DEGREE ESTIMATE FOR TWO-GENERATED SUBALGEBRAS

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Abstract. We develop a new combinatorial method to deal with degree estimate for two-generated subalgebras in different environments. We obtain a lower degree bound for elements in two-generated subalgebras of a free associative algebra over a field of zero characteristic. We also reproduce a somewhat refined degree estimate of Shestakov and Umirbaev for the polynomial algebra, that plays an essential role in recent celebrated solution of the Nagata conjecture and the strong Nagata conjecture.

1. **Introduction and main results**

Suppose A is an algebra on which a degree function deg with usual properties is defined. Let B be a subalgebra of A which is generated by two elements f and g.

The question what are the possible degrees for the elements of B seems to be both natural and interesting. To be more specific, regarding $P \in B$ as a polynomial in f and g what can we say about the degree of $P(f, g)$ through degrees of P relative to f and g? It is rather clear that the upper bound of $\deg(P(f, g))$ can be easily found, but is it possible to find a meaningful lower bound?

The question is motivated by a result in [7] where a lower bound is discovered when A is a polynomial ring over a field of zero characteristic, and f, g are algebraically independent. This result plays a crucial rule in the recent celebrated solution of the Nagata conjecture [8] and the Strong Nagata conjecture [10].

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The cases when f and g are algebraically dependent and algebraically independent are apparently quite different. Say, even when A is the polynomial algebra in one variable (then, of course f and g are algebraically dependent) there is no useful estimate known to us of the smallest degree of non-constant polynomials in B and this question is very interesting indeed. For instance the well-known AMS Theorem (Abhyankar's high-school lemma, [1, 9]), is equivalent to the statement that this degree is one implies $\min\{\deg(f), \deg(g)\}$ divides $\max\{\deg(f), \deg(g)\}$.

From now on we assume that f and g are algebraically independent.

The degree estimate in [7] depends linearly on $\deg_f(P)$ and it seems that with right assumptions it should be so in far greater generality. We replace $\deg_f(P)$ by the 'weighted degree' $w_{\deg(f),\ \deg(g)}(P)$ (As the reader will see it is more natural to use $w_{\deg(f),\ \deg(g)}(P)$ instead of $\deg_f(P)$). In this paper we develop a new combinatorial method based on **Lemma on radicals**, that can be used in different environments. In particular, by this method we obtain a sharp degree estimate for the "free" case, namely when A is either a free associative algebra or a polynomial algebra over a field of zero characteristic. For a free associate algebra our degree estimate is new. We also reproduce the degree estimate for polynomial case with some improvement (see [7]) by somewhat simpler means.

Here is our main result:

Theorem 1.1. *Let $A = F\langle x_1, \dots, x_k\rangle$ be a free associative algebra over an arbitrary field F of zero characteristic, $f, g \in A$ be algebraically independent, f^+ and g^+ are algebraically independent, or f^+ and g^+ are algebraically dependent and neither* $\deg(f) \nmid \deg(g)$ *nor* $\deg(g) \nmid \deg(f)$, *$P \in F\langle x, y\rangle$. Then*

$$\deg(P(f,g)) \geq \frac{\deg([f,g])}{\deg(fg)} w_{\deg(f),\ \deg(g)}(P).$$

Here deg *is the homogeneous (total) degree of the corresponding element, $w_{\deg(f),\ \deg(g)}(P)$ is the weighted degree of P when the weight of the first variable is* $\deg(f)$ *and the weight of the second variable is* $\deg(g)$, *f^+ and g^+ are the highest homogeneous forms of f and g respectively, and $[f,g] = fg - gf$ is the commutator of f and g.*

We also reproduce the degree estimate obtained in [7].

Proposition 1.2. *Let $A = F[x_1, \dots, x_k]$ be a polynomial algebra over a field F of zero characteristic, $f, g \in A$ be algebraically independent, $P \in F[x, y]$. Then*

$\deg(P(f, g)) \geq$
$w_{\deg(f),\ \deg(g)}(P)[1 - \frac{(\deg(f),\deg(g))(\deg(f)+\deg(g)-\deg(J(f,g))-2)}{\deg(f)\deg(g)}]$.

Here deg *is the homogeneous (total) degree of the corresponding polynomial,* $w_{\deg(f),\ \deg(g)}(P)$ *is the weighted degree of* P *when the weight of the first variable is* $\deg(f)$ *and the weight of the second variable is* $\deg(g)$*,* $(\deg(f), \deg(g))$ *is the greatest common divisor of* $\deg(f)$ *and* $\deg(g)$*,* $\deg(J(f, g))$ *is the largest degree of nonzero Jacobian determinants of* f *and* g *with respect to two of* $x_1, \dots, x_k$*.*

Remark 1.3. Note that in Proposition 1.2 there are no restrictions for the highest homogeneous forms of f and g, unlike the estimate in [7], where $(f,\ g)$ is required to be a so-called $*$-pair, i. e. neither f^+ is a power of g^+ nor g^+ is a power of f^+. Also the estimate in [7] follows from the estimate above since $P(f, g)$ contains a monomial $f^{\deg_1(P)} g^j$ where $j \geq 0$; hence $w_{\deg(f),\ \deg(g)}(P(f, g)) \geq \deg_1(P)w(f)$ and replacing $w_{\deg(f),\ \deg(g)}(P)$ by $\deg_1(P)w(f)$ one gets the estimate from [7]. Therefore the estimate obtained in Proposition 1.2 can be viewed as a refinement of the estimate in [7].

2. **Reduction by Lemma on radicals**

As the first step extend the algebra A to the algebra $\mathcal{A}$ of the Maltsev-Neumann power series.

Here is the definition of Matlsev-Neumann power series. Let G be a linearly ordered group (of course the order should agree with the group operations). $\mathcal{A}$ is defined as the set of all sums $\sum_{g \in \Delta} c_g g$ where $g \in G$, $c_g \in F \setminus 0$, and Δ is a well-ordered subset of G. Both addition and multiplication are naturally defined though it requires some efforts to prove that the multiplication is well-defined.

These algebras were introduced by Maltsev and Neumann [4, 6] in order to show that a free associative algebra of any rank can be embedded into a ring with division.

In our context it is more convenient to have $\Delta' s$ to be well-ordered relative to the opposite order, that is, any non-empty subset of Δ should have the largest element.

Let us start with polynomial algebras. In this case we take as G the free abelian group G on generators $x_1, \dots, x_k$. The total degree can be defined on the elements of G and it gives a partial order on G. This partial order can be refined to a linear order on G, say by adding lexicographic order $x_1 >> \cdots >> x_k > 1$.

For a free associative algebra take the free group G on generators $x_1, \dots, x_k$. The total degree can be defined on the elements of G and again it gives a partial order on G. It is possible to refine (in many different ways) this partial order to a linear order on G (so that the order agrees with the group operations), but the description of these orders is too involved, and the reader should consult [4, 6] for details, if interested.

The algebra $\mathcal{A}$ has a valuation $v(a) = \max\{g \in \Delta(a)\}$, where $\Delta(a)$ is the support of a, i. e. the set of all g appearing in a with non-zero coefficients.

It is also possible to define the leading forms of elements of algebras $\mathcal{A}$ introduced above. If $a = \sum_{g\in\Delta(a)} c_g g$ then $a^+ = \sum_{g\in\delta(a)} c_g g$ where $\delta(a) = \{g \in \Delta(a) | \deg(g) = \deg(v(a))$. Here deg is the total degree.

The Maltsev-Neumann algebras ([4, 6]) are, as mentioned, algebras with division which satisfy

Lemma on radicals. If $a \in \mathcal{A}$ and the monomial $v(a)$ has a root: $v(a) = (c\mathbf{m})^k$ where $c \in F$ and $\mathbf{m}$ is a monomial, then there exists an $\alpha \in \mathcal{A}$ such that $a = \alpha^k$ if k is not divisible by characteristic of F. (It can be shown using "approximations" relative to the valuation function on $\mathcal{A}$.)

See [2, 3] for a proof of the above lemma for the case A is a free associative algebra. The proof is similar to the proofs given in [4, 6] that $\mathcal{A}$ is a ring with division, that is, for a non-zero element of $\mathcal{A}$ there exists a root of degree -1. See also [5] where further applications of this technique are given.

In the case A is a polynomial algebra, the Lemma can be proved directly by using Newton's binomial theorem for general degree.

Call an element $a \in \mathcal{A}$ homogeneous if all monomials of a have the same total degree. We are going to use the following obvious observation: If $a, b \in \mathcal{A}$ are algebraically dependent and homogeneous then one is a fractional power of another (with a coefficient from F). Indeed, if a and b are algebraically dependent then $v(a)$ and $v(b)$ are algebraically dependent. Since these are just monomials with coefficients they are powers of a third monomial (with coefficients from F). We can assume that $v(a) = \alpha^q$, $v(b) = \alpha^p$. Let $a_1 = a^{q^{-1}}$. Then $b = a_1^p + b_1$. If $b_1 \neq 0$ then $v(b_1)$ is algebraically independent with $v(b)$ since $v(b_1) < v(b)$. In this case a_1 and b_1 generate a free subalgebra of $\mathcal{A}$ and $P(a, b) = P(a_1^q, a_1^p + b_1) \neq 0$ if P is a non-zero polynomial.

Now we can reduce f and g. Let $v(f) = \alpha$ and $v(g) = \beta$. If α and β are algebraically independent then obviously $\deg(P(f, g)) = w_{\deg(f),\ \deg(g)}(P)$. Also then $\deg([f, g]) = \deg(fg)$ in the free case and $\deg(J(f, g)) + 2 = \deg(fg)$ in the commutative case and both estimates have $w_{\deg(f),\ \deg(g)}(P)$ in the right sides.
So from now on let us assume that α and β are algebraically dependent. Then $\alpha = \gamma^q$, $\beta = \gamma^p$ where γ is a monomial without a root. (As above, we are assuming without loss of generality that the coefficients of α and β are equal to 1.)

If F has zero characteristic, we can by the Lemma on radicals put $f = \tau^q$ and $g = \tau^p + \sum_{i=l}^{p-1} c_i\tau^i + s$, $c_i \in F$, where $v(\tau)$ and $v(s)$ are algebraically independent. Note that in the sum i can be less than zero.

Now we can find $\deg(s)$. Indeed, in the free case $[f, g] = [f, s]$ and $\deg([f, s]) = \deg(f) + \deg(s)$. So $\deg(s) = \deg([f, g]) - \deg(f)$. Similarly in commutative case take derivations ∂_i on $\mathcal{A}$ which are given by $\partial_i(x_j) = \delta_{i,j}$ (where $\delta_{i,j}$ is the Kronecker delta). Then $\partial_i(f)\partial_j(g) - \partial_j(f)\partial_i(g) = \partial_i(f)\partial_j(s) - \partial_j(f)\partial_i(s)$ by elementary calculus rules. It is also clear that

$$v(\partial_i(f)\partial_j(s) - \partial_j(f)\partial_i(s)) = \partial_i(v(f))\partial_j(v(s)) - \partial_j(v(f))\partial_i(v(s))$$

if $\partial_i(v(f))\partial_j(v(s)) - \partial_j(v(f))\partial_i(v(s)) \neq 0$ and that if $\partial_i(v(f))\partial_j(v(s)) - \partial_j(v(f))\partial_i(v(s)) = 0$ for all i, j then $v(f)$ and $v(s)$ are algebraically dependent. Since it is not the case, $\deg(s) = D + 2 - \deg(f)$, where D is the largest degree of $J_{i,j}(f, g)$ i. e. the Jacobians with respect to x_i, x_j.

Consider subalgebra B of $\mathcal{A}$ which is generated by τ, τ^{-1}, and s. Clearly $f, g \in B$. Take the weight degree function on B given by $w(\tau) = 1$, $w(s) = p$. Then $w(f) = q$, $w(g) = p$ and we can write $P(f, g) = \sum_{i=M}^{N} P_i(f, g)$ where P_i are homogeneous polynomials relative to these weights and $w(P_i) = i$. Let a^+ be the leading form of the element $a \in A$ relative to w. Then $f^+ = \tau^q$ and $g^+ = \tau^p + s$. Since these forms are algebraically independent, we can conclude that

$(P_i(f,g))^+ = P_i(f^+,g^+)$. All monomials of $P_i(f^+,g^+)$ have weight i. Therefore monomials in $P_i(f^+,g^+)$ and $P_j(f^+,g^+)$ are different if $i \neq j$. This implies that $\deg(P(f,g)) \geq \deg(P_N(f^+,g^+))$ where P_N is "the heaviest" homogeneous component of P.

We have reduced our main problem to the following

Set-up. Estimate $\deg(P(f,g))$ for an n,m-homogeneous polynomial P and $f = t^{n_1}$, $g = t^{m_1} + s$ where t and s generate either a free associative algebra or a polynomial algebra. Here $n_1 = n/d$, $m_1 = m/d$ where $d = (n,m)$. Function deg is given by $\deg(f) = n$, $\deg(g) = m$. So $\deg(t) = d$. In the free case $\deg(s) = \deg([f,g]) - \deg(f)$ and in the polynomial case $\deg(s) = D + 2 - \deg(f)$, where $D = \max(\deg(J_{i,j}(f,g)))$. Since $mq = np$ element $t \in \mathcal{A}$ is a power of τ. We replaced τ by t since it is more convenient for further computations to have $\deg_t(f)$ and $\deg_t(g)$ relatively prime.

We will also use another degree function w given by $w(t) = 1$, $w(s) = m_1$. So $w(f) = n_1$, $w(g) = m_1$, and P is w-homogeneous.

3. Degree estimate for free associative algebras

In this section we assume that $F\langle t,s\rangle$ is a free associative algebra of rank two. In view of the hypothesis of Theorem 1.1 we add one more condition that $m_1 > n_1 > 1$ (When $n_1 = 1$ the pair f,g can be replaced by a 'smaller' pair $f, g - cf^{m_1}$ which generates the same algebra).

Assume $w(P) = N$. We will show that $Q(t,s) = P(t^{n_1}, t^{m_1}+s)$ contains a monomial with s-degree not exceeding $\frac{N}{n_1+m_1}$.

We can write $P(f,g) = \sum P_{I,J}(f,g)$ where all monomials of $P_{I,J}$ have the same total degrees I and J relative to f and g correspondingly. Of course $N = n_1 I + m_1 J$.

Denote the integral part of $\frac{N}{n_1+m_1}$ by q. Let R be the non-zero $P_{I,J}(f,g)$ where J is the largest possible. If $J = \deg_g(R) \leq q$ our claim is obviously correct since all monomials of $Q(t,s)$ will have s-degree at most $\deg_g(R)$. So assume that $\deg_g(R) > q$ and that all monomials of $Q(t,s)$ have s-degree larger than q.

Let $\mu = f^{i_1}g^{j_1}\dots f^{i_k}g^{j_k}$ be one of the monomials of R. After the substitution $f \to t^{n_1}$, $g \to t^{m_1} + s$ the image of monomial μ will be a sum of monomials obtained by multiplication of t^{n_1}, t^{m_1}, and s. Among these monomials we consider only monomials with the total degree q relative to s and denote this set of monomials by $\varphi(\mu)$.

Suppose that $I+J \geq 2q+2$. Then $N = n_1 I + m_1 J = n_1(I+J) + (m_1 - n_1)J \geq n_1(2q+2) + (m_1-n_1)(q+1) = (n_1+m_1)(q+1)$ and $\frac{N}{n_1+m_1} \geq q+1$ which is impossible since q is the integer part of $\frac{N}{n_1+m_1}$. So $I+J \leq 2q+1$.

If $\mu = f^I g^J$ and $2q - J \geq 0$ take $\nu = t^{In_1}(st^{m_1})^{J-q}s^{2q-J} \in \varphi(f^I g^J)$. Since $I+J \leq 2q+1$ if $2q-J<0$ then $J = 2q+1$, $I=0$. In this case $\mu = g^J$ and we take $\nu = t^{m_1}(st^{m_1})^q$.

It is clear that if $\varphi(\xi) \ni t^{m_1}(st^{m_1})^q$ then $\xi = g^J$, since neither s nor t^{m_1} can be in the image of f^i because n_1 does not divide m_1. Similarly if $t^{In_1}(st^{m_1})^{J-q}s^{2q-J} \in \varphi(\xi)$ then $\xi = \xi_1 g^J$ since neither s nor t^{m_1} cannot be in the image of f^i. If $\xi_1 \neq f^I$ then ξ_1 contains g and $\deg_g(\xi) > J$. But by assumption on R such a monomial does not belong to P. So ν cannot cancel out and belongs to Q with a non-zero coefficient. Therefore by our assumption on monomials of Q neither g^J nor $f^I g^J$ belong to R (with a non-zero coefficient).

Suppose now that μ is the largest monomial of R in a lexicographic ordering given by $f >> g$. As above, let $\mu = f^{i_1}g^{j_1}\dots f^{i_k}g^{j_k}$. If $j_r = 2\sigma$ replace $g^{j_r} \to (st^{m_1})^\sigma$, if $j_r = 2\sigma+1$ replace $g^{j_r} \to (st^{m_1})^\sigma s$, replace $f \to t^{n_1}$. The s-degree of obtained monomial π is $\sum_r [\frac{j_r+1}{2}] \leq \frac{J+k}{2}$. Since $i_1 \geq 0$ and $i_r > 0$ if $r > 1$ it is clear that $I \geq k-1$. So $J + k \leq J + I + 1 \leq 2q+2$. Therefore $\deg_s(\pi) \leq q+1$.

If $\deg_s(\pi) = q$ take $\nu = \pi \in \varphi(\mu)$.

If $\deg_s(\pi) < q$ to obtain $\nu \in \varphi(\mu)$ replace the corresponding number of t^{m_1} in π by s. The choice of the terms for replacement is arbitrary.

If $\deg_s(\pi) = q+1$ then $I+J = 2q+1$, all j_s are odd, and $k = I+1$. Therefore $i_1 = 0$, $i_2 = \dots = i_k = 1$. To obtain $\nu \in \varphi(\mu)$ in this case replace the image of g^{j_1} in π by $(t^{m_1}s)^\sigma t^{m_1}$.

It is easy to see that $\nu \notin \varphi(\xi)$ for a monomial $\xi \neq \mu$ of $P(f,g)$ and so μ cannot appear in R. Indeed, any s in ν as well as t^{m_1} (surrounded by s) comes from g in ξ. The structure of ν is such that it may contain $t^{m_1+n_1}$ and large powers of t, which correspond to products gf and powers of f in μ. Any $t^{m_1+n_1}$ corresponds to gf or fg so $\deg_g(\xi) \geq q$ and if any t^{kn_1} of ν is replaced by a monomial containing g then $\deg_g(\xi) > q$ and so ξ does not belong to P. So any t^{kn_1} should be replaced by f^k in order for ξ to be a monomial in $P(f,g)$. Finally, if any $t^{m_1+n_1}$ is replaced by fg the resulting monomial will be larger than μ in the lexicographic order and again cannot belong to $P(f,g)$.

Estimate. Let $N = w(P(f,g) = w_{n_1,\ m_1}(P)$. As we checked $Q(t,s)$ contains a monomial ν for which $\deg_s(\nu) \le \frac{N}{n_1+m_1}$. Let ξ be a monomial of $Q(t,s)$. If $i = \deg_t(\xi)$ and $j = \deg_s(\xi)$ then $N = i + jm_1$ and $\deg(\xi) = id + j(\deg([f,g]) - \deg(f))$ since $\deg(s) = \deg([f,g]) - \deg(f)$. So $\deg(\xi) = dN + j(\deg([f,g] - n - m)$. Therefore $\deg(P(f,g)) = \deg(Q(s,t)) \ge dN + \frac{N}{n_1+m_1}(\deg([f,g] - n - m) = \frac{\deg([f,g])}{n_1+m_1}N$ $= \frac{\deg([f,g])}{n+m} w_{n,m}(P)$. Theorem 1.1 is proved.

Example 3.1. $f = x^n$, $g = x^m + y$, $P = [x,y]^k$, $\deg(P(f,g)) = k(n+1) = \frac{w_{n,m}(P)}{\deg(fg)} \deg([f,g])$ shows that the lower bound in Theorem 1.1 cannot be improved.

Remark 3.2. If $n_1 = 1$ the estimate does not work. Take e. g. $f = x$, $g = x^m + y$, and $P = g - f^m$. It happens because the degree drop for $g^{n_1} - f^{m_1}$ in this case is larger than the degree drop of $[f,g]$.

4. **Degree estimate for polynomial algebras**

From now let us assume that $F[t,s]$ is a polynomial algebra of rank two. Note that unlike the noncommutative case, here we do not need any restriction for n_1.

Let $f_1 = f^{m_1} - g^{n_1}$. We can write $P(f,g) = \sum c_{i,j,k} f^i g^j f_1^k$ where $0 \le i < m_1$. Since P is w-homogeneous $in_1 + jm_1 + kn_1m_1 = w(P)$ does not depend on monomial ($w(f) = n_1$, $w(g) = m_1$, $w(f_1) = n_1m_1$) and we can conclude that i is the same in all monomials. So $P = f^i \sum g^j f_1^k$ where $0 \le i < m_1$.

Now $\deg(f_1) = m(n_1 - 1) + \deg(s)$, $\deg(s) = D + 2 - \deg(f)$, and $\deg(f^i g^j f_1^k) = in + jm + k(mn_1 - m - n + D + 2) = dw(P) - k(m + n - D - 2)$ and it is different for the different monomials of P. (Since i is fixed different monomials have different k.) The maximal possible value of k is $[\frac{w(P)}{n_1m_1}]$ (the integral part of the corresponding fraction) and the minimal value of $\deg(P(t^n, t^m + s))$ is $w(P)(d - \frac{m+n-D-2}{n_1m_1}) = w_{\deg(f),\ \deg(g)}(P)[1 - \frac{(\deg(f),\deg(g))(\deg(f)+\deg(g)-\deg(J(f,g))-2}{\deg(f)\deg(g)}]$. Proposition 1.2 (a) is proved.

Example 1 in [7] shows the lower bound in Proposition 1.2 cannot be improved.

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