Opial-Type Inequalities for Differential Operators

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Abstract

Some continuous and discrete versions of Opial-type inequalities which are readily applicable to differential and difference operators are established. These generalize earlier results of Anastassiou-Pečarić and Koliha-Pečarić.

Keywords: Opial-type inequalities, differential operators
Mathematics Subject Classification (2000) : 26D15, 26D10, 26A33

* Corresponding author. Research partially supported by the Research Grants Council of the Hong Kong SAR (Project No. HKU7017/05P)

1. Introduction

It is well recognized that integral inequalities in general provide an effective tool to the study of quantitative as well as qualitative properties of solutions of differential and integral equations. Among these, the following Opial's inequality has been one of the most useful few and has continuously drawn people's attentions over the past few decades.

Theorem (Opial [17]). *If $f \in \mathcal{L}^1[0,h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0,h)$, then*

$$\int_0^h \left|f(x)\right| \left|f'(x)\right| dx \le \frac{h}{4} \int_0^h \left|f'(x)\right|^2 dx \ .$$

Over the years, Opial's inequality has been generalized to many different situations and settings. For example, in [3, 22] it was generalized to the case of many functions of 1 variable, in [9, 10, 11, 18, 19] to the case of many functions of many variables, and in [5, 6, 7, 12, 13, 14, 18, 21] to the case involving higher order derivatives. Meanwhile, in [4], Agarwal first introduced and established some general Opial-type inequalities involving general ρ-derivatives, that is, for the class of differential operators $D_\rho^{(n)}$, which properly contains the class of disconjugate linear operators

$$L := D^{(n)} + \sum_{i=1}^{n} a_i(t) D^{(n-i)} \ .$$

Anastassiou [1] then established for the first time Opial-type inequalities for general linear differential operators, and his results have later on been generalized in many directions by Anastassiou-Pečarić [2] and Koliha-Pečarić [16].

This work is a further generalization of the works in [2] and [16].

Let $I \subset \mathbb{R}$ be a closed interval, $a \in I$, $n \in \mathbb{N}$, and we consider linear differential operators

$$L_i = D^m + \sum_{\alpha=0}^{m-1} \phi_{i,\alpha}(t) D^\alpha \ , \quad \phi_{i,\alpha} \in C(I) \ , \quad i = 1, \ldots, n \ .$$

Let $G_i(x,t)$ be the Green's function for L_i. For any $h \in C(I)$, it is well-known (see, e.g. [15]) that for each $i = 1, \ldots, n$,

$$y_i(x) := \int_a^x G_i(x,t) h(t) dt$$

is the unique solution for the initial value problem

$$\begin{cases} L_i y = h \\ y^{(j)}(a) = 0 \ , \qquad j = 0, 1, \ldots, m-1 \ . \end{cases}$$

Collectively, if we consider $L = L_1 \otimes \cdots \otimes L_n$ as operating on $C(I) \times \cdots \times C(I)$ (n copies), then

$$\mathbf{y} := (y_1 \cdots y_n)$$

is the unique solution for the initial value problem

$$\begin{cases} L\mathbf{y} = \mathbf{h} \\ \mathbf{y}^{(j)}(a) = 0 \ , \quad j = 0, 1, \ldots, m-1 \ , \end{cases}$$

where $\mathbf{h} = (h, \ldots, h)$ and for each j, $\mathbf{y}^{(j)} = (y_1^{(j)}, \ldots, y_n^{(j)})$.

In [16], the case $n = 1$ was considered and estimates for the integral

$$\int_a^x U(s) \left|y(s)\right|^\beta \left|h(s)\right|^\alpha ds$$

for any function $0 \le U \in C(I)$ and any real numbers $\alpha, \beta > 0$ were established. These provide valuable information to the unique solution of the aforesaid initial value problem for the case

$n = 1$. In this paper, we extend the results of [16] to the case where $n > 1$, and also obtain discrete analogues which are equally useful in discrete initial value problems. As a far-reaching application, these are applied to obtain estimates of fractional derivatives.

2. General Weighted Opial-type Inequalities

In the sequel, let I be a closed and bounded interval in $\mathbb{R}$, and $a \in I$. Let $N \in \mathbb{N}$ be a fixed integer and for the sake of simplicity, the index i will always run from 1 to N.

The following is a generalization of Theorem 2.4 in [16] and Theorem 1, Theorem 2, and Corollary 1 in [2].

Theorem 2.1. *Let $K_i \in C(I \times I)$ be positive, and $y_i,\ h \in C(I)$ satisfy*

$$|y_i(t)| \le \left| \int_a^t K_i(t,s)|h(s)|ds \right| \quad \text{for all } t \in I\ .$$

Then for any constants $p_i \ge 0$, $p := \sum\limits_{i=1}^{N} p_i > 0$, $q > 0$, $r > \max\{1, q\}$, and any $u, v \in C(I)$ with $u \ge 0$ and $v > 0$,

$$\left| \int_a^x u(t)|h(t)|^q \prod_{i=1}^{N} |y_i(t)|^{p_i} dt \right| \le A(x) \left| \int_a^x v(t)|h(t)|^r dt \right|^{\frac{p+q}{r}}$$

for all $x \in I$, where

$$A(x) = \Big(\frac{q}{p+q}\Big)^{\frac{q}{r}} \left| \int_a^x u(t)^{\frac{r}{r-q}} v(t)^{-\frac{q}{r-q}} \prod_{i=1}^{N} |P_i(t)|^{\frac{p_i(r-1)}{r-q}} dt \right|^{\frac{r-q}{r}} ,$$

$$P_i(t) = \int_a^t v(s)^{-\frac{1}{r-1}} K_i(t,s)^{\frac{r}{r-1}} ds\ .$$

Proof. Consider first $t \ge a$. By assumption and Hölder's Inequality with indices $\frac{r}{r-1}$, r,

$$\begin{aligned} |y_i(t)| &\le \int_a^t K_i(t,s)|h(s)|ds \\ &= \int_a^t v(s)^{-\frac{1}{r}} K_i(t,s) v(s)^{\frac{1}{r}} |h(s)| ds \\ &\le \Big(\int_a^t v(s)^{-\frac{1}{r-1}} K_i(t,s)^{\frac{r}{r-1}} ds \Big)^{\frac{r-1}{r}} \cdot \Big(\int_a^t v(s)|h(s)|^r ds \Big)^{\frac{1}{r}} \\ &= |P_i(t)|^{\frac{r-1}{r}} \Big(\int_a^t v(s)|h(s)|^r ds \Big)^{\frac{1}{r}} , \end{aligned}$$

hence

$$\prod_{i=1}^{N} |y_i(t)|^{p_i} \le \Big(\int_a^t v(s)|h(s)|^r ds \Big)^{\frac{p}{r}} \prod_{i=1}^{N} |P_i(t)|^{\frac{p_i(r-1)}{r}} .$$

Therefore, by applying Hölder's Inequality with indices $\frac{r}{r-q}$, $\frac{r}{q}$,

$$\begin{aligned} &\int_a^x u(t) \cdot \prod_{i=1}^{N} |y_i(t)|^{p_i} \cdot |h(t)|^q dt \\ \le\ & \int_a^x u(t) \cdot \prod_{i=1}^{N} |P_i(t)|^{\frac{p_i(r-1)}{r}} \cdot \Big(\int_a^t v(s)|h(s)|^r ds \Big)^{\frac{p}{r}} |h(t)|^q dt \\ =\ & \int_a^x u(t) v(t)^{-\frac{q}{r}} \prod_{i=1}^{N} |P_i(t)|^{\frac{p_i(r-1)}{r}} \Big(\int_a^t v(s)|h(s)|^r ds \Big)^{\frac{p}{r}} v(t)^{\frac{q}{r}} |h(t)|^q dt \\ \le\ & \Big(\int_a^x u(t)^{\frac{r}{r-q}} v(t)^{-\frac{q}{r-q}} \prod_{i=1}^{N} |P_i(t)|^{\frac{p_i(r-1)}{r-q}} dt \Big)^{\frac{r-q}{r}} \cdot \Big(\int_a^x \Big(\int_a^t v(s)|h(s)|^r ds \Big)^{\frac{p}{q}} v(t)|h(t)|^r dt \Big)^{\frac{q}{r}} \end{aligned}$$

for any $x \geq a$. Let

$$\Psi(t) = \int_a^t v(s)\left|h(s)\right|^r ds \ .$$

Then

$$\left[\int_a^x \left(\int_a^t v(s)\left|h(s)\right|^r ds\right)^{\frac{p}{q}} v(t)\left|h(s)\right|^r dt\right]^{\frac{q}{r}} = \left(\int_a^x \Psi(t)^{\frac{p}{q}}\Psi'(t)dt\right)^{\frac{q}{r}} = \left(\frac{q}{p+q}\right)^{\frac{q}{r}}\Psi(x)^{\frac{p+q}{r}} \ .$$

Hence

$$\int_a^x u(t)\left(\prod_{i=1}^N \left|y_i(t)\right|^{p_i}\right)\left|h(t)\right|^q dt \leq A(x)\left(\int_a^x v(s)\left|h(s)\right|^r ds\right)^{\frac{p+q}{r}}$$

for any $x > a$. The case for $x < a$ can be obtained immediately from the relation

$$\int_x^a (\cdot)ds = -\int_a^x (\cdot)ds \ .$$

Hence the theorem. □

Corollary 2.1. *Let $K_i \in C(I \times I)$ be positive, and y_i, $h \in C(I)$ satisfy*

$$\left|y_i(t)\right| \leq \left|\int_a^t K_i(t,s)\left|h(s)\right|ds\right| \quad \textit{for all } t \in I \ .$$

Then for any constants $p_i \geq 0$, $p := \sum_{i=1}^N p_i > 0$, $q > 0$, with $p+q > 1$, and any $u, v \in C(I)$ with $u \geq 0$ and $v > 0$,

$$\left|\int_a^x u(t)\left|h(t)\right|^q \prod_{i=1}^N \left|y_i(t)\right|^{p_i} dt\right| \leq A(x)\left|\int_a^x v(t)\left|h(t)\right|^{p+q} dt\right|$$

for all $x \in I$, where

$$A(x) = \left(\frac{q}{p+q}\right)^{\frac{q}{p+q}}\left|\int_a^x u(t)^{\frac{p+q}{p}} v(t)^{-\frac{q}{p}} \prod_{i=1}^N \left|Q_i(t)\right|^{\frac{p_i(p+q-1)}{p}}\right|^{\frac{p}{p+q}} ,$$

$$Q_i(t) = \int_a^t v(s)^{-\frac{1}{p+q-1}} K_i(t,s)^{\frac{p+q}{p+q-1}} ds \ .$$

Proof. It follows from Theorem 2.1 by taking $r = p+q > \max\{1, q\}$. □

Theorem 2.2. *Under the same conditions as in Corollary 2.1, for any $u, v \in C(I)$ with $u \geq 0$ and $v > 0$,*

$$\left|\int_a^x u(t)\prod_{i=1}^N \left|y_i(t)\right|^{p_i} \cdot \left|h(t)\right|^q dt\right| \leq \|v\|_\infty^p \|h\|_\infty^{p+q} \left|\int_a^x u(t)\prod_{i=1}^N Q_i(t)^{p_i} dt\right|$$

for all $x \in I$, where

$$Q_i(t) := \left|\int_a^t v(s)^{-1}K_i(s,t)ds\right| \ .$$

Proof. For any $t \geq a$ and any $i = 1, \ldots, N$,

$$\left|y_i(t)\right| \leq \int_a^t K_i(t,s)\left|h(s)\right|ds = \int_a^t v(s)^{-1}K_i(t,s)v(s)\left|h(s)\right|ds \leq Q_i(t)\|v\|_\infty\|h\|_\infty \ ,$$

thus

$$\prod_{i=1}^N \left|y_i(t)\right|^{p_i} \leq \|v\|_\infty^p \|h\|_\infty^p \prod_{i=1}^N Q_i(t)^{p_i} \ .$$

So for all $x \geq a$,

$$\int_a^x u(t) \cdot \prod_{i=1}^{N} \big|y_i(t)\big|^{p_i} \cdot \big|h(t)\big|^q dt$$

$$\leq \int_a^x u(t) \cdot \prod_{i=1}^{N} Q_i(t)^{p_i} \cdot \|v\|_\infty^p \|z\|_\infty^p \cdot \big|h(t)\big|^q dt$$

$$= \Big(\int_a^x u(t) \prod_{i=1}^{N} Q_i(t)^{p_i} dt \Big) \|v\|_\infty^p \|h\|_\infty^{p+q} \ .$$

The case for $x < a$ can be obtained by the relation

$$\int_x^a (\cdot) ds = - \int_a^x (\cdot) ds \ .$$

□

3. Discrete Versions

This section deals with discrete analogs of the results in Section 2.

As above, $N \in \mathbb{N}$ is a fixed integer and i runs 1 to N. Let $m \in \mathbb{N}$ be another fixed integer.

The following results generalize the main results of [8].

Theorem 3.1. For any $i = 1, \dots, N$ and $\alpha, \beta = 0, \dots, m-1$, let $K^i_{\alpha\beta} > 0$ and a^i_α, b_β be real numbers such that

$$|a^i_\alpha| \leq \sum_{\beta=0}^{\alpha-1} K^i_{\alpha\beta} |b_\beta| \ .$$

Then for any constants $p_i \geq 0$, $p := \sum_{i=1}^{N} p_i > 0$, $q > 0$, $r > \max\{1, q\}$, and any $U_\alpha \geq 0$ and $V_\alpha > 0$,

$$\sum_{\alpha=0}^{m-1} U_\alpha \Big(\prod_{i=1}^{N} |a^i_\alpha|^{p_i} \Big) |b_\alpha|^q \leq C_m^{\frac{q}{r}} A_m \Big(\sum_{\beta=0}^{m-1} V_\beta |b_\beta|^r \Big)^{\frac{p+q}{r}} \ ,$$

where

$$C_1 = 0 \ ,$$

$$C_\alpha = \frac{q}{p+q} \Big(\frac{1}{p+q} \Big)^{\frac{p}{q}} (1 - C_{\alpha-1})^{-\frac{p}{q}} \ , \quad \alpha = 2, 3, \dots, m \ ,$$

$$A_m = \Big[\sum_{\alpha=0}^{m-1} U_\alpha^{\frac{r}{r-q}} V_\alpha^{-\frac{q}{r-q}} \prod_{i=1}^{N} (P^i_\alpha)^{\frac{p_i(r-1)}{r-q}} \Big]^{\frac{r-q}{r}} \ , \text{ and}$$

$$P^i_\alpha = \sum_{\beta=0}^{\alpha-1} V_\beta^{-\frac{1}{r-1}} (K^i_{\alpha\beta})^{\frac{r}{r-1}} \ .$$

Proof. For any $i = 1, \dots, N$ and $\alpha = 0, \dots, m-1$,

$$|a^i_\alpha|^{p_i} \leq \Big(\sum_{\beta=0}^{\alpha-1} K^i_{\alpha\beta} |b_\beta| \Big)^{p_i}$$

$$= \Big(\sum_{\beta=0}^{\alpha-1} V_\beta^{-\frac{1}{r}} K^i_{\alpha\beta} V_\beta^{\frac{1}{r}} |b_\beta| \Big)^{p_i}$$

$$\leq \Big[\sum_{\beta=0}^{\alpha-1} V_\beta^{-\frac{1}{r-1}} (K^i_{\alpha\beta})^{\frac{r}{r-1}} \Big]^{\frac{p_i(r-1)}{r}} \Big[\sum_{\beta=0}^{\alpha-1} V_\beta |b_\beta|^r \Big]^{\frac{p_i}{r}} \ .$$

Thus

$$\prod_{i=1}^{N} |a^i_\alpha|^{p_i} \leq \Big\{ \prod_{i=1}^{N} \Big[\sum_{\beta=0}^{\alpha-1} V_\beta^{-\frac{1}{r-1}} (K^i_{\alpha\beta})^{\frac{r}{r-1}} \Big]^{\frac{p_i(r-1)}{r}} \Big\} \cdot \Big\{ \sum_{\beta=0}^{\alpha-1} V_\beta |b_\beta|^r \Big\}^{\frac{p}{r}}$$

$$= \Big\{ \prod_{i=1}^{N} (P^i_\alpha)^{\frac{p_i(r-1)}{r}} \Big\} \cdot \Big\{ \sum_{\beta=0}^{\alpha-1} V_\beta |b_\beta|^r \Big\}^{\frac{p}{r}}$$

and so

$$
\begin{aligned}
&\sum_{\alpha=0}^{m=1} U_\alpha \Big(\prod_{i=1}^{N} |a_\alpha^i|^{p_i} \Big) |b_\alpha|^q \\
\le\ &\sum_{\alpha=0}^{m-1} U_\alpha V_\alpha^{-\frac{q}{r}} \Big[\prod_{i=1}^{N} (P_\alpha^i)^{\frac{p_i(r-1)}{r}} \Big] \cdot V_\alpha^{\frac{q}{r}} |b_\alpha|^q \Big[\sum_{\beta=0}^{\alpha-1} V_\beta |b_\beta|^r \Big]^{\frac{p}{r}} \\
\le\ &\Big[\sum_{\alpha=0}^{m-1} U_\alpha^{\frac{r}{r-q}} V_\alpha^{-\frac{q}{r-q}} \prod_{i=1}^{N} (P_\alpha^i)^{\frac{p_i(r-1)}{r-q}} \Big]^{\frac{r-q}{r}} \cdot \Big[\sum_{\alpha=0}^{m-1} \Big(\sum_{\beta=0}^{\alpha-1} V_\beta |b_\beta|^r \Big)^{\frac{p}{q}} V_\alpha |b_\alpha|^r \Big]^{\frac{q}{r}} .
\end{aligned}
$$

Write

$$
X_\alpha = \sum_{\beta=0}^{\alpha-1} V_\beta |b_\beta|^r \ ,
$$

then

$$
\Delta X_\alpha = V_\alpha |b_\alpha|^r
$$

and

$$
\Big[\sum_{\alpha=0}^{m-1} \Big(\sum_{\beta=0}^{\alpha-1} V_\beta |b_\beta|^r \Big)^{\frac{p}{q}} V_\alpha |b_\alpha|^r \Big]^{\frac{q}{r}} = \Big[\sum_{\alpha=0}^{m-1} (X_\alpha)^{\frac{p}{q}} \Delta X_\alpha \Big]^{\frac{q}{r}} \le \big[C_m X_m^{\frac{p}{q}+1} \big]^{\frac{q}{r}} = C_m^{\frac{q}{r}} X_m^{\frac{p+q}{r}}
$$

by Lemma 2 in [7]. Thus

$$
\sum_{\alpha=0}^{m-1} U_\alpha \Big(\prod_{i=1}^{N} |a_\alpha^i|^{p_\alpha} \Big) |b_\alpha|^q \le C_m^{\frac{q}{r}} A_m X_m^{\frac{p+q}{r}} = C_m^{\frac{q}{r}} A_m \Big(\sum_{\beta=0}^{m-1} V_\beta |b_\beta|^r \Big)^{\frac{p+q}{r}} .
$$

□

Corollary 3.2. For any $i = 1, \dots, N$ and $\alpha, \beta = 0, \dots, m-1$, let $K_{\alpha\beta}^i > 0$ and a_α^i, b_β be real numbers such that

$$
|a_\alpha^i| \le \sum_{\beta=0}^{\alpha-1} K_{\alpha\beta}^i |b_\beta| \ .
$$

Then for any constants $p_i \ge 0$, $p := \sum_{i=1}^{N} p_i > 0$, $q > 0$, with $p+q > 1$, and any $U_\alpha \ge 0$ and $V_\alpha > 0$,

$$
\sum_{\alpha=0}^{m-1} U_\alpha \Big(\prod_{i=1}^{N} |a_\alpha^i|^{p_\alpha} \Big) |b_\alpha|^q \le C_m^{\frac{q}{p+q}} A_m \sum_{\beta=0}^{m-1} V_\beta |b_\beta|^{p+q} \ ,
$$

where

$$
\begin{aligned}
C_1 &= 0 \ , \\
C_\alpha &= \frac{q}{p+q} \Big(\frac{1}{p+q} \Big)^{\frac{p}{q}} (1 - C_{\alpha-1})^{-\frac{p}{q}} \ , \quad \alpha = 2, 3, \dots, m \ , \\
A_m &= \Big[\sum_{\alpha=0}^{m-1} U_\alpha^{\frac{p+q}{p}} V_\alpha^{-\frac{q}{p}} \prod_{i=1}^{N} (P_\alpha^i)^{\frac{p_i(p+q-1)}{p}} \Big]^{\frac{p}{p+q}} \ , \text{ and} \\
P_\alpha^i &= \sum_{\beta=0}^{\alpha-1} V_\beta^{-\frac{1}{p+q-1}} (K_{\alpha\beta}^i)^{\frac{p+q}{p+q-1}} \ .
\end{aligned}
$$

Proof. It follows from Theorem 3.1 by setting $r = p+q > \max\{1, q\}$. □

Remark. Let $\{u_\alpha\}$ be a sequence of real numbers,

$$
\begin{aligned}
\Delta^0 u_\alpha &= u_\alpha \ , \\
\Delta^1 u_\alpha &= \Delta u_\alpha = u_{\alpha+1} - u_\alpha \ , \\
\Delta^i u_\alpha &= \Delta^{i-1} u_{\alpha+1} - \Delta^{i-1} u_\alpha \ , \quad i = 2, 3, \dots \ ,
\end{aligned}
$$

if we write

$$\begin{cases} a_\alpha^i = \Delta^i u_\alpha \ , & i = 1, \dots, N-1 \\ b_\alpha = a_\alpha^N = \Delta^N u_\alpha \end{cases}$$

then clearly there exist $K_{\alpha\beta}^i > 0$ such that

$$|a_\alpha^i| \le \sum_{\beta=0}^{\alpha-1} K_{\alpha\beta}^i |b_\beta|$$

and so Theorem 3.1 and Corollary 3.2 apply. In particular, observe that these generalize the main result of Alzer in [8].

4. Applications to Fractional Derivatives

We first recall the following basic notations and facts (see, e.g. [20]): For any $x > 0$,

$C^n[0, x] = \{$functions on $[0, x]$ with continuous mth order derivative$\}$,
$AC[0, x] = \{$absolutely continuous functions on $[0, x]\}$,
$AC^n[0, x] = \{$functions $f \in C^n[0, x]$ with $f^{(n-1)} \in AC[0, x]\}$,
$\mathcal{L}(0, x) = \{$Lebesgue integrable functions on $(0, x)\}$,
$\mathcal{L}^\infty(0, x) = \{$Lebesgue measurable functions that are essentially bounded on $[0, x]\}$.

Let $\alpha > 0$ and $f \in \mathcal{L}(0, x)$. The Riemann-Liouville fractional integral $I^\alpha f$ of f of order α is defined by

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \ , \quad t \in [0, x] \ , \tag{4.1}$$

where Γ is the usual Gamma function. Note that the RHS of (4.1) exists for almost all $t \in [0, x]$, and $I^\alpha f \in \mathcal{L}(0, x)$ (see [20]). The Riemann-Liouville fractional derivative of f of order α is defined by

$$D^\alpha f(t) := \Big(\frac{d}{ds}\Big)^m I^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \Big(\frac{d}{ds}\Big)^m \int_0^t (t-s)^{m-\alpha-1} f(s) ds \ ,$$

where $m = [\alpha] + 1$, $[\alpha] =$ the largest integral part of α, provided that the derivative exists. As a convention, we define

$$\begin{aligned} D^0 f &:= f \ , \\ I^0 f &:= f \ , \\ I^{-\alpha} f &:= D^\alpha f \quad \text{for } \alpha > 0 \ , \\ D^{-\alpha} f &:= I^\alpha f \quad \text{for } 0 < \alpha \le 1 \ . \end{aligned}$$

Observe that Riemann-Liouville fractional derivatives are generalizations of regular derivatives, that is, if $\alpha \in \mathbb{N}$, then

$$D^\alpha f = \frac{d^\alpha f}{ds^\alpha} \ .$$

Let $m = [\alpha] + 1$. A function $f \in \mathcal{L}(0, x)$ is said to have an integrable fractional derivative $D^\alpha f$ if

$$\begin{cases} D^{\alpha-k} f \in C[0, x] \ , & k = 1, \dots, m \\ D^{\alpha-1} f \in AC[0, x] \ . \end{cases}$$

Lemma 4.1. *Let $\nu > \mu \ge 0$. If $f \in \mathcal{L}(0, x)$ has an integrable fractional derivative $D^\nu f$, and*

$$D^{\nu-j} f(0) = 0 \quad \textit{for all } j = 1, \dots, [\nu] + 1 \ ,$$

then

$$D^\mu f(t) = \frac{1}{\Gamma(\nu-\mu)} \int_0^t (t-s)^{\nu-\mu-1} D^\nu f(s) ds$$

for all $t \in [0, x]$.

Proof. By the Index Law for fractional derivatives [20],

$$I^{\nu-\mu} D^{\nu} f = I^{\nu-\mu} I^{-\nu} f = I^{-\mu} f = D^{\mu} f \ .$$

Hence the result. □

Theorem 4.1. *Let p_i, q, μ_i, ν, r $(i = 1, \ldots, N)$ be real numbers such that $p_i \geq 0$, $p := \sum_{i=1}^{N} p_i > 0$, $q > 0$, $\nu > \mu_i + 1 \geq 0$ for all $i = 1, \ldots, N$, and*

$$r > \max\Big\{1, q, \frac{1}{\nu - \mu_i} : i = 1, \ldots, N\Big\}.$$

Suppose $f \in \mathcal{L}(0, x)$ has an integrable fractional derivative $D^{\nu} f \in \mathcal{L}^{\infty}(0, x)$, and

$$D^{\nu - j} f(0) = 0 \quad \text{for all } j = 1, \ldots, [\nu] + 1 \ ,$$

then for any $u, v \in C[0, x]$ with $u \geq 0$ and $v > 0$,

$$\int_0^x u(t) \big|D^{\nu} f(t)\big|^q \prod_{i=1}^{N} \big|D^{\mu_i} f(t)\big|^{p_i} dt \leq A(x) \Big[\int_0^x v(t) \big|D^{\nu} f(t)\big|^r dt\Big]^{\frac{p+q}{r}} \ , \tag{4.2}$$

where

$$\begin{aligned}
A(x) &= \Big(\frac{q}{p+q}\Big)^{\frac{q}{r}} \Big\{\int_0^x u(t)^{\frac{r}{r-q}} v(t)^{-\frac{q}{r-q}} \prod_{i=1}^{N} \big|P_i(t)\big|^{\frac{p_i(r-1)}{r-q}} dt\Big\}^{\frac{r-q}{r}} \ , \\
P_i(t) &= \int_0^t v(s)^{-\frac{1}{r-1}} K_i(t, s)^{\frac{r}{r-1}} ds \ , \quad 0 \leq t \leq x \ , \\
K_i(t, s) &= \frac{(t-s)_+^{\nu - \mu_i - 1}}{\Gamma(\nu - \mu_i)} \ , \quad 0 \leq t, s \leq x \ , \quad \textit{and} \\
(t - s)_+ &= \max\{t - s, 0\} \ .
\end{aligned}$$

Proof. First observe that
(i) since $\nu > \mu_i + 1$, $K_i \in C\big([0, x]^2\big)$ for all i;
(ii) since $r > \frac{1}{\nu - \mu_i}$ for all i, we have

$$\frac{(\nu - \mu_i - 1) r}{r - 1} > -1 \quad \text{for all } i$$

and so for any fixed $t \in [0, x]$,

$$K_i(t, s)^{\frac{r}{r-1}} = \frac{(t-s)_+^{(\nu - \mu_i - 1) r/(r-1)}}{\Gamma(\nu - \mu_i)^{r/(r-1)}}$$

is integrable over $[0, t]$. Since $v > 0$ and is continuous, $P_i(t)$ is well-defined;
(iii) since $v > 0$, and by assumption $D^{\nu} f \in \mathcal{L}^{\infty}(0, x)$, $v(t)\big|D^{\nu} f(t)\big|^r$ is integrable over $[0, x]$, so the RHS of (4.2) is well-defined.

By Lemma 4.1, for each $i = 1, \ldots, N$,

$$D^{\mu_i} f(t) = \frac{1}{\Gamma(\nu - \mu_i)} \int_0^t (t - s)^{\nu - \mu_i - 1} D^{\nu} f(s) ds$$

for all $t \in [0, x]$. Letting

$$\begin{aligned}
h &= D^{\nu} f \ , \\
y_i &= D^{\mu_i} f \ , \\
K_i(t, s) &= \frac{(t-s)_+^{\nu - \mu_i - 1}}{\Gamma(\nu - \mu_i)} \ , \quad 0 \leq t, s \leq x \ ,
\end{aligned}$$

we have

$$\big|y_i(t)\big| \le \Big|\int_0^t K_i(t,s)\big|h(s)\big|ds\Big|$$

for all $i = 1, \dots, N$ and all $t \in [0,x]$. Hence by Theorem 1, for any $u, v \in C[0,x]$,

$$\int_0^x u(t)\big|h(t)\big|^q \prod_{i=1}^N \big|y_i(t)\big|^{p_i} dt \le A(x)\Big[\int_0^x v(t)\big|h(t)\big|^r dt\Big]^{\frac{p+q}{r}},$$

that is

$$\int_0^x u(t)\big|D^\nu f(t)\big|^q \prod_{i=1}^N \big|D^{\mu_i} f(t)\big|^{p_i} dt \le A(x)\Big[\int_0^x v(t)\big|D^\nu f(t)\big|^r dt\Big]^{\frac{p+q}{r}}.$$

□

Remark. Note that Theorem 4.1 reduces to Theorem 4.2 in [16] when $N = 1$.

An interesting special case of Theorem 4.2 is the following

Corollary 4.2. Under the same conditions as in Theorem 4.1,

$$\int_0^x \big|D^\nu f(t)\big|^q \prod_{i=1}^N \big|D^{\mu_i} f(t)\big|^{p_i} dt \le \frac{\prod_{i=1}^N \big(\frac{r-1}{\sigma_i}\big)^{\frac{p_i(r-1)}{r}}}{\prod_{i=1}^N \Gamma(\nu-\mu_i)^{p_i}} \cdot \left(\frac{r-q}{\sum_{i=1}^N \sigma_i p_i + r - q}\right)^{\frac{r-q}{r}} \cdot x^{[\sum_{i=1}^N \sigma_i p_i + r - q]/r},$$

where for each $i = 1, \dots, N$, $\sigma_i := \nu r - \mu_i r - 1$.

Proof. By taking $u = v \equiv 1$ in Theorem 4.1, we have

$$\int_0^x \big|D^\nu f(t)\big|^q \prod_{i=1}^N \big|D^{\mu_i} f(t)\big|^{p_i} dt \le A(x)\Big[\int_0^x \big|D^\nu f(t)\big|^r dt\Big]^{\frac{p+q}{r}},$$

where

$$\begin{aligned}
A(x) &= \Big(\frac{q}{p+q}\Big)^{\frac{q}{r}} \Big\{\int_0^x \prod_{i=1}^N \Big|\int_0^t \Big(\frac{(t-s)_+^{\nu-\mu_i-1}}{\Gamma(\nu-\mu_i)}\Big)^{\frac{r}{r-1}} ds\Big|^{\frac{p_i(r-1)}{r-q}} dt\Big\}^{\frac{r-q}{r}} \\
&= \frac{1}{\prod_{i=1}^N \Gamma(\nu-\mu_i)^{p_i}} \cdot \Big\{\int_0^x \prod_{i=1}^N \Big(\frac{r-1}{\sigma_i} \cdot t^{\frac{\sigma_i}{r-1}}\Big)^{\frac{p_i(r-1)}{r-q}} dt\Big\}^{\frac{r-q}{r}} \\
&= \frac{\prod_{i=1}^N \big(\frac{r-1}{\sigma_i}\big)^{\frac{p_i(r-1)}{r}}}{\prod_{i=1}^N \Gamma(\nu-\mu_i)^{p_i}} \cdot \Big\{\int_0^x t^{\frac{1}{r-q}\sum_{i=1}^N \sigma_i p_i} dt\Big\}^{\frac{r-q}{r}} \\
&= \frac{\prod_{i=1}^N \big(\frac{r-1}{\sigma_i}\big)^{\frac{p_i(r-1)}{r}}}{\prod_{i=1}^N \Gamma(\nu-\mu_i)^{p_i}} \cdot \left(\frac{r-q}{\sum_{i=1}^N \sigma_i p_i + r - q}\right)^{\frac{r-q}{r}} \cdot x^{[\sum_{i=1}^N \sigma_i p_i + r - q]/r},
\end{aligned}$$

hence the result follows. □

References

[1] G.A. Anastassiou, Opial Type Inequalities for Linear Differential Operators, *Math. Ineq. Appl.* **1**, No.2 (1998), 193-200.

[2] G.A. Anastassiou and J. Pečarić, General Weighted Opial Inequalities for Linear Differential Operators, *J. Math. Anal. Appl.* **239** (1999), 402-418.

[3] R.P. Agarwal and P.Y.H. Pang, Remarks on the Generalization of Opial's Inequality, *J. Math. Anal. Appl.* **190** (1995), 559-577.

[4] R.P. Agarwal and P.Y.H. Pang, "Opial Inequalities with Applications in Differential and Difference Equations," Kluwer, Dordrecht, 1995.

[5] R.P. Agarwal and P.Y.H. Pang, Opial-type Inequalities Involving Higher Order Derivatives, *J. Math. Anal. Appl.* **189** (1995), 85-103.

[6] R.P. Agarwal and E. Thandapani, On Some New Integrodifferential Inequalities, *Analete stiintifice ale Universitatic, "Al. I. Cuza" din Iasi Tomal XXVIII, S. Ia* (1982), 123-126.

[7] H. Alzer, On Some Inequalities of Opial-type, *Arch. Math.* **63** (1994), 431-436.

[8] H. Alzer, Note on a Discrete Opial-type Inequality, *Arch. Math.* **65** (1995), 267-270.

[9] W.S. Cheung, Generalizations of Opial-type Inequalities in two Variables, *Tamkang Journal of Mathematics* **22**, No.1 (1991), 43-50.

[10] W.S. Cheung, On Opial-type Inequalities in two Variables, *Aequationes Mathematicae* **38** (1989), 236-244.

[11] W.S. Cheung, Opial-type Inequalities with m Functions in n Variables, *Mathematika* **39** (1992), 319-326.

[12] W.S. Cheung, Some Generalized Opial-type Inequalities, *Journal of Mathematical Analysis and Applications* **162**, No.2 (1991), 317-321.

[13] W.S. Cheung, Some New Opial-type Inequalities, *Mathematika* **37** (1990), 136-142.

[14] K.M. Das, An Inequality Similar to Opial's Inequality, *Proc. Amer. Math. Soc.* **22** (1969), 258-261.

[15] D. Kreider, R. Kuller, D. Ostberg and F. Perkins, "An Introduction to Linear Analysis," Addison-Wesley, Reading, 1966.

[16] J.J. Koliha and J. Pečarić, Weighted Opial Inequalities, *Tamkang J. Math.* **33**, No.1 (2002), 93-102.

[17] Z. Opial, Sur une Inégalité, *Ann. Polon, Math.* **8** (1960), 29-32.

[18] B.G. Pachpatte, On Opial-type Inequalities, *J. Math. Anal. Appl.* **120** (1986), 547-556.

[19] B.G. Pachpatte, On Opial-type Inequalities in Two Independent Variables, *Proc. Royal Soc. Edinburgh* **100** (1985), 263-270.

[20] S.G. Samko, A.A. Kilbas and O.I. Marichev, "Fractional Integrals and Derivatives: Theory and Applications," Gordon and Breach, Reading, 1993.

[21] D. Willet, The Existence-Uniqueness Theorem for an nth Order Linear Ordinary Differential Equation, *Amer. Math. Monthly* **75** (1968), 174-178.

[22] G.S. Yang, On a Certain Result of Z. Opial, *Proc. Japan Acad.* **42** (1966), 78-83.

Appendix

Direct proof of Corollary 2.1.

Proof. $t \geq a$:

$$\begin{aligned}
\left|y_i(t)\right|^{p_i} &\leq \Big(\int_a^t K_i(t,s)\left|h(s)\right|ds\Big)^{p_i} \\
&= \Big[\int_a^t v(s)^{-\frac{1}{p+q}} K_i(t,s) v(s)^{\frac{1}{p+q}}\left|h(s)\right|ds\Big]^{p_i} \\
&\leq \left\{\Big[\int_a^t v(s)^{-\frac{1}{p+q-1}} K_i(t,s)^{\frac{p+q}{p+q-1}} ds\Big]^{\frac{p+q-1}{p+q}} \cdot \Big[\int_a^t v(s)\left|h(s)\right|^{p+q} ds\Big]^{\frac{1}{p+q}}\right\}^{p_i} \\
&\qquad \text{(Hölder with indices } \tfrac{p+q}{p+q-1},\ p+q)
\end{aligned}$$

$$\prod_{i=1}^N \left|y_i(t)\right|^{p_i} \leq \Big[\prod_{i=1}^N \Big(\int_a^t v(s)^{-\frac{1}{p+q-1}} K_i(t,s)^{\frac{p+q}{p+q-1}} ds\Big)^{\frac{p_i(p+q-1)}{p+q}}\Big] \cdot \Big[\int_a^t v(s)\left|h(s)\right|^{p+q} ds\Big]^{\frac{p}{p+q}}$$

$$\begin{aligned}
&\int_a^x u(t)\Big(\prod_{i=1}^N \left|y_i(t)\right|^{p_i}\Big)\left|h(t)\right|^q dt \\
&\leq \int_a^x \Big\{u(t)v(t)^{-\frac{q}{p+q}}\Big[\prod_{i=1}^N \Big(\int_a^t v(s)^{-\frac{1}{p+q-1}} K_i(t,s)^{\frac{p+q}{p+q-1}} ds\Big)^{\frac{p_i(p+q-1)}{p+q}}\Big] \\
&\qquad \cdot v(t)^{\frac{q}{p+q}}\left|h(t)\right|^q \Big(\int_a^t v(s)\left|h(s)\right|^{p+q} ds\Big)^{\frac{p}{p+q}}\Big\} dt \\
&\leq \Big\{\int_a^x u(t)^{\frac{p+q}{p}} v(t)^{-\frac{q}{p}} \prod_{i=1}^N \Big(\int_a^t v(s)^{-\frac{1}{p+q-1}} K_i(t,s)^{\frac{p+q}{p+q-1}} ds\Big)^{\frac{p_i(p+q-1)}{p}} dt\Big\}^{\frac{q}{p+q}} \\
&\qquad \cdot \Big\{\int_a^x v(t)\left|h(t)\right|^{p+q}\Big(\int_a^t v(s)\left|h(s)\right|^{p+q} ds\Big)^{\frac{p}{q}} dt\Big\}^{\frac{q}{p+q}} \\
&\qquad \text{(Hölder with indices } \tfrac{p+q}{p},\ \tfrac{p+q}{q}) .
\end{aligned}$$

Let

$$\Phi(t) = \int_a^t v(s)\left|h(s)\right|^{p+q} ds .$$

Then

$$\Phi'(t) = v(t)\left|h(t)\right|^{p+q} .$$

So

$$\begin{aligned}
&\Big\{\int_a^x v(t)\left|h(t)\right|^{p+q}\Big(\int_a^t v(s)\left|h(s)\right|^{p+q} ds\Big)^{\frac{p}{q}} dt\Big\}^{\frac{q}{p+q}} \\
&= \Big\{\int_a^x \Phi'(t)\Phi(t)^{\frac{p}{q}} dt\Big\}^{\frac{q}{p+q}} \\
&= \Big\{\frac{q}{p+q}\Phi(x)^{\frac{p+q}{q}}\Big\}^{\frac{q}{p+q}} \\
&= \Big(\frac{q}{p+q}\Big)^{\frac{q}{p+q}} \Phi(x) .
\end{aligned}$$

Hence

$$\int_a^x u(t)\Big(\prod_{i=1}^N \left|y_i(t)\right|^{p_i}\Big)\left|h(t)\right|^q dt \leq C\ A(x) \int_a^x v(s)\left|h(s)\right|^{p+q} ds .$$

The case for $x < a$ can be obtained immediately from the relation

$$\int_x^a (\cdot) ds = -\int_a^x (\cdot) ds .$$

□