On Low-Dimensional Cancellation Problems

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Abstract. A well-known cancellation problem of Zariski asks when, for two given domains (fields) K_1 and K_2 over a field k, a k-isomorphism of $K_1[t]$ ($K_1(t)$) and $K_2[t]$ ($K_2(t)$) implies a k-isomorphism of K_1 and K_2. The main results of this article give affirmative answer to the two low-dimensional cases of this problem:

1. *Let K be an affine field over an algebraically closed field k of any characteristic. Suppose $K(t) \simeq k(t_1, t_2, t_3)$, then $K \simeq k(t_1, t_2)$.*

2. *Let M be a 3-dimensional affine algebraic variety over an algebraically closed field k of any characteristic. Let $A = K[x, y, z, w]/M$ be the coordinate ring of M. Suppose $A[t] \simeq k[x_1, x_2, x_3, x_4]$, then $frac(A) \simeq k(x_1, x_2, x_3)$, where $frac(A)$ is the field of fractions of A.*

In the case of zero characteristic these results were obtained by Kang in [14] and [15]. However, the case of finite characteristic is first settled in this article, that answered the questions proposed by Kang in [14, 15].

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1 Introduction

There are well-known cancellation problems of Zariski:

Problem 1. *Let K_1 and K_2 be affine domains over a field k and let $K_1[t] \simeq K_2[t]$ over k. Is it true that $K_1 \simeq K_2$ over k?*

Problem 2. *Let K_1 and K_2 be an affine fields over a field k and let $K_1(t) \simeq K_2(t)$ over k. Is it true that $K_1 \simeq K_2$ over k? In particular, if $K(t)$ is a field of rational functions over k, is it true that K is also a field of rational functions?*

The answers of these problems are **no** in general, even if $k = \mathbb{C}$. See, for instance, [4] and references therein.

However, for some low dimensional cases Problem 2 has a positive solution. See, for instance, [14].

Since there are well-known counterexamples for Problem 1 (See, [8] and [4] and references therein), a special case of that problem (when K_2 is a polynomial ring over k) has been brought more attentions:

Cancellation Conjecture of Zariski. *Let D be a domain over a field k. If $D[t]$ is k-isomorphic to $k[x_1, \dots, x_n][t]$ where k is a field. Then D is k-isomorphic to $k[x_1, \dots, x_n]$.*

The Zariski conjecture is settled for $n = 1$ by S. S. Abyankar, P. Eakin and W. J. Heinzer in [1] and M. Miyanishi in [17] for an arbitrary field k and for $n = 2$ by T. Fujita in the case $k = \mathbb{C}$ [9]. For $n \geq 3$, the Conjecture remains open, to the best of our knowledge. See [16], [18], [13], [8], [3], [21] for cancellation conjecture and problems of Zariski and related problems.

In the sequel all rings (fields) are commutative over a field k, all ring and field embeddings (isomorphisms) are k-embeddings (k-isomorphisms). Recall that an *affine domain* is a domain of finite transcendence degree over a field k; and an *affine field* is a field of finite transcendence degree over a field k.

For an affine field of transcendence degree two over a field k of characteristics zero, Problem 2 was solved in the positive by Kang [14].

In this article we shall prove the following new results for low dimensional cases of the Cancellation Problem, that answer a question of Kang in [14] positively.

Theorem 1.1 *Let K be an affine field over an algebraically closed field k of arbitrary characteristic. Suppose $K(t) \simeq k(t_1, t_2, t_3)$, then $K \simeq k(t_1, t_2)$.*

Theorem 1.1 has the following generalization.

Theorem 1.1' Let K be an affine field over an algebraically closed field k of arbitrary characteristic. Suppose $K(t_1, \ldots, t_{n+2}) \simeq k(t_1, \ldots, t_n)$, then $K \simeq k(t_1, t_2)$.

In the case of zero characteristic (e.g. for $k = \mathbb{C}$) both Theorem 1.1 and Theorem 1.1' were proved by Kang [14]. The following examples show that there is no much room left for any possible improvement of both Theorem 1.1 and Theorem 1.1'.

Example 1.1 ([5], Theorem 1 and Theorem 2) *Let k be a nonalgebraic closed field with $char(k) \neq 2$. Consider the field extension K over k,*

$$K = k(x, y, z) \quad \text{where } x^2 - ay^2 = f(z)$$

where x, y are independent indeterminates over k, such that

- *$f(z) \in k[z]$ is irreducible of degree three, and*
- *$a = disc(f(z)) \in k \backslash \{0\}$ is square-free.*

Then K is not isomorphic to $k(t_3, t_4, t_5)$ over k, but $K(t_1, t_2)$ is isomorphic to $k(t_1, t_2, t_3, t_4, t_5)$ over k.

Example 1.2 ([5], Theorem 1' and Theorem 3) *Let k be an algebraic closed field with $char(k) \neq 2$. Consider the field extension K over k,*

$$K = k(x, y, z, w) \quad \text{where } x^2 - a(w)y^2 = f(w, z)$$

where x, y, z are independent indeterminates over k, such that

- *$f(w, z) \in k[w, z]$ is irreducible of degree three in z, and*

- $a(w) = disc_z(f(w,z)) \in k[w]\backslash\{0\}$ *is square-free of degree* ≥ 5.

Then K *is not isomorphic to* $k(t_3, t_4, t_5, t_6)$ *over* k*, but* $K(t_1, t_2)$ *is isomorphic to* $k(t_1, t_2, t_3, t_4, t_5, t_6)$ *over* k*.*

The examples give a negative answer to an analog of the Concellation Conjecture of Zariski (for rational fields) orginally proposed also by Zariski. Based on the above examples, the general mathematical community believe that there should be a counterexample also for the *Cancellation Conjecture of Zariski* for polynomial rings whenever $n \geq 3$, although it is still an open problem, to the best of our knowledge.

However, if we replace equivalence by a weaker condition of birational equivalence, we can obtain the following new positive result that answers a question of Kang in [15] positively:

Theorem 1.2 *Let* M *be a* 3*-dimensional affine algebraic variety over algebraically closed field* k *of any characteristic. Let* $A = K[x, y, z, w]/M$ *be the coordinate ring of* M*. Suppose* $R[t] \simeq k[x_1, x_2, x_3, x_4]$*, then* $frac(R) \simeq k(x_1, x_2, x_3)$*, where* $frac(A)$ *is the field of fractions of* A*.*

In the case $k = \mathbb{C}$, Theorem 1.2 was also established by Kang [15].

2 Proofs of Main Results

In [4] the following main results was obtained by the authors of this article together with Makar-Limanov:

Proposition 2.1 *Let* K_1 *and* K_2 *be affine domains over an arbitrary field* k *and* $K_1[t]$ *can be embedded into* $K_2[t]$*. Then* K_1 *can be embedded into* K_2*.*

Proposition 2.2 *Let* K_1 *and* K_2 *be affine fields over an arbitrary field* k *and* $K_1(t)$ *can be embedded into* $K_2(t)$*. Then* K_1 *can be embedded into* K_2*.*

Let $K_1 = k(x_1, x_2)$, where k is an algebraically closed field of characteristic zero. Then any subfield of K_1 with transcendence degree two over k is isomorphic to K_1 (It is a variation of Lüroth's Theorem. See, for example, [20]). Suppose $K(t_1, t_2) \simeq k(x_1, x_2, x_3, x_4)$. Then by Proposition 2.2, K can be embedded into $k(x_1, x_2)$ so using the Lüroth theorem obtained by Castelnuovo in 1894 (all references can be found in [20]) we conclude that $K \simeq k(x_1, x_2)$. Similarly Kang's previous results (the zero characteristic case of Theorem 1.1) can also be deduced this way.

Unfortunately, Lüroth's Theorem does not hold in general (See [20]), hence we are not able to conclude that unirationally equivalence implies isomorphism in general. Therefore, we cannot answer Problem 2 affirmatively in general. From our discussion, it is obvious that the Cancellation type of problems are closely related to Lüroth's Theorem.

We recall that Lüroth theorem holds in two dimensional case for an algebraic closed field k of any characteristic if field K_2 is separable over $\varphi(K_1)$ (Namely, if $\phi : K_1 \to K_2$ and $\psi : K_2 \to K_1$ are both k-embeddings between fields K_1 and K_2 over k, the transcendental degree of K_i over k is two, K_2 is a separable extension over $\phi(K_1)$ then K_1 is isomorphic to K_2 over k, see [20]). Hence in order to obtain 2-dimensional cancellation theorem for any characteristic, we need establish the main results in [4] for separable embeddings.

Definition Let A and B be two affine domains over a field k. We call an embedding $\varphi : A \to B$ *good* if B is a separable extension of a pure transcendental extension of image of A under φ. Two affine k-fields K and L are *good unirationally equivalent* if there exists two good embedings $\varphi : K \to L$ and $\psi : L \to K$. A transcendence basis $z_1, \ldots, z_l$ of A is *good* if A is a pure separable algebraic extension of $k[z_1, \ldots, z_l]$.

Every affine domain has a good transcendence basis. (This is because every affine domain of transcendence degree d has d differentials with non-zero internal product. See, [22]).

Remarks.

1. An equivalent definition: an embedding $\varphi : A \to B$ is *good* if there exist for some s an extension map $\hat{\varphi} : A[t_1, \ldots, t_s]$ such that B is a separable extension of image of the embedding $\hat{\varphi}$.

2. An embedding $\varphi : A \to B$ may be not good but the fractional field of B may not contain proper non separable extension of fraction field of A (this

is because there exists a pure inseparable extension of $\mathbb{Z}_p[x]$ which is not generated by one element, but any algebraic extension of $\mathbb{Z}_p$ is separable.
3. Certainly an isomorphism is a good embedding.

4. A composition of good embeddings is a good embedding.

The following two theorems are similar to the main results in [4]:

Theorem 2.1 *Suppose there exists a good k-embedding $\varphi : A[t] \to B[\tau]$. Then there exist a good k-embedding $\psi : A \to B$.*

The similar fact is true for fields:

Theorem 2.2 *Suppose there exists a good k-embedding $\varphi : K_1(t) \to K_2(\tau)$. Then there exist a good k-embedding $\psi : K_1 \to K_2$.*

Now, we shall prove the main results of this paper by using the above two theorems first, then prove the above two theorems, as the latter proofs are quite long.

By repeatly using Theorem 2.2 first, then applying the 2-dimensional Lüroth theorem we obtain the following

Theorem 2.3 *Let K_1 be an extension field of an algebraically closed field k and let K_2 be a pure transcendental extension field of k with $tr.deg_k(K_2) = 2$. Then $K_1(t_1, \ldots, t_n) \simeq K_2(\tau_1, \ldots, \tau_n)$ implies $K_1 \simeq K_2$.*

Proof of Theorem 1.1 and Theorem 1.1'
Both theorems are direct consequences of the above theorem. □

Proof of Theorem 1.2
We identify $A[t]$ with $k[x_1, x_2, x_3, x_4]$ and A with a subring of $k[x_1, x_2, x_3, x_4]$. Let $B = k[x_1, x_2, x_3]$, K_A and K_B be the fields of fractions of A and B respectively. Recall Proposition 7.4 in [1]:

Proposition 2.3 *Let A be an integral domain and suppose A is of finite transcendence degree over the subring A_u generated by units of A (in particular, affine ring). Suppose that $A[x] = B[y]$ and let K and L denote the fields of fractions of A and B, respectively. If $A \neq B$, then K and L are both ruled over the quotient field k of R_u — i.e., K and L are both simple transcendental extensions of fields containing k.*

If $A = B$ in our case then of course $K_A = K_B$ is a field of rational functions, therefore my may assume that $A \neq B$. By Proposition 2.3 K_R is ruled, i.e. $K_A = K'(\tau)$ is a simple transcendental extension of some field K' of transcendence degree 2. Hence $K'(\tau, t) = L(x_3, x_4)$ where $L = k(x_1, x_2)$. By Theorem 2.4 the fields K' and L are good unirationally equivalent over k. By Lüroth theorem the Proof of Theorem 1.2 is completed. $\square$

Proof of Theorems 2.1 and 2.2

To prove Theorem 2.1 and Theorem 2.2 we need the following two propositions:

Proposition 2.4 *Let $\xi_1, \ldots, \xi_s$ be algebraic over $k[x_1, \ldots, x_m]$. Suppose that the rational function (in particular, it could be a polynomial) $Q(R, \vec{x}, \vec{\xi}\,) \not\equiv 0$, then there exists a polynomial $R \in k[x_1]$, such that $Q(R, \vec{x}, \vec{\xi}\,) \not\equiv 0$. Moreover, if $n \gg 1$ is sufficiently large, we may assume $R = x_1^n$.*

Proof.

Step 1. Consider a basis ξ_i' of field extension $k(\vec{x})(\vec{\xi})$ over $k(\vec{x})$. Then $Q(R, \vec{x}, \vec{\xi}\,)$ can be presented in the following form:

$$Q(R, \vec{x}, \vec{\xi}\,) = \sum_i Q_i(R, \vec{x})/T_i(R, \vec{x})\xi_i'$$

$Q \not\equiv 0$ iff $\exists i : Q_i \not\equiv 0$ and $\forall i,\ T_i \not\equiv 0$. Hence it is enough to prove that for each finite set of nonzero polynomials $\{P_j(R, \vec{x}) \mid j = 1, \ldots, l\}$, if n is sufficiently large, any polynomial in the set $\{P_j(x_1^n, \vec{x})\}_{j=1}^{l}$ is not identically zero.

Step 2. If we can find for each j such n_j that for all $n > n_j$ polynomial $R_j(x_1^n, \vec{x}) \not\equiv 0$. Then we can choose $N = \max(n_j)$ and such n satisfies conditions of step 1. Hence we may assume $l = 1$ in Step 1, so we only need to deal with one polynomial

$$Q(R, \vec{x}) = \sum_{j=0}^{\deg_R(Q)} R^j S_j(\vec{x}).$$

Step 3. Because $Q(R, \vec{x}) \not\equiv 0$, $S_j \not\equiv 0$ for some j. Let $N = \max_j\{\deg(S_j)\} + 1$. It is easy to see that for any $n \geq N$, $Q(x_1^n, \vec{x}) \not\equiv 0$. $\square$

The above proposition has the following consequence.

Proposition 2.5 *Let* $\xi_1, \dots, \xi_s$ *be algebraic and separable over* $k[x_1, \dots, x_m]$*, and let the external product of differentials of the polynomials*

$$\bigwedge_{i=1}^{m} dQ_i(\vec{t}, \vec{x}, \vec{\xi}\ ) \not\equiv 0$$

Then there exist a specialization of $t_1, \dots, t_s$*,* $t_i \to R_i$ *such that*

$$\bigwedge_{i=1}^{m} dQ_i(\vec{R}, \vec{x}, \vec{\xi}\ ) \not\equiv 0$$

Proof. In case $\mathrm{char}(k) = 0$, as the external product of the differentials of the polynomials is not zero if and only if the polynomials are algebraically independent, the conclusion follows from the proofs of main results (Proposition 2.1 and Proposition 2.2 in this paper) in [4]. Hence we may assume that $\mathrm{char}(k) = p > 0$. By induction it is enough to consider the case $s = 1$, i.e., the case of just one parameter (denoted by t). The external product can be presented in the following form:

$$\bigwedge_{i=1}^{m} dQ_i(\vec{t}, \vec{x}, \vec{\xi}\ ) = \\ = R dx_1 \cdots dx_m + \sum_{i=1}^{m} T_i dt dx_1 \cdots \widehat{dx_i} \cdots dx_n \quad (1)$$

The convention $\widehat{dx_i}$ means that this factor is omitted.

Consider two cases.

Case 1. $R \not\equiv 0$. Then we can substitute $x_1^{k \cdot p} \to t$. dt goes to 0 and apply Proposition 2.4 to R.

Case 2. $R \equiv 0$. Then $T_i \not\equiv 0$ for some i. Due to renumeration we can assume that $T_1 \not\equiv 0$. Let $k \not\equiv 0 \mathrm{mod}\ p$. Substitute $x_1^k \to t$. Then all terms of sum $\sum_{i=1}^{m} T_i dt dx_1 \cdots \widehat{dx_i} \cdots dx_n$ except first one vanishes and we get

$$T_1|_{x^k \to t} \cdot k \cdot \bigwedge_{i=1}^{m} dx_i.$$

According to Proposition 2.4, $T_1|_{x^k \to t} \not\equiv 0$ for sufficiently large k.

Now, we continue **Proof of Theorem 2.1 and Theorem 2.2**.

In the sequel we proceed only for rings. The case of fields is similar, i.e., we only need to replace rings of polynomials by fields of rational functions. First of all we deduce the proof to the case when $\mathrm{Trdeg}(A) = \mathrm{Trdeg}(B)$. If $\mathrm{Trdeg}(A) < \mathrm{Trdeg}(B)$, then $\mathrm{Trdeg}(A[t]) < \mathrm{Trdeg}(B[\tau])$. Hence there exists a good embedding $A[y,t] \to B[\tau]$ sending y to some element in a good transcendence basis, by additing this element to a completes good transcendence basis of $\mathrm{Im}(A[t])$, we obtain a good transcendence basis of $B[\tau]$. Now we may replace $A' = A[z]$ and conclude by induction on $\mathrm{Trdeg}(B) - \mathrm{Trdeg}(A)$. Therefoe without of loss of generality, we may assume that $\mathrm{Trdeg}(A) = \mathrm{Trdeg}(B)$.

There exists a transcendence basis $\{x_1, \ldots, x_m\}$ of A such that A is a separable extension of $k[x_1, \ldots, x_m]$. (Indeed every affine field has a separable transcendence basis $\{T_i = P_i/Q_i\}$, see p.57-77 in [22]). Hence $\bigwedge_i dT_i \neq 0$. It implies that for some sequence $\{R_i\}$ such that for each i either $R_i = P_i$ or $R_i = Q_i$ the external product $\bigwedge_i dR_i \neq 0$. Therefore $A[t]$ is a separable extension of $k[t, x_1, \ldots, x_m]$. Hence $B[\tau]$ is a separable extension of $\varphi(k[t, x_1, \ldots, x_m])$. According to [22], this means that the external product

$$d\varphi(t) \wedge d\varphi(x_1) \wedge \cdots \wedge d\varphi(x_m) \neq 0.$$

In particular,

$$\bigwedge_{i=1}^{m} d\varphi(x_i) \neq 0.$$

By Proposition 2.5, there exists a specialization of $t \to x_i^q$ for some q, such that

$$\bigwedge_{i=1}^{m} d\varphi(x_i)' \neq 0 \tag{2}$$

where $\varphi(x_i)' \in B$, the element corresponding to $\varphi(x_i)$ after this specialization. The equality (2) means that B is a separable extension of image of $k[x_1, \ldots, x_m]$ under φ'. Hence B is separable over $\varphi'(A)$. $\square$

Remark. The following theorem generalizes Proposition 2.4. It is very useful because it allows to find an 'elements of general position' in case of finite fields and it is very important in the dimension theory (Proposition 2.4 corresponds to the case $r = 0$ of this theorem).

Theorem 2.4 *Let $\xi_1, \ldots, \xi_s$ be algebraic over $k[x_1, \ldots, x_m]$, the polynomials $Q_i(\vec{t}, \vec{x}, \vec{\xi}\ ); i = 1, \ldots, n$ are algebraically independent for some value of set of parameter $\vec{t} = (t_1, \ldots, t_r)$ in some extension field k_1 of the ground field k. Then there exist polynomials $R_i \in \Phi[x_1]; i = 1, \ldots, r$, $\vec{R} = (R_1, \ldots, R_r)$ such that the set of polynomials*

$$\{Q_1(\vec{R}, \vec{x}, \vec{\xi}\ ), \ldots, Q_n(\vec{R}, \vec{x}, \vec{\xi}\ )\}$$

is algebraically independent.

Moreover, if the growth of the sequence $n_1 \ll n_2 \ll \cdots \ll n_r$ is sufficiently fast, we may be assumed $R_i = x_1^{n_i}$.

The similar fact is hold for fields and rational functions Q_i. In this case we also can put $R_i = x_1^{-n_i}$ (as well as $x_1^{n_i}$).

Instead of x_1 one can take any other variable x_i; $\Phi = \mathbb{Z}_p$ if $\mathrm{Char}(k) = p$ and $\Phi = \mathbb{Z}$ if $\mathrm{Char}(k) = 0$.

Because we do not use this theorem in full generality (for our purpose Proposition 2.4 is enough), we omit the proof.

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