12]. It improves on the power of $\log_2 t$ in the previous result of Hafner and Ivic [5], from $(3+2\log 2)/4 = 1.0965...$ to the present value of $\frac{3}{4}(2^{4/3}-1) = 1.13988....$ As was pointed out in [12], this improvement is of significance since, on some probabilistic grounds, (1.2) may represent the true order of $E(t)$ up to $(\log_2 t)^{o(1)}$. On the other hand, due to the presence of the phase angle $\pi/4$ inside the cosine function in (1.1), the method we use does not yield the more precise Ω_+ or Ω_--result.

Our argument utilises the following two ideas.

First, if $\Delta^*(x)$ denotes the error term

$$\Delta^*(x) := \sum_{n\le 4x}\frac{(-1)^n}{2}d(n) - x\log x - (2\gamma-1)x,$$

then there is also a Voronoi type series for $\Delta^*(x)$ (see [11], Theorem 2 or [8], eq.(1.22)), namely,

$$\Delta^*(x) = \frac{x^{1/4}}{\pi\sqrt{2}}\sum_{n\le N}(-1)^n d(n)n^{-\frac{3}{4}}\cos(4\pi\sqrt{nx}-\frac{\pi}{4}) + O(x^{\frac{1}{2}+\varepsilon}N^{-\frac{1}{2}}) \quad (\text{for} \quad 1\le N\ll x). \tag{1.3}$$

Furthermore, we have the following relationship between $\Delta(x)$ and $\Delta^*(x)$:

$$\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x) \tag{1.4}$$

(see [8]). This functional relation immediately gives, for any upper bound for $\Delta(x)$, a corresponding upper bound of the same order of magnitude for $\Delta^*(x)$. It turns out that the converse of this is also true (see Theorem 1.2 below). Thus, both $\Delta(x)$ and $\Delta^*(x)$ have the same omega result.

Second, as already observed by Jutila [8], [9], $\Delta^*(x)$ is the right approximation to $(2\pi)^{-1}E(2\pi x)$. This is due to the fact that the above Voronoi series (1.3) for $\Delta^*(x)$, like the Atkinson formula for $(2\pi)^{-1}E(2\pi t)$, also has the oscillating factor $(-1)^n$ attached to each of its terms. The relation (1.4) then provides the link between $E(x)$ and $\Delta(x)$ to which Soundararajan's method applies. However, for our argument, $\Delta^*(x)$ is not close enough to $(2\pi)^{-1}E(2\pi x)$. Hence, we cannot deduce our omega result directly from that of $\Delta(x)$. Instead, we have to first smooth the functions by taking convolutions with a suitable kernel (at present, the Fejer kernel). The details of all these are given in the next section.

Theorem 1.2. *Suppose* $\Delta^*(x) \ll x^\sigma$ *for some* $\sigma \le \frac{1}{3}$*, then we also have*

$$\Delta(x) \ll x^\sigma.$$

2 Proofs of Theorem 1.1 and Theorem 1.2

In the sequel we use ε to denote an arbitrarily small positive number, which may not be the same at each occurrence. Let $e(y) := e^{2\pi iy}$.

First we define

$$E_1(x) := (2x)^{-\frac{1}{2}} E(2\pi x^2) \quad \text{for} \quad x \geq 1, \tag{2.1}$$

and consider its convolution with the Fejer kernel

$$K(u) := (\pi u)^{-2} \sin^2(\pi u).$$

Lemma 2.1. *Let $x \geq 200$ and $2 \leq \tau \leq x^{1/7}$. We have*

$$\int_{-\tau}^{\tau} E_1(x + u\tau^{-1}) e(-u) K(u) du$$

$$= \frac{1}{2} e(-\frac{1}{8}) \sum_{n \leq \tau^2} (-1)^n d(n) n^{-\frac{3}{4}} e(2\sqrt{n}x) k(1 - 2\tau^{-1}\sqrt{n}) + O(\log \tau),$$

where $k(y) := \max(0, 1 - |y|)$ is the Fourier transform of $K(u)$.

Proof. This is essentially Lemma 1 in [5], where parts of the analysis there could be simplified by the argument used in the proof of Lemma 1 in [6]. □

Let

$$\Delta_1(x) := x^{-\frac{1}{2}} \Delta(x^2), \quad \Delta_1^*(x) := x^{-\frac{1}{2}} \Delta^*(x^2) \quad \text{for} \quad x \geq 1.$$

The relation (1.4) then becomes

$$\Delta_1^*(x) = -\Delta_1(x) + 2^{\frac{5}{4}} \Delta_1(\sqrt{2}x) - 2^{-\frac{1}{2}} \Delta_1(2x), \ \ x \geq 1. \tag{2.2}$$

From the Voronoi series (1.3) for $\Delta^*(x^2)$ (with $N = x^2$), we have

$$\Delta_1^*(x) = \frac{1}{\pi\sqrt{2}} \sum_{n \leq x^2} (-1)^n d(n) n^{-\frac{3}{4}} \cos(4\pi\sqrt{n}x - \frac{\pi}{4}) + O(x^{-\frac{1}{2}+\varepsilon}).$$

Substituting for $\Delta_1(x)$ in (2.2) by the Voronoi series (1.1), we get

$$\begin{aligned} \sum_{n \leq x^2} (-1)^n d(n) n^{-\frac{3}{4}} \cos(4\pi\sqrt{n}x - \frac{\pi}{4}) \ &= \ - \sum_{n \leq x^2} d(n) n^{-\frac{3}{4}} \cos(4\pi\sqrt{n}x - \frac{\pi}{4}) + \\ &+ \ 2^{\frac{5}{4}} \sum_{n \leq x^2} d(n) n^{-\frac{3}{4}} \cos(4\pi\sqrt{2n}x - \frac{\pi}{4}) \\ &- \ 2^{-\frac{1}{2}} \sum_{n \leq x^2} d(n) n^{-\frac{3}{4}} \cos(8\pi\sqrt{n}x - \frac{\pi}{4}) \\ &+ \ O(x^{-\frac{1}{2}+\varepsilon}). \end{aligned} \tag{2.3}$$

Recall that $k(y)$ is the Fourier transform of $K(u)$. We easily prove that, for any positive x, ω and τ,

$$\int_{|u|<x} \cos(2\pi\omega(x+u) - \frac{\pi}{4}) e(-\tau u) K(\tau u) du = (2\tau)^{-1} e(\omega x - \frac{1}{8}) k(\frac{\omega}{\tau} - 1) + O(\tau^{-2} x^{-1}).$$

Hence, on multiplying (2.3) throughout by $e(-\tau u)K(\tau u)$ and then integrating with respect to u over the range $(-\frac{1}{2}x, \frac{1}{2}x)$, we get

$$\begin{aligned}\sum_{n\le x^2}(-1)^n d(n)n^{-\frac{3}{4}}e(2\sqrt{n}x-\frac{1}{8})k(\frac{2\sqrt{n}}{\tau}-1) &= -\sum_{n\le x^2} d(n)n^{-\frac{3}{4}}e(2\sqrt{n}x-\frac{1}{8})k(\frac{2\sqrt{n}}{\tau}-1)\\ &+ 2^{\frac{5}{4}}\sum_{n\le x^2} d(n)n^{-\frac{3}{4}}e(2\sqrt{2n}x-\frac{1}{8})k(\frac{2\sqrt{2n}}{\tau}-1)\\ &- 2^{-\frac{1}{2}}\sum_{n\le x^2} d(n)n^{-\frac{3}{4}}e(4\sqrt{n}x-\frac{1}{8})k(\frac{4\sqrt{n}}{\tau}-1)\\ &+ O(\tau^{-1}x^{-\frac{1}{2}+\varepsilon}).\end{aligned}$$

Multiplying throughout by $e(\frac{1}{8})$ and then taking the real parts, this becomes

$$\begin{aligned}\sum_{n\le \tau^2}(-1)^n d(n)n^{-\frac{3}{4}}\cos(4\pi\sqrt{n}x)k(\frac{2\sqrt{n}}{\tau}-1) &= -\sum_{n\le \tau^2} d(n)n^{-\frac{3}{4}}\cos(4\pi\sqrt{n}x)k(\frac{2\sqrt{n}}{\tau}-1)\\ &+ 2^{\frac{5}{4}}\sum_{n\le \tau^2/2} d(n)n^{-\frac{3}{4}}\cos(4\pi\sqrt{2n}x)k(\frac{2\sqrt{2n}}{\tau}-1)\\ &- 2^{-\frac{1}{2}}\sum_{n\le (\tau/2)^2} d(n)n^{-\frac{3}{4}}\cos(8\pi\sqrt{n}x)k(\frac{4\sqrt{n}}{\tau}-1)\\ &+ O(\tau^{-1}x^{-\frac{1}{2}+\varepsilon}), \qquad (2.4)\end{aligned}$$

for any $\tau \le x$.

Remark. The identity (1.4) can be easily derived by writing

$$\sum_{n\le 4x}(-1)^n d(n) = -\sum_{\substack{n\le 4x\\ n\ odd}} d(n) + \sum_{\substack{2^l m\le 4x\\ l\ge 1}} d(2^l m) = -\sum_{n\le 4x} d(n) + 2\sum_{\substack{2^l m\le 4x\\ l\ge 1}} d(2^l m)$$

and then applying the simple identity $d(2^l m) = 2d(2^{l-1}m) - d(2^{l-2}m)$ to the last sum. This same argument can also be applied to obtain (2.4) directly.

Define

$$\begin{aligned}P(x,\tau) &:= \sum_{n\le \tau^2}(-1)^n d(n)n^{-\frac{3}{4}}\cos(4\pi\sqrt{n}x)(1-|\frac{2\sqrt{n}}{\tau}-1|), \qquad (2.5)\\ Q(x,\tau) &:= \sum_{n\le \tau^2} d(n)n^{-\frac{3}{4}}\cos(4\pi\sqrt{n}x)(1-|\frac{2\sqrt{n}}{\tau}-1|).\end{aligned}$$

Then (2.4) is

$$P(x,\tau) = -Q(x,\tau) + 2^{\frac{5}{4}}Q(\sqrt{2}x, \sqrt{2}^{-1}\tau) - 2^{-\frac{1}{2}}Q(2x, 2^{-1}\tau) + O(\tau^{-1}x^{-\frac{1}{2}+\varepsilon}). \qquad (2.6)$$

If we write

$$R(x,\tau) = Q(x,\tau) - \alpha Q(\sqrt{2}x, \sqrt{2}^{-1}\tau), \qquad (2.7)$$

(2.6) can be rewritten as

$$P(x,\tau) = -R(x,\tau) + \beta R(\sqrt{2}x, \sqrt{2}^{-1}\tau) + O(\tau^{-1}x^{-\frac{1}{2}+\varepsilon}), \tag{2.8}$$

where α and β satisfy $\alpha\beta = 2^{-1/2}$ and $\alpha + \beta = 2^{5/4}$. Hence, we set $\alpha = 2^{1/4} - 2^{-1/4} = 0.348\cdots$, $\beta = 2^{1/4} + 2^{-1/4} = 2.030\cdots$.

Repeated application of (2.7) then yields

$$Q(x,\tau) = \sum_{j=0}^{J} \alpha^j R(\sqrt{2}^j x, \sqrt{2}^{-j}\tau) + \alpha^{J+1} Q(\sqrt{2}^{J+1}x, \sqrt{2}^{-J-1}\tau), \quad \text{for} \quad \tau \le x \tag{2.9}$$

and any $J \ge 0$. Similarly, from (2.8) we have, for any $I \ge 1$ and $\tau \le 2^{-I}x$,

$$R(x,\tau) = \sum_{i=1}^{I} \beta^{-i} P(\sqrt{2}^{-i}x, \sqrt{2}^{i}\tau) + \beta^{-I} R(\sqrt{2}^{-I}x, \sqrt{2}^{I}\tau) + O(\tau^{-1}x^{-\frac{1}{2}+\varepsilon}).$$

Substituting this into (2.9) then yields

$$\begin{aligned} Q(x,\tau) \quad &= \quad \sum_{j=0}^{J}\sum_{i=1}^{I} \alpha^j\beta^{-i} P(\sqrt{2}^{j-i}x, \sqrt{2}^{-j+i}\tau) + \sum_{j=0}^{J} \alpha^j \beta^{-I} R(\sqrt{2}^{j-I}x, \sqrt{2}^{-j+I}\tau) \\ &+ \quad \alpha^{J+1} Q(\sqrt{2}^{J+1}x, \sqrt{2}^{-J-1}\tau) + O(\tau^{-1}x^{-\frac{1}{2}+\varepsilon} \sum_{j=0}^{J} (\alpha 2^{\frac{1}{4}})^j). \end{aligned}$$

Finally, by substituting $R(x,\tau)$ from (2.7), we get

$$\begin{aligned} Q(x,\tau) \quad &= \quad \sum_{j=0}^{J}\sum_{i=1}^{I} \alpha^j\beta^{-i} P(\sqrt{2}^{j-i}x, \sqrt{2}^{-j+i}\tau) + \beta^{-I} Q(\sqrt{2}^{-I}x, \sqrt{2}^{I}\tau) \\ &- \quad \alpha^{J+1}\beta^{-I} Q(\sqrt{2}^{J+1-I}x, \sqrt{2}^{-J-1+I}\tau) + \alpha^{J+1} Q(\sqrt{2}^{J+1}x, \sqrt{2}^{-J-1}\tau) \\ &+ \quad O(\tau^{-1}x^{-\frac{1}{2}+\varepsilon}), \end{aligned} \tag{2.10}$$

which essentially expresses Q in terms of P. Choosing now $I = J = 2\log_2 \tau$, and using the trivial bound

$$|Q(x,\tau)| \le \sum_{n \le \tau^2} d(n) n^{-3/4} \ll \sqrt{\tau}\log\tau$$

to bound the last three terms involving Q on the right hand side of (2.10), we have

$$\begin{aligned} Q(x,\tau) \quad &= \quad \sum_{j=0}^{J}\sum_{i=1}^{I} \alpha^j\beta^{-i} P(\sqrt{2}^{j-i}x, \sqrt{2}^{-j+i}\tau) + \beta^{-I} Q((\log\tau)^{-\log 2}x, \tau(\log\tau)^{\log 2}) \\ &- \quad \alpha^{J+1}\beta^{-I} Q(\sqrt{2}x, \sqrt{2}^{-1}\tau) + \alpha^{J+1} Q(\sqrt{2}(\log\tau)^{\log 2}x, \sqrt{2}^{-1}(\log\tau)^{-\log 2}\tau) + O(\tau^{-1}x^{-\frac{1}{2}+\varepsilon}) \\ &= \quad \sum_{j=1}^{J}\sum_{i=1}^{I} \alpha^j\beta^{-i} P(\sqrt{2}^{j-i}x, \sqrt{2}^{-j+i}\tau) + O(\sqrt{\tau}). \end{aligned} \tag{2.11}$$

We now employ the idea of Soundararajan in [12] to produce an Ω_+-result for $Q(x,\tau)$. Note that our $Q(x,\tau)$ is indeed the function $F_1(x)$ considered in Lemma 3 of [12], with $\lambda_n = 2\sqrt{n}$, $\lambda_N = \tau$ and $f(n) = 2d(n)n^{-3/4}$. Following the arguments in Lemma 3 and the proof of Theorem 1 in §3 of [12], we obtain the following.

Lemma 2.2. *Let $\lambda > 0$ be a constant. For any $X \geq 2$ and*

$$\tau \ll (\log X)^{\frac{1}{2}} (\log_2 X)^{\frac{1}{2}(1-\lambda+\lambda\log\lambda)} (\log_3 X)^{-\frac{1}{4}},$$

there exists $x \in [X, X^{\frac{3}{2}}]$ and a positive constant c such that

$$Q(x,\tau) > c\sqrt{\tau}(\log\tau)^{\lambda\log(\frac{2e}{\lambda})-1}(\log_2\tau)^{-\frac{1}{2}}.$$

In particular, with the optimal choice of $\lambda = 2^{\frac{4}{3}}$ and

$$\tau \asymp (\log X)^{\frac{1}{2}} (\log_2 X)^{\frac{1}{2}(1-\lambda+\lambda\log\lambda)} (\log_3 X)^{-\frac{1}{4}},$$

we have the lower bound

$$Q(x,\tau) > c(\log X)^{\frac{1}{4}} (\log_2 X)^{\frac{3}{4}(2^{4/3}-1)} (\log_3 X)^{-\frac{5}{8}}$$

for a suitable $x \in [X, X^{\frac{3}{2}}]$.

Now, by considering the largest $P(\sqrt{2}^{j-i}x, \sqrt{2}^{-j+i}\tau)$ in the first double sum on the right hand side of (2.11), we deduce from Lemma 2.2 the following large value estimate for $P(x,\tau)$.

Lemma 2.3. *For any $X > 2$, there exists $x \in [X, X^{\frac{3}{2}}\log_2^2 X]$ and a $\tau \ll (\log X)^{\frac{1}{2}}(\log_2 X)^{1.1}$ such that*

$$P(x,\tau) > c(\log X)^{\frac{1}{4}} (\log_2 X)^{\frac{3}{4}(2^{4/3}-1)} (\log_3 X)^{-\frac{5}{8}}.$$

To finish the proof of Theorem 1.1 we note from Lemma 2.1 that,

$$\begin{aligned}
\sup_{|y-x|\leq 1} |E_1(y)| &\geq \mathfrak{Re}\left(\int_{-\tau}^{\tau} E_1(x+u\tau^{-1})e(\frac{1}{8}-u)K(u)du\right) \\
&= \frac{1}{2}\sum_{n\leq\tau^2}(-1)^n d(n) n^{\frac{3}{4}}\cos(4\pi\sqrt{n}x)k(2\frac{\sqrt{n}}{\tau}-1) + O(\log\tau).
\end{aligned}$$

The sum on the last line is just $P(x,\tau)$ defined in (2.5). Our Theorem 1.1 now follows readily from Lemma 2.3 and (2.1).

We now prove our Theorem 1.2. Similar to (2.7), (2.8), we can rewrite (1.4) as

$$\Delta^*(x) = W(x) - bW(2x),\ W(x) = -\Delta(x) + a\Delta(2x),$$

where a and b satisfy $a+b=2$, $ab=1/2$. Hence, we set $a = 1-2^{-1/2} = 0.29289\cdots, b = 1+2^{-1/2} = 1.7071\cdots$. Then similar to the deduction in (2.9) to (2.10), we find that

$$\Delta(x) = \sum_{j=0}^{J}\sum_{i=1}^{I} a^j b^{-i}\Delta^*(2^{j-i}x) - \sum_{j=0}^{J} a^j b^{-I} W(2^{j-I}x) + a^{J+1}\Delta(2^{J+1}x) \tag{2.12}$$

for $J \geq 0,\ I \geq 1$. Since $\Delta(y) \ll y^{1/3}$, we have $W(y) \ll y^{1/3}$ too. By choosing $I = J \asymp \log^2 x$ (notice that $2^\sigma a \leq 2^{1/3}a < 0.37$ and $2^\sigma b > b > 1.7$), we deduce from (2.12) that

$$\Delta(x) \ll \sum_{j=0}^{J}(2^\sigma a)^j \sum_{i=1}^{I}(2^\sigma b)^{-i}x^\sigma + (2^{\frac{1}{3}}b)^{-I}x^{\frac{1}{3}} + (2^{\frac{1}{3}}a)^J x^{\frac{1}{3}} \ll x^\sigma,$$

as desired. This finishes the proof of Theorem 1.2.

Acknowledgements. The work described in this paper was fully supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (HKU 7042/04P).

References

[1] F. V. Atkinson, The mean value of the zeta-function on the critical line, Quart. J. Math. Oxford 10 (1939), 122-128.

[2] F. V. Atkinson, The mean value of the Riemann zeta-function, Acta Math. 81 (1949), 353-376.

[3] K. Girstmair, M. Kühleitner, W. Müller and W. G. Nowak, The Piltz divisor problem in number fields: An improved lower bound by Soundararajan's method, Acta Arith. 117 (2005), 187-206.

[4] J. L. Hafner, New omega theorems for two classical lattice point problems, Invent. Math. 63 (1981), 181-186.

[5] J. L. Hafner and A. Ivić, On the mean-square of the Riemann zeta-function on the critical line, J. Number Theory 32 (1989), 151-191.

[6] D. R. Heath-Brown and K. Tsang, Sign changes of $E(T), \Delta(x)$ and $P(x)$, J. Number Theory, 49 (1994), 73-83.

[7] A. Ivić, The Riemann Zeta-Function, Wiley, New York, 1985.

[8] M. Jutila, Riemann's zeta-function and the divisor problem, Arkiv för Matematik, 21 (1983), 76-96.

[9] M. Jutila, Riemann's zeta-function and the divisor problem. II, Arkiv för Matematik, 31 (1993), 61-70.

[10] M. Kühleitner and W. G. Nowak, The lattice point discrepancy of a body of revolution: improving the lower bound by Soundararajan's method, Arch. Math. (Basel) 83 (2004), 208-216.

[11] T. Meurman, A generalization of Atkinson's formula to L-functions, Acta Arith. 47 (1986), 351-370.

[12] K. Soundararajan, Omega results for the divisor and circle problems, Int. Math. Res. Not. (2003), 1987-1998.

[13] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd ed. Oxford University Press, 1986.

Yuk-Kam Lau
yklau@maths.hku.hk

Kai-Man Tsang
kmtsang@maths.hku.hk

Department of Mathematics
University of Hong Kong
Pokfulam Road
Hong Kong
HONG KONG SAR