AUTOMORPHIC ORBIT PROBLEM FOR POLYNOMIAL ALGEBRAS

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Abstract. It is proved that every endomorphism preserving the automorphic orbit of a nontrivial element of the rank two polynomial algebra over the complex number field is an automorphism.

1. Introduction and the main results

In [13], Shpilrain raised the following

Problem 1.1. (**Automorphic orbit problem for free groups**) *Let F_n be the free group of rank n, $u \in F_n - \{e\}$, ϕ an endomorphism of F_n preserving the automorphic orbit of u in F_n, i.e. for each automorphism α of F_n, there exists an automorphism β of F_n, such that $\phi(\alpha(u)) = \beta(u)$. Is ϕ an automorphism of F_n?*

Problem 1.1 is solved affirmatively for $n = 2$ by Shpilrain [14] and Ivanov [7], and completely solved in the positive by D.Lee [6]. The automorphic orbit problem is solved affirmatively by A.A.Mikhalev and J.-T.Yu [12] for free Lie algebras, and solved affirmatively by A.A.Mikhalev, U.Umirbaev and J.-T.Yu for free non-associative algebras.

In the sequel all automorphisms (endomorphisms) of a polynomial algebra over a field K are always K-automorphisms (K-endomorphisms). In view of Problem 1.1, it is natural and interesting to raise

Problem 1.2. (**Automorphic orbit problem for polynomial algebras**) *Let P_n be the polynomial algebra of rank n over a field K, $p \in P_n - K$, ϕ an endomorphism of P_n preserving the automorphic orbit of p in P_n. Is ϕ an automorphism of P_n?*

Recall that a polynomial $p \in P_n$ is a *coordinate* if there exists an automorphism ψ of P_n taking x_1 to p. A special case of Problem 1.1 when u is a coordinate of P_n is the following

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Problem 1.3 (**Coordinate preserving problem**)**.** *Let P_n be the polynomial algebra of rank n over a field K. Is every endomorphism ϕ of P_n taking all coordinates of P_n to coordinates an automorphism ?*

Problem 1.3 is solved affirmatively for $n = 2$ when K is an arbitrary field by van den Essen and Shpilrain [3], and is solved affirmatively for arbitrary n when K is an algebraically closed field of zero characteristic by Jelonek [8].
In this paper we solve Problem 1.2 for $n = 2$ when K is the complex number field:

Theorem 1.4. *Let $p \in \mathbb{C}[x, y] - \mathbb{C}$, ϕ an endomorphism of $\mathbb{C}[x, y]$ preserving the automorphic orbit of p. Then ϕ is an automorphism of $\mathbb{C}[x, y]$.*

Recall the outer rank k of a polynomial $p \in P_n$ is the minimal number k such that under an automorphism ϕ of P_n, $\phi(p) \in P_k$. See Shpilrain and J.-T. Yu [15]. In our proof of Theorem 1.4, it is crucial to use the result below based on a theorem of Shpilrain and J.-T.Yu [17], which has its own interest.

Theorem 1.5. *Let $p \in \mathbb{C}[x, y]$ has outer rank 2. Then p is a test polynomial recognizing automorphisms among injective endomorphisms of $\mathbb{C}[x, y]$. Or, more precisely, if ϕ is an injective endomorphism of $\mathbb{C}[x, y]$ such that $\phi(p) = p$, then ϕ is an automorphism.*

The above theorem can be viewed as an analogue of a result of Turner [18] for free groups.

2. **Preliminaries**

First let us recall test polynomials and retracts of polynomial algebras. See [4, 5, 9, 10, 16, 17].
A polynomial $p \in P_n$ is called a test polynomial, if, for any endomorphism ϕ of P_n, $\phi(p) = p$ implies that ϕ is an automorphism. A subalgebra R of P_n is called a retract if there is a idempotent homomorphism (π is called the retraction from P_n to R) π of P_n such that $\pi(P_n) = R$. By a theorem of Costa [1], every proper retract of $K[x, y]$ (a retract of $K[x, y]$ different from K and $K[x, y]$) is of the form $K[p]$ for some $p \in K[x, y]$ for arbitrary field K. Recently Shpilrain and J.-T.Yu [16, 17] have shown the close connection among test polynomials, retracts, and the Jacobian conjecture. See also [2, 10].

Lemma 2.1 (Shpilrain and J.-T.Yu [16])**.** *Let K be a field of zero characteristic. A polynomial $r \in K[x, y]$ generates a proper retract* *of* $K[x,y]$ *if and only if there is an automorphism* α *of* $K[x,y]$ *such that* $\alpha(r) = x + yq$ *for some* $q \in K[x,y]$*. Moreover, under the above condition the retraction from* $\mathbb{C}[x,y]$ *to* $\mathbb{C}[r]$ *is* $\alpha^{-1}\pi\alpha$*, where* π *is the retraction of* $\mathbb{C}[x,y]$ *to* $\mathbb{C}[x+yq]$ *defined by* $\pi(x) = x+yq$ *and* $\pi(y) = 0$*.*

The next lemma is based on the main theorem and its proof in Drensky and J.-T.Yu [4].

Lemma 2.2. *A polynomial* $p \in \mathbb{C}[x,y]$ *belongs to a proper retract* $\mathbb{C}[r]$ *if and only if* p *is fixed by a non-injective endomorphism* ϕ *of* $\mathbb{C}[x,y]$*. Moreover, under the above condition, if* $p = f(r)$*,* $f(t) \in \mathbb{C}[t] - \mathbb{C}$*,* $\deg(f) = m$*, then* $\pi = \phi^m$ *is the retraction from* $\mathbb{C}[x,y]$ *to* $\mathbb{C}[r]$*.*

Proof. The first sentence is just the Theorem in [4]. Moreover, in the proof of the Theorem in [4], it is actually proved that $\pi = \phi^m$ is the retraction from $\mathbb{C}[x,y]$ to $\mathbb{C}[r]$ with $m = [\mathbb{C}(r) : \mathbb{C}(p)]$. By elementary algebra, $m = \deg(f)$, where $f \in K[t]$, and $p = f(r)$. □

Lemma 2.3. *Let* K *be an arbitrary field,* $u \in K[x,y]$ *with outer rank* 1*,* ϕ *an endomorphism preserving the automorphic orbit of* u*. Then* ϕ *is an automorphism.*

Proof. Write $u = f(p)$, where $f \in K[t]$, p is a coordinate of $K[x,y]$. We may assume $p = x$. For any automorphism α, $\phi\alpha(f(x)) = \beta(f(x))$ for some automorphism β. Hence $\beta^{-1}\phi\alpha(f(x)) = f(x)$, therefore $f(\beta^{-1}\phi\alpha(x)) = f(x)$. Let $\beta^{-1}\phi\alpha(x) = g(x,y)$. Compare the degrees of y in both sides of $f(g(x,y)) = f(x)$, $g(x,y) = g(x,0) = h(x) \in K[x]$. Compare the degrees in both sides of $f(h(x)) = f(x)$, $\deg(h(x)) = 1$, that forces $h(x) = \beta^{-1}\phi\alpha(x) = cx$, hence $\phi\alpha(x) = \beta(cx)$ for some $c \in K^*$ (in fact c can only be some m-th root of unity, $m = \deg(f)$, but we do not need that). Therefore ϕ preserves coordinates of $K[x,y]$. By a result of Shpilrain and van den Essen [3], ϕ is an automorphism. □

Lemma 2.4. *Let* K *be an arbitrary field,* $p \in P_n = K[x_1, \dots, x_n]$ *a test polynomial. Then every endomorphism* ϕ *of* P_n *preserving the automorphic orbit of* p *is an automorphism.*

Proof. Since $\phi(p) = \alpha(p)$ for some automorphism α of P_n, $\alpha^{-1}\phi(p) = p$, as p is a test polynomial, $\alpha^{-1}\phi$, hence ϕ, is an automorphism. □

The following lemma is the main result of Shpilrain and J.-T. Yu [17].

Lemma 2.5. *A polynomial* $p \in \mathbb{C}[x,y]$ *is a test polynomial if and only if* p *does not belong to any proper retract of* $\mathbb{C}[x,y]$*.*

3. **Proof of the main results**

Proof of Theorem 1.5. Let $p \in \mathbb{C}[x,y]$ has outer rank 2, ϕ an injective endomorphism such that $\phi(p) = p$. Suppose on the contrary, ϕ is not an automorphism, then by Theorem 2 in [17], p has outer rank 1. This contradiction completes the proof.

Proof of Theorem 1.4. We may assume $\phi(p) = p$. By Lemma 2.4, we may assume p is not a test polynomial. By Lemma 2.5, we may assume p belongs to a proper retract $\mathbb{C}[r]$ of $\mathbb{C}[x,y]$. By Lemma 2.3, we may assume p has outer rank 2. By Theorem 1.5, we may assume ϕ is non-injective. Suppose $p = f(r)$, where $f \in \mathbb{C}[t] - \mathbb{C}$, $\deg(f) = m$. By Lemma 2.2, $\pi = \phi^m$ is the retraction from $\mathbb{C}[x,y]$ to $\mathbb{C}[r]$. As ϕ preserves the automorphic orbit of p, so does $\pi = \phi^m$. Applying Lemma 2.1 (suppose $\alpha(r) = x + yq(x,y)$, where $q(x,y) \notin K[y]$, α is some automorphism of $\mathbb{C}[x,y]$, replace r by $\alpha(r)$, and π by $\alpha\pi\alpha^{-1}$), we have reduced our proof to the proof of the following

Lemma 3.1. *Let $r = x + yq(x,y)$, where $q(x,y) \in \mathbb{C}[x,y]$, $q(x,y) \notin \mathbb{C}[y]$, π the retraction of $\mathbb{C}[x,y]$ to $\mathbb{C}[r]$ defined by $\pi(x) = x + yq(x,y)$, $\phi(y) = 0$, $f \in \mathbb{C}[t] - \mathbb{C}$. Then π does not preserve the automorphic orbit of $f(r)$.*

Proof. Suppose on the contrary, π preserves the automorphic orbit of $f(r)$. Then for any automorphism α of $\mathbb{C}[x,y]$, $\pi\alpha(f(r)) = \beta(f(r)) \in \mathbb{C}[r]$ for some automorphim β of $\mathbb{C}[x,y]$. Note that $\pi\beta(f(r)) = \beta(f(r))$. By Lemma 2.2, $\pi^{\deg(f)} = \pi$ is the retraction from $\mathbb{C}[x,y]$ to the retract $\mathbb{C}[\beta(r)]$ taking $\beta(r)$ to $\beta(r)$. By hypothesis, π is also the retraction of $\mathbb{C}[x,y]$ to the retract $\mathbb{C}[r]$ taking r to r. This forces that $\beta(r) = r$. Therefore $\beta(x + yq(x,y)) = x + yq(x,y)$. Subsituting $y = 0$, $\beta(x) = x$. Hence $\beta(yq(x,y)) = yq(x,y)$. But β is an automorphism, so $\beta(y) = cy + h(x)$ where $c \in \mathbb{C}^*$, $h(x) \in \mathbb{C}[x]$. It follows easily that $\beta(y) = y$, β is the identity automorphism. We have conculde that for all automorphisms α of $\mathbb{C}[x,y]$, $\pi\alpha(f(r)) = f(r)$. Let M be a positive integer greater than $\deg(q(x,y))$, it is easy to see that $x^M - y$ does not divide $q(x,y)$ in $\mathbb{C}[x,y]$. Let α be the automorphism of $\mathbb{C}[x,y]$ defined by $\alpha(x) = x,\ \alpha(y) = y + x^M$. Then easy calculation shows that $\pi\alpha(f(r)) = f(r + r^M q(r, r^M))$. As $x^M - y$ does not divide $q(x,y)$, $q(r, r^M) \neq 0$. Therefore $\pi\alpha(f(r)) = f(r + r^M q(r, r^M)) \neq f(r)$. This contradiction completes the proof. $\square$

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