On Convergent Probability of a Random Walk

Yiu-Fai Lee and Wai-Ki Ching*

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Abstract

We introduce an interesting random walk on a straight path with cards of random numbers based on [1]. By using the method of recurrent relation, we obtain the convergent probability of the random walk with different initial positions.

Key Words: Random Walk, Recurrent Relation, Probability, Card Games.

1 Introduction

Random walks have been studied by many researchers and have a lot of applications [2, 6, 7]. In this short note, we consider the random walk with cards studied in Ching and Lee [1]. In [1], they prove the probability that a walker will return to a fixed position (they call it the convergent probability) will tend to one as the length of the path is allowed to go to infinity. The proof therein [1] is based on some probability bounds. Here we would like to calculate the exact convergent probability. The results here require knowledge in elementary probability theory and difference equations. We remark that this random walk game with cards is equivalent to the Kruskal Count which is a card trick invented by Martin D. Kruskal [3, 4], also see the book by Haigh [5, pp. 133-135]. A webpage by Haga and Robins on Kruskal's principle with computer simulation on the card game and detailed Markov chain analysis of the problem can be found in [8].

*Advanced Modeling and Applied Computing Laboratory and Department of Mathematics, The University of Hong Kong, Pokfulam road, Hong Kong. Research supported in part by RGC Grant No. HKU 7126/02P and HKU CRCG Grant and Hung Hing Ying Physical Sciences Research Fund. (Email: leeyiufai@gmail.com, wkc@hkusua.hku.hk)

We consider a random walk on the path with grid points labeled from mN down to 1.

$$\bullet^{mN} \rightarrow \bullet^{mN-1} \rightarrow \bullet^{mN-2} \rightarrow \cdots\cdots \rightarrow \bullet^{m} \rightarrow \bullet^{m-1} \rightarrow \cdots\cdots \rightarrow \bullet^{1}.$$

For the ease of presentation, we adopt the above labelling. At each grid "$\bullet$" $i \in M$ where

$$M = \{1, 2, \ldots, mN\},$$

there is a card with possible number from 1 to m. We may assume that the numbers on the cards are uniformly distributed. Then the walker is allowed to choose any one of the positions in

$$S = \{mN, mN-1, mN-2, \ldots, m(N-1)+1\}$$

as a starting position. Each time, the walker picks up the card at the position and walks in a forward direction with the number of steps equals to the number on the card. The process stops when the walker first steps on any one of the positions in

$$T = \{1, 2, \ldots, m\}.$$

We are interested in obtaining the probability that two walkers with positions i and j will eventually come to the same position in T.

The rest of the paper is organized as follows. In Section 2, we give the recurrent relation for the calculation of convergent probability of the random walk. In Sections 3 and 4, we discuss the cases of $m = 2$ and 3 and we compare the results with [1].

2 The Recurrent Relation

We define $p(i, j)$ to be the probability that two walkers meet eventually at the same position in T, given that they are currently in the positions i and j. Then we have the followings.

The boundary conditions:

For $1 \leq i, j \leq m$

$$p(i, j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Moreover, we have for all i, j,

$$p(i, j) = 1 \quad \text{if} \quad i = j.$$

The recurrent relation:

$$p(i,j) = \frac{1}{m}\left(p(i-1,j) + p(i-2,j) + \ldots + p(i-m,j)\right)$$

and

$$p(j,i) = p(i,j).$$

MATLAB implementation:

```
m = 2;
N = 4;
p = eye(m * N, m * N);
for i = m + 1 : m * N,
for j = 1 : i - 1,
p(i, i - j) = 0;
for k = 1 : m,
p(i, i - j) = p(i, i - j) + p(i - k, i - j);
end;
p(i, i - j) = p(i, i - j)/m;
p(i - j, i) = p(i, i - j);
end;
end;
```

3 The Analysis of the Case $m = 2$

In this section, we give an example for the case of $m = 2$ and we also give the convergent probability for it. Suppose $m = 2$ and $N = 4$ then by using the MATLAB program, one can get

$p(1,1) = 1;$

$p(1,2) = 0;$

$p(2,1) = 0;$

$p(2,2) = 1;$

$p(3,2) = p(2,3) = \frac{1}{2}(p(2,2) + p(1,2)) = 0.5;$

$p(3,1) = p(1,3) = \frac{1}{2}(p(2,1) + p(1,1)) = 0.5;$

$p(4,3) = p(3,4) = \frac{1}{2}(p(3,3) + p(2,3)) = 0.75;$

$p(4,2) = p(2,4) = \frac{1}{2}(p(3,2) + p(2,2)) = 0.75;$

$p(5,4) = p(4,5) = \frac{1}{2}(p(4,4) + p(3,4)) = 0.875;$

$p(5,3) = p(3,5) = \frac{1}{2}(p(3,2) + p(2,2)) = 0.875;$

$p(6,5) = p(5,6) = \frac{1}{2}(p(5,5) + p(4,5)) = 0.9375;$

$p(6,4) = p(4,6) = \frac{1}{2}(p(5,4) + p(4,4)) = 0.9375;$

$p(7,6) = p(6,7) = \frac{1}{2}(p(6,6) + p(5,6)) = 0.96875;$

$p(7,5) = p(5,7) = \frac{1}{2}(p(6,5) + p(5,5)) = 0.96875;$

$p(8,7) = p(7,8) = \frac{1}{2}(p(7,7) + p(6,7)) = 0.984375;$

$p(8,6) = p(6,8) = \frac{1}{2}(p(7,6) + p(6,6)) = 0.984375.$

One can also compute all the probabilities $p(i,j) = P_{ij}$ as follows:

$$P = \begin{pmatrix} \boxed{1.0000} & \boxed{0.0000} & 0.5000 & 0.2500 & 0.3750 & 0.3125 & 0.3438 & 0.3281 \\ \boxed{0.0000} & \boxed{1.0000} & \boxed{0.5000} & 0.7500 & 0.6250 & 0.6875 & 0.6562 & 0.6719 \\ 0.5000 & 0.5000 & \boxed{1.0000} & \boxed{0.7500} & 0.8750 & 0.8125 & 0.8438 & 0.8281 \\ 0.2500 & 0.7500 & 0.7500 & \boxed{1.0000} & \boxed{0.8750} & 0.9375 & 0.9062 & 0.9219 \\ 0.3750 & 0.6250 & 0.8750 & 0.8750 & \boxed{1.0000} & \boxed{0.9375} & 0.9688 & 0.9531 \\ 0.3125 & 0.6875 & 0.8125 & 0.9375 & 0.9375 & \boxed{1.0000} & \boxed{0.9688} & 0.9844 \\ 0.3438 & 0.6562 & 0.8438 & 0.9062 & 0.9688 & 0.9688 & \boxed{1.0000} & \boxed{0.9844} \\ 0.3281 & 0.6719 & 0.8281 & 0.9219 & 0.9531 & 0.9844 & 0.9844 & \boxed{1.0000} \end{pmatrix}. \quad (1)$$

In general, for each j we have the following recurrent relations:

$$p(i,j) = \frac{1}{2}(p(i-1,j) + p(i-2,j)) \quad i = j-1, j, \dots.$$

The difference equations have a general solution of the form

$$p(i,j) = A_j(\frac{-1}{2})^i + B_j.$$

In particular, we have

$$p(i,1) = \frac{1}{2}\left(p(i-1,1) + p(i-2,1)\right), \quad p(1,1) = 1 \quad \text{and} \quad p(2,1) = 0. \qquad (2)$$

and

$$p(i,2) = \frac{1}{2}\left(p(i-1,2) + p(i-2,2)\right), \quad p(1,2) = 0 \quad \text{and} \quad p(2,2) = 1. \qquad (3)$$

Solving the difference equations (2) and (3) we get respectively

$$p(i,1) = \frac{1}{3}(1 - (-\frac{1}{2})^{i-2}) \quad \text{and} \quad p(i,2) = \frac{1}{3}(2 + (-\frac{1}{2})^{i-2}).$$

To compute $p(2N, 2N-1)$, for each i, we let $q(i) = p(i+1, i)$. Since $p((i+1, i) = p(i, i+1)$, then it is easy to see that

$$q(i+1) = \frac{q(i)+1}{2} \quad \text{with} \quad q(1) = 0.$$

Solving the recurrent relations, we have

$$q(i) = 1 - \frac{1}{2^{i-1}}$$

Therefore we have

$$p(2N, 2N-1) = q(2N-1) = 1 - \frac{1}{2^{2N-2}}.$$

This means if two walkers randomly choose two different positions $2N$ or $2N-1$ as their starting positions then the probability that they will meet in T is given by

$$\frac{(1 - \frac{1}{2^{2N-2}}) + (1 - \frac{1}{2^{2N-2}})}{2} = 1 - \frac{1}{2^{2N-2}}.$$

This result is consistent with the convergent probability in [1] when $m = 2$ and there are $2N$ grid points in the circular path.

4 The Case of $m = 3$

For the case of $m = 3$, for each j we have the following recurrent relations

$$p(i,j) = \frac{1}{3}(p(i-1,j) + p(i-2,j) + p(i-3,j)), \quad i = j-2, j-1, \ldots.$$

The difference equations have a general solution of the form

$$p(i,j) = (\frac{1}{3})^i (A_j \cos(i\theta) + B_j \sin(i\theta)) + C_j$$

where $\theta = 2.1863$. In particular, we have

$$\begin{cases} p(i,1) = \frac{1}{3}\left(p(i-1,1) + p(i-2,1) + p(i-3,1)\right), \ p(1,1) = 1, \ p(2,1) = 0, \ p(3,1) = 0, \\ p(i,2) = \frac{1}{3}\left(p(i-1,2) + p(i-2,2) + p(i-3,2)\right), \ p(1,2) = 0, \ p(2,2) = 1, \ p(3,2) = 0, \\ p(i,3) = \frac{1}{3}\left(p(i-1,3) + p(i-2,3) + p(i-3,3)\right), \ p(1,3) = 0, \ p(2,3) = 0, \ p(3,3) = 1, \\ p(i,4) = \frac{1}{3}\left(p(i-1,4) + p(i-2,4) + p(i-3,4)\right), \ p(2,4) = \frac{1}{3}, \ p(3,4) = \frac{1}{3}, \ p(4,4) = 1. \end{cases}$$

Solving the above difference equations, we get

$$\begin{cases} p(i,1) = (\frac{1}{\sqrt{3}})^i(-\frac{7}{6}\cos(i\theta) + \frac{2\sqrt{2}}{3}\sin(i\theta)) + \frac{1}{6}, \\ p(i,2) = (\frac{1}{\sqrt{3}})^i(-\frac{4}{3}\cos(i\theta) - \frac{7\sqrt{2}}{6}\sin(i\theta)) + \frac{1}{3}, \\ p(i,3) = (\frac{1}{\sqrt{3}})^i(\frac{5}{2}\cos(i\theta) + \frac{1}{\sqrt{2}}\sin(i\theta)) + \frac{1}{2}, \\ p(i,4) = (\frac{1}{\sqrt{3}})^i(-\frac{7}{3}\cos(i\theta) + \frac{4\sqrt{2}}{3}\sin(i\theta)) + \frac{2}{3}. \end{cases}$$

Hence we can find the probability of two walkers starting at any two initial positions eventually meet each other. In particular, for $m = 3$ and $N = 5$, the convergent probability of two walkers at two different starting positions is 0.9822 by using the MATLAB program. While an lower bound of this probability is given in [1] as $1 - (5/9)^4 = 0.9047$.

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