# PARTITIONS OF THE WONDERFUL GROUP COMPACTIFICATION

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Abstract. We define and study a family of partitions of the wonderful compactification $\overline{G}$ of a semi-simple algebraic group $G$ of adjoint type. The partitions are obtained from subgroups of $G \times G$ associated to triples $(A_1, A_2, a)$, where $A_1$ and $A_2$ are subgraphs of the Dynkin graph $\Gamma$ of $G$ and $a\colon A_1 \to A_2$ is an isomorphism. The partitions of $\overline{G}$ of Springer and Lusztig correspond respectively to the triples $(\emptyset, \emptyset, \mathrm{id})$ and $(\Gamma, \Gamma, \mathrm{id})$.

## 1. Introduction

Let $G$ be a connected semi-simple algebraic group over an algebraically closed field $k$. De Concini and Procesi [5, 6] constructed a wonderful compactification $\overline{G}$ of $G$, which is a smooth irreducible $(G \times G)$-variety with finitely many $(G \times G)$-orbits. Let $G_{\mathrm{diag}}$ be the diagonal subgroup of $G \times G$. In his study of parabolic character sheaves on $\overline{G}$ in [14, 15], Lusztig introduced (by an inductive procedure) a partition of $\overline{G}$ by finitely many $G_{\mathrm{diag}}$-stable pieces. The closure of a $G_{\mathrm{diag}}$-stable piece was shown by X.-H. He [8] to be a union of such pieces. Let $B$ be a Borel subgroup of $G$. Then $\overline{G}$ is also partitioned into finitely many $(B \times B)$-orbits. The $(B \times B)$-orbits in $\overline{G}$, as well as their closures, were studied by T. Springer in [18]. In [8], X.-H. He gave a second description of Lusztig's $G_{\mathrm{diag}}$-stable pieces using $(B \times B)$-orbits in $\overline{G}$, which then enabled him to give [9] an equivalent definition of Lusztig's character sheaves on $\overline{G}$. Further properties and applications of the $G_{\mathrm{diag}}$-stable pieces were obtained by X.-H. He and J. F. Thomsen in [7, 9, 10].

Both $G_{\mathrm{diag}}$ and $B \times B$ are special examples of subgroups $R_{\mathcal{A}}$ of $G \times G$ associated to triples $(A_1, A_2, a)$, where $A_1$ and $A_2$ are subgraphs of the Dynkin graph $\Gamma$ of $G$, and $a$ is an isomorphism from $A_1$ to $A_2$. If $P_{A_1}$ and $P_{A_2}$ are the standard parabolic subgroups of $G$ corresponding to $A_1$ and $A_2$ respectively, then, roughly speaking, $R_{\mathcal{A}}$ is a subgroup of $P_{A_1} \times P_{A_2}$, obtained by identifying the Levi subgroups of $P_{A_1}$ and $P_{A_2}$ via the map $a$. The precise definition of $R_{\mathcal{A}}$ is given in §2.1. For example, every stabilizer subgroup of $G \times G$ in $\overline{G}$ is conjugate to a group of this form. Moreover, $G_{\mathrm{diag}}$ is associated to the triple $(\Gamma, \Gamma, \mathrm{id})$ and $B \times B$ to the triple $(\emptyset, \emptyset, \mathrm{id})$, where $\emptyset$ is the empty set.

In this paper, for any subgroup $R_{\mathcal{A}}$ of $G \times G$ associated to a triple $(A_1, A_2, a)$, we study a partition of $\overline{G}$ into finitely many $R_{\mathcal{A}}$-stable pieces indexed by a subset of the Weyl group of $G \times G$. Our definition of the $R_{\mathcal{A}}$-stable pieces is based on our earlier paper [12] on $(R_{\mathcal{A}}, R_{\mathcal{C}})$-double cosets in $G \times G$ for any pair of subgroups $R_{\mathcal{A}}$ and $R_{\mathcal{C}}$ associated to triples $(A_1, A_2, a)$ and $(C_1, C_2, c)$. We give two additional descriptions of the $R_{\mathcal{A}}$-stable pieces, which for the case of $G_{\mathrm{diag}}$-stable pieces, reduce to Lusztig's inductive description in [14, 15] and He's description in [8] using $(B \times B)$-orbits. In particular, we show that the $R_{\mathcal{A}}$-stable pieces are smooth, irreducible, locally closed subsets of $\overline{G}$, fibered over flag varieties of Levi subgroups of $G$. We also show that the closure in $\overline{G}$ of an $R_{\mathcal{A}}$-stable piece is a union of such pieces. We then describe the combinatorics for the closures of the $R_{\mathcal{A}}$-stable pieces, generalizing both the result of He [8] for the $G_{\mathrm{diag}}$-stable pieces and that of Springer for the $(B \times B)$-orbits. The closure relations of the $R_{\mathcal{A}}$-stable pieces are expressed in terms of intersections of the closures with the unique closed $(G \times G)$-orbit in the boundary of $\overline{G}$.

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Our motivation for studying the $R_{\mathcal{A}}$-stable pieces in $\overline{G}$ for an arbitrary triple $(A_1, A_2, a)$ comes from Poisson geometry. In [13], we study a class of Poisson structures on $\overline{G}$ induced by Belavin–Drinfeld $r$-matrices [1]. The triples $(A_1, A_2, a)$ needed there are precisely the Belavin–Drinfeld triples for the $r$-matrices. The $R_{\mathcal{A}}$-stable pieces in $\overline{G}$ as well as their closures are Poisson subvarieties of $\overline{G}$ for the corresponding Poisson structures. To understand these Poisson structures, one needs to first understand the geometry of the $R_{\mathcal{A}}$-stable pieces.

In §2 - §5, we in fact assume that $G_1$ and $G_2$ are any two reductive algebraic groups over an algebraically closed field, and that $R_{\mathcal{A}}$ and $R_{\mathcal{C}}$ are subgroups of $G_1 \times G_2$ associated to two triples $(A_1, A_2, a)$ and $(C_1, C_2, c)$ for $G_1 \times G_2$. The precise definitions of the subgroups $R_{\mathcal{A}}$ and $R_{\mathcal{C}}$ are given in §2.1. Each pair $(R_{\mathcal{A}}, R_{\mathcal{C}})$ of such subgroups gives rise to a decomposition of $G_1 \times G_2$ into $(R_{\mathcal{A}}, R_{\mathcal{C}})$-stable subsets of the form $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$, where $(v_1, v_2)$ runs over a subset of the Weyl group for $G_1 \times G_2$ (see (2.4) for detail). In §4, we give a description of the set $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ as iterated fiber bundles. The closures of the sets $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ in $G_1 \times G_2$ are described in §5. In §6 and §7, the results in §5 are used to prove our main theorems on the $R_{\mathcal{A}}$-stable pieces in $\overline{G}$ for a semi-simple algebraic group $G$ of adjoint type. Precise statements of our results are summarized in §2. In the appendix we collect a few facts on the Bruhat order on Weyl groups that we use in the paper.

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## 2. Notation and statements of results

2.1. **Admissible quadruples.** For $i = 1, 2$, let $G_i$ be a connected reductive algebraic group over an algebraically closed field k. Let $B_i$ and $B_i^-$ be a fixed pair of opposite Borel subgroups of $G_i$. Set $T_i = B_i \cap B_i^-$, and let $\Gamma_i$ be the set of simple roots determined by $(B_i, T_i)$. For $\alpha \in \Gamma_i$, denote by $U_i^\alpha$ the one-parameter unipotent subgroup of $G_i$ defined by $\alpha$. For a subset $A_i$ of $\Gamma_i$, let $P_{A_i}$ and $P_{A_i}^-$ be the standard parabolic subgroups of $G_i$ containing respectively $B_i$ and $B_i^-$. Let $M_{A_i} = P_{A_i} \cap P_{A_i}^-$ be the common Levi factor of $P_{A_i}^{\pm}$, and let $Z_{A_i}$ be the center of $M_{A_i}$. The unipotent radicals of $P_{A_i}$ and $P_{A_i}^-$ will be denoted by $U_{A_i}$ and $U_{A_i}^-$ respectively. Let $W_i$ be the Weyl group of $\Gamma_i$ and $W_{A_i}$ the subgroup of $W_i$ generated by reflections defined by simple roots in $A_i$. Let $W_i^{A_i}$ and ${}^{A_i}W_i$ be the sets of minimal length representatives of cosets from $W_i/W_{A_i}$ and $W_{A_i}\backslash W_i$ respectively. For each $w_i \in W_i$, we also fix a choice $\dot{w}_i$ of a representative of $w_i$ in the normalizer of $T_i$ in $G_i$. The length function on $W_i$ will be denoted by $l$. If a group $G$ acts on a set $X$, $g.x$ denote the action of $g \in G$ on $x \in X$. For an element $g \in G$, the map $G \to G : h \mapsto ghg^{-1}$ will be denoted by $\mathrm{Ad}_g$. The identity element of a group will be denoted by $e$ or 1.

For subsets $A_i$ of the Dynkin graphs $\Gamma_i$, $i = 1, 2$, we call a bijective map $a\colon A_1 \to A_2$ an isomorphism, if it preserves the type of each arrow.

**Definition 2.1.** An admissible quadruple for $G_1 \times G_2$ is a quadruple $\mathcal{A} = (A_1, A_2, a, K)$ consisting of subsets $A_1$ of $\Gamma_1$ and $A_2$ of $\Gamma_2$, an isomorphism $a\colon A_1 \to A_2$, and a closed subgroup $K$ of $M_{A_1} \times M_{A_2}$ of the form

$$K = \{(m_1, m_2) \in M_{A_1} \times M_{A_2} \mid \theta_a(m_1 Z_1) = m_2 Z_2\}, \tag{2.1}$$

where, for $i = 1, 2$, $Z_i$ is a closed subgroup of $Z_{A_i}$ and $\theta_a : M_{A_1}/Z_1 \to M_{A_2}/Z_2$ is an isomorphism mapping $T_1/Z_1$ to $T_2/Z_2$ and $U_1^\alpha$ to $U_2^{a(\alpha)}$ for each $\alpha \in A_1$. Here we identify $U_1^\alpha$ and $U_2^{a(\alpha)}$ with their images in $M_{A_1}/Z_1$ and $M_{A_2}/Z_2$ respectively. Given an admissible quadruple $\mathcal{A} = (A_1, A_2, a, K)$ of $G_1 \times G_2$, define

$$R_{\mathcal{A}} = K(U_{A_1} \times U_{A_2}) \subset P_{A_1} \times P_{A_2}, \qquad R_{\mathcal{A}}^- = K(U_{A_1}^- \times U_{A_2}) \subset P_{A_1}^- \times P_{A_2}. \tag{2.2}$$

Note that when $G_1 = G_2 = G$, the diagonal subgroup $G_{\rm diag}$ and $B \times B$ for a Borel subgroup $B$ are examples of the groups $R_{\mathcal{A}}$. If further $G$ is of adjoint type, all stabilizer subgroups of $G \times G$ in the De Concini–Procesi [5] compactification $\overline{G}$ of $G$ are conjugate to groups of the type $R_{\mathcal{A}}^-$ (see §2.3).

2.2. **An $(R_{\mathcal{A}}, R_{\mathcal{C}})$-stable partition of $G_1 \times G_2$.** In [12] we obtained a classification of $(R_{\mathcal{A}}, R_{\mathcal{C}})$-double cosets of $G_1 \times G_2$ for two arbitrary admissible quadruples $\mathcal{A} = (A_1, A_2, a, K)$ and $\mathcal{C} = (C_1, C_2, c, L)$ for $G_1 \times G_2$. Given $v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2$, set

$$C_2(v_1, v_2) = \{\beta \in C_2 \mid (v_2^{-1} a v_1 c^{-1})^n \beta \text{ is defined and is in } C_2 \text{ for } n = 1, 2, \ldots\}. \tag{2.3}$$

In other words, $C_2(v_1, v_2)$ is the largest subset of $C_2$ that is stable under $v_2^{-1} a v_1 c^{-1}$. We proved [12] that each $(R_{\mathcal{A}}, R_{\mathcal{C}})$-double coset of $G_1 \times G_2$ is of the form $R_{\mathcal{A}}(\dot{v}_1, \dot{v}_2 m_2) R_{\mathcal{C}}$ for some $v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2$ and $m_2 \in M_{C_2(v_1,v_2)}$. Two such double cosets $R_{\mathcal{A}}(\dot{v}_1, \dot{v}_2 m_2) R_{\mathcal{C}}$ and $R_{\mathcal{A}}(\dot{v}_1', \dot{v}_2' m_2') R_{\mathcal{C}}$ coincide if and only if $v_i' = v_i$ for $i = 1, 2$, and $m_2$ and $m_2'$ are in the same $(v_2^{-1} a v_1 c^{-1})$-twisted conjugacy class in $M_{C_2(v_1,v_2)}$, see Theorem 3.1 for details.

For $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$, let

$$[v_1, v_2]_{\mathcal{A},\mathcal{C}} = R_{\mathcal{A}}(v_1, v_2 M_{C_2(v_1,v_2)}) R_{\mathcal{C}} \subset G_1 \times G_2. \tag{2.4}$$

Then by the above result from [12], we have the decomposition

$$G_1 \times G_2 = \bigsqcup_{v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2} [v_1, v_2]_{\mathcal{A},\mathcal{C}}. \tag{2.5}$$

Here and below $\bigsqcup$ stands for disjoint union. Note that (2.5) is constructed in such a way that the $(R_{\mathcal{A}}, R_{\mathcal{C}})$-double cosets of $G_1 \times G_2$ corresponding to the same discrete parameters $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$ but possibly different continuous parameters $m_2 \in M_{C_2(v_1,v_2)}$ are put together in a single stratum. Alternatively we have the decomposition

$$(G_1 \times G_2)/R_{\mathcal{C}} = \bigsqcup_{v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2} [v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}} \tag{2.6}$$

of $(G_1 \times G_2)/R_{\mathcal{C}}$ into $R_{\mathcal{A}}$-stable subsets.

The main objects of study in this paper are the sets $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ for $v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2$. We describe their geometry, as well as their closure relations. The results are then applied to the wonderful group compactifications.

The following theorem summarizes our results for the decompositions (2.5) and (2.6), see Corollary 4.9, Proposition 4.10, Proposition 4.11, and Theorem 5.2.

**Theorem 2.2.** *Given any two admissible quadruples $\mathcal{A}$ and $\mathcal{C}$ for $G_1 \times G_2$, the following hold for every $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$.*

*(i) $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ is locally closed, smooth, and irreducible. Its projection $[v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}$ to $(G_1 \times G_2)/R_{\mathcal{C}}$ fibers over the flag variety $M_{A_1}/(M_{A_1} \cap P_{A_1(v_1,v_2)})$ with fibers isomorphic to the product of $M_{C_2(v_1,v_2)}/Y_2$ and the affine space of dimension*

$$\dim U_{A_1(v_1,v_2)} - \dim(U_1 \cap v_1(U_{C_1})) + l(v_2),$$

*where $A_1(v_1, v_2) = v_1 c^{-1} C_2(v_1, v_2) \subset A_1$, and $Y_2 = \{m \in M_{C_2} \mid (e, m) \in L\} \subset Z_{C_2}$.*

*(ii) Alternatively the set $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ is given by*

$$[v_1, v_2]_{\mathcal{A},\mathcal{C}} = R_{\mathcal{A}}(B_1 \times B_2)(v_1, v_2) R_{\mathcal{C}}. \tag{2.7}$$

*(iii) The Zariski closure of $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ in $G_1 \times G_2$ consists of those $[w_1, w_2]_{\mathcal{A},\mathcal{C}}$ with $w_1 \in W_1^{C_1}$ and $w_2 \in {}^{A_2}W_2$ for which there exist $x_1 \in W_{A_1}$ and $y_1 \in W_{C_1}$ such that*

$$x_1 w_1 y_1 \le v_1 \quad \text{and} \quad a(x_1) w_2 c(y_1) \le v_2.$$

*(iv) If $\mathcal{A}' = (A_1, A_2, a, L')$ and $\mathcal{C} = (C_1, C_2, c, L')$ are two other admissible quadruples containing the same triples $(A_1, A_2, a)$ and $(C_1, C_2, c)$, then there exist $t_2, s_2 \in T_2$ such that*

$$[v_1, v_2]_{\mathcal{A}',\mathcal{C}'} = (e, t_2)[v_1, v_2]_{\mathcal{A},\mathcal{C}}(e, s_2), \qquad \forall v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2.$$

**Example 2.3.** When $\mathcal{A} = \mathcal{C} = (\emptyset, \emptyset, \mathrm{id}, T_1 \times T_2)$ so that $R_{\mathcal{A}} = R_{\mathcal{C}} = B_1 \times B_2$, we have

$$[v_1, v_2]_{\mathcal{A},\mathcal{C}} = (B_1 \times B_2)(v_1, v_2)(B_1 \times B_2), \qquad \forall (v_1, v_2) \in W_1 \times W_2.$$

Thus (2.5) reduces to the Bruhat decomposition

$$G_1 \times G_2 = \bigsqcup_{v_1 \in W_1, v_2 \in W_2} (B_1 \times B_2)(v_1, v_2)(B_1 \times B_2).$$

Part (iii) of Theorem 2.2 in this case is the well-known statement for the closures of Bruhat cells.

2.3. **Partitions of the wonderful group compactification.** Now we specialize to the case when $G_1$ and $G_2$ are both isomorphic to a connected semisimple algebraic group $G$ of adjoint type. All data for $G$ will be denoted as in §2.1, omitting the index $i$. In particular, $B$ and $B^-$ will be two fixed opposite Borel subgroups of $G$, $T = B \cap B^-$, and $\Gamma$ will be the set of simple roots determined by $(B, T)$.

For $J \subset \Gamma$, let $\pi_J : M_J \to M_J/Z_J$ be the natural projection. By abuse of notation, we will denote by $J$ the quadruple $(J, J, \mathrm{id}, L_J)$, where

$$L_J = \{(m_1, m_2) \in M_J \times M_J \mid \pi_J(m_1) = \pi_J(m_2)\};$$

so

$$R_J^- = L_J(U_J^- \times U_J) = \{(p_1, p_2) \in P_J^- \times P_J \mid \pi_J(p_1) = \pi_J(p_2)\}. \tag{2.8}$$

Recall that the wonderful compactification $\overline{G}$ of $G$ is a smooth irreducible projective $(G \times G)$-variety containing $G$ as an open $(G \times G)$-orbit. It was defined and studied (in the more general framework of symmetric spaces) by De Concini and Procesi [5] in the complex case and by De Concini and Springer [6] for the general case of an algebraically closed field k. The $(G \times G)$-orbits on $\overline{G}$ are parameterized by the subsets $J$ of $\Gamma$. A base point $h_J$ of the $(G \times G)$-orbit corresponding to $J \subset \Gamma$ has stabilizer subgroup $R_J^-$.

The partitions (2.5) induce partitions of $\overline{G}$ as follows. Given an arbitrary admissible quadruple $\mathcal{A} = (A_1, A_2, a, K)$ for $G \times G$, by setting

$$[J, v_1, v_2]_{\mathcal{A}} = R_{\mathcal{A}}(B \times B)(v_1, v_2).h_J, \quad J \subset \Gamma, v_1 \in W^J, v_2 \in {}^{A_2}W, \tag{2.9}$$

we obtain the following partition of the wonderful compactification

$$\overline{G} = \bigsqcup_{J \subset \Gamma, v_1 \in W^J, v_2 \in {}^{A_2}W} [J, v_1, v_2]_{\mathcal{A}}, \tag{2.10}$$

cf. (2.6) and (ii) of Theorem 2.2. We will refer to the sets $[J, v_1, v_2]_{\mathcal{A}}$ in (2.10) as $R_{\mathcal{A}}$-stable pieces of $\overline{G}$. If the subgroup $K$ in $\mathcal{A}$ is changed to $K'$, the partition (2.10) is changed by an overall left translation by $(e, t)$ for some $t \in T$ (see (iv) of Theorem 2.2 and Proposition 4.11).

**Example 2.4.** Let $\mathcal{A}$ be the trivial admissible quadruple $(\emptyset, \emptyset, \mathrm{Id}, T \times T)$. Then $R_{\mathcal{A}} = B \times B$ and we recover Springer's partition [18] of $\overline{G}$ by $(B \times B)$-orbits:

$$\overline{G} = \bigsqcup_{J \subset \Gamma, v_1 \in W^J, v_2 \in W} (B \times B)(v_1, v_2).h_J. \tag{2.11}$$

On the other hand, let $\mathcal{A}$ be the quadruple $\Gamma_{\mathrm{diag}} := (\Gamma, \Gamma, \mathrm{id}, G_{\mathrm{diag}})$ where $G_{\mathrm{diag}}$ is the diagonal subgroup of $G \times G$. By [8], we recover Lusztig's partition [14, 15] of $\overline{G}$ :

$$\overline{G} = \bigsqcup_{J \subset \Gamma, v_1 \in W^J} [J, v_1, 1]_{\Gamma_{\mathrm{diag}}} = G_{\mathrm{diag}}(B \times B)(v_1, 1).h_J. \tag{2.12}$$

For a general admissible quadruple $\mathcal{A}$ for $G \times G$, our partition (2.10) is a discrete interpolation between Springer's partition (2.11) and Lusztig's partition (2.12) of $\overline{G}$.

The closure relations of the strata in (2.10) can be derived directly from part (iii) in Theorem 2.2. This is stated in Proposition 7.3. However, we give a more elegant description of the closures using the unique closed $(G \times G)$-orbit on the boundary of $\overline{G}$, namely the orbit $(G \times G).h_\emptyset \cong G/B^- \times G/B$, where $\emptyset$ is the empty subset of $\Gamma$. More precisely, for any $J \subset \Gamma$ and $v_1 \in W^J$, $v_2 \in {}^{A_2}W$, let $\overline{[J, v_1, v_2]_\mathcal{A}}$ be the closure of $[J, v_1, v_2]_\mathcal{A}$ in $\overline{G}$, and set

$$\partial_\emptyset \overline{[J, v_1, v_2]_\mathcal{A}} = \overline{[J, v_1, v_2]_\mathcal{A}} \cap (G \times G).h_\emptyset.$$

The following is a summary of Theorem 7.4 and Theorem 7.6.

**Theorem 2.5.** *Let $\mathcal{A}$ be an admissible quadruple for $G \times G$. Let $I, J \subset \Gamma$, $v_1 \in W^J, v_1' \in W^I$, and $v_2, v_2' \in {}^{A_2}W$. Then*

*(i) $[I, v_1', v_2']_\mathcal{A} \subset \overline{[J, v_1, v_2]_\mathcal{A}}$ if and only if $I \subset J$ and $[\emptyset, v_1', v_2']_\mathcal{A} \subset \partial_\emptyset \overline{[J, v_1, v_2]_\mathcal{A}}$;*

*(ii) $[I, v_1', v_2']_\mathcal{A} \subset \overline{[J, v_1, v_2]_\mathcal{A}}$ if and only if $I \subset J$ and there exist $x \in W_{A_1}$ and $z \in W_J$ such that $xv_1' \geq v_1 z$ and $a(x)v_2' \leq v_2 z$.*

**Example 2.6.** Consider the case when $\mathcal{A} = (\emptyset, \emptyset, \mathrm{Id}, T \times T)$, so that $R_\mathcal{A} = B \times B$. Theorem 2.5 implies that for any $I, J \subset \Gamma$, $v_1 \in W^J, v_1' \in W^I$, and $v_2, v_2' \in W$,

$$(B \times B)(v_1', v_2').h_I \subset \overline{(B \times B)(v_1, v_2).h_J}$$

if and only if $I \subset J$ and there exists $z \in W_J$ such that $v_1' \geq v_1 z$ and $v_2' \leq v_2 z$. This description of the $(B \times B)$-orbit closures in $\overline{G}$, which is simpler than what is given in [18], was independently obtained in [11]. When $\mathcal{A} = \Gamma_{\mathrm{diag}}$ as in Example 2.4, Theorem 2.5 implies that for any $I, J \subset \Gamma$, $v_1 \in W^J, v_1' \in W^I$,

$$G_{\mathrm{diag}}(B \times B)(v_1', 1).h_I \subset \overline{G_{\mathrm{diag}}(B \times B)(v_1, 1).h_J}$$

if and only if $I \subset J$ and there exists $x \leq z \in W_J$ such that $xv_1' \geq v_1 z$. It follows from [8, Corollary 3.4] (see Lemma 8.7 in the Appendix) that the above description of the closures of the $G_{\mathrm{diag}}$-stable subsets is the same as that given in [8] by X.-H. He.

## 3. $(R_\mathcal{A}, R_\mathcal{C})$-double cosets in $G_1 \times G_2$ and their stablizer subgroups

In this section, we first recall some results from [12].

3.1. **Classification of $(R_\mathcal{A}, R_\mathcal{C})$-double cosets in $G_1 \times G_2$.** Fix two arbitrary admissible quadruples $\mathcal{A} = (A_1, A_2, a, K)$ and $\mathcal{C} = (C_1, C_2, c, L)$ for $G_1 \times G_2$. In [12] we obtained a classification of the $(R_\mathcal{A}, R_\mathcal{C})$-double cosets of $G_1 \times G_2$.

For $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$, recall the definition (2.3) of the set $C_2(v_1, v_2) \subset C_2$. Similarly, let

$$A_1(v_1, v_2) = \{\alpha \in A_1 \mid (v_1 c^{-1} v_2^{-1} a)^n \alpha \text{ is defined and is in } A_1 \text{ for } n = 1, 2, \ldots\}, \tag{3.1}$$

so $A_1(v_1, v_2)$ is the largest subset of $A_1$ that is stable under $v_1 c^{-1} v_2^{-1} a$. Note that

$$v_2^{-1} a \quad \text{and} \quad c v_1^{-1} \colon \;\; A_1(v_1, v_2) \longrightarrow C_2(v_1, v_2) \tag{3.2}$$

are isomorphisms. Define

$$K_{(v_1,v_2)} = \left(M_{A_1(v_1,v_2)} \times M_{C_2(v_1,v_2)}\right) \cap \mathrm{Ad}^{-1}_{(e,\dot{v}_2)} K, \tag{3.3}$$

$$L_{(v_1,v_2)} = \left(M_{A_1(v_1,v_2)} \times M_{C_2(v_1,v_2)}\right) \cap \mathrm{Ad}_{(\dot{v}_1,e)} L, \tag{3.4}$$

$$\begin{aligned} Q_{(v_1,v_2)} = \{(m, m') \in M_{C_2(v_1,v_2)} \times M_{C_2(v_1,v_2)} \mid \exists n \in M_{A_1(v_1,v_2)} \\ \text{such that } (n, m) \in K_{(v_1,v_2)} \text{ and } (n, m') \in L_{(v_1,v_2)}\}, \end{aligned} \tag{3.5}$$

and let $Q_{(v_1,v_2)}$ act on $M_{C_2(v_1,v_2)}$ from the left by

$$(m, m') \cdot m_2 := m m_2 (m')^{-1}, \qquad (m, m') \in Q_{(v_1,v_2)}, \; m_2 \in M_{C_2(v_1,v_2)}. \tag{3.6}$$

**Theorem 3.1.** [12] *Let $\mathcal{A} = (A_1, A_2, a, K)$ and $\mathcal{C} = (C_1, C_2, c, L)$ be two admissible quadruples for $G_1 \times G_2$. Then every $(R_\mathcal{A}, R_\mathcal{C})$-double coset of $G_1 \times G_2$ is of the form*

$$R_\mathcal{A}(\dot{v}_1, \dot{v}_2 m_2)R_\mathcal{C} \quad \text{for some } v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2,\ m_2 \in M_{C_2(v_1,v_2)}.$$

*Two such double cosets $R_\mathcal{A}(\dot{v}_1, \dot{v}_2 m_2)R_\mathcal{C}$ and $R_\mathcal{A}(\dot{v}_1', \dot{v}_2' m_2')R_\mathcal{C}$ coincide if and only if $v_i' = v_i$ for $i = 1, 2$ and $m_2$ and $m_2'$ are in the same $Q_{(v_1,v_2)}$-orbit in $M_{C_2(v_1,v_2)}$, cf.* (3.6).

In [12] we dealt with a slightly more general class of the groups $R_\mathcal{A}$ for which the projections $K \to M_{A_i}$, for $i = 1, 2$, do not have to be surjective.

Recall from (2.4) that given any two admissible quadruples $\mathcal{A}$ and $\mathcal{C}$ for $G_1 \times G_2$, we have

$$[v_1, v_2]_{\mathcal{A},\mathcal{C}} = R_\mathcal{A}(v_1, v_2 M_{C_2(v_1,v_2)})R_\mathcal{C} \subset G_1 \times G_2$$

for $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$. Theorem 3.1 immediately implies:

**Corollary 3.2.** *For any two admissible quadruples $\mathcal{A}$ and $\mathcal{C}$ for $G_1 \times G_2$,*

$$G_1 \times G_2 = \bigsqcup_{(v_1,v_2) \in W_1^{C_1} \times {}^{A_2}W_2} [v_1, v_2]_{\mathcal{A},\mathcal{C}} \qquad (\textit{disjoint union}).$$

**Lemma 3.3.** *For any $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$,*

$$[v_1, v_2]_{\mathcal{A},\mathcal{C}} = R_\mathcal{A}(v_1, v_2(B_2 \cap M_{C_2(v_1,v_2)}))R_\mathcal{C}, \tag{3.7}$$

*and $R_\mathcal{A}(v_1, v_2 T_2)R_\mathcal{C}$ is dense in $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$, where $T_2 = B_2 \cap B_2^-$.*

*Proof.* It is easy to see that $\sigma_{(v_1,v_2)} = \mathrm{Ad}_{\dot{v}_2}^{-1} \theta_a \mathrm{Ad}_{\dot{v}_1} \theta_c^{-1}$ defines an automorphism of $G_{C_2(v_1,v_2)}$. The action of $Q_{(v_1,v_2)}$ on $M_{C_2(v_1,v_2)}$ descends to an action on $G_{C_2(v_1,v_2)}$. The orbits of the latter are exactly the $\sigma_{(v_1,v_2)}$-twisted conjugacy classes of $G_{C_2(v_1,v_2)}$. (Here and below for a group $F$ and $\sigma \in \mathrm{Aut}(F)$, by a $\sigma$-twisted conjugacy class of $F$ we mean an orbit of the action $g.x = gx(\sigma(g))^{-1}$ of $F$ on itself.) Lemma 3.3 now follows from the following results in [16] and [20, Lemma 7.3]: If $G$ is a connected reductive algebraic group, $\sigma \in \mathrm{Aut}(G)$, and $B$ is a $\sigma$-stable Borel subgroup $G$, then

(1) all $\sigma$-twisted conjugacy classes of $G$ meet $B$;

(2) for every maximal torus $T$ of $G$ inside $B$ the union of all $\sigma$-twisted conjugacy classes that meet $T$ is a Zariski open subset of $G$. □

3.2. **Stabilizer subgroups.** For $(g_1, g_2) \in G_1 \times G_2$, set

$$\mathrm{Stab}_{\mathcal{A},\mathcal{C}}(g_1, g_2) = R_\mathcal{A} \cap \mathrm{Ad}_{(g_1,g_2)} R_\mathcal{C}. \tag{3.8}$$

An explicit description of the subgroups $\mathrm{Stab}_{\mathcal{A},\mathcal{C}}(g_1, g_2)$ was given in [12]. We only recall some facts about these groups that will be used in this paper (see the proof of Proposition 4.8).

For $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$, let $A_2(v_1, v_2) = aA_1(v_1, v_2) \subset A_2$.

**Proposition 3.4.** [12] *For $q = (\dot{v}_1, \dot{v}_2 m)$, where $v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2$, and $m \in M_{C_2(v_1,v_2)}$, $\mathrm{Stab}_{\mathcal{A},\mathcal{C}}(q) \subset P_{A_1(v_1,v_2)} \times P_{A_2(v_1,v_2)}$, and $\mathrm{Stab}_{\mathcal{A},\mathcal{C}}(q) = \mathrm{Stab}^{\mathrm{red}}_{\mathcal{A},\mathcal{C}}(q)\, \mathrm{Stab}^{\mathrm{uni}}_{\mathcal{A},\mathcal{C}}(q)$, where*

$$\begin{aligned}\mathrm{Stab}^{\mathrm{red}}_{\mathcal{A},\mathcal{C}}(q) =& \mathrm{Stab}_{\mathcal{A},\mathcal{C}}(q) \cap (M_{A_1(v_1,v_2)} \times M_{A_2(v_1,v_2)})\\ =& (M_{A_1(v_1,v_2)} \times M_{A_2(v_1,v_2)}) \cap K \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2 m)} L\end{aligned}$$

*and $\mathrm{Stab}^{\mathrm{uni}}_{\mathcal{A},\mathcal{C}}(q) = \mathrm{Stab}_{\mathcal{A},\mathcal{C}}(q) \cap (U_{A_1(v_1,v_2)} \times U_{A_2(v_1,v_2)})$ has dimension equal to*

$$\dim(U_1 \cap v_1(U_{C_1})) + \dim(U_{A_2} \cap v_2(U_2)).$$

## 4. An inductive description and the geometry of the sets $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$

In this section, we modify the inductive arguments in [12] to give an inductive description of the sets $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$. Consequently, we will be able to describe the sets $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ as iterated bundles. Our arguments generalize those of Lusztig's in [14, 15] for the special case when $G_1 = G_2 = G$ and $R_{\mathcal{A}}$ is the diagonal subgroup of $G$.

4.1. **Construction of new admissible quadruples.** For $i = 1, 2$ and $E_i, F_i \subset \Gamma_i$, let

$$^{E_i}W_i^{F_i} = {}^{E_i}W_i \cap W_i^{F_i}.$$

Given an admissible quadruple $\mathcal{C} = (C_1, C_2, c, L)$ for $G_1 \times G_2$, a subset $E_1$ of $\Gamma_1$, and any $y_1 \in {}^{E_1}W_1^{C_1}$, we construct another admissible quadruple $\mathcal{C}^{(E_1,y_1)}$ for $G_1 \times G_2$ as follows: set

$$C_1^{(E_1,y_1)} = E_1 \cap y_1(C_1), \quad C_2^{(E_1,y_1)} = c(C_1 \cap y_1^{-1}(E_1))$$
$$L^{(E_1,y_1)} = \left(M_{C_1^{(E_1,y_1)}} \times M_{C_2^{(E_1,y_1)}}\right) \cap \mathrm{Ad}_{(\dot{y}_1,e)}L.$$

**Lemma 4.1.** *In the above setting the following hold:*

*(i) The quadruple* $\mathcal{C}^{(E_1,y_1)} := \left(C_1^{(E_1,y_1)}, C_2^{(E_1,y_1)}, cy_1^{-1}, L^{(E_1,y_1)}\right)$ *is an admissible quadruple for* $G_1 \times G_2$.

*(ii) Let* $R_{\mathcal{C}^{(E_1,y_1)}}$ *be the subgroup of* $G_1 \times G_2$ *defined by* $\mathcal{C}^{(E_1,y_1)}$ *as in Definition 2.1. Then* $(P_{E_1} \times G_2) \cap \mathrm{Ad}_{(\dot{y}_1,e)}R_{\mathcal{C}} \subset R_{\mathcal{C}^{(E_1,y_1)}}$.

*Proof.* Part (i) follows directly from the definition of admissible quadruples. To prove (ii), let $(p, g_2) \in P_{E_1} \times G_2$ be such that $(\dot{y}_1^{-1}p\dot{y}_1, g_2) \in R_{\mathcal{C}}$. For notational simplicity, set

$$D_1 = C_1 \cap y_1^{-1}(E_1) = y_1^{-1}(C_1^{(E_1,y_1)}), \quad D_2 = c(D_1) = C_2^{(E_1,y_1)}.$$

Then by [12, (4.11) in Lemma 4.2],

$$\dot{y}_1^{-1}p\dot{y}_1 \in P_{C_1} \cap \mathrm{Ad}_{\dot{y}_1}^{-1}P_{E_1} = M_{D_1}(U_{D_1} \cap \mathrm{Ad}_{\dot{y}_1}^{-1}U_{C_1^{(E_1,y_1)}}) \subset P_{D_1}. \tag{4.1}$$

It follows from [12, (3) in Lemma 3.4] that $g_2 \in P_{c(D_1)} = P_{D_2} = P_{C_2^{(E_1,y_1)}}$, and

$$(\dot{y}_1^{-1}p\dot{y}_1, g_2) \in (P_{D_1} \times P_{D_2}) \cap R_{\mathcal{C}} = ((P_{D_1} \times P_{D_2}) \cap L)(U_{C_1} \times U_{C_2}) \tag{4.2}$$
$$\subset ((M_{D_1} \times M_{D_2}) \cap L)(U_{D_1} \times U_{D_2}). \tag{4.3}$$

Thus by (4.1), $(\dot{y}_1^{-1}p\dot{y}_1, g_2) \in ((M_{D_1} \times M_{D_2}) \cap L)(\mathrm{Ad}_{\dot{y}_1}^{-1}U_{C_1^{(E_1,y_1)}} \times U_{D_2})$ and hence $(p, g_2) \in R_{\mathcal{C}^{(E_1,y_1)}}$. □

4.2. **The first step of the induction.** Fix two admissible quadruples $\mathcal{A}$ and $\mathcal{C}$ for $G_1 \times G_2$. Given $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$, we will use Lemma 4.1 to construct a sequence of admissible quadruples for $G_1 \times G_2$ which will be used to give an inductive description of $[v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}} \subset (G_1 \times G_2)/R_{\mathcal{C}}$. In this subsection, we present the first step of the inductive description.

Consider the projection

$$\rho_0 : \ (G_1 \times G_2)/R_{\mathcal{C}} \longrightarrow (G_1 \times G_2)/(P_{C_1} \times P_{C_2}).$$

By [12, Proposition 8.1 and Proposition 4.1], every $R_{\mathcal{A}}$-orbit in $(G_1 \times G_2)/(P_{C_1} \times P_{C_2})$ contains exactly one point of the form $(x_1, x_2).(P_{C_1} \times P_{C_2})$, where

$$x_2 \in {}^{A_2}W_2^{C_2} \quad \text{and} \quad x_1 \in {}^{a^{-1}(A_2 \cap x_2(C_2))}W_1^{C_1}. \tag{4.4}$$

Let $\Omega_0$ be the set that consists of all pairs $(x_1, x_2)$ satisfying (4.4). For $(x_1, x_2) \in \Omega_0$, let

$$\mathcal{O}^{(x_1,x_2)} = R_{\mathcal{A}}(x_1, x_2).(P_{C_1} \times P_{C_2}) \subset (G_1 \times G_2)/(P_{C_1} \times P_{C_2}),$$
$$\mathcal{X}^{(x_1,x_2)} = \rho_0^{-1}(\mathcal{O}^{(x_1,x_2)}) \subset (G_1 \times G_2)/R_{\mathcal{C}}.$$

Then clearly,

$$\mathcal{X}^{(x_1,x_2)} = \{(r_1\dot{x}_1,\ \ r_2\dot{x}_2 m).R_{\mathcal{C}} \mid (r_1,r_2) \in R_{\mathcal{A}},\ m \in M_{C_2}\}.$$

By letting $E_1 = a^{-1}\,(A_2 \cap x_2(C_2))$ and $y_1 = x_1$ in Lemma 4.1, we get the admissible quadruple

$$\mathcal{C}^{(x_1,x_2)} := \mathcal{C}^{(E_1,\,x_1)} = (C_1^{(x_1,x_2)},\ C_2^{(x_1,x_2)},\ c^{(x_1,x_2)},\ L^{(x_1,x_2)}) \tag{4.5}$$

and the corresponding subgroup $R_{\mathcal{C}^{(x_1,x_2)}}$ of $G_1 \times G_2$, where in particular,

$$C_1^{(x_1,x_2)} = a^{-1}(A_2 \cap x_2(C_2)) \cap x_1(C_1), \tag{4.6}$$

$$C_2^{(x_1,x_2)} = c(C_1 \cap x_1^{-1}a^{-1}(A_2 \cap x_2(C_2))), \tag{4.7}$$

$$c^{(x_1,x_2)} = cx_1^{-1}\colon\ \ C_1^{(x_1,x_2)} \longrightarrow C_2^{(x_1,x_2)}. \tag{4.8}$$

**Lemma 4.2.** *For any $(x_1,x_2) \in \Omega_0$, the map $f_0\colon\ \mathcal{X}^{(x_1,x_2)} \to (G_1 \times G_2)/R_{\mathcal{C}^{(x_1,x_2)}}$ given by*

$$f_0\colon\ \ (r_1\dot{x}_1,\ \ r_2\dot{x}_2 m).R_{\mathcal{C}} \longmapsto (r_1,\ \ r_2\dot{x}_2 m).R_{\mathcal{C}^{(x_1,x_2)}},\ \ \ (r_1,r_2) \in R_{\mathcal{A}},\ m \in M_{C_2} \tag{4.9}$$

*is a well-defined $R_{\mathcal{A}}$-equivariant morphism of algebraic varieties.*

*Proof.* Assume that $(r_1\dot{x}_1, r_2\dot{x}_2 m).R_{\mathcal{C}} = (r_1'\dot{x}_1, r_2'\dot{x}_2 m').R_{\mathcal{C}}$ for $(r_1,r_2), (r_1', r_2') \in R_{\mathcal{A}}$ and $m, m' \in M_{C_2}$. Then

$$((r_1')^{-1}r_1,\ (r_2')^{-1}r_2) \in R_{\mathcal{A}} \quad \text{and} \quad (\dot{x}_1^{-1}(r_1')^{-1}r_1\dot{x}_1,\ (m')^{-1}\dot{x}_2^{-1}(r_2')^{-1}r_2\dot{x}_2 m) \in R_{\mathcal{C}}. \tag{4.10}$$

It follows from (4.10) that $(r_2')^{-1}r_2 \in P_{A_2} \cap x_2(P_{C_2}) \subset P_{A_2 \cap x_2(C_2)}$, which, together with (4.10) and [12, (3) of Lemma 3.4], imply that $(r_1')^{-1}r_1 \in P_{a^{-1}(A_2 \cap x_2(C_2))}$. By (ii) of Lemma 4.1, we know that $((r_1')^{-1}r_1,\ (m')^{-1}\dot{x}_2^{-1}(r_2')^{-1}r_2\dot{x}_2 m) \in R_{\mathcal{C}^{(x_1,x_2)}}$. Thus $f_0$ is well-defined.

Clearly $f_0$ is $R_{\mathcal{A}}$-equivariant. To see that $f_0$ is a morphism, note that since the fiber of $\rho_0$ over the point $(\dot{x}_1, \dot{x}_2).(P_{C_1} \times P_{C_2})$ is $(\dot{x}_1, \dot{x}_2)(P_{C_1} \times P_{C_2})/R_{\mathcal{C}}$, we have

$$\mathcal{X}^{(x_1,x_2)} \cong R_{\mathcal{A}} \times_{R_1} (\dot{x}_1, \dot{x}_2)(P_{C_1} \times P_{C_2})/R_{\mathcal{C}},$$

where $R_1 = R_{\mathcal{A}} \cap \mathrm{Ad}_{(\dot{x}_1,\dot{x}_2)}(P_{C_1} \times P_{C_2})$ is the stabilizer subgroup of $R_{\mathcal{A}}$ at the point $(\dot{x}_1, \dot{x}_2).(P_{C_1} \times P_{C_2})$. Let $Y_2 = \{m \in M_{C_2} \mid (e,m) \in L\} \subset Z_{C_2}$. Then the inclusion map of $\{e\} \times M_{C_2} \hookrightarrow P_{C_1} \times P_{C_2}$ induces an isomorphism

$$\phi_1:\ \ (\dot{x}_1, \dot{x}_2)(\{e\} \times M_{C_2})/(\{e\} \times Y_2) \xrightarrow{\cong} (\dot{x}_1, \dot{x}_2)(P_{C_1} \times P_{C_2})/R_{\mathcal{C}}.$$

Note that $\{e\} \times Y_2 \subset L^{(x_1,x_2)} \subset R_{\mathcal{C}^{(x_1,x_2)}}$ so we have the projection

$$\phi_2:\ \ (G_1 \times G_2)/(\{e\} \times Y_2) \to (G_1 \times G_2)/R_{\mathcal{C}^{(x_1,x_2)}}.$$

Consider the morphism

$$\phi_3 : R_{\mathcal{A}} \times (G_1 \times G_2)/(\{e\} \times Y_2) \longrightarrow (G_1 \times G_2)/(\{e\} \times Y_2) :$$
$$((r_1,r_2),(g_1,g_2)(\{e\} \times Y_2)) \longmapsto (r_1\dot{x}_1^{-1}g_1,\ r_2 g_2)(\{e\} \times Y_2).$$

The composition of $\phi_3$ with $\phi_2$ gives rises to a morphism

$$R_{\mathcal{A}} \times (\dot{x}_1, \dot{x}_2)(\{e\} \times M_{C_2})/(\{e\} \times Y_2) \longrightarrow (G_1 \times G_2)/R_{\mathcal{C}^{(x_1,x_2)}}.$$

Let $R_1$ act on $(\dot{x}_1, \dot{x}_2)(\{e\} \times M_{C_2})/(\{e\} \times Y_2)$ so that the isomorphism $\phi_1$ is $R_1$-equivariant. Then the well-definedness of $f_0$ implies that

$$f_0:\ \ \chi^{(x_1,x_2)} \cong R_{\mathcal{A}} \times_{R_1} (\dot{x}_1, \dot{x}_2)(\{e\} \times M_{C_2})/(\{e\} \times Y_2) \longrightarrow (G_1 \times G_2)/R_{\mathcal{C}^{(x_1,x_2)}}$$

is a morphism of varieties. □

Let now $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$. By [3, Proposition 2.7.5] (see Lemma 8.2 in the Appendix), $v_2$ can be uniquely written as

$$v_2 = x_2 u_2, \quad \text{where} \quad x_2 \in {}^{A_2}W_2^{C_2}, \text{ and } u_2 \in {}^{C_2 \cap x_2^{-1}(A_2)}W_{C_2}. \tag{4.11}$$

Given the decomposition $v_2 = x_2 u_2$, write $v_1$ as

$$v_1 = u_1 x_1, \quad \text{where} \quad x_1 \in {}^{a^{-1}(A_2 \cap x_2(C_2))}W_1^{C_1}, \text{ and } u_1 \in W_{a^{-1}(A_2\cap x_2(C_2))}^{C_1^{(x_1,x_2)}} \tag{4.12}$$

with $C_1^{(x_1,x_2)}$ given in (4.6). Here and below, for $i = 1, 2$ and $E_i \subset D_i \subset \Gamma_i$, we let

$$W_{D_i}^{E_i} = W_{D_i} \cap W_i^{E_i}.$$

It is easy to see that $[v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}} \subset \chi^{(x_1,x_2)}$. Since $u_1 \in W_{a^{-1}(A_2\cap x_2(C_2))}^{C_1^{(x_1,x_2)}} \subset W_1^{C_1^{(x_1,x_2)}}$, we have the set

$$[u_1, v_2]_{\mathcal{A},\mathcal{C}^{(x_1,x_2)}}/R_{\mathcal{C}^{(x_1,x_2)}} = R_{\mathcal{A}}\left(u_1, v_2 M_{C_2^{(x_1,x_2)}(u_1,v_2)}\right) \cdot R_{\mathcal{C}^{(x_1,x_2)}} \subset (G_1 \times G_2)/R_{\mathcal{C}^{(x_1,x_2)}},$$

where $C_2^{(x_1,x_2)}(u_1, v_2)$ is the largest subset of $C_2^{(x_1,x_2)}$ that is stable under the map

$$v_2^{-1} a u_1 (c^{(x_1,x_2)})^{-1} = v_2^{-1} a v_1 c^{-1}.$$

An argument similar to that in the proof of [12, Lemma 5.3] shows that $C_2^{(x_1,x_2)}(u_1, v_2) = C_2(v_1, v_2)$. The following Proposition 4.3 now follows directly from Theorem 3.1.

**Proposition 4.3.** *Let $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$ have the decompositions* (4.12) *and* (4.11)*. Then $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ consists of those $(g_1, g_2) \in G_1 \times G_2$, for which*

$$(g_1, g_2).R_{\mathcal{C}} \in \mathcal{X}^{(x_1,x_2)} \quad \text{and} \quad f_0((g_1, g_2).R_{\mathcal{C}}) \in [u_1, v_2]_{\mathcal{A},\mathcal{C}^{(x_1,x_2)}}/R_{\mathcal{C}^{(x_1,x_2)}}.$$

*Moreover, $f_0\colon [v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}} \to [u_1, v_2]_{\mathcal{A},\mathcal{C}^{(x_1,x_2)}}/R_{\mathcal{C}^{(x_1,x_2)}}$ induces a one-to-one correspondence between the $R_{\mathcal{A}}$-orbits in $[v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}$ and the $R_{\mathcal{A}}$-orbits in $[u_1, v_2]_{\mathcal{A},\mathcal{C}^{(x_1,x_2)}}/R_{\mathcal{C}^{(x_1,x_2)}}$.*

4.3. **Inductive description of the sets $[v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}$.** Given $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$, by repeating the construction in §4.2, we get a sequence

$$\mathcal{C}^{(i)} = (C_1^{(i)}, C_2^{(i)}, c^{(i)}, L^{(i)}), \quad i = 0, 1, 2, \ldots$$

of admissible quadruples for $G_1 \times G_2$ and a sequence $u_1^{(i)} \in W_1^{C_1^{(i)}}$, $i = 0, 1, 2, \ldots$ which gives rise to the sequence of double cosets

$$\mathcal{Z}^{(i)} = [u_1^{(i)},\ v_2]_{\mathcal{A},\mathcal{C}^{(i)}}/R_{\mathcal{C}^{(i)}} \subset (G_1 \times G_2)/R_{\mathcal{C}^{(i)}}, \qquad i = 0, 1, 2, \ldots.$$

Here $\mathcal{C}^{(0)} = \mathcal{C}$, $u_1^{(0)} = v_1$, $\mathcal{Z}^{(0)} = [v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}$, $\mathcal{C}^{(1)} = \mathcal{C}^{(x_1,x_2)}$ as in (4.5), and $u_1^{(1)} = u_1$ as in (4.12). In general, once $(\mathcal{C}^{(i)}, u_1^{(i)})$ is given, $(\mathcal{C}^{(i+1)}, u_1^{(i+1)})$ is constructed from $(\mathcal{C}^{(i)}, u_1^{(i)})$ in the same way as $(\mathcal{C}^{(x_1,x_2)}, u_1)$ was from $(\mathcal{C}, v_1)$ in §4.2. Namely, first decompose $v_2$ as the unique product $v_2 = x_2^{(i)} u_2^{(i)}$ with

$$x_2^{(i)} \in {}^{A_2}W_2^{C_2^{(i)}}, \quad u_2^{(i)} \in {}^{C_2^{(i)} \cap (x_2^{(i)})^{-1}(A_2)}W_{C_2^{(i)}}. \tag{4.13}$$

Then decompose $u_1^{(i)}$ as the unique product $u_1^{(i)} = u_1^{(i+1)} x_1^{(i)}$, where

$$x_1^{(i)} \in {}^{a^{-1}(A_2 \cap x_2^{(i)}(C_2^{(i)}))}W_1^{C_1^{(i)}}, \quad u_1^{(i+1)} \in W_{a^{-1}(A_2 \cap x_2^{(i)}(C_2^{(i)}))}^{a^{-1}(A_2 \cap x_2^{(i)}(C_2^{(i)})) \cap x_1^{(i)}(C_1^{(i)})}.$$

The admissible quadruple $\mathcal{C}^{(i+1)}$ is constructed as in §4.1 by taking $\mathcal{C}$ to be $\mathcal{C}^{(i)}$, $E_1$ to be $a^{-1}(A_2 \cap x_2^{(i)}(C_2^{(i)}))$ and $y_1$ to be $x_1^{(i)}$.

For $i \geq 0$, let $\rho_i$ be the natural projection

$$\rho_i\colon\ (G_1 \times G_2)/R_{\mathcal{C}^{(i)}} \longrightarrow (G_1 \times G_2)/(P_{C_1^{(i)}} \times P_{C_2^{(i)}}),$$

$\mathcal{O}^{(i)} := R_{\mathcal{A}}(x_1^{(i)}, x_2^{(i)}).(P_{C_1^{(i)}} \times P_{C_2^{(i)}})$, and

$$\mathcal{X}^{(i)} := \rho_i^{-1}(\mathcal{O}^{(i)}) \subset (G_1 \times G_2)/R_{\mathcal{C}^{(i)}}.$$

Then $\mathcal{X}^{(i)}$ is a locally closed subset of $(G_1 \times G_2)/R_{\mathcal{C}^{(i)}}$. By Lemma 4.2 and Proposition 4.3, we have a well-defined $R_{\mathcal{A}}$-equivariant morphism $f_i\colon\ \mathcal{X}^{(i)} \longrightarrow (G_1 \times G_2)/R_{\mathcal{C}^{(i+1)}}$ such that

$$\mathcal{Z}^{(i)} \subset \mathcal{X}^{(i)} \quad \text{and} \quad \mathcal{Z}^{(i)} = f_i^{-1}\left(\mathcal{Z}^{(i+1)}\right), \quad \forall i \geq 0. \tag{4.14}$$

Let $i_0 \geq 0$ be the smallest integer such that $C_2^{(i_0+1)} = C_2^{(i_0)}$. It is then easy to see that

$$\mathcal{C}^{(i_0)} = \mathcal{C}^{(\infty)} := (A_1(v_1, v_2),\, C_2(v_1, v_2),\, cv_1^{-1},\, L_{(v_1,v_2)}) \text{ and } u_1^{(i)} = e, \quad \forall i \geq i_0 + 1, \tag{4.15}$$

where $L_{(v_1,v_2)}$ is given by (3.4). Set $\mathcal{Z}^{(\infty)} = \mathcal{Z}^{(i_0+1)}$ and $\mathcal{X}^{(\infty)} = \mathcal{X}^{(i_0+1)}$. It follows from (4.15) that

$$\mathcal{Z}^{(i)} = \mathcal{Z}^{(\infty)} = \mathcal{X}^{(\infty)} = [e, v_2]_{\mathcal{A},\mathcal{C}^{(\infty)}}/R_{\mathcal{C}^{(\infty)}}, \quad \forall i \geq i_0 + 1.$$

**Proposition 4.4.** *For every $i \geq 0$, $\mathcal{Z}^{(i)}$ is locally closed in $(G_1 \times G_2)/R_{\mathcal{C}^{(i)}}$.*

*Proof.* By the inductive description of the sets $\mathcal{Z}^{(i)}$ in (4.14), if for some $i \geq 0$, $\mathcal{Z}^{(i+1)}$ is locally closed in $(G_1 \times G_2)/R_{\mathcal{C}^{(i+1)}}$, then $\mathcal{Z}^{(i)}$ is locally closed in $(G_1 \times G_2)/R_{\mathcal{C}^{(i)}}$ because $f_i$ is a morphism. Since $\mathcal{Z}^{(\infty)} = \mathcal{X}^{(\infty)}$ is locally closed in $(G_1 \times G_2)/R_{\mathcal{C}^{(\infty)}}$, it follows that $\mathcal{Z}^{(i)}$ is locally closed in $(G_1 \times G_2)/R_{\mathcal{C}^{(i)}}$ for every $i \geq 0$. □

Let $\rho_\infty := \rho_{i_0+1}\colon\ (G_1 \times G_2)/R_{\mathcal{C}^{(\infty)}} \to (G_1 \times G_2)/(P_{A_1(v_1,v_2)} \times P_{C_2(v_1,v_2)})$, and

$$\mathcal{O}^{(\infty)} := \mathcal{O}^{(i_0+1)} = R_{\mathcal{A}}(e, v_2).(P_{A_1(v_1,v_2)} \times P_{C_2(v_1,v_2)}),$$

so that $\mathcal{Z}^{(\infty)} = \mathcal{X}^{(\infty)} = \rho_\infty^{-1}(\mathcal{O}^{(\infty)})$. Then we have the projection $\rho_\infty\colon \mathcal{Z}^{(\infty)} \to \mathcal{O}^{(\infty)}$. Note that the stabilizer subgroup of $R_{\mathcal{A}}$ at the point

$$(e, v_2).(P_{A_1(v_1,v_2)} \times P_{C_2(v_1,v_2)}) \in (G_1 \times G_2)/(P_{A_1(v_1,v_2)} \times P_{C_2(v_1,v_2)})$$

is

$$R_{\mathcal{A}} \cap \big(P_{A_1(v_1,v_2)} \times v_2(P_{C_2(v_1,v_2)})\big) \subset R_{\mathcal{A}} \cap (P_{A_1(v_1,v_2)} \times P_{A_2(v_1,v_2)}), \tag{4.16}$$

where $A_2(v_1, v_2) = aA_1(v_1, v_2)$. For notational simplicity, set

$$R_{(v_1,v_2)} = R_{\mathcal{A}} \cap (P_{A_1(v_1,v_2)} \times P_{A_2(v_1,v_2)}). \tag{4.17}$$

Then we have the projection $\rho'_\infty\colon \mathcal{O}^{(\infty)} \to R_{\mathcal{A}}/R_{(v_1,v_2)}$ induced by the inclusion map in (4.16). Thus we have the sequence of $R_{\mathcal{A}}$-equivariant morphisms

(4.18)

$$[v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}} = \mathcal{Z}^{(0)} \xrightarrow{f_0} \mathcal{Z}^{(1)} \xrightarrow{f_1} \cdots \xrightarrow{f_{i_0-1}} \mathcal{Z}^{(i_0)} \xrightarrow{f_{i_0}} \mathcal{Z}^{(\infty)} \xrightarrow{\rho_\infty} \mathcal{O}^{(\infty)} \xrightarrow{\rho'_\infty} R_{\mathcal{A}}/R_{(v_1,v_2)}.$$

**Lemma 4.5.** *The quotient $R_{\mathcal{A}}/R_{(v_1,v_2)}$ is isomorphic to the flag variety $M_{A_1}/(M_{A_1} \cap P_{A_1(v_1,v_2)})$ of $M_{A_1}$, and the fibers of the fibration $\rho'_\infty \circ \rho_\infty : \mathcal{Z}^{(\infty)} \to R_{\mathcal{A}}/R_{(v_1,v_2)}$ are isomorphic to the product $(U_2 \cap v_2(U_2^-)) \times (M_{C_2(v_1,v_2)}/Y_2)$, where $Y_2 = \{m \in M_{C_2} \mid (e, m) \in L\} \subset Z_{C_2}$.*

*Proof.* Consider the group homomorphism

$$p\colon\ R_{\mathcal{A}} \longrightarrow M_{A_1}\colon\ (r_1, r_2) \longmapsto m_1 \quad \text{if} \quad r_1 = m_1u_1 \quad \text{for} \quad m_1 \in M_{A_1}, u_1 \in U_{A_1}.$$

Since $p$ is surjective, the action of $R_{\mathcal{A}}$ on $M_{A_1}/(M_{A_1} \cap P_{A_1(v_1,v_2)})$ through the homomorphism $p$ is transitive. The stabilizer subgroup of $R_{\mathcal{A}}$ at the point $e.(M_{A_1} \cap P_{A_1(v_1,v_2)})$ is $R_{(v_1,v_2)}$ by [12, (3.7) in Lemma 3.4]. Thus $R_{\mathcal{A}}/R_{(v_1,v_2)}$ is isomorphic to $M_{A_1}/(M_{A_1} \cap P_{A_1(v_1,v_2)})$. It is easy to see that the fibers of $\rho_\infty$ are isomorphic to $M_{C_2(v_1,v_2)}/Y_2$ and the fibers of $\rho'_\infty$ are isomorphic to $U_2 \cap v_2(U_2^-)$. □

4.4. **The set $[v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}$ as an iterated fiber bundle.** Assume the setting from §4.3. We will show in this subsection that each morphism $f_i : \mathcal{Z}^{(i)} \to \mathcal{Z}^{(i+1)}$ in (4.18) has fibers isomorphic to an affine space.

**Notation 4.6.** Fix $m \in M_{C_2(v_1,v_2)}$. For $i \geq 0$, let $z_m^{(i)} = (\dot{u}_1^{(i)}, \dot{v}_2 m).R_{\mathcal{C}^{(i)}} \in \mathcal{Z}^{(i)}$ and

$$S_m^{(i)} = R_{\mathcal{A}} \cap \mathrm{Ad}_{(\dot{u}_1^{(i)}, \dot{v}_2 m)} R_{\mathcal{C}^{(i)}}$$

be the stabilizer subgroup of $R_{\mathcal{A}}$ at $z_m^{(i)} \in \mathcal{Z}^{(i)}$. Set

$$F_m^{(i)} = f_i^{-1}(z_m^{(i+1)}) \subset \mathcal{Z}^{(i)}.$$

**Lemma 4.7.** *For any $m \in M_{C_2(v_1,v_2)}$ and $i \geq 0$, $S_m^{(i)} \subset S_m^{(i+1)}$,*

$$F_m^{(i)} = S_m^{(i+1)}.z_m^{(i)} \cong S_m^{(i+1)}/S_m^{(i)}. \tag{4.19}$$

*Proof.* Let $i \geq 0$. Since $f_i(z_m^{(i)}) = z_m^{(i+1)}$ and since $f_i$ is $R_{\mathcal{A}}$-equivariant, we clearly have $S_m^{(i)} \subset S_m^{(i+1)}$, and $S_m^{(i+1)}.z_m^{(i)} \subset F_m^{(i)}$. It suffices to prove (4.19) for $i = 0$, and we only need to show that $F_m^{(0)} \subset S_m^{(1)}.z_m^{(0)}$.

Recall that $\mathcal{C}^{(1)} = \mathcal{C}^{(x_1,x_2)}$ as in (4.5), and $u^{(1)} = u_1$ as in the $v_1 = u_1 x_1$ decomposition in (4.12). To show that $F_m^{(0)} \subset S_m^{(1)}.z_m^{(0)}$, assume that $(r_1, r_2) \in R_{\mathcal{A}}$ and $m' \in M_{C_2(v_1,v_2)}$ are such that $(r_1\dot{v}_1, r_2\dot{v}_2 m').R_{\mathcal{C}} \in F_m^{(0)}$. Then

$$R_{\mathcal{A}}(\dot{u}_1, \dot{v}_2 m').R_{\mathcal{C}^{(x_1,x_2)}} = R_{\mathcal{A}}(\dot{u}_1, \dot{v}_2 m).R_{\mathcal{C}^{(x_1,x_2)}}.$$

By applying Theorem 3.1 to the quadruples $(\mathcal{A}, \mathcal{C}^{(x_1,x_2)})$, noting that the largest subset of $A_1$ that is stable under $u_1(c^{(x_1,x_2)})^{-1}v_2^{-1}a = v_1 c^{-1} v_2^{-1} a$ is $A_1(v_1, v_2)$ and that

$$(M_{A_1(v_1,v_2)} \times M_{C_2(v_1,v_2)}) \cap \mathrm{Ad}_{(\dot{u}_1,e)} L^{(x_1,x_2)} = L_{(v_1,v_2)},$$

we obtain that there exist $m_1 \in M_{A_1(v_1,v_2)}$ and $n, n' \in M_{C_2(v_1,v_2)}$ with $(m_1, n) \in K_{(v_1,v_2)}$ and $(m_1, n') \in L_{(v_1,v_2)}$ such that $m' = nm(n')^{-1}$, where $K_{(v_1,v_2)}$ and $L_{(v_1,v_2)}$ are respectively given by (3.3) and (3.4). Thus

$$\begin{aligned}(r_1\dot{v}_1, r_2\dot{v}_2 m').R_{\mathcal{C}} &= (r_1\dot{v}_1,\ r_2\dot{v}_2 nm(n')^{-1}).R_{\mathcal{C}} = (r_1 m_1\dot{v}_1,\ r_2\dot{v}_2 nm).R_{\mathcal{C}} \\ &= (r_1, r_2)(m_1, \mathrm{Ad}_{\dot{v}_2}(n))(\dot{v}_1,\ \dot{v}_2 m).R_{\mathcal{C}} = (r_1', r_2') z_m^{(0)},\end{aligned}$$

where $(r_1', r_2') = (r_1, r_2)(m_1, \mathrm{Ad}_{\dot{v}_2}(n)) \in R_{\mathcal{A}}$. On the other hand, since $(r_1\dot{v}_1, r_2\dot{v}_2 m').R_{\mathcal{C}} \in F_m^{(0)}$, we have

$$z_m^{(1)} = (r_1\dot{u}_1,\ r_2\dot{v}_2 m').R_{\mathcal{C}^{(x_1,x_2)}} = (r_1\dot{u}_1,\ r_2\dot{v}_2 nm(n')^{-1}).R_{\mathcal{C}^{(x_1,x_2)}} = (r_1', r_2') z_m^{(1)},$$

so $(r_1', r_2') \in S_m^{(1)}$, and thus $(r_1\dot{v}_1, r_2\dot{v}_2 m').R_{\mathcal{C}} \in S_m^{(1)}.z_m^{(0)}$. Hence $F_m^{(0)} \subset S_m^{(1)}.z_m^{(0)}$. □

**Proposition 4.8.** *For each $i \geq 0$, the fibers of the morphism $f_i : \mathcal{Z}^{(i)} \to \mathcal{Z}^{(i+1)}$ are isomorphic to the affine space of dimension*

$$\dim\left(U_1 \cap u_1^{(i+1)}\left(U_{C_1^{(i+1)}}\right)\right) - \dim\left(U_1 \cap u_1^{(i)}\left(U_{C_1^{(i)}}\right)\right). \tag{4.20}$$

*Proof.* Let $i \geq 0$ and $m \in M_{C_2(v_2,v_2)}$. By Proposition 3.4, we have the semi-direct product decompositions $S_m^{(i)} = S_m^{(i),\mathrm{red}} S_m^{(i),\mathrm{uni}}$, where

$$S_m^{(i),\mathrm{red}} = S_m^{(i)} \cap (M_{A_1(v_1,v_2)} \times M_{A_2(v_1,v_2)}), \qquad S_m^{(i),\mathrm{uni}} = S_m^{(i)} \cap (U_{A_1(v_1,v_2)} \times U_{A_2(v_1,v_2)}).$$

It also follows from Proposition 3.4 that $S_m^{(i),\mathrm{red}} = S_m^{(i+1),\mathrm{red}}$. Thus by (4.19) and by Proposition 3.4, $F_m^{(i)} \cong S_m^{(i+1),\mathrm{uni}}/S_m^{(i),\mathrm{uni}}$ is isomorphic to an affine space of dimension (4.20). □

For $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$, set

$$X_{(v_1,v_2)} \stackrel{\mathrm{def}}{=} R_{(v_1,v_2)}(v_1, v_2 M_{C_2(v_1,v_2)}).R_{\mathcal{C}} \subset [v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}.$$

**Corollary 4.9.** *For any* $v_1 \in W_1^{C_1}$ *and* $v_2 \in {}^{A_2}W_2$,

*(i)* $X_{(v_1,v_2)}$ *is a smooth locally closed subset of* $(G_1 \times G_2)/R_{\mathcal{C}}$ *isomorphic to the product of* $M_{C_2(v_1,v_2)}/Y_2$ *and the affine space of dimension* $\dim U_{A_1(v_1,v_2)} - \dim(U_1 \cap v_1(U_{C_1})) + l(v_2)$, *where* $Y_2 = \{m \in M_{C_2} \mid (e,m) \in L\} \subset Z_{C_2}$, *and* $l(v_2)$ *is the length of* $v_2$.

*(ii) the action map of* $R_{\mathcal{A}}$ *on* $[v_1,v_2]_{\mathcal{A},\mathcal{C}}$ *gives rise to an isomorphism*

$$R_{\mathcal{A}} \times_{R_{(v_1,v_2)}} X_{(v_1,v_2)} \longrightarrow [v_1,v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}.$$

*In particular,* $[v_1,v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}$ *is a smooth irreducible locally closed subset of* $(G_1 \times G_2)/R_{\mathcal{C}}$.

Since $G_1 \times G_2 \to (G_1 \times G_2)/R_{\mathcal{C}}$ is a locally trivial fibration, $[v_1,v_2]_{\mathcal{A},\mathcal{C}}$ is a smooth irreducible locally closed subset of $G_1 \times G_2$ which establishes the first part (i) of Theorem 2.2.

*Proof.* Consider the composition

$$p_0 = \rho'_\infty \circ \rho_\infty \circ f_{i_0} \circ f_{i_0-1} \circ \cdots \circ f_1 \circ f_0 : \; [v_1,v_2]_{\mathcal{A},\mathcal{C}} \longrightarrow R_{\mathcal{A}}/R_{(v_1,v_2)}.$$

It is easy to see that $X_{(v_1,v_2)}$ is precisely the fiber of $p_0$ over the point $e.R_{(v_1,v_2)} \in R_{\mathcal{A}}/R_{(v_1,v_2)}$. Since $p_0$ is $R_{\mathcal{A}}$-equivariant, by [17, Lemma 4], the action map of $R_{\mathcal{A}}$ on $[v_1,v_2]_{\mathcal{A},\mathcal{C}}$ induces an isomorphism

$$R_{\mathcal{A}} \times_{R_{(v_1,v_2)}} X(v_1,v_2) \longrightarrow [v_1,v_2]_{\mathcal{A},\mathcal{C}}$$

of $R_{\mathcal{A}}$-varieties. Since $p_0$ is a fibration, $X_{(v_1,v_2)}$ is a smooth locally closed subset of $(G_1 \times G_2)/R_{\mathcal{C}}$. The rest of the claim in 1) follows from Lemma 4.5 and Proposition 4.8. □

### 4.5. **Another description of the strata** $[v_1,v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}$.

**Proposition 4.10.** *For any* $v_1 \in W_1^{C_1}$, $v_2 \in {}^{A_2}W_2$, *one has*

$$[v_1,v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}} = R_{\mathcal{A}}(B_1 \times B_2)(v_1,v_2).R_{\mathcal{C}}. \tag{4.21}$$

*Proof.* By Lemma 3.3, we have

$$[v_1,v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}} = R_{\mathcal{A}}(v_1, v_2(B_2 \cap M_{C_2(v_1,v_2)})).R_{\mathcal{C}} \subset R_{\mathcal{A}}(v_1, v_2B_2).R_{\mathcal{C}} = R_{\mathcal{A}}(B_1 \times B_2)(v_1,v_2).R_{\mathcal{C}}.$$

It remains to show that $R_{\mathcal{A}}(B_1 \times B_2)(v_1,v_2).R_{\mathcal{C}} \subset [v_1,v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}$. Write $v_2 = x_2u_2$ and $v_1 = u_1x_1$ as in (4.11) and (4.12) and let $f_0\colon \mathcal{X}^{(x_1,x_2)} \to (G_1 \times G_2)/R_{\mathcal{C}^{(x_1,x_2)}}$ be as in Lemma 4.2. Then

$$\begin{aligned} R_{\mathcal{A}}(B_1 \times B_2)(v_1,v_2).R_{\mathcal{C}} &= R_{\mathcal{A}}(v_1, v_2(B_2 \cap M_{C_2})).R_{\mathcal{C}} \subset \mathcal{X}^{(x_1,x_2)}, \\ f_0(R_{\mathcal{A}}(B_1 \times B_2)(v_1,v_2).R_{\mathcal{C}}) &= R_{\mathcal{A}}(u_1, v_2(B_2 \cap M_{C_2})).R_{\mathcal{C}^{(x_1,x_2)}} \\ &= R_{\mathcal{A}}(u_1, v_2(B_2 \cap M_{C_2^{(x_1,x_2)}})).R_{\mathcal{C}^{(x_1,x_2)}} \\ &= R_{\mathcal{A}}(B_1 \times B_2)(u_1,v_2).R_{\mathcal{C}^{(x_1,x_2)}}. \end{aligned}$$

By Proposition 4.3, if $R_{\mathcal{A}}(B_1 \times B_2)(u_1,v_2).R_{\mathcal{C}^{(x_1,x_2)}} \subset [u_1,v_2]_{\mathcal{A},\mathcal{C}^{(x_1,x_2)}}/R_{\mathcal{C}^{(x_1,x_2)}}$, it would follow that $R_{\mathcal{A}}(B_1 \times B_2)(v_1,v_2).R_{\mathcal{C}} \subset [v_1,v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}$. Consider the sequence of fibrations in (4.18). Since

$$R_{\mathcal{A}}(B_1 \times B_2)(e,v_2).R_{\mathcal{C}^{(\infty)}} = R_{\mathcal{A}}(e, v_2(B_2 \cap M_{C_2(v_1,v_2)})).R_{\mathcal{C}^{(\infty)}} = \mathcal{Z}^{(\infty)},$$

we see inductively that $R_{\mathcal{A}}(B_1 \times B_2)(u_1^{(i)}, v_2).R_{\mathcal{C}^{(i)}} \subset \mathcal{Z}^{(i)}$ holds for all $i \geq 0$. □

**Proposition 4.11.** *Given two admissible triples* $\mathcal{A} = (A_1, A_2, a, K)$ *and* $\mathcal{C} = (C_1, C_2, c, L)$, *suppose that* $\mathcal{A}' = (A_1, A_2, a, K')$ *and* $\mathcal{C} = (C_1, C_2, c, L')$ *are two other admissible quadruples containing the same triples* $(A_1, A_2, a)$ *and* $(C_1, C_2, c)$. *Then there exist* $t_2, s_2 \in T_2$ *such that*

$$[v_1,v_2]_{\mathcal{A}',\mathcal{C}'} = (e,t_2)[v_1,v_2]_{\mathcal{A},\mathcal{C}}(e,s_2), \qquad \forall v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2.$$

*Proof.* Let $K$ be given as in (2.1), and assume that

$$K' = \{(m_1, m_2) \in M_{A_1} \times M_{A_2} \mid \theta_a'(m_1 Z_1') = m_2 Z_2'\},$$

where, for $i = 1, 2$, $Z_1'$ is a closed subgroup of $Z_{A_i}$, and $\theta_1' : M_{A_1}/Z_1' \to M_{A_2}/Z_2'$ is an isomorphism having the same properties as $\theta_a$. The isomorphisms from $M_{A_1}/Z_{A_1}$ to $M_{A_2}/Z_{A_2}$ induced by $\theta_a$ and $\theta_a'$ will still be denoted by the same symbols. Then our assumptions imply that the automorphism $\theta := (\theta_a')^{-1}\theta_a$ of $M_{A_1}/Z_{A_1}$ is inner. Since $\theta$ leaves both $T_1/Z_{A_1}$ and $(B_1 \cap M_{A_1})/Z_{A_1})$ invariant, $\theta = \mathrm{Ad}_{t_1}$ for some $t_1 \in T_1$. It follows that $K'(B_1 \times B_2) = (e, t_2)K(B_1 \times B_2)$ for some $t_2 \in T_2$. Similarly, $(B_1 \times B_2)L' = (B_1 \times B_2)L(e, s_2)$ for some $s_2 \in T_2$. Proposition 4.11 now follows from Proposition 4.10. □

**Example 4.12.** Consider the case when $G_1 = G_2 = G$. Take $A_1 = A_2 = \Gamma$ and $a = \mathrm{id}$, where $\Gamma$ is the set of all simple roots for a pair $(B, T)$ of Borel subgroup $B$ of $G$ and maximal torus $T \subset B$. Take $R_{\mathcal{A}} = K = G_{\mathrm{diag}}$ and for some $t \in T$, take $R_{\mathcal{A}'} = K' = \{(g, tgt^{-1}) : g \in G\}$. Let $R_{\mathcal{C}} = R_{\mathcal{C}'} = B \times B$. Then it is easy to see that

$$R_{\mathcal{A}'}(v, 1)(B \times B) = (e, t)R_{\mathcal{A}}(v, 1)(B \times B)$$

for all $v$ in the Weyl group of $(G, T)$.

## 5. Closures of the sets $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ in $G_1 \times G_2$

5.1. **The set $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ for any $v_1 \in W_1$ and $v_2 \in W_2$.** Let $\mathcal{A} = (A_1, A_2, a, K)$ and $\mathcal{C} = (C_1, C_2, c, L)$ be two arbitrary admissible quadruples for $G_1 \times G_2$. Extending (4.21), let

$$[v_1, v_2]_{\mathcal{A},\mathcal{C}} = R_{\mathcal{A}}(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)R_{\mathcal{C}} \subset G_1 \times G_2, \quad \forall v_1 \in W_1, v_2 \in W_2. \tag{5.1}$$

**Lemma 5.1.** *When $v_1 \in W_1^{C_1}$,*

$$(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)R_{\mathcal{C}} = (B_1 \times B_2)(v_1, v_2)R_{\mathcal{C}}, \quad \forall v_2 \in W_2. \tag{5.2}$$

*Proof.* Since $v_1 \in W_1^{C_1}$, one has $B_1 v_1 (B_1 \cap M_{C_1}) = B_1 v_1$, and thus

$$(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)R_{\mathcal{C}} = (B_1 \times B_2)(v_1, v_2)(B_1 \cap M_{C_1} \times \{e\})R_{\mathcal{C}} = (B_1 \times B_2)(v_1, v_2)R_{\mathcal{C}}.$$

□

Thus when $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$, the set $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ in (5.1) is the same as what we defined before. Our main result in this section is the following Theorem 5.2 which describes the closure of $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ in $G_1 \times G_2$ for any $v_1 \in W_1, v_2 \in W_2$. The proof of Theorem 5.2, which uses a series of lemmas proved in §5.2, will be given in §5.3. In this section, if $X$ is a subset of $G_1 \times G_2$, $\overline{X}$ always denotes the closure of $X$ in $G_1 \times G_2$.

**Theorem 5.2.** *For any $v_1 \in W_1, v_2 \in W_2$,*

$$\overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}} = \bigsqcup_{\substack{v_1' \in W_1^{C_1}, v_2' \in {}^{A_2}W_2 : \\ \exists x_1 \in W_{A_1}, y_1 \in W_{C_1} \text{ s.t.} \\ x_1 v_1' y_1 \leq v_1 \\ a(x_1) v_2' c(y_1) \leq v_2}} [v_1', v_2']_{\mathcal{A},\mathcal{C}} \quad (\textit{disjoint union}). \tag{5.3}$$

**Remark 5.3.** A special case of (2.6) says that for any admissible quadruple $\mathcal{C}$, the $(B_1 \times B_2)$-orbits in $(G_1 \times G_2)/R_{\mathcal{C}}$ are precisely of the form $(B_1 \times B_2)(v_1, v_2).R_{\mathcal{C}}$, where $v_1 \in W_1^{C_1}$ and $v_2 \in W_2$. Combining Lemma 5.1 and Theorem 5.2, we see that for any $(B_1 \times B_2)$-orbit $\mathcal{O}$ in $(G_1 \times G_2)/R_{\mathcal{C}}$, the closure of the set $R_{\mathcal{A}}\mathcal{O}$ in $(G_1 \times G_2)/R_{\mathcal{C}}$ is a disjoint union of some sets of the form $R_{\mathcal{A}}(B_1 \times B_2)(v_1, v_2).R_{\mathcal{C}}$ with $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$.

5.2. **The closures of $(B_1 \times B_2, R_{\mathcal{C}})$-double cosets in $G_1 \times G_2$.** In this section, we will describe the $(B_1 \times B_2, R_{\mathcal{C}})$-double cosets in the set

$$\overline{(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)R_{\mathcal{C}}} \subset G_1 \times G_2$$

for any $v_1 \in W_1, v_2 \in W_2$. Note again that by Lemma 5.1, $(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)R_{\mathcal{C}}$ is a single $(B_1 \times B_2, R_{\mathcal{C}})$-double coset when $v_1 \in W_1^{C_1}$.

For $y_1, z_1 \in W_1$, let

$$\mathcal{W}_1(y_1, z_1) = \{x_1 \in W_1 \mid B_1x_1B_1 \subset B_1y_1B_1z_1B_1\}.$$

The following Lemma 5.4 on $\mathcal{W}_1(x_1, z_1)$ can be either proved by induction on the length of $z_1$ or can be seen as a direct consequence of the explicit description of $\mathcal{W}_1(y_1, z_1)$ in [2, Remark 3.19].

**Lemma 5.4.** *Let $y_1, z_1 \in W_1$ be arbitrary. Then every $x_1 \in \mathcal{W}_1(y_1, z_1)$ is of the form $x_1 = y_1u_1$ for some $u_1 \in W_1, u_1 \le z_1$.*

**Lemma 5.5.** *Let $y_1 \in W_1, y_2 \in W_2$. Then every $(B_1 \times B_2, R_{\mathcal{C}})$-double coset in*

$$(B_1 \times B_2)(y_1, y_2)(B_1 \times B_2)R_{\mathcal{C}}$$

*is of the form $(B_1 \times B_2)(y_1, u_2)R_{\mathcal{C}}$ for some $u_2 \in W_2$, $u_2 \le y_2$.*

*Proof.* Write $y_2 = w_2z_2$, where $w_2 \in W_2^{C_2}, z_2 \in W_{C_2}$. Then

$$\begin{aligned}
(B_1 \times B_2)(y_1, y_2)(B_1 \times B_2)R_{\mathcal{C}} &= ((B_1y_1B_1) \times (B_2w_2z_2))R_{\mathcal{C}} \\
&= ((B_1y_1B_1c^{-1}(z_2^{-1})) \times (B_2w_2))R_{\mathcal{C}} \\
&= ((B_1y_1B_1c^{-1}(z_2^{-1})) \times (B_2w_2B_2))R_{\mathcal{C}} \\
&= ((B_1y_1B_1c^{-1}(z_2^{-1})B_1) \times (B_2w_2))R_{\mathcal{C}} \\
&= \bigcup_{x_1 \in \mathcal{W}_1(y_1, c^{-1}(z_2^{-1}))} ((B_1x_1B_1) \times (B_2w_2))R_{\mathcal{C}} \\
&= \bigcup_{x_1 \in \mathcal{W}_1(y_1, c^{-1}(z_2^{-1}))} (B_1 \times B_2)(x_1, w_2)R_{\mathcal{C}}.
\end{aligned}$$

By Lemma 5.4, every $x_1 \in \mathcal{W}_1(y_1, c^{-1}(z_2^{-1}))$ is of the form $x_1 = y_1u_1$ for some $u_1 \in W_1$ such that $u_1 \le c^{-1}(z_2^{-1})$, i.e. $c(u_1^{-1}) \le z_2$. For such an $x_1 \in \mathcal{W}_1(y_1, c^{-1}(z_2^{-1}))$,

$$(B_1 \times B_2)(x_1, w_2)R_{\mathcal{C}} = (B_1 \times B_2)(y_1, w_2c(u_1^{-1}))R_{\mathcal{C}},$$

and $w_2c(u_1^{-1}) \le w_2z_2 = y_2$. This completes the proof the Lemma. □

**Lemma 5.6.** *For any $v_1 \in W_1$ and $v_2 \in W_2$, one has*

$$\overline{(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)R_{\mathcal{C}}} = \overline{(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)}\, R_{\mathcal{C}}.$$

*Proof.* The Lemma follows from [19, Lemma 2, P. 68] (see Lemma 8.1 in the Appendix) by noting that $R_{\mathcal{C}}/((B_1 \times B_2) \cap R_{\mathcal{C}})$ is isomorphic to the full flag variety of $M_{C_1}$ (see Lemma 4.5) and is hence complete. □

**Lemma 5.7.** *For any $v_1 \in W_1$ and $v_2 \in W_2$, one has*

$$\overline{(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)R_{\mathcal{C}}} = \bigcup_{\substack{w_1 \in W_1, w_2 \in W_2 : \\ w_1 \le v_1, w_2 \le v_2}} (B_1 \times B_2)(w_1, w_2)R_{\mathcal{C}}. \tag{5.4}$$

*Proof.* By Lemma 5.6 and the Bruhat decomposition,

$$\overline{(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)R_{\mathcal{C}}} = \bigcup_{\substack{y_1 \in W_1, y_2 \in W_2 : \\ y_1 \leq v_1, y_2 \leq v_2}} (B_1 \times B_2)(y_1, y_2)(B_1 \times B_2)R_{\mathcal{C}}.$$

Let $y_1 \in W_1, y_2 \in W_2$ be such that $y_1 \leq v_1, y_2 \leq v_2$. By Lemma 5.5, every $(B_1 \times B_2, R_{\mathcal{C}})$-double coset in $(B_1 \times B_2)(y_1, y_2)(B_1 \times B_2)R_{\mathcal{C}}$ is of the form $(B_1 \times B_2)(y_1, u_2)R_{\mathcal{C}}$ with $u_2 \in W_2, u_2 \leq y_2 \leq v_2$. Thus $\overline{(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)R_{\mathcal{C}}}$ is contained in the right hand side of (5.4). Conversely, let $w_1 \in W_1, w_2 \in W_2$ be such that $w_1 \leq v_1, w_2 \leq v_2$. Then

$$(B_1 \times B_2)(w_1, w_2)R_{\mathcal{C}} \subset (B_1 \times B_2)(w_1, w_2)(B_1 \times B_2)R_{\mathcal{C}} \subset \overline{(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)}\, R_{\mathcal{C}}.$$

Thus the right hand side of (5.4) is contained in $\overline{(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)R_{\mathcal{C}}}$. □

**Proposition 5.8.** *For any $v_1 \in W_1$ and $v_2 \in W_2$, one has the disjoint union*

$$\overline{(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2)R_{\mathcal{C}}} = \bigsqcup_{\substack{w_1 \in W_1^{C_1}, w_2 \in W_2 : \\ \exists u_1 \in W_{C_1} \text{ s.t.} \\ w_1 u_1 \leq v_1, w_2 c(u_1) \leq v_2}} (B_1 \times B_2)(w_1, w_2)R_{\mathcal{C}}.$$

*Proof.* Proposition 5.8 follows from Lemma 5.7 by decomposing the element $w_1 \in W_1$ in the right hand side of (5.4) according to the decomposition $W_1 = W_1^{C_1} W_{C_1}$ and by the fact that $(T_1 \times T_2)(u_1, c(u_1))R_{\mathcal{C}} = (T_1 \times T_2)R_{\mathcal{C}}$ for any $u_1 \in W_{C_1}$, where $T_1$ and $T_2$ are respectively the maximal tori of $G_1$ and $G_2$ as fixed in §2.1. □

**Corollary 5.9.** *For any $v_1 \in W_1^{C_1}, v_2 \in W_2$, one has the disjoint union*

$$\overline{(B_1 \times B_2)(v_1, v_2)R_{\mathcal{C}}} = \bigsqcup_{\substack{w_1 \in W_1^{C_1}, w_2 \in W_2 : \\ \exists u_1 \in W_{C_1} \text{ s.t.} \\ w_1 u_1 \leq v_1, w_2 c(u_1) \leq v_2}} (B_1 \times B_2)(w_1, w_2)R_{\mathcal{C}}.$$

**Remark 5.10.** By Lemma 8.3 in the Appendix, Corollary 5.9 is equivalent to the following description of closures of $(B_1 \times B_2, R_{\mathcal{C}}^-)$-double cosets in $G_1 \times G_2$ as given in [18, Lemma 2.2]: for $v_1 \in W_1^{C_1}, v_2 \in W_2$,

$$\overline{(B_1 \times B_2)(v_1, v_2)R_{\mathcal{C}}^-} = \bigsqcup_{\substack{w_1 \in W_1^{C_1}, w_2 \in W_2 : \\ \exists u_1 \in W_{C_1} \text{ s.t.} \\ v_1 u_1^{-1} \leq w_1, w_2 c(u_1) \leq v_2}} (B_1 \times B_2)(w_1, w_2)R_{\mathcal{C}}^-.$$

**Lemma 5.11.** *Let $w_1 \in W_1, w_2 \in W_2, v_1' \in W_1^{C_1}$, and $v_2' \in {}^{A_2}W_2$. Suppose that*

$$(R_{\mathcal{A}}(B_1 \times B_2)(w_1, w_2)R_{\mathcal{C}}) \cap [v_1', v_2']_{\mathcal{A},\mathcal{C}} \neq \emptyset. \tag{5.5}$$

*Then there exist $x_1 \in W_{A_1}$ and $y_1 \in W_{C_1}$ such that*

$$x_1 v_1' y_1 \leq w_1 \quad \text{and} \quad a(x_1) v_2' c(y_1) \leq w_2.$$

*Proof.* One sees from (5.5) that $(w_1, w_2)R_{\mathcal{C}} \subset (B_1 \times B_2)R_{\mathcal{A}}(B_1 \times B_2)(v_1', v_2')(B_1 \times B_2)R_{\mathcal{C}}$. Since

$$(B_1 \times B_2)R_{\mathcal{A}}(B_1 \times B_2) = \bigcup_{x_1 \in W_{A_1}} (B_1 \times B_2)(x_1, a(x_1))(B_1 \times B_2),$$

we have

$$\begin{aligned}(w_1, w_2)R_{\mathcal{C}} &\subset \bigcup_{x_1 \in W_{A_1}} ((B_1 x_1 B_1 v_1' B_1) \times (B_2 a(x_1) B_2 v_2' B_2))\, R_{\mathcal{C}} \\ &= \bigcup_{x_1 \in W_{A_1}} ((B_1 x_1 B_1 v_1' B_1) \times (B_2 a(x_1) v_2' B_2))\, R_{\mathcal{C}},\end{aligned}$$

where in the last step we used the fact that $l(a(x_1)v_2') = l(a(x_1)) + l(v_2')$. Thus there exists $x_1 \in W_{A_1}$ such that $(w_1, w_2) \in ((B_1 x_1 B_1 v_1' B_1) \times (B_2 a(x_1) v_2' B_2)) R_{\mathcal{C}}$. Since

$$R_{\mathcal{C}} \subset \bigcup_{y_1 \in W_{C_1}} ((B_1 y_1 B_1) \times (B_2 c(y_1) B_2)),$$

there exists $y_1 \in W_{C_1}$ such that

$$\begin{aligned}(w_1, w_2) &\in (B_1 x_1 B_1 v_1' B_1 y_1 B_1) \times (B_2 a(x_1) v_2' B_2 c(y_1) B_2) \\ &= (B_1 x_1 B_1 v_1' y_1 B_1) \times (B_2 a(x_1) v_2' B_2 c(y_1) B_2).\end{aligned}$$

By Lemma 8.4 in the Appendix, $w_1 \geq x_1 v_1' y_1$ and $w_2 \geq a(x_1) v_2' c(y_1)$. □

5.3. **Proof of Theorem 5.2.** Fix $v_1 \in W_1$ and $v_2 \in W_2$. Let

$$\mathcal{J}(v_1, v_2) = \{(v_1', v_2') \in W_1^{C_1} \times {}^{A_2}W_2 \mid \overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}} \cap [v_1', v_2']_{\mathcal{A},\mathcal{C}} \neq \emptyset\}.$$

Then

$$\overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}} = \bigsqcup_{(v_1', v_2') \in \mathcal{J}(v_1, v_2)} \overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}} \cap [v_1', v_2']_{\mathcal{A},\mathcal{C}}.$$

We will first show that

$$\begin{aligned}\mathcal{J}(v_1, v_2) = \{&(v_1', v_2') \in W_1^{C_1} \times {}^{A_2}W_2 \mid \\ &\exists x_1 \in W_{A_1}, y_1 \in W_{C_1} \ s.t. \ x_1 v_1' y_1 \leq v_1,\ a(x_1) v_2' c(y_1) \leq v_2\}.\end{aligned} \tag{5.6}$$

Indeed, by Lemma 8.1 in the Appendix,

$$\overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}} = R_{\mathcal{A}}\, \overline{(B_1 \times B_2)(v_1, v_2)(B_1 \times B_2) R_{\mathcal{C}}}.$$

Thus by Lemma 5.7,

$$\overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}} = \bigsqcup_{\substack{w_1 \in W_1, w_2 \in W_2: \\ w_1 \leq v_1, w_2 \leq v_2}} R_{\mathcal{A}}(B_1 \times B_2)(w_1, w_2) R_{\mathcal{C}}.$$

Suppose that $(v_1', v_2') \in \mathcal{J}(v_1, v_2)$. Then there exists $(w_1, w_2) \in W_1 \times W_2$ with $w_1 \leq v_1, w_2 \leq v_2$ such that

$$(R_{\mathcal{A}}(B_1 \times B_2)(w_1, w_2) R_{\mathcal{C}}) \cap [v_1', v_2']_{\mathcal{A},\mathcal{C}} \neq \emptyset.$$

By Lemma 5.11, there exist $x_1 \in W_{A_1}, y_1 \in W_{C_1}$ such that

$$x_1 v_1' y_1 \leq w_1 \leq v_1 \quad \text{and} \quad a(x_1) v_2' c(y_1) \leq w_2 \leq v_2.$$

Thus $(v_1', v_2')$ is in the set of the right hand side of (5.6). Conversely, suppose that $(v_1', v_2') \in W_1^{C_1} \times {}^{A_2}W_2$ are such that $x_1 v_1' y_1 \leq v_1$ and $a(x_1) v_2' c(y_1) \leq v_2$ for some $x_1 \in W_{A_1}$ and $y_1 \in W_{C_1}$. Let $w_1 = x_1 v_1' y_1$ and $w_2 = a(x_1) v_2' c(y_1)$ so that

$$v_1' = x_1^{-1} w_1 y_1^{-1} \quad \text{and} \quad v_2' = a(x_1^{-1}) w_2 c(y_1^{-1}).$$

It follows that $(v_1' T_1,\ v_2' T_2) = (x_1^{-1} w_1 y_1^{-1} T_1,\ a(x_1^{-1}) w_2 c(y_1^{-1}) T_2)$ and hence

$$(v_1' T_1,\ v_2' T_2) \subset R_{\mathcal{A}}(B_1 \times B_2)(w_1, w_2) R_{\mathcal{C}} \subset \overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}}, \tag{5.7}$$

where $T_i = B_i^- \cap B_i$ for $i = 1, 2$. Thus $\overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}} \cap [v_1',\ v_2']_{\mathcal{A},\mathcal{C}} \neq \emptyset$, and so $(v_1',\ v_2') \in \mathcal{J}(v_1, v_2)$. This completes the Theorem 5.2.

For any $(v_1',\ v_2') \in \mathcal{J}(v_1, v_2)$, it follows from (5.7) that

$$R_{\mathcal{A}}(v_1',\ v_2'T_2)R_{\mathcal{C}} = R_{\mathcal{A}}(v_1'T_1,\ v_2'T_2)R_{\mathcal{C}} \subset \overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}}.$$

By Lemma 3.3, $R_{\mathcal{A}}(v_1',\ v_2'T_2)R_{\mathcal{C}}$ is dense in $[v_1',\ v_2']_{\mathcal{A},\mathcal{C}}$. Hence $[v_1',\ v_2']_{\mathcal{A},\mathcal{C}} \subset \overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}}$, and $\overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}} \cap [v_1', v_2']_{\mathcal{A},\mathcal{C}} = [v_1', v_2']_{\mathcal{A},\mathcal{C}}$. This completes the proof of Theorem 5.2.

## 6. The $(R_{\mathcal{A}}, R_{\mathcal{C}}^-)$-stable subsets $[v_1, v_2]_{\mathcal{A},\mathcal{C}}^-$

6.1. **The subsets $[v_1, v_2]_{\mathcal{A},\mathcal{C}}^-$ of $G_1 \times G_2$.** Let again $\mathcal{A}$ and $\mathcal{C}$ be two admissible quadruples for $G_1 \times G_2$. Let $w_{0,\Gamma_1}$ and $w_{0,C_1}$ be the longest element in $W_1$ and $W_{C_1}$ respectively. Associated to $\mathcal{C}$, we have the admissible quadruple $\mathcal{C}^* \stackrel{\text{def}}{=} (C_1^*, C_2^*, c^*, L^*)$, where

$$C_1^* = -w_{0,\Gamma_1}(C_1), \quad C_2^* = C_2, \quad c^* = c(w_{0,\Gamma_1}w_{0,C_1})^{-1}, \quad L^* = \mathrm{Ad}_{(\dot{w}_{0,\Gamma_1}\dot{w}_{0,C_1},\, e)}L. \tag{6.8}$$

It is easy to see that $R_{\mathcal{C}^*} = \mathrm{Ad}_{(\dot{w}_{0,\Gamma_1}\dot{w}_{0,C_1},\, e)}R_{\mathcal{C}}^-$. Moreover,

$$W_1^{C_1}(w_{0,\Gamma_1}w_{0,C_1})^{-1} = W_1^{C_1^*} \tag{6.9}$$

**Proposition 6.1.** *For $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$, let*

$$[v_1, v_2]_{\mathcal{A},\mathcal{C}}^- = R_{\mathcal{A}}(v_1, v_2M_{C_2(v_1,v_2)})R_{\mathcal{C}}^- = R_{\mathcal{A}}(B_1 \times B_2)(v_1, v_2)R_{\mathcal{C}}^- \subset G_1 \times G_2.$$

*Then (i)*

$$G_1 \times G_2 = \bigsqcup_{v_1 \in W_1^{C_1}, v_2 \in {}^{A_2}W_2} [v_1, v_2]_{\mathcal{A},\mathcal{C}}^- \qquad (\textit{disjoint union});$$

*(ii) $[v_1, v_2]_{\mathcal{A},\mathcal{C}}^- = R_{\mathcal{A}}(B_1 \times B_2)(v_1, v_2)R_{\mathcal{C}}^-$ for every $v_1 \in W_1^{C_1}$ and $v_2 \in {}^{A_2}W_2$;*

*(iii) $[v_1, v_2]_{\mathcal{A},\mathcal{C}}$ is locally closed, smooth, and irreducible. Its projection $[v_1, v_2]_{\mathcal{A},\mathcal{C}}/R_{\mathcal{C}}^-$ to $(G_1 \times G_2)/R_{\mathcal{C}}^-$ fibers over the flag variety $M_{A_1}/(M_{A_1} \cap P_{A_1(v_1,v_2)})$ with fibers isomorphic to the product of $M_{C_2(v_1,v_2)}/Y_2$ and the affine space of dimension $\dim U_{A_1(v_1,v_2)} - l(v_1) + l(v_2)$, where $Y_2$ is as in Theorem 2.2.*

*Proof.* Let $v_1^* = v_1(w_{0,\Gamma_1}w_{0,C_1})^{-1} \in W_1^{C_1^*}$. All the statements in Proposition 6.1 follow from the fact that

$$[v_1, v_2]_{\mathcal{A},\mathcal{C}}^- = R_{\mathcal{A}}(B_1 \times B_2)(v_1^*, v_2)(B_1 \times B_2)R_{\mathcal{C}^*}(\dot{w}_{0,\Gamma_1}\dot{w}_{0,C_1}, e).$$

□

6.2. **Closures of the sets $[v_1, v_2]_{\mathcal{A},\mathcal{C}}^-$ in $G_1 \times G_2$.** For each $v_1 \in W_1$ and $v_2 \in W_2$, set

$$[v_1, v_2]_{\mathcal{A},\mathcal{C}}^- = R_{\mathcal{A}}(B_1 \times B_2)(v_1, v_2)(w_{0,C_1}(B_1^-) \times B_2)R_{\mathcal{C}}^- \subset G_1 \times G_2. \tag{6.10}$$

It follows from Lemma 5.1 that

$$[v_1, v_2]_{\mathcal{A},\mathcal{C}}^- = R_{\mathcal{A}}(B_1 \times B_2)(v_1, v_2)R_{\mathcal{C}}^-, \qquad \text{when } v_1 \in W_1^{C_1},\ v_2 \in W_2. \tag{6.11}$$

**Theorem 6.2.** *For any $v_1 \in W_1$ and $v_2 \in W_2$, with $[v_1, v_2]_{\mathcal{A},\mathcal{C}}^-$ given in (6.10), one has*

$$\overline{[v_1, v_2]_{\mathcal{A},\mathcal{C}}^-} = \bigsqcup_{\substack{v_1' \in W_1^{C_1},\, v_2' \in {}^{A_2}W_2 :\\ \exists x_1 \in W_{A_1},\, y_1 \in W_{C_1} \text{ s.t.}\\ x_1v_1'y_1w_{0,C_1} \geq v_1w_{0,C_1}\\ a(x_1)v_2'c(y_1) \leq v_2}} [v_1', v_2']_{\mathcal{A},\mathcal{C}}^-.$$

*Proof.* For $v_1 \in W_1$, let again $v_1^* = v_1(w_{0,\Gamma_1} w_{0,C_1})^{-1}$. Then again we have

$$[v_1, v_2]^-_{\mathcal{A},\mathcal{C}} = R_{\mathcal{A}}(B_1 \times B_2)(v_1^*, v_2)(B_1 \times B_2) R_{\mathcal{C}^*}(\dot{w}_{0,\Gamma_1} \dot{w}_{0,C_1}, e), \qquad \forall v_1 \in W_1,\ v_2 \in W_2.$$

For $y_1 \in W_{C_1}$, set $(y_1)_* = (w_{0,\Gamma_1} w_{0,C_1}) y_1 (w_{0,\Gamma_1} w_{0,C_1})^{-1} \in W_{C_1^*}$. Then by Theorem 5.2,

$$\overline{[v_1, v_2]^-_{\mathcal{A},\mathcal{C}}} = \bigsqcup_{\substack{v_1' \in W_1^{C_1},\, v_2' \in {}^{A_2}W_2 : \\ \exists x_1 \in W_{A_1}, y_1 \in W_{C_1} \ s.t. \\ x_1 (v_1')^* (y_1)_* \leq v_1^* \\ a(x_1) v_2' c(y_1) \leq v_2}} [v_1', v_2']^-_{\mathcal{A},\mathcal{C}} = \bigsqcup_{\substack{v_1' \in W_1^{C_1},\, v_2' \in {}^{A_2}W_2 : \\ \exists x_1 \in W_{A_1}, y_1 \in W_{C_1} \ s.t. \\ x_1 v_1' y_1 w_{0,C_1} \geq v_1 w_{0,C_1} \\ a(x_1) v_2' c(y_1) \leq v_2}} [v_1', v_2']^-_{\mathcal{A},\mathcal{C}}.$$

□

## 7. The $R_{\mathcal{A}}$-stable pieces in $\overline{G}$ and their closures

We retain the notation in §2.3. In particular, $G$ is a semi-simple algebraic group of adjoint type, and $\overline{G}$ denotes the De Concini-Procesi compactification of $G$. In this section, unless otherwise stated, if $X$ is a subset of $\overline{G}$, $\overline{X}$ always denotes the closure of $X$ in $\overline{G}$.

For each $J \subset G$, recall that $h_J$ is a point in $\overline{G}$ such that the stabilizer subgroup of $G \times G$ at $h_J$ is $R_J^-$ given in (2.8). Let $\mathcal{A} = (A_1, A_2, a, K)$ be any admissible quadruple for $G \times G$. Then we have the decomposition of $\overline{G}$ into $R_{\mathcal{A}}$-stable pieces

$$\overline{G} = \bigsqcup_{J \subset \Gamma, v_1 \in W^J, v_2 \in {}^{A_2}W} [J, v_1, v_2]_{\mathcal{A}},$$

where for $J \subset \Gamma$ and for $v_1 \in W^J$ and $v_2 \in {}^{A_2}W$, the subset $[J, v_1, v_2]_{\mathcal{A}}$ of $\overline{G}$ is defined by (2.9), namely

$$[J, v_1, v_2]_{\mathcal{A}} = R_{\mathcal{A}}(B \times B)(v_1, v_2).h_J.$$

By part (iii) of Proposition 6.1, we have

**Proposition 7.1.** *Let $A$ be any admissible quadruple for $G \times G$. Then for any $J \subset \Gamma$ and $v_1 \in W^J, v_2 \in {}^{A_2}W$, $[J, v_1, v_2]_{\mathcal{A}}$ is a locally closed smooth subset of $\overline{G}$. It fibers over the flag variety $M_{A_1}/(M_{A_1} \cap P_{A_1(v_1,v_2)})$ with fibers isomorphic to the product of $M_{J(v_1,v_2)}/Z_J$ and the affine space of dimension $\dim U_{A_1(v_1,v_2)} - l(v_1) + l(v_2)$, where $J(v_1, v_2)$ is the smallest subset of $J$ stable under $v_2^{-1} a v_1$ and $A_1(v_1, v_2) = v_1 J(v_1, v_2) \subset A_1$.*

When $R_{\mathcal{A}} = G_{\rm diag}$, the Proposition 7.1 coincides with the description of the geometry of Lusztig's $G_{\rm diag}$-stable pieces given in [8].

In this section, we study the closures of the subsets $[J, v_1, v_2]_{\mathcal{A}}$ in $\overline{G}$.

7.1. **The first description.** For $J \subset \Gamma$ and $v_1 \in W^J$, $v_2 \in W$, let

$$[J, v_1, v_2] = (B \times B)(v_1, v_2).h_J \subset \overline{G}.$$

The following Lemma 7.2 follows immediately from Lemma 8.1 in the Appendix.

**Lemma 7.2.** *For any admissible triple $\mathcal{A}$ for $G \times G$ and all $J \subset \Gamma$, $v_1 \in W^J$, $v_2 \in {}^{A_2}W$,*

$$\overline{[J, v_1, v_2]_{\mathcal{A}}} = R_{\mathcal{A}} \overline{[J, v_1, v_2]}.$$

**Proposition 7.3.** *Let $\mathcal{A}$ be an admissible quadruple for $G \times G$. Then for any $J \subset \Gamma$ and $(v_1, v_2) \in W^J \times {}^{A_2}W$,*

$$\overline{[J, v_1, v_2]_{\mathcal{A}}} = \bigsqcup_{I \subset J} \bigsqcup_{\substack{v_1' \in W^I,\, v_2' \in {}^{A_2}W : \\ \exists x \in W_{A_1}, y \in W_I, z \in W_J^I \ s.t. \\ l(v_2 z) = l(v_2) + l(z) \\ x v_1' y w_{0,I} \geq v_1 z w_{0,I} \\ a(x) v_2' y \leq v_2 z}} [I,\ v_1',\ v_2']_{\mathcal{A}}. \tag{7.1}$$

*Proof.* It is well-known [5] that for $I_1, I_2 \subset \Gamma$,

$$\overline{(G \times G).h_{I_1}} \,\cap\, (G \times G).h_{I_2} \neq \emptyset$$

if and only if $I_2 \subset I_1$. Thus

$$\overline{[J, v_1, v_2]_{\mathcal{A}}} = \bigsqcup_{I \subset J} \left( \overline{[J, v_1, v_2]_{\mathcal{A}}} \,\cap\, (G \times G).h_I \right).$$

Fix $I \subset J$. By Lemma 7.2,

$$\overline{[J, v_1, v_2]_{\mathcal{A}}} \,\cap\, (G \times G).h_I = R_{\mathcal{A}} \left( \overline{[J, v_1, v_2]} \,\cap\, (G \times G).h_I \right).$$

By [18, Lemma 2.3],

$$\overline{[J, v_1, v_2]} \,\cap\, (G \times G).h_I = \bigsqcup_{\substack{z \in W_J^I, \\ l(v_2 z) = l(v_2) + l(z)}} \overline{[I,\ v_1 z,\ v_2 z]}^I,$$

where, for a subsets $Y$ of $(G \times G).h_I$, $\overline{Y}^I$ denotes the closure of $Y$ in $(G \times G).h_I$. Thus

$$\overline{[J, v_1, v_2]_{\mathcal{A}}} \,\cap\, (G \times G).h_I = \bigsqcup_{\substack{z \in W_J^I, \\ l(v_2 z) = l(v_2) + l(z)}} R_{\mathcal{A}}\, \overline{[I,\ v_1 z,\ v_2 z]}^I.$$

By Lemma 8.1, $R_{\mathcal{A}}\, \overline{[I,\ v_1 z,\ v_2 z]}^I = \overline{R_{\mathcal{A}}[I,\ v_1 z,\ v_2 z]}^I$ for every $z \in W_J^I$. The decomposition in (7.1) now follows from Theorem 6.2. □

In the following §7.2 and §7.3, we will simplify the descriptions of the the closure relations in Proposition 7.3.

7.2. **The second description.** Recall that $(G \times G).h_\emptyset \cong (G \times G)/(B_- \times B)$ is the unique closed $(G \times G)$-orbit in $\overline{G}$. For $J \subset \Gamma$ and $v_1 \in W^J$, $v_2 \in {}^{A_2}W$, let

$$\partial_\emptyset \overline{[J, v_1, v_2]_{\mathcal{A}}} = \overline{[J, v_1, v_2]_{\mathcal{A}}} \cap (G \times G).h_\emptyset.$$

**Theorem 7.4.** *Let $\mathcal{A}$ be an admissible quadruple for $G \times G$. Let $I, J \subset \Gamma$, $v_1 \in W^J$, $v_1' \in W^I$, and $v_2, v_2' \in {}^{A_2}W$. Then the following are equivalent:*

*(i) $[I, v_1', v_2']_{\mathcal{A}} \subset \overline{[J, v_1, v_2]_{\mathcal{A}}}$;*

*(ii) $I \subset J$ and $\partial_\emptyset \overline{[I, v_1', v_2']_{\mathcal{A}}} \subset \partial_\emptyset \overline{[J, v_1, v_2]_{\mathcal{A}}}$;*

*(iii) $I \subset J$ and $[\emptyset, v_1', v_2']_{\mathcal{A}} \subset \partial_\emptyset \overline{[J, v_1, v_2]_{\mathcal{A}}}$.*

*Proof.* It is clear that (i) implies (ii). Since $[\emptyset, v_1', v_2']_{\mathcal{A}} \subset \partial_\emptyset \overline{[I, v_1', v_2']_{\mathcal{A}}}$, one sees that (ii) implies (iii). It remains to show that (iii) implies (i).

Assume (iii). By Proposition 7.3, there exist $x \in W_{A_1}$ and $z \in W_J$ with $l(v_2 z) = l(v_2) + l(z)$ such that $x v_1' \geq v_1 z$ and $a(x) v_2' \leq v_2 z$. Write $z = uy$ with $u \in W_J^I$ and $y \in W_I$. Then the set

$$S = \{(x', y') \in W_{A_1} \times W_I \mid x' v_1' \geq v_1 u y',\ \ a(x') v_2' \leq v_2 u y'\}$$

is non-empty. Let $(x_0, y_0) \in W_{A_1} \times W_I$ be a minimal element in $S$. We claim that

$$(x_0 v_1')^I \geq v_1 u y_0 (x_0 v_1')_I^{-1} \quad \text{and} \quad a(x_0) v_2' y_0^{-1} \leq v_2 u, \tag{7.2}$$

where $(x_0 v_1')^I \in W^I$ and $(x_0 v_1')_I \in W_I$ are such that $x_0 v_1' = (x_0 v_1')^I (x_0 v_1')_I$. By Lemma 8.3, it would follow from (7.2) that

$$x_0 v_1' y_0^{-1} w_{0,I} \geq v_1 u w_{0,I} \quad \text{and} \quad a(x_0) v_2' y_0^{-1} \leq v_2 u,$$

and, since $l(v_2 u) = l(v_2) + l(u)$, we would see by Proposition 7.3 that $[I, v_1', v_2']_{\mathcal{A}} \subset \overline{[J, v_1, v_2]_{\mathcal{A}}}$.

It remains to prove (7.2). We first show that $a(x_0)v_2'y_0^{-1} \leq v_2u$. Indeed, since $a(x_0)v_2' \leq v_2uy_0$, it follows from Lemma 8.7 in the Appendix that there exists $y_1 \leq y_0$ such that

$$v_2u = v_2uy_0y_0^{-1} \geq a(x_0)v_2'y_1^{-1}. \tag{7.3}$$

Again by Lemma 8.7, there exists $y_2 \leq y_1$ such that

$$a(x_0)v_2' = a(x_0)v_2'y_1^{-1}y_1 \leq v_2uy_2.$$

Since $y_2 \leq y_1 \leq y_0$, we have $x_0v_1' \geq v_1uy_0 \geq v_1uy_2$. Thus $(x_0, y_2) \in S$. Since $(x_0, y_2) \leq (x_0, y_0)$ and since $(x_0, y_0)$ is minimal in $S$, we must have $y_2 = y_0$. Hence $y_1 = y_0$, and $a(x_0)v_2'y_0^{-1} \leq v_2u$ by (7.3).

We now show that $l(x_0v_1') = l(x_0) + l(v_1')$. Indeed, if $l(x_0v_1') < l(x_0) + l(v_1')$, then by Lemma 8.6 in the Appendix, there exists $x_1 < x_0$ such that $x_1v_1' > x_0v_1' \geq v_1uy_0$. Since $a(x_1)v_2' \leq a(x_0)v_2' \leq v_2uy_0$, we have $(x_1, y_0) \in S$. Since $(x_1, y_0) < (x_0, y_0)$, this is a contradiction to the minimality of $(x_0, y_0)$ in $S$. Hence $l(x_0v_1') = l(x_0) + l(v_1')$.

By Lemma 8.7 in the Appendix, there exists $y_1 \leq (x_1v_1')_I$ such that

$$(x_0v_1')^I = (x_0v_1')(x_0v_1')_I^{-1} \geq v_1uy_0y_1^{-1}.$$

By Lemma 8.8 in the Appendix, there exists $x_2 \leq x_0$ such that $(x_0v_1')^I y_1 = x_2v_1'$. Now

$$x_2v_1' = (x_0v_1')^I y_1 \geq v_1uy_0y_1^{-1}y_1 = v_1uy_0,$$

and $a(x_2)v_2' \leq a(x_0)v_2' \leq v_2uy_0$. Hence $(x_2, y_0) \in S$. By the minimality of $(x_0, y_0)$ in $S$, we have $x_2 = x_0$, so $y_1 = (x_1v_1')_I$, and $(x_0v_1')^I \geq v_1uy_0(x_1v_1')_I^{-1}$. This proves (7.2). $\square$

As a corollary of (iii) in Theorem 7.4 and Proposition 7.3, we get the following second description of the closure relations on the $R_{\mathcal{A}}$-stable pieces in $\overline{G}$.

**Corollary 7.5.** *Let $\mathcal{A}$ be an admissible quadruple for $G \times G$. Then for any $J \subset \Gamma$ and $(v_1, v_2) \in W^J \times {}^{A_2}W$,*

$$\overline{[J, v_1, v_2]_{\mathcal{A}}} = \bigsqcup_{I \subset J} \bigsqcup_{\substack{v_1' \in W^I,\, v_2' \in {}^{A_2}W:\\ \exists x \in W_{A_1},\, z \in W_J \ s.t.\\ l(v_2z) = l(v_2) + l(z)\\ xv_1' \geq v_1z,\ a(x)v_2' \leq v_2z}} [I,\ v_1',\ v_2']_{\mathcal{A}}. \tag{7.4}$$

7.3. **The third description.** In this section, we show that the condition $l(v_2z) = l(v_2)+l(z)$ in (7.4) can be dropped. Namely, we have

**Theorem 7.6.** *Let $\mathcal{A}$ be an admissible quadruple for $G \times G$. Then for any $J \subset \Gamma$ and $(v_1, v_2) \in W^J \times {}^{A_2}W$,*

$$\overline{[J, v_1, v_2]_{\mathcal{A}}} = \bigsqcup_{I \subset J} \bigsqcup_{\substack{v_1' \in W^I,\, v_2' \in {}^{A_2}W:\\ \exists x \in W_{A_1},\, z \in W_J \ s.t.\\ xv_1' \geq v_1z,\ a(x)v_2' \leq v_2z}} [I,\ v_1',\ v_2']_{\mathcal{A}}. \tag{7.5}$$

*Proof.* Fix $J \subset \Gamma$ and $(v_1, v_2) \in W^J \times {}^{A_2}W$. It is enough to show that the right hand side of (7.5) is contained in the right hand side of (7.4). To this end, let $I \subset J$ and let $(v_1', v_2') \in W^I \times {}^{A_2}W$ be such that there exist $x \in W_{A_1}$ and $z \in W_J$ with $xv_1' \geq v_1z$ and $a(x)v_2' \leq v_2z$. Choose such an $x \in W_{A_1}$ and let

$$Z = \{z' \in W_J \mid\ xv_1' \geq v_1z',\ \ a(x)v_2' \leq v_2z'\}.$$

Then $Z \neq \emptyset$. Let $z_0 \in Z$ be a minimal element. We claim that $l(v_2z_0) = l(v_2) + l(z_0)$. Indeed, if $l(v_2z_0) < l(v_2) + l(z_0)$, then by Lemma 8.6 in the Appendix, there exists $z_1 < z_0$ such that $v_2z_1 > v_2z_0$. Thus $v_2z_1 > a(x)v_2'$. Since $v_1 \in W^J$ and $z_1, z_0 \in W_J$, we also have

$v_1z_1 \leq v_1z_0 \leq xv_1'$. Thus $z_1 \in Z$, which contradicts to the fact the $z_0$ is a minimal element in $Z$. This shows that $l(v_2z_0) = l(v_2) + l(z_0)$, and thus $[I, v_1', v_2']_{\mathcal{A}}$ is contained in the right hand side of (7.4). □

**Remark 7.7.** Note that in the proofs of Proposition 7.3, Corollary 7.5, and Theorem 7.6, we did not use the fact that $v_2 \in {}^{A_2}W$. In fact, the decomposition formulas (7.1), (7.4), and (7.5) hold for any $v_1 \in W^J$ and $v_2 \in W_2$. Thus the closure of $R_{\mathcal{A}}\mathcal{O}$ in $\overline{G}$ for any $(B\times B)$-orbit $\mathcal{O}$ in $\overline{G}$ is a union of the sets of the form $[I, v_1', v_2']_{\mathcal{A}}$ for $I \subset \Gamma$, $v_1' \in W^I$, and $v_2' \in {}^{A_2}W$. Proposition 7.3, Corollary 7.5, and Theorem 7.6 give three equivalent descriptions of the decomposition. See also Remark 5.3.

## 8. Appendix

8.1. **A lemma from** [19]**.** The following is [19, Lemma 2, P. 68]. We state it here for the convenience of the reader.

**Lemma 8.1.** *Let $G$ be an algebraic group acting on a variety $V$. Let $H$ be a closed subgroup of $G$ and let $U \subset V$ be a closet subset of $V$, invariant under the action of $H$. Assume that $G/H$ is complete. Then $G.U$ is closed.*

8.2. **A few facts on the Weyl group.** Let $G$ be a any connected reductive algebraic group over an algebraically closed field. Let $T$ be a maximal torus of $G$, let $B$ a Borel subgroup of $G$ containing $T$, and let $\Gamma$ be the set of simple roots for $(B, T)$. Let $W$ be the Weyl group of $\Gamma$. For the convenience of the reader, we collect in this section a few facts on $W$ that are used in this paper. Recall that for two subsets $A$ and $C$ of $\Gamma$ we denote ${}^AW^C = {}^AW \cap W^C$.

**Lemma 8.2.** [3, Proposition 2.7.5] *For any $A, C \subset \Gamma$, every $v \in {}^AW$ can be uniquely written as a product $v = xu$, where $x \in {}^AW^C$ and $u \in {}^{C\cap x^{-1}(A)}W_C$.*

For $C \subset \Gamma$, let $w_{0,C}$ be the longest element of $W_C$.

**Lemma 8.3.** [18, Page 79] *Let $x, w \in W^C$ and $u \in W_C$. Then $xu^{-1} \leq w$ if and only if $xw_{0,C} \leq wuw_{0,C}$.*

*Proof.* If $xu^{-1} \leq w$, then $xw_{0,C} = xu^{-1}uw_{0,C} \leq wuw_{0,C}$. Conversely, assume that $xw_{0,C} \leq wuw_{0,C}$. Then there exist $w_1 \leq w$ and $y \leq uw_{0,C}$ such that $xw_{0,C} = w_1y$. It follows from $y \leq uw_{0,C}$ that $yw_{0,C} \geq u$, so $u^{-1} \leq w_{0,C}y^{-1}$. Thus $xu^{-1} \leq xw_{0,C}y^{-1} = w_1 \leq w$. □

For $y, z \in W$, let $\mathcal{W}(y, z) = \{x \in W \mid BxB \subset ByBzB\}$.

**Lemma 8.4.** *Let $y, z \in W$. Then $x \geq yz$ for any $x \in \mathcal{W}(y, z)$.*

*Proof.* Let $x \in \mathcal{W}(y, z)$. Then $(yBz) \cap (BxB) = ((yz)(z^{-1}Bz)) \cap (BxB) \neq \emptyset$. By $z^{-1}Bz = ((z^{-1}Bz) \cap B^-)((z^{-1}Bz) \cap B)$, we know that $(yz((z^{-1}Bz) \cap B^-)) \cap (BxB) \neq \emptyset$. Since

$$(ByzB^-) \cap (BxB) \supset (yzB^-) \cap (BxB) \supset (yz((z^{-1}Bz) \cap B^-)) \cap (BxB),$$

we know that $(ByzB^-) \cap (BxB) \neq \emptyset$. By [4], $yz \leq x$. □

**Lemma 8.5.** [8, Lemma 3.3] *For any $u, w \in W$, the subset $\{vw \mid v \leq u\}$ of $W$ contains a unique maximal element $u_1w$. Moreover, $l(u_1w) = l(u_1) + l(w)$.*

**Lemma 8.6.** *If $u, w \in W$ are such that $l(uw) < l(u) + l(w)$, then there exists $u_1$ such that $u_1 < u$ and $u_1w > uw$.*

*Proof.* The $u_1$ such that $u_1w$ is the maximal element in the set $\{vw \mid v \leq u\}$ is as required. □

**Lemma 8.7.** [8, Corollary 3.4] *Let $u, w, w' \in W$ and assume that $w' \leq w$. Then*

*(i) there exists $u_1 \leq u$ such that $w'u_1 \leq wu$;*

*(ii) there exists $u_2 \leq u$ such that $w'u \leq wu_2$.*

**Lemma 8.8.** [8, Lemma 3.10] *Let $J \subset \Gamma, w \in W^J$, and $u \in W$ be such that $l(uw) = l(u) + l(w)$. Write $uw = xv$ with $x \in W^J$ and $v \in W_J$. Then for any $v' \leq v$, there exists $u' \leq u$ such that $u'w = xv'$.*

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