GROUP ORBITS AND REGULAR PARTITIONS OF POISSON MANIFOLDS

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Abstract. We study a large class of Poisson manifolds, derived from Manin triples, for which we construct explicit partitions into regular Poisson submanifolds by intersecting certain group orbits. Examples include all varieties $\mathcal{L}$ of Lagrangian subalgebras of reductive quadratic Lie algebras $\mathfrak{d}$ with Poisson structures defined by Lagrangian splittings of $\mathfrak{d}$. In the special case of $\mathfrak{g} \oplus \mathfrak{g}$, where $\mathfrak{g}$ is a complex semi-simple Lie algebra, we explicitly compute the ranks of the Poisson structures on $\mathcal{L}$ defined by arbitrary Lagrangian splittings of $\mathfrak{g} \oplus \mathfrak{g}$. Such Lagrangian splittings have been classified by P. Delorme, and they contain the Belavin–Drinfeld splittings as special cases.

1. Introduction

Lie theory provides a rich class of examples of Poisson manifolds/varieties. In this paper, we study a class of Poisson manifolds of the form $(D/Q, \Pi_{\mathfrak{u},\mathfrak{u}'})$, where D is an even dimensional connected real or complex Lie group whose Lie algebra $\mathfrak{d}$ is quadratic, i.e. $\mathfrak{d}$ is equipped with a nondegenerate invariant symmetric bilinear form $\langle \, , \, \rangle$; the closed subgroup Q of D corresponds to a subalgebra $\mathfrak{q}$ of $\mathfrak{d}$ that is coisotropic with respect to $\langle \, , \, \rangle$, and $(\mathfrak{u}, \mathfrak{u}')$ is a pair of complementary Lagrangian subalgebras of $\mathfrak{d}$. A Lie subalgebra $\mathfrak{l}$ of $\mathfrak{d}$ will be called Lagrangian if $\mathfrak{l}^\perp = \mathfrak{l}$ with respect to $\langle \, , \, \rangle$. We will call such a splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$ a Lagrangian splitting. The Poisson structure $\Pi_{\mathfrak{u},\mathfrak{u}'}$ is obtained from the r-matrix

$$r_{\mathfrak{u},\mathfrak{u}'} = \frac{1}{2} \sum_{i=1}^{n} \xi_i \wedge x_i \in \wedge^2 \mathfrak{d},$$

where $\{x_1, \ldots, x_n\}$ and $\{\xi_1, \ldots, \xi_n\}$ are pairs of dual bases of $\mathfrak{u}$ and $\mathfrak{u}'$ with respect to $\langle \, , \, \rangle$. We refer the reader to § 2.2 for the precise definition of $\Pi_{\mathfrak{u},\mathfrak{u}'}$.

Let U and U' be the connected subgroups of D with Lie algebras $\mathfrak{u}$ and $\mathfrak{u}'$ respectively. Our first main result, see Theorem 2.7 and Proposition 2.13, is that when

$$[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}^\perp, \tag{1.1}$$

all intersections of U and U'-orbits in D/Q are regular Poisson submanifolds. In fact, if $N(\mathfrak{u})$ and $N(\mathfrak{u}')$ denote the normalizers of $\mathfrak{u}$ and $\mathfrak{u}'$ in D respectively, we also show that all intersections of $N(\mathfrak{u})$ and $N(\mathfrak{u}')$-orbits in D/Q are regular Poisson submanifolds. Note that the condition (1.1) is an intrinsic property of the coisotropic subalgebra $\mathfrak{q}$ of $(\mathfrak{d}, \langle\, ,\, \rangle)$ and does not depend on the Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$. Once this condition is verified for a given $\mathfrak{q}$, the above result provides "regular" partitions for the Poisson structures $\Pi_{\mathfrak{u},\mathfrak{u}'}$ on D/Q for any Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$.

Our second main result shows that the condition (1.1) is satisfied when $\mathfrak{d}$ is reductive and $\mathfrak{q}$ is the normalizer subalgebra in $\mathfrak{d}$ of any Lagrangian subalgebra of $(\mathfrak{d}, \langle\, ,\, \rangle)$. In fact, we show in Proposition 3.3 that in this case $[\mathfrak{q}, \mathfrak{q}] = \mathfrak{q}^{\perp}$. Let $\mathcal{L}(\mathfrak{d}, \langle\, ,\, \rangle)$ be the variety of Lagrangian subalgebras of $(\mathfrak{d}, \langle\, ,\, \rangle)$. All D-orbits in $\mathcal{L}(\mathfrak{d}, \langle\, ,\, \rangle)$ are of the form $D/N(\mathfrak{l})$, where $N(\mathfrak{l})$ is the normalizer subgroup in D of an $\mathfrak{l} \in \mathcal{L}(\mathfrak{d}, \langle\, ,\, \rangle)$, and every Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$ defines a Poisson structure $\Pi_{\mathfrak{u},\mathfrak{u}'}$ on $\mathcal{L}(\mathfrak{d}, \langle\, ,\, \rangle)$. As a corollary of the second main result we obtain that, if $\mathfrak{d}$ is an even dimensional reductive quadratic Lie algebra, then every non-empty intersection of an $N(\mathfrak{u})$-orbit and an $N(\mathfrak{u}')$-orbit on $\mathcal{L}(\mathfrak{d}, \langle\, ,\, \rangle)$ is a regular Poisson submanifold with respect to $\Pi_{\mathfrak{u},\mathfrak{u}'}$

In §4, we take the special case when $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ for a complex semi-simple Lie algebra $\mathfrak{g}$ and

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \ll x_1, y_1 \gg - \ll x_2, y_2 \gg, \quad x_1, x_2, y_1, y_2 \in \mathfrak{g},$$

where $\ll .,. \gg$ is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ whose restriction to a compact real form of $\mathfrak{g}$ is negative definite. Lagrangian splittings of $(\mathfrak{g} \oplus \mathfrak{g}, \langle\, ,\, \rangle)$ have been classified by Delorme [4]. In particular, one has the Belavin–Drinfeld splittings $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\mathrm{diag}} + \mathfrak{l}$, where $\mathfrak{g}_{\mathrm{diag}}$ is the diagonal of $\mathfrak{g} \oplus \mathfrak{g}$. For any $\mathfrak{l} \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$, $N(\mathfrak{l})$-orbits in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ can be described by using results in [14]. For an arbitrary Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ we prove that the intersection of each $N(\mathfrak{l}_1)$ and $N(\mathfrak{l}_2)$-orbit on $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ is connected. Further, using [14], we compute the rank of all corresponding Poisson structures $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ on $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$. This result extends the dimension formulas for symplectic leaves in the second author's classification [21] of symplectic leaves of Belavin–Drinfeld Poisson structures on complex reductive Lie groups. Our result also generalizes the rank formulas of S. Evens and the first author for the standard Poisson structure on $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$.

As have been shown in [7, 8], all real and complex semi-simple symmetric spaces, as well as certain of their compactifications can be embedded into suitable varieties of Lagrangian subalgebras. Out results show that all such spaces carry Poisson structures and natural partitions into regular Poisson subvarieties.

All manifolds and vector spaces in this paper, unless otherwise stated, are assumed to be either complex or real.

A submanifold N of a Poisson manifold (M, π) will be called a *complete Poisson submanifold* if it is closed under all Hamiltonian flows or equivalently it is a union of symplectic leaves of π.

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2. The Poisson spaces D/Q

Recall that a quadratic Lie algebra is a pair $(\mathfrak{d}, \langle\,,\,\rangle)$, where $\mathfrak{d}$ is a Lie algebra and $\langle\,,\,\rangle$ is an invariant symmetric nondegenerate bilinear form on $\mathfrak{d}$. Throughout this section, we fix a quadratic Lie algebra $(\mathfrak{d}, \langle\,,\,\rangle)$ and a connected Lie group D with Lie algebra $\mathfrak{d}$. For a subspace V of $\mathfrak{d}$, set

$$V^\perp = \{x \in \mathfrak{d} \mid \langle x,\, y\rangle = 0, \forall y \in V\}. \tag{2.1}$$

2.1. **Lagrangian splittings.** A *coisotropic* (resp. *Lagrangian, isotropic*) subalgebra of $\mathfrak{d}$ (with respect to $\langle\,,\,\rangle$) is by definition a Lie subalgebra $\mathfrak{q}$ of $\mathfrak{d}$ such that $\mathfrak{q}^\perp \subset \mathfrak{q}$ (resp. $\mathfrak{q}^\perp = \mathfrak{q}$, $\mathfrak{q} \subset \mathfrak{q}^\perp$).

Definition 2.1. A Lagrangian splitting of $\mathfrak{d}$ is a vector space direct sum decomposition $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$, where $\mathfrak{u}$ and $\mathfrak{u}'$ are both Lagrangian subalgebras of $\mathfrak{d}$. The triple $(\mathfrak{d}, \mathfrak{u}, \mathfrak{u}')$ is also called a Manin triple [11].

Given a Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$, for a subspace $W \subset \mathfrak{u}$, set

$$W^0 = \{\xi \in \mathfrak{u}' \mid \langle \xi,\, x\rangle = 0, \forall x \in W\} = W^\perp \cap \mathfrak{u}'. \tag{2.2}$$

We now recall how Lagrangian splittings give rise to Poisson Lie groups. Recall that a Poisson Lie group is a pair (G, π), where G is a Lie group and π is a Poisson structure on G such that the group multiplication $G \times G \to G$ is a Poisson map. When a (not necessarily closed) subgroup H of G is also a Poisson submanifold with respect to π, (H, π) is itself a Poisson Lie group and is called a Poisson Lie subgroup of (G, π). If (G, π) is a Poisson Lie group, then $\pi(e) = 0$, where $e \in G$ is the identity element. Let $\mathfrak{g}$ be the Lie algebra of G, and let $d_e\pi : \mathfrak{g} \to \wedge^2\mathfrak{g}$ be the linearization of π at e defined by

$$(d_e\pi)(x) = (L_{\widetilde{x}}\pi)(e),$$

where for $x \in \mathfrak{g}, \widetilde{x}$ is any local vector fields with $\widetilde{x}(e) = x$ and $L_{\widetilde{x}}\pi$ is the Lie derivative of π at e. Then $(\mathfrak{g}, d_e\pi)$ is a Lie bialgebra [11] called the tangential Lie bialgebra of (G, π).

Assume that $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$ is a Lagrangian splitting. The bilinear form $\langle\, , \,\rangle$ induces a non-degenerate pairing between $\mathfrak{u}$ and $\mathfrak{u}'$. Define

$$\delta_{\mathfrak{u}} : \mathfrak{u} \longrightarrow \wedge^2 \mathfrak{u} : \langle \delta_{\mathfrak{u}}(x), y \wedge z \rangle = \langle x, [y, z] \rangle, \qquad x \in \mathfrak{u}, y, z \in \mathfrak{u}', \tag{2.3}$$

$$\delta_{\mathfrak{u}'} : \mathfrak{u}' \longrightarrow \wedge^2 \mathfrak{u}' : \langle \delta_{\mathfrak{u}'}(x), y \wedge z \rangle = -\langle x, [y, z] \rangle, \qquad x \in \mathfrak{u}', y, z \in \mathfrak{u}. \tag{2.4}$$

Then $(\mathfrak{u}, \delta_{\mathfrak{u}})$ and $(\mathfrak{u}', \delta_{\mathfrak{u}'})$ are Lie bialgebras [11]. Associated to the splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$ we also have the r-matrix

$$R_{\mathfrak{u},\mathfrak{u}'} = \frac{1}{2} \sum_{j=1}^{n} \xi_j \wedge x_j \in \wedge^2 \mathfrak{d}, \tag{2.5}$$

where $\{x_1, x_2, \cdots, x_n\}$ and $\{\xi_1, \xi_2, \cdots, \xi_n\}$ are bases of $\mathfrak{u}$ and $\mathfrak{u}'$, respectively, such that $\langle x_i, \xi_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$. It is easy to see that $R_{\mathfrak{u},\mathfrak{u}'}$ is independent of the choice of the bases. Moreover, the Schouten bracket $[R_{\mathfrak{u},\mathfrak{u}'}, R_{\mathfrak{u},\mathfrak{u}'}] \in \wedge^3 \mathfrak{d}$ is given by

$$\langle [R_{\mathfrak{u},\mathfrak{u}'}, R_{\mathfrak{u},\mathfrak{u}'}], a \wedge b \wedge c \rangle = 2 \langle a, [b, c] \rangle, \qquad a, b, c \in \mathfrak{d}. \tag{2.6}$$

Recall that D is a connected Lie group with Lie algebra $\mathfrak{d}$. Denote by U and U' the connected subgroups of D with Lie algebras $\mathfrak{u}$ and $\mathfrak{u}'$, respectively. Let $R^l_{\mathfrak{u},\mathfrak{u}'}$ and $R^r_{\mathfrak{u},\mathfrak{u}'}$ be the left and the right invariant bi-vector fields on D with values $R_{\mathfrak{u},\mathfrak{u}'}$ at the identity element. Set

$$\pi^D_{\mathfrak{u},\mathfrak{u}'} := R^r_{\mathfrak{u},\mathfrak{u}'} - R^l_{\mathfrak{u},\mathfrak{u}'}. \tag{2.7}$$

The following fact can be found in [6, 11].

Proposition 2.2. *The bivector field $\pi^D_{\mathfrak{u},\mathfrak{u}'}$ is a Poisson structure on D and $(D, \pi^D_{\mathfrak{u},\mathfrak{u}'})$ is a Poisson Lie group. Both U and U' are Poisson Lie subgroups of $(D, \pi^D_{\mathfrak{u},\mathfrak{u}'})$. Let*

$$\pi_U = \pi^D_{\mathfrak{u},\mathfrak{u}'}|_U, \qquad \pi_{U'} = -\pi^D_{\mathfrak{u},\mathfrak{u}'}|_{U'}. \tag{2.8}$$

Then the tangential Lie bialgebras of the Poisson Lie groups (U, π_U) and $(U', \pi_{U'})$ are respectively $(\mathfrak{u}, \delta_{\mathfrak{u}})$ and $(\mathfrak{u}', \delta_{\mathfrak{u}'})$.

2.2. **The Poisson spaces D/Q.** Assume that Q is a closed subgroup of D whose Lie algebra $\mathfrak{q}$ is a coisotropic subalgebra of $(\mathfrak{d}, \langle\, , \,\rangle)$. For an integer $k \geq 1$, let $\chi^k(D/Q)$ be the space of k-vector fields on D/Q. Then the left action of D on D/Q gives rise to the Lie algebra anti-homomorphism

$$\kappa : \mathfrak{d} \longrightarrow \chi^1(D/Q)$$

whose multi-linear extension $\wedge^k \mathfrak{d} \to \chi^k(D/Q)$ will be denoted by the same letter.

Given a Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$, define the bivector field $\Pi_{\mathfrak{u},\mathfrak{u}'}$ on D/Q by

$$\Pi_{\mathfrak{u},\mathfrak{u}'} := \kappa(R_{\mathfrak{u},\mathfrak{u}'}), \tag{2.9}$$

recall (2.5). The following theorem is the main result for this subsection.

Theorem 2.3. *For every Lagrangian splitting* $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$ *and every closed subgroup* Q *of* D *whose Lie algebra* $\mathfrak{q}$ *is coisotropic in* $\mathfrak{d}$,

1) $\Pi_{\mathfrak{u},\mathfrak{u}'}$ *is a Poisson bi-vector field on* D/Q;

2) all U *and* U'*-orbits in* D/Q *are complete Poisson submanifolds of* $(D/Q, \Pi_{\mathfrak{u},\mathfrak{u}'})$.

Proof. 1) The Lie algebra of the stabilizer subgroup of each point of D/Q is a coisotropic subalgebra of $\mathfrak{d}$. To prove that $\Pi_{\mathfrak{u},\mathfrak{u}'}$ is Poisson, it suffices to show that

$$[R_{\mathfrak{u},\mathfrak{u}'},\, R_{\mathfrak{u},\mathfrak{u}'}] \in \mathfrak{q} \wedge \mathfrak{d} \wedge \mathfrak{d}$$

for each coisotropic subalgebra $\mathfrak{q}$ of $\mathfrak{d}$. This is equivalent to

$$\langle [R_{\mathfrak{u},\mathfrak{u}'},\, R_{\mathfrak{u},\mathfrak{u}'}], a \wedge b \wedge c \rangle = 0, \quad \forall a, b, c \in \mathfrak{q}^{\perp}$$

which follows from (2.6) because $\mathfrak{q}^{\perp}$ is an isotropic subalgebra of $\mathfrak{d}$.

Denote by $\kappa_{\mathfrak{q}} : \mathfrak{d} \to \mathfrak{d}/\mathfrak{q}$ the canonical projection and its induced map $\wedge^2 \mathfrak{d} \to \wedge^2(\mathfrak{d}/\mathfrak{q})$. The second part of Theorem 2.3 now follows from Lemma 2.4 below.

Q.E.D.

Lemma 2.4. *For every coisotropic subalgebra* $\mathfrak{q}$ *of* $\mathfrak{d}$, *one has*

$$\kappa_{\mathfrak{q}}(R_{\mathfrak{u},\mathfrak{u}'}) \in \left(\kappa_{\mathfrak{q}}(\wedge^2 \mathfrak{u})\right) \cap \left(\kappa_{\mathfrak{q}}(\wedge^2 \mathfrak{u}')\right).$$

Proof. It is sufficient to show that $\kappa_{\mathfrak{q}}(R_{\mathfrak{u},\mathfrak{u}'}) \in \kappa_{\mathfrak{q}}(\wedge^2 \mathfrak{u})$. Let $\{x_1, x_2, \cdots, x_l\}$ be a basis for $\mathfrak{u} \cap \mathfrak{q}$. Extend it to a basis $\{x_1, x_2, \cdots, x_l, x_{l+1}, \cdots, x_n\}$ of $\mathfrak{u}$. Let $\{\xi_1, \cdots, \xi_n\}$ be the dual basis of $\mathfrak{u}'$ with respect to $\langle\, , \rangle$. It is easy to see that $(\mathfrak{u} \cap \mathfrak{q})^0 = \mathrm{Span}\{\xi_{l+1}, \xi_{l+1}, \cdots, \xi_n\} = p_{\mathfrak{u}'}(\mathfrak{q}^{\perp})$, recall (2.2), where $p_{\mathfrak{u}'} : \mathfrak{d} \to \mathfrak{u}'$ is the projection along $\mathfrak{u}$. Choose $y_j \in \mathfrak{u}$ such that $y_j + \xi_j \in \mathfrak{q}^{\perp}$ for $l+1 \le j \le n$ and write

$$R_{\mathfrak{u},\mathfrak{u}'} = \frac{1}{2}\sum_{j=1}^{l} \xi_j \wedge x_j + \frac{1}{2}\sum_{j=l+1}^{n} (y_j + \xi_j) \wedge x_j - \frac{1}{2}\sum_{j=l+1}^{n} y_j \wedge x_j.$$

Since $\kappa_{\mathfrak{q}}(x_j) = 0$ for $1 \le j \le l$ and $\kappa_{\mathfrak{q}}(y_j + \xi_j) = 0$ for $l+1 \le j \le n$, we have

$$\kappa_{\mathfrak{q}}(R_{\mathfrak{u},\mathfrak{u}'}) = -\frac{1}{2}\sum_{j=l+1}^{n} \kappa_{\mathfrak{q}}(y_j) \wedge \kappa_{\mathfrak{q}}(x_j) \in \kappa_{\mathfrak{q}}(\wedge^2 \mathfrak{u}). \tag{2.10}$$

Q.E.D.

In the setting of Theorem 2.3, the action map

$$(D, \pi^D_{\mathfrak{u},\mathfrak{u}'}) \times (D/Q,\, \Pi_{\mathfrak{u},\mathfrak{u}'}) \longrightarrow (D/Q,\, \Pi_{\mathfrak{u},\mathfrak{u}'})$$

is easily seen to be Poisson. Thus $(D/Q, \Pi_{\mathfrak{u},\mathfrak{u}'})$ is a Poisson homogeneous space [5] of $(D, \pi^D_{\mathfrak{u},\mathfrak{u}'})$. By part 2) of Theorem 2.3, each U and U'-orbit in D/Q is a Poisson homogeneous space of (U, π_U) and $(U', -\pi_{U'})$, respectively.

2.3. **Rank of the Poisson structure $\Pi_{\mathfrak{u},\mathfrak{u}'}$ on D/Q.** For $d \in D$ set $\underline{d} = dQ \in D/Q$. Consider the U-orbit $U.\underline{d}$ through $\underline{d}$. Then $(U.\underline{d}, \Pi_{\mathfrak{u},\mathfrak{u}'})$ is a Poisson homogeneous space of (U, π_U). Denote by $\mathfrak{l}_{\underline{d}}$ the *Drinfeld Lagrangian subalgebra* of $\mathfrak{d}$, associated to the base point $\underline{d}$ of $U.\underline{d}$, cf. [5]. It is defined as follows: identify $T_{\underline{d}}(U.\underline{d}) \cong \mathfrak{u}/(\mathfrak{u} \cap \mathrm{Ad}_d q)$ and regard $\Pi_{\mathfrak{u},\mathfrak{u}'}(\underline{d})$ as an element in $\wedge^2(\mathfrak{u}/(\mathfrak{u} \cap \mathrm{Ad}_d q))$. For $\xi \in (\mathfrak{u} \cap \mathrm{Ad}_d q)^0$, recall (2.2), let $\iota_\xi \Pi_{\mathfrak{u},\mathfrak{u}'}(\underline{d}) \in \mathfrak{u}/(\mathfrak{u} \cap \mathrm{Ad}_d q)$ be such that $\langle \iota_\xi \Pi_{\mathfrak{u},\mathfrak{u}'}(\underline{d}),\ \eta \rangle = \Pi_{\mathfrak{u},\mathfrak{u}'}(\underline{d})(\xi, \eta)$ for all $\eta \in (\mathfrak{u} \cap \mathrm{Ad}_d q)^0$. Then $\mathfrak{l}_{\underline{d}} \subset \mathfrak{d}$ is given by

$$\mathfrak{l}_{\underline{d}} = \{x + \xi \mid x \in \mathfrak{u}, \xi \in (\mathfrak{u} \cap \mathrm{Ad}_d q)^0,\ \iota_\xi \Pi_{\mathfrak{u},\mathfrak{u}'}(\underline{d}) = x + \mathfrak{u} \cap \mathrm{Ad}_d q\}.$$

If $\mathrm{Rank}_{\Pi_{\mathfrak{u},\mathfrak{u}'}}(\underline{d})$ denotes the rank of $\Pi_{\mathfrak{u},\mathfrak{u}'}$ at $\underline{d}$, it is easy to see from the definition of $\mathfrak{l}_{\underline{d}}$ that

$$\mathrm{Rank}_{\Pi_{\mathfrak{u},\mathfrak{u}'}}(\underline{d}) = \dim(U.\underline{d}) - \dim(\mathfrak{u}' \cap \mathfrak{l}_{\underline{d}}). \tag{2.11}$$

Proposition 2.5. *For any $d \in D$, the Drinfeld Lagrangian subalgebra $\mathfrak{l}_{\underline{d}}$ is*

$$\mathfrak{l}_{\underline{d}} = \mathrm{Ad}_d \mathfrak{q}^\perp + \mathfrak{u} \cap \mathrm{Ad}_d q. \tag{2.12}$$

Proof. Since the stabilizer subalgebra of $\mathfrak{d}$ at $\underline{d}$ is $\mathrm{Ad}_d \mathfrak{q}$, it is enough to prove that $\mathfrak{l}_{\underline{e}} = \mathfrak{q}^\perp + \mathfrak{u} \cap \mathfrak{q}$, where e is the identity element of D. Note that since $\mathfrak{q}^\perp \subset \mathfrak{q}$,

$$(\mathfrak{q}^\perp + \mathfrak{u} \cap \mathfrak{q})^\perp = \mathfrak{q} \cap (\mathfrak{u} \cap \mathfrak{q})^\perp = \mathfrak{q} \cap (\mathfrak{u} + \mathfrak{q}^\perp) = \mathfrak{u} \cap \mathfrak{q} + \mathfrak{q}^\perp.$$

Thus $\mathfrak{q}^\perp + \mathfrak{u} \cap \mathfrak{q}$ is a Lagrangian subspace of $\mathfrak{d}$. Since $\mathfrak{l}_{\underline{e}}$ is Lagrangian in $\mathfrak{d}$ and $\mathfrak{u} \cap \mathfrak{q} \subset \mathfrak{l}_{\underline{e}}$, it is sufficient to show that $\mathfrak{q}^\perp \subset \mathfrak{l}_{\underline{e}}$.

For $1 \leq i \leq n$ and $l+1 \leq j \leq n$, let $x_i \in \mathfrak{u}, \xi_i \in \mathfrak{u}'$ and $y_j \in \mathfrak{u}$ be as in the proof of Lemma 2.4. If $y + \xi \in \mathfrak{q}^\perp$, for some $y \in \mathfrak{u}, \xi \in \mathfrak{u}'$, then $\xi = \sum_{j=l+1}^n \lambda_j \xi_j$. Thus $y + \xi - \sum_{j=l+1}^n \lambda_j (y_j + \xi_j) \in \mathfrak{u} \cap \mathfrak{q}^\perp \subset \mathfrak{l}$. The proposition will now follow if we show that $y_j + \xi_j \in \mathfrak{l}_{\underline{e}}$ for every $l+1 \leq j \leq n$. By (2.10)

$$\Pi_{\mathfrak{u},\mathfrak{u}'}(\underline{e}) = -\frac{1}{2} \sum_{j=l+1}^n (y_j + \mathfrak{u} \cap \mathfrak{q}) \wedge (x_j + \mathfrak{u} \cap \mathfrak{q}) \in \wedge^2(\mathfrak{u}/(\mathfrak{u} \cap \mathfrak{q})) \cong \wedge^2 T_{\underline{e}} D/Q.$$

Thus for each $l+1 \leq j \leq n$,

$$\iota_{\xi_j} \Pi_{\mathfrak{u},\mathfrak{u}'}(\underline{e}) = \frac{1}{2} y_j - \frac{1}{2} \sum_{k=l+1}^n \langle \xi_j, y_k \rangle x_k + \mathfrak{u} \cap \mathfrak{q}.$$

Since $0 = \langle y_j + \xi_j, y_k + \xi_k \rangle = \langle \xi_j, y_k \rangle + \langle \xi_k, y_j \rangle$ for $l+1 \leq k \leq n$ and since $x_j \in \mathfrak{u} \cap \mathfrak{q}$ for $1 \leq j \leq l$, we have

$$\begin{aligned}\iota_{\xi_j} \Pi_{\mathfrak{u},\mathfrak{u}'}(\underline{e}) &= \frac{1}{2} y_j + \frac{1}{2} \sum_{k=l+1}^n \langle \xi_k, y_j \rangle x_k + \mathfrak{u} \cap \mathfrak{q} = \frac{1}{2} y_j + \frac{1}{2} \sum_{k=1}^n \langle \xi_k, y_j \rangle x_k + \mathfrak{u} \cap \mathfrak{q} \\ &= \frac{1}{2} y_j + \frac{1}{2} y_j + \mathfrak{u} \cap \mathfrak{q} = y_j + \mathfrak{u} \cap \mathfrak{q},\end{aligned}$$

we see that $y_j + \xi_j \in \mathfrak{l}_{\underline{e}}$.

Q.E.D.

Since $\mathfrak{u}+\mathfrak{u}' = \mathfrak{d}$, U-orbits and U'-orbits in D/Q intersect transversally. By Theorem 2.3, any such non-empty intersection is a Poisson submanifold of $\Pi_{\mathfrak{u},\mathfrak{u}'}$. The following corollary gives the corank of $\Pi_{\mathfrak{u},\mathfrak{u}'}$ in $U.\underline{d} \cap U'.\underline{d}$ at $\underline{d}$ for every $d \in D$.

Corollary 2.6. *For any $d \in D$,*

$$\mathrm{Rank}_{\Pi_{\mathfrak{u},\mathfrak{u}'}}(\underline{d}) = \dim(U.\underline{d} \cap U'.\underline{d}) + \dim(D/Q) - \dim(U'.\underline{d}) - \dim(\mathfrak{u}' \cap \mathfrak{l}_{\underline{d}}),$$

where $\mathfrak{l}_{\underline{d}}$ is the Drinfeld Lagrangian subalgebra given by (2.12).

Proof. The statement follows immediately from (2.11) and the fact that

$$\dim(U.\underline{d}) + \dim(U'.\underline{d}) - \dim(D/Q) = \dim(U.\underline{d} \cap U'.\underline{d}).$$

Q.E.D.

2.4. **First main theorem.** Recall that a manifold with a Poisson structure of constant rank is called a regular Poisson manifold.

Theorem 2.7. *If $\mathfrak{q}$ is a coisotropic subalgebra of $\mathfrak{d}$ such that $[\mathfrak{q},\mathfrak{q}] \subset \mathfrak{q}^{\perp}$, then for any closed subgroup Q of D with Lie algebra $\mathfrak{q}$ and for any Lagrangian splitting $\mathfrak{d} = \mathfrak{u}+\mathfrak{u}'$ of $\mathfrak{d}$, the intersection of any U-orbit with any U'-orbit in D/Q is a regular Poisson submanifold for the Poisson structure $\Pi_{\mathfrak{u},\mathfrak{u}'}$.*

Proof. Let again e be the identity in D. Since $[\mathrm{Ad}_d\mathfrak{q}, \mathrm{Ad}_d\mathfrak{q}] \subset (\mathrm{Ad}_d\mathfrak{q})^{\perp}$ for any $d \in D$, it suffices to show that $U.\underline{e} \cap U'.\underline{e}$ is a regular Poisson manifold of $\Pi_{\mathfrak{u},\mathfrak{u}'}$. Let $a \in U$ and $a' \in U'$ be such that $\underline{a} = \underline{a}' \in U.\underline{e} \cap U'.\underline{e}$. Then there exists $b \in Q$ such that $a = a'b$. Thus

$$\begin{aligned}\dim(\mathfrak{u}' \cap \mathfrak{l}_{\underline{a}}) &= \dim(\mathfrak{u}' \cap \mathrm{Ad}_a(\mathfrak{q}^{\perp} + \mathfrak{u}\cap\mathfrak{q})) = \dim(\mathfrak{u}' \cap \mathrm{Ad}_{a'b}(\mathfrak{q}^{\perp} + \mathfrak{u}\cap\mathfrak{q})) \\ &= \dim(\mathfrak{u}' \cap \mathrm{Ad}_b(\mathfrak{q}^{\perp} + \mathfrak{u}\cap\mathfrak{q})).\end{aligned}$$

Since $[\mathfrak{q},\mathfrak{q}] \subset \mathfrak{q}^{\perp}$, we have $[\mathfrak{q}, \mathfrak{q}^{\perp}+\mathfrak{u}\cap\mathfrak{q}] \subset [\mathfrak{q},\mathfrak{q}] \subset \mathfrak{q}^{\perp} \subset \mathfrak{q}^{\perp}+\mathfrak{u}\cap\mathfrak{q}$. Thus Q normalizes $\mathfrak{q}^{\perp}+\mathfrak{u}\cap\mathfrak{q}$, and $\mathrm{Ad}_b(\mathfrak{q}^{\perp}+\mathfrak{u}\cap\mathfrak{q}) = \mathfrak{q}^{\perp}+\mathfrak{u}\cap\mathfrak{q}$. Hence $\dim(\mathfrak{u}' \cap \mathfrak{l}_{\underline{a}}) = \dim(\mathfrak{u}' \cap \mathfrak{l}_{\underline{e}})$. It follows from Corollary 2.6 that the rank of $\Pi_{\mathfrak{u},\mathfrak{u}'}$ at $\underline{a}$ is the same as that at $\underline{e}$.

Q.E.D.

Remark 2.8. Note that if $\mathfrak{q} \subset \mathfrak{d}$ is a coisotropic subalgebra such that $\mathfrak{n}(\mathfrak{q}) = \mathfrak{q}$, where $\mathfrak{n}(\mathfrak{q})$ is the normalizer of $\mathfrak{q}$ in $\mathfrak{d}$, then $[\mathfrak{q},\mathfrak{q}] \supset \mathfrak{q}^{\perp}$. Indeed, if $x \in [\mathfrak{q},\mathfrak{q}]^{\perp}$, then $\langle x, [\mathfrak{q},\mathfrak{q}]\rangle = 0$ which implies that $\langle [x,\mathfrak{q}], \mathfrak{q}\rangle = 0$, so $[x,\mathfrak{q}] \subset \mathfrak{q}^{\perp} \subset \mathfrak{q}$. Thus $x \in \mathfrak{n}(\mathfrak{q}) = \mathfrak{q}$. This shows that $[\mathfrak{q},\mathfrak{q}]^{\perp} \subset \mathfrak{q}$, so $[\mathfrak{q},\mathfrak{q}] \supset \mathfrak{q}^{\perp}$. We thus conclude that, if $\mathfrak{q}$ is coisotropic such that $\mathfrak{n}(\mathfrak{q}) = \mathfrak{q}$ and $[\mathfrak{q},\mathfrak{q}] \subset \mathfrak{q}^{\perp}$, then $[\mathfrak{q},\mathfrak{q}] = \mathfrak{q}^{\perp}$.

This remark will be used in §3.2.

Corollary 2.9. *Let L be a closed subgroup of D whose Lie algebra $\mathfrak{l} \subset \mathfrak{d}$ is Lagrangian. Then for any Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$, symplectic leaves of $\Pi_{\mathfrak{u},\mathfrak{u}'}$ in D/L are precisely the connected components of the intersections of U and U'-orbits in D/L.*

Proof. Again it is enough to prove that $U.\underline{e} \cap U'.\underline{e}$ is symplectic. By Corollary 2.6 and Theorem 2.7, the corank of $\Pi_{\mathfrak{u},\mathfrak{u}'}$ in $U.\underline{e} \cap U'.\underline{e}$ is equal to

$$\dim(U'.\underline{e}) + \dim(\mathfrak{u}' \cap \mathfrak{l}) - \dim(D/L) = 0.$$

Q.E.D.

Remark 2.10. Corollary 2.9 describes the symplectic leaves for a large class of Poisson homogeneous spaces. Indeed, by [5], every Poisson homogeneous space of (U, π_U) is of the form U/H, where H is a subgroup of U whose Lie algebra is $\mathfrak{u} \cap \mathfrak{l}$ for a Lagrangian subalgebra $\mathfrak{l}$ of $\mathfrak{d}$. In the case when $H = U \cap L$, where L is a closed subgroup of D with Lie algebra $\mathfrak{l}$, we can embed U/H into D/L as the U-orbit through $\underline{e}$. This embedding is also Poisson. Thus the symplectic leaves of U/H are the connected components of the intersections of U/H with the U'-orbits in D/L, i.e. with the $(U', -\pi_{U'})$ Poisson homogeneous spaces inside $(D/L, \Pi_{\mathfrak{u},\mathfrak{u}'})$.

We conclude this subsection with an example showing that the statement of Theorem 2.7 is incorrect if the condition $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{q}^\perp$ is dropped.

Example 2.11. Let G a connected complex simple Lie group with a pair of opposite Borel subgroups B and B^-. Set $\mathfrak{g} = \mathrm{Lie}\, G$, $T = B \cap B^-$, and $\mathfrak{h} = \mathrm{Lie}\, T$. Then $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h}$ is a quadratic Lie algebra with the bilinear form

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \ll x_1, y_1 \gg - \ll x_2, y_2 \gg, \qquad x_1, y_1, \in \mathfrak{g}, x_2, y_2 \in \mathfrak{h}$$

where $\ll .,. \gg$ is a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$. Let $D = G \times T$. Given a parabolic subgroup $P \supset B$ of G, the Lie algebra $\mathfrak{q}$ of $Q = P \times T$ is coisotropic but does not satisfy the condition $\mathfrak{q}^\perp \subset [\mathfrak{q}, \mathfrak{q}]$. The following subalgebras provide a Lagrangian splitting of $\mathfrak{g} \oplus \mathfrak{h}$:

$$\mathfrak{u} = \{(x + h, h) \mid x \in \mathfrak{n},\ h \in \mathfrak{h}\}, \qquad \mathfrak{u}' = \{(x + h, -h) \mid x \in \mathfrak{n}^-,\ h \in \mathfrak{h}\},$$

where $\mathfrak{n}$ and $\mathfrak{n}^-$ are the nilpotent radicals of $\mathrm{Lie}\, B$ and $\mathrm{Lie}\, B^-$. Under the identification $(G \times T)/(P \times T) \cong G/P$ the Poisson structure $\Pi_{\mathfrak{u},\mathfrak{u}'}$ corresponds to the Poisson structure

$$\pi = \kappa \left(\sum_{\alpha \in \Delta^+} f_\alpha \wedge e_\alpha \right),$$

where, Δ^+ is the set of positive roots of $\mathfrak{g}$ corresponding to $\mathfrak{n}$, $\{e_\alpha\}$ and $\{f_\alpha\}$ are sets of root vectors of $\mathfrak{g}$, normalized by $\ll e_\alpha, f_\alpha \gg = 1$, and κ is the extension to $\wedge^2 \mathfrak{g}$ of the infinitesimal action of $\mathfrak{g}$ on G/P. It was shown in [9] that the partition of

$(G/P, \pi)$ by T-orbits of leaves (which is a partition by regular Poisson submanifolds) coincides with Lusztig's partition [16] of G/P. The strata of this partition are

$$\mathrm{pr}_P \left(Bw_1.B \cap B^- w_2.B \right), \quad w_1 \in W, w_2 \in W^{W_P}_{\max}. \tag{2.13}$$

Here $\mathrm{pr}_P : G/B \to G/P$ denotes the standard projection, W the Weyl group of (G, T), and $W^{W_P}_{\max}$ the set of the maximal length representatives of cosets in W/W_P where, W_P is the parabolic subgroup of W corresponding to P.

Under the identification $(G \times T)/(P \times T) \cong G/P$ the U and U'-orbits on $(G \times T)/(P \times T)$ correspond respectively to the B and B^--orbits on G/P. The coarser partition of (2.13) by intersecting B and B^--orbits on $(G/P, \pi)$ is no longer a partition by regular Poisson submanifolds if $P \neq B$ or G. This is easily seen by applying Theorem 4.10 below and the Poisson embedding [9, (1.10)].

2.5. **Intersections of $N(\mathfrak{u})$ and $N(\mathfrak{u}')$-orbits.** For a Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$, let $N(\mathfrak{u})$ and $N(\mathfrak{u}')$ be the normalizer subgroups of $\mathfrak{u}$ and $\mathfrak{u}'$ in D, respectively. Both $N(\mathfrak{u})$ and $N(\mathfrak{u}')$ are closed subgroups of D, and sometimes $N(\mathfrak{u})$ and $N(\mathfrak{u}')$-orbits in a space D/Q are easier to determine than the U and U'-orbits. This is the case for the examples considered in §4. In this subsection, we prove some facts on $N(\mathfrak{u})$ and $N(\mathfrak{u}')$-orbits.

It is clear from Theorem 2.3 that for any closed subgroup Q of D with coisotropic Lie subalgebra $\mathfrak{q}$ and for any Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$, all $N(\mathfrak{u})$ and $N(\mathfrak{u}')$-orbits in D/Q are complete Poisson submanifolds with respect to the Poisson structure $\Pi_{\mathfrak{u},\mathfrak{u}'}$, cf. Theorem 2.3. Recall the Poisson structure $\pi^D_{\mathfrak{u},\mathfrak{u}'}$ on D from (2.7).

Lemma 2.12. *For any Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$, the Poisson structure $\pi^D_{\mathfrak{u},\mathfrak{u}'}$ on D vanishes at all points in $N(\mathfrak{u}) \cap N(\mathfrak{u}')$. Consequently, for any closed subgroup Q of D with coisotropic Lie subalgebra $\mathfrak{q}$, $N(\mathfrak{u}) \cap N(\mathfrak{u}')$ leaves the Poisson structure $\Pi_{\mathfrak{u},\mathfrak{u}'}$ on D/Q invariant.*

Proof. Let $d \in N(\mathfrak{u}) \cap N(\mathfrak{u}')$. If $\{x_1, \ldots, x_n\}$ and $\{\xi_1, \ldots, \xi_n\}$ is a pair of dual bases for $\mathfrak{u}$ and $\mathfrak{u}'$ with respect to $\langle \, , \, \rangle$, then so are $\{\mathrm{Ad}_d(x_1), \ldots \mathrm{Ad}_d(x_n)\}$ and $\{\mathrm{Ad}_d(\xi_1), \ldots, \mathrm{Ad}_d(x_n)\}$. Thus

$$\pi^D_{\mathfrak{u},\mathfrak{u}'}(d) = L_d(r_{\mathfrak{u},\mathfrak{u}'}) - R_d(r_{\mathfrak{u},\mathfrak{u}'}) = R_d \left(\sum_{i=1}^n \mathrm{Ad}_d(\xi_i) \wedge \mathrm{Ad}_d(x_i) - \sum_{i=1}^n \xi_i \wedge x_i \right) = 0.$$

Since the $(D, \pi^D_{\mathfrak{u},\mathfrak{u}'})$-action on $(D/Q, \Pi_{\mathfrak{u},\mathfrak{u}'})$ is Poisson, $N(\mathfrak{u}) \cap N(\mathfrak{u}')$ leaves $\Pi_{\mathfrak{u},\mathfrak{u}'}$ invariant.

Q.E.D.

Next we generalize Theorem 2.7 to intersections of arbitrary $N(\mathfrak{u})$-orbits and $N(\mathfrak{u}')$-orbits in D/Q. Its proof is similar to the one of Theorem 2.7 and is left to the reader.

Proposition 2.13. *If $\mathfrak{q}$ is a coisotropic subalgebra of $\mathfrak{d}$ such that $[\mathfrak{q},\mathfrak{q}] \subset \mathfrak{q}^{\perp}$, then for any closed subgroup Q with Lie algebra $\mathfrak{q}$ and for any Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$ of $\mathfrak{d}$, the intersection of any $N(\mathfrak{u})$-orbit with any $N(\mathfrak{u}')$-orbit in D/Q is a regular Poisson submanifold for the Poisson structure $\Pi_{\mathfrak{u},\mathfrak{u}'}$.*

Although Proposition 2.13 provides a stronger result than Theorem 2.7, it is apriori possible that the geometry of the strata of the coarser partition from Proposition 2.13 is more complicated than that of the strata of the finer partition from Theorem 2.7. The next result, Proposition 2.15, shows that this is not the case. First we prove an auxiliary lemma.

Lemma 2.14. *Let $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$ be any Lagrangian splitting of $\mathfrak{d}$. Assume that $N(\mathfrak{u})$ is connected. Then $N(\mathfrak{u}) = (U' \cap N(\mathfrak{u}))^o U$, where $(U' \cap N(\mathfrak{u}))^o$ denotes the identity component of the group $U' \cap N(\mathfrak{u})$. Moreover, $N(\mathfrak{u})$ is a Poisson Lie subgroup of (D, π).*

Proof. This is because $(U' \cap N(\mathfrak{u}))^o U$ is a connected subgroup of D with Lie algebra $\mathfrak{u}' \cap \mathfrak{n}(\mathfrak{u}) + \mathfrak{u}$ which is equal to $\mathfrak{n}(\mathfrak{u})$ because $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$.

Q.E.D.

Proposition 2.15. *Let $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$ be any Lagrangian splitting of $\mathfrak{d}$, and assume that $N(\mathfrak{u})$ and $N(\mathfrak{u}')$ are both connected. Let X be any Poisson space with a Poisson $(D, \pi^D_{\mathfrak{u},\mathfrak{u}'})$-action. Let $x \in X$ such that $N(\mathfrak{u})x \cap N(\mathfrak{u}')x \neq \emptyset$. Then the group $N(\mathfrak{u}) \cap N(\mathfrak{u}')$ acts transitively on the set of intersections of U-orbits and U'-orbits in $N(\mathfrak{u})x \cap N(\mathfrak{u}')x$.*

Proof. Using Lemma 2.14 we obtain

$$\begin{aligned} N(\mathfrak{u})x \cap N(\mathfrak{u}')x &= \bigcup_{\alpha \in (U' \cap N(\mathfrak{u}))^o, \beta \in (U \cap N(\mathfrak{u}'))^o} (\alpha U x) \cap (U' \beta x) \\ &= \bigcup_{\alpha \in (U' \cap N(\mathfrak{u}))^o, \beta \in (U \cap N(\mathfrak{u}'))^o} \alpha (U x \cap U' \beta x) \\ &= \bigcup_{\alpha \in (U' \cap N(\mathfrak{u}))^o, \beta \in (U \cap N(\mathfrak{u}'))^o} \alpha \beta (U x \cap U' x). \end{aligned}$$

Q.E.D.

We finish this section with a formula for the corank of $\Pi_{\mathfrak{u},\mathfrak{u}'}$ to be used in §4. As before $Q \subset D$ is assumed to be a closed subgroup with coisotropic Lie subalgebra $\mathfrak{q}$. For any $d \in D$, let

$$\mathrm{Corank}_{\Pi_{\mathfrak{u},\mathfrak{u}'}}(\underline{d}) = \dim(N(\mathfrak{u}).\underline{d} \cap N(\mathfrak{u}')'\!.\underline{d}) - \mathrm{Rank}_{\Pi_{\mathfrak{u},\mathfrak{u}'}}(\underline{d})$$

be the corank of $\Pi_{\mathfrak{u},\mathfrak{u}'}$ in $N(\mathfrak{u}).\underline{d} \cap N(\mathfrak{u}').\underline{d}$ at $\underline{d} \in D/Q$.

Lemma 2.16. *In the above setting, for any $d \in D$,*

$$\begin{aligned}\mathrm{Corank}_{\Pi_{\mathfrak{u},\mathfrak{u}'}}(\underline{d}) = &\dim \mathfrak{n}(\mathfrak{u}') - \dim(D/Q) + \dim \mathfrak{n}(\mathfrak{u}) - \dim \mathfrak{u} \\ &- \dim(\mathfrak{n}(\mathfrak{u}) \cap \mathrm{Ad}_d \mathfrak{q}) + \dim(\mathfrak{u} \cap \mathrm{Ad}_d \mathfrak{q}) \\ &- \dim(\mathfrak{n}(\mathfrak{u}') \cap \mathrm{Ad}_d \mathfrak{q}) + \dim(\mathfrak{u}' \cap \mathfrak{l}_{\underline{d}}),\end{aligned}$$

where $\mathfrak{l}_{\underline{d}} = \mathrm{Ad}_d \mathfrak{q}^\perp + \mathfrak{u} \cap \mathrm{Ad}_d q$ *is the Drinfeld Lagrangian subalgebra as in* (2.12).

Proof. Since $N(\mathfrak{u}).\underline{d}$ and $N(\mathfrak{u}').\underline{d}$ intersect transversally,

$$\begin{aligned}\dim(N(\mathfrak{u}).\underline{d} \cap N(\mathfrak{u}')'\underline{d}) &= \dim(N(\mathfrak{u}).\underline{d}) + \dim(N(\mathfrak{u}').\underline{d}) - \dim(D/Q) \\ &= \dim \mathfrak{n}(\mathfrak{u}) - \dim(\mathfrak{n}(\mathfrak{u}) \cap \mathrm{Ad}_d \mathfrak{q}) \\ &\quad + \dim \mathfrak{n}(\mathfrak{u}') - \dim(\mathfrak{n}(\mathfrak{u}') \cap \mathrm{Ad}_d \mathfrak{q}) - \dim(D/Q).\end{aligned}$$

By (2.11), $\mathrm{Rank}_{\Pi_{\mathfrak{u},\mathfrak{u}'}}(\mathfrak{d}) = \dim \mathfrak{u} - \dim(\mathfrak{u} \cap \mathrm{Ad}_d \mathfrak{q}) - \dim(\mathfrak{u}' \cap \mathfrak{l}_{\underline{d}})$. The formula for $\mathrm{Corank}_{\Pi_{\mathfrak{u},\mathfrak{u}'}}(\underline{d})$ in Lemma 2.16 thus follows.

Q.E.D.

3. The variety of Lagrangian subalgebras associated to a reductive Lie algebra

3.1. **General case.** Let $(\mathfrak{d}, \langle\, , \rangle)$ be a $2n$-dimensional quadratic Lie algebra and let D be a connected Lie group with Lie algebra $\mathfrak{d}$. We will denote by $\mathcal{L}(\mathfrak{d})$ the variety of all Lagrangian subalgebras of $\mathfrak{d}$. It is an algebraic subvariety of the Grassmannian $\mathrm{Gr}(n, \mathfrak{d})$ of n-dimensional subspaces of $\mathfrak{d}$. The group D acts on $\mathcal{L}(\mathfrak{d})$ through the adjoint action.

Fix a Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$, recall $R_{\mathfrak{u},\mathfrak{u}'} \in \wedge^2 \mathfrak{d}$ given by (2.5). Let again $\kappa : \mathfrak{d} \to \chi^1(\mathcal{L}(\mathfrak{d}))$ be the Lie algebra anti-homomorphism from $\mathfrak{d}$ to the Lie algebra of vector fields on $\mathcal{L}(\mathfrak{d})$, and define the bi-vector field $\Pi_{\mathfrak{u},\mathfrak{u}'} = \kappa(R_{\mathfrak{u},\mathfrak{u}'})$ on $\mathcal{L}(\mathfrak{d})$. For $l \in \mathcal{L}(\mathfrak{d})$, let $N(\mathfrak{l})$ and $\mathfrak{n}(\mathfrak{l})$ be respectively the normalizer subgroup of $\mathfrak{l}$ in D and the normalizer subalgebra of $\mathfrak{l}$ in $\mathfrak{d}$. Then the D-orbit in $\mathcal{L}(\mathfrak{d})$ through $\mathfrak{l}$ is isomorphic to $D/N(\mathfrak{l})$. Clearly, $\mathfrak{n}(\mathfrak{l})$ is coisotropic in $\mathfrak{d}$ because it contains $\mathfrak{l}$. Thus it follows from Theorem 2.3 that $\Pi_{\mathfrak{u},\mathfrak{u}'}$ is a Poisson structure on $\mathcal{L}(\mathfrak{d})$, see also [7]. The following proposition now follows immediately from Theorem 2.7.

Proposition 3.1. *Assume that $(\mathfrak{d}, \langle\, , \rangle)$ is an even dimensional quadratic Lie algebra and D is a connected Lie group with Lie algebra $\mathfrak{d}$ such that for every $\mathfrak{l} \in \mathcal{L}(\mathfrak{d})$*

$$[\mathfrak{n}(\mathfrak{l}), \mathfrak{n}(\mathfrak{l})] \subset (\mathfrak{n}(\mathfrak{l}))^\perp.$$

Then for any Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$, the intersection of an $N(\mathfrak{u})$-orbit and an $N(\mathfrak{u}')$-orbit in $\mathcal{L}(\mathfrak{d})$ is a regular Poisson submanifold for the Poisson structure $\Pi_{\mathfrak{u},\mathfrak{u}'}$.

3.2. **Second main theorem: the case of a reductive Lie algebra.** When $(\mathfrak{d}, \langle\, , \,\rangle)$ is a reductive quadratic Lie algebra, we have the following second main theorem of the paper.

Theorem 3.2. *If D is a connected complex or real reductive Lie group and $\langle\, , \,\rangle$ is a nondegenerate symmetric invariant bilinear form on $\mathfrak{d} = \mathrm{Lie}\, D$, then for any Lagrangian splitting $\mathfrak{d} = \mathfrak{u} + \mathfrak{u}'$, the intersection of any $N(\mathfrak{u})$-orbit and any $N(\mathfrak{u}')$-orbit in $\mathcal{L}(\mathfrak{d})$ is a regular Poisson submanifold for the Poisson structure $\Pi_{\mathfrak{u},\mathfrak{u}'}$ on $\mathcal{L}(\mathfrak{d})$.*

To prove Theorem 3.2 we need to check that in the setting of Theorem 3.2 the condition of Proposition 3.1 is satisfied. In fact, we prove a stronger statement.

Proposition 3.3. *If $\mathfrak{d}$ is an even dimensional complex or real reductive Lie algebra and $\langle\, , \,\rangle$ is a nondegenerate symmetric invariant bilinear form on $\mathfrak{d}$, then for all Lagrangian subalgebras $\mathfrak{l}$ of $(\mathfrak{d}, \langle\, , \,\rangle)$, $[\mathfrak{n}(\mathfrak{l}), \mathfrak{n}(\mathfrak{l})] = (\mathfrak{n}(\mathfrak{l}))^{\perp}$.*

The real case in Proposition 3.3 follows from the complex one. Indeed, let $(\mathfrak{d}, \langle\, , \,\rangle)$ be a quadratic real reductive Lie algebra. Then $(\mathfrak{d}_{\mathbb{C}}, \langle\, , \,\rangle_{\mathbb{C}})$ is a quadratic complex reductive Lie algebra. Let $\mathfrak{l}$ be a Lagrangian subalgebra of $(\mathfrak{d}, \langle\, , \,\rangle)$, and let $\mathfrak{n}(\mathfrak{l}_{\mathbb{C}})$ be the normalizer subalgebra of $\mathfrak{l}_{\mathbb{C}}$ in $\mathfrak{d}_{\mathbb{C}}$. Then $\mathfrak{n}(\mathfrak{l}_{\mathbb{C}}) = (\mathfrak{n}(\mathfrak{l}))_{\mathbb{C}}$. Assume the validity of Proposition 3.3 in the complex case. We get

$$\begin{aligned}[\mathfrak{n}(\mathfrak{l}), \mathfrak{n}(\mathfrak{l})] &= [\mathfrak{n}(\mathfrak{l}_{\mathbb{C}}), \mathfrak{n}(\mathfrak{l}_{\mathbb{C}})] \cap \mathfrak{d} = (\mathfrak{n}(\mathfrak{l}_{\mathbb{C}}))^{\perp} \cap \mathfrak{d} \\ &= ((\mathfrak{n}(\mathfrak{l}))_{\mathbb{C}})^{\perp} \cap \mathfrak{d} = (\mathfrak{n}(\mathfrak{l}))^{\perp}\end{aligned}$$

where $(.)^{\perp}$ denotes orthogonal complements in $\mathfrak{d}$ and $\mathfrak{d}_{\mathbb{C}}$. This proves the real case in Proposition 3.3.

To obtain the complex case in Proposition 3.3 we need the following result of Delorme [4].

Theorem 3.4. *[Delorme] Assume that $(\mathfrak{d}, \langle\, , \,\rangle)$ is an even dimensional reductive quadratic Lie algebra. For each Lagrangian subalgebra $\mathfrak{l}$ of $(\mathfrak{d}, \langle\, , \,\rangle)$ the normalizer of the nilpotent radical $\mathfrak{n}$ of $\mathfrak{l}$ is a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{d}$. In addition, $\mathfrak{p}$ has a Levi subalgebra $\mathfrak{m}$ whose derived subalgebra $\bar{\mathfrak{m}} = [\mathfrak{m}, \mathfrak{m}]$ decomposes as $\bar{\mathfrak{m}} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ and for which there exists an isomorphism $\theta\colon \mathfrak{m}_1 \to \mathfrak{m}_2$. If $\mathfrak{z}$ denotes the center of $\mathfrak{m}$ then*

$$\bar{\mathfrak{m}}^{\theta} + \mathfrak{n} \subset \mathfrak{l} \subset \left(\bar{\mathfrak{m}}^{\theta} \oplus \mathfrak{z}\right) + \mathfrak{n} \tag{3.1}$$

where $\bar{\mathfrak{m}}^{\theta} = \{x + \theta(x) \mid x \in \mathfrak{m}_1\} \subset \mathfrak{m}_1 \oplus \mathfrak{m}_2$.

Proof of Proposition 3.3 in the complex case. Let $\mathfrak{l}$ be a Lagrangian subalgebra of $(\mathfrak{d}, \langle\, , \,\rangle)$ as in Theorem 3.4.

First we claim that $\mathfrak{n}(\mathfrak{l}) \subset \mathfrak{p}$. Indeed, the normalizer of $\mathfrak{l}$ lies inside the normalizer of the nilpotent radical $\mathfrak{n}$ of $\mathfrak{l}$ which is $\mathfrak{p}$: if $y \in \mathfrak{n}(\mathfrak{l})$, then for small t, $\exp(t\,\mathrm{ad}_y)$

is an automorphism of $\mathfrak{l}$ and thus of its nilpotent radical $\mathfrak{n}$. Taking derivative at $t = 0$, we get that y normalizes $\mathfrak{n}$.

Next we will show that

$$\mathfrak{n}(\mathfrak{l}) = \left(\bar{\mathfrak{m}}^\theta \oplus \mathfrak{z}\right) + \mathfrak{n}. \tag{3.2}$$

The inclusion $\mathfrak{n}(\mathfrak{l}) \supset \left(\bar{\mathfrak{m}}^\theta \oplus \mathfrak{z}\right) + \mathfrak{n}$ is clear from (3.1). Define the subspace

$$\bar{\mathfrak{m}}^- = \{x - \theta(x) \mid x \in \mathfrak{m}_1\} \subset \bar{\mathfrak{m}}.$$

Under the adjoint action of $\bar{\mathfrak{m}}^\theta$ we have the direct sum decomposition of $\bar{\mathfrak{m}}^\theta$-modules

$$\mathfrak{p} = \bar{\mathfrak{m}}^\theta \oplus \bar{\mathfrak{m}}^- \oplus \mathfrak{z} \oplus \mathfrak{n}.$$

If $\mathfrak{n}(\mathfrak{l}) \neq \left(\bar{\mathfrak{m}}^\theta \oplus \mathfrak{z}\right) + \mathfrak{n}$, then there exists a nonzero $Y = y - \theta(y) \in \bar{\mathfrak{m}}^-$ which belongs to $\mathfrak{n}(\mathfrak{l})$. Since $\bar{\mathfrak{m}}^\theta$ normalizes $\bar{\mathfrak{m}}^-$ we get that $\mathrm{ad}_Y(\bar{\mathfrak{m}}^\theta) = 0$ and thus $\mathrm{ad}_y(\mathfrak{m}_1) = 0$. This is a contradiction since $\mathfrak{m}_1$ is semi-simple and $y \neq 0$. This completes the proof of (3.2).

Repeating the proof with $\mathfrak{n}(\mathfrak{l})$ in the place of $\mathfrak{l}$, which also satisfies (3.1) as shown above, we get that $\mathfrak{n}(\mathfrak{l})$ coincides with its normalizer.

Now (3.2) implies $[\mathfrak{n}(\mathfrak{l}), \mathfrak{n}(\mathfrak{l})] \subset \bar{\mathfrak{m}}^\theta + \mathfrak{n}$. Because $\mathfrak{l}$ is a Lagrangian subalgebra of $(\mathfrak{d}, \langle\, , \,\rangle)$, it is clear that $\bar{\mathfrak{m}}^\theta + \mathfrak{n} \subset \mathfrak{n}(\mathfrak{l})^\perp$. Thus, $[\mathfrak{n}(\mathfrak{l}), \mathfrak{n}(\mathfrak{l})] \subset \mathfrak{n}(\mathfrak{l})^\perp$. By Remark 2.8, $[\mathfrak{n}(\mathfrak{l}), \mathfrak{n}(\mathfrak{l})] = \mathfrak{n}(\mathfrak{l})^\perp$.

Q.E.D.

4. Ranks of Poisson structures on the variety $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$

4.1. **The quadratic Lie algebra $(\mathfrak{g} \oplus \mathfrak{g}, \langle\, , \,\rangle)$.** Assume that $\mathfrak{g}$ is a complex semi-simple Lie algebra and $\ll\, , \gg$ is a fixed nondegenerate invariant symmetric bilinear form whose restriction to a compact real form of $\mathfrak{g}$ is negative definite. Let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ be the direct sum Lie algebra and let $\langle\, , \,\rangle$ be the bilinear form on $\mathfrak{d}$ given by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \ll x_1, y_1 \gg - \ll x_2, y_2 \gg, \quad x_1, x_2, y_1, y_2 \in \mathfrak{g}. \tag{4.1}$$

In this section, we will study in more detail the Poisson structure $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ on $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ defined by an arbitrary Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$.

A classification of Lagrangian subalgebras of $\mathfrak{g} \oplus \mathfrak{g}$ was first obtained by Karolinsky [10]. It also follows from the more general results of Delorme [4], where Lagrangian splittings of an arbitrary reductive quadratic Lie algebras were classified. We will recall Delorme's classification in §4.2. Let G be the adjoint group of $\mathfrak{g}$. For $\mathfrak{l} \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ denote by $N(\mathfrak{l})$ the normalizer subgroup of $\mathfrak{l}$ in $G \times G$. Let $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ be an arbitrary Lagrangian splitting. By Theorem 3.2 the intersection of any $N(\mathfrak{l}_1)$-orbit and any $N(\mathfrak{l}_2)$-orbit in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ is a regular Poisson submanifold for the Poisson structure $\Pi_{\mathfrak{l}_1, \mathfrak{l}_1}$. Using results from [14], we will describe the $N(\mathfrak{l}_1)$ and $N(\mathfrak{l}_2)$-orbits in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ and will obtain an explicit formula for the rank of $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ at an arbitrary $\mathfrak{l} \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$.

4.2. **Lagrangian splittings of $(\mathfrak{g} \oplus \mathfrak{g}, \langle\, , \,\rangle)$.** Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a choice Δ^+ of positive roots in the set Δ of all roots for $(\mathfrak{g}, \mathfrak{h})$. Let Γ be the set of simple roots in Δ^+. For each $\alpha \in \Delta$, let $H_\alpha \in \mathfrak{h}$ be such that $\ll x, H_\alpha \gg = \alpha(x)$ for all $x \in \mathfrak{h}$. We will also fix a root vector E_α for each $\alpha \in \Delta$ such that $[E_\alpha, E_{-\alpha}] = H_\alpha$. Following [20, 8], we define a *generalized Belavin–Drinfeld (gBD) triple* to be a triple (S, T, d), where S and T are subsets of Γ and $d : S \to T$ is a bijection such that $\ll H_{d\alpha}, H_{d\beta} \gg = \ll H_\alpha, H_\beta \gg$ for all $\alpha \in S$.

For a subset S of Γ, let Δ_S be the set of roots in the linear span of S. Set

$$\mathfrak{m}_S = \mathfrak{h} + \sum\nolimits_{\alpha \in \Delta_S} \mathfrak{g}_\alpha, \quad \mathfrak{n}_S = \sum\nolimits_{\alpha \in \Delta^+ - \Delta_S} \mathfrak{g}_\alpha, \quad \mathfrak{n}_S^- = \sum_{\alpha \in \Delta^+ - \Delta_S} \mathfrak{g}_{-\alpha}$$

and $\mathfrak{p}_S = \mathfrak{m}_S + \mathfrak{n}_S$ and $\mathfrak{p}_S^- = \mathfrak{m}_S + \mathfrak{n}_S^-$. We set $\bar{\mathfrak{m}}_S = [\mathfrak{m}_S, \mathfrak{m}_S]$ and

$$\mathfrak{h}_S = \mathfrak{h} \cap \bar{\mathfrak{m}}_S = \mathrm{Span}_{\mathbb{C}}\{H_\alpha : \alpha \in \Delta_S\}, \qquad \mathfrak{z}_S = \{x \in \mathfrak{h} \mid \alpha(x) = 0, \ \forall \alpha \in S\}.$$

Then we have the decompositions $\mathfrak{h} = \mathfrak{z}_S + \mathfrak{h}_S$, $\mathfrak{m}_S = \mathfrak{z}_S + \bar{\mathfrak{m}}_S$ and

$$\mathfrak{p}_S = \mathfrak{z}_S + \bar{\mathfrak{m}}_S + \mathfrak{n}_S, \qquad \mathfrak{p}_S^- = \mathfrak{z}_S + \bar{\mathfrak{m}}_S + \mathfrak{n}_S^-.$$

Recall that G denotes the adjoint group of $\mathfrak{g}$. The connected subgroups of G with Lie algebras $\mathfrak{p}_S$, $\mathfrak{p}_S^-$, $\mathfrak{m}_S$, $\mathfrak{n}_S$ and $\mathfrak{n}_S^-$ will be respectively denoted by P_S, P_S^-, M_S, N_S and N_S^-. Correspondingly we have the Levi decompositions $P_S = M_S N_S$, $P_S^- = M_S N_S^-$. Let Z_S be the center of M_S, and let $\chi_S : P_S \to M_S/Z_S$ be the natural projection by first projecting to M_S along N_S and then to M_S/Z_S. We also denote by χ_S the similar projection from P_S^- to M_S/Z_S.

For a generalized Belavin–Drinfeld triple (S, T, d), let $\mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$ be the set of all Lagrangian subspaces of $\mathfrak{z}_S \oplus \mathfrak{z}_T$ with respect to the (nondegenerate) restriction of $\langle\, , \,\rangle$ to $\mathfrak{z}_S \oplus \mathfrak{z}_T$. Let $\theta_d : \bar{\mathfrak{m}}_S \to \bar{\mathfrak{m}}_T$ be the unique Lie algebra isomorphism satisfying

$$\theta_d(H_\alpha) = H_{d\alpha}, \quad \theta_d(E_\alpha) = E_{d\alpha}, \quad \forall \alpha \in S.$$

For every $V \in \mathcal{L}_{space}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$, define

$$\mathfrak{l}_{S,T,d,V} = V + \{(x, \theta_d(x)) \mid x \in \bar{\mathfrak{m}}_S\} + (\mathfrak{n}_S \oplus \mathfrak{n}_T) \subset \mathfrak{p}_S \oplus \mathfrak{p}_T, \tag{4.2}$$

$$\mathfrak{l}'_{S,T,d,V} = V + \{(x, \theta_d(x)) \mid x \in \bar{\mathfrak{m}}_S\} + (\mathfrak{n}_S \oplus \mathfrak{n}_T^-) \subset \mathfrak{p}_S \oplus \mathfrak{p}_T^-, \tag{4.3}$$

$$\mathfrak{l}''_{S,T,d,V} = V + \{(x, \theta_d(x)) \mid x \in \bar{\mathfrak{m}}_S\} + (\mathfrak{n}_S^- \oplus \mathfrak{n}_T) \subset \mathfrak{p}_S^- \oplus \mathfrak{p}_T. \tag{4.4}$$

It is easy to see that $\mathfrak{l}_{S,T,d,V}$, $\mathfrak{l}'_{S,T,d,V}$, and $\mathfrak{l}''_{S,T,d,V}$ are all in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$. The subalgebras $\mathfrak{l}'_{S,T,d,V}$ and $\mathfrak{l}''_{S,T,d,V}$ are of course conjugate to ones of the type $\mathfrak{l}_{S,T,d,V}$. Indeed, let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$, and let w_0 be the longest element in W. For $A \subset \Gamma$, let W_A be the subgroup of W generated by simple reflections with respect to roots in A, and let $x_A = w_0 w_{0,A}$ where $w_{0,A}$ denotes the longest element of W_A. Then it is easy to see that

$$\mathfrak{l}'_{S,T,d,V} = \mathrm{Ad}^{-1}_{(e,\dot{x}_T)} \mathfrak{l}_{S,-w_0(T),\, x_T d, \mathrm{Ad}_{(e,\dot{x}_T)} V}, \tag{4.5}$$

$$\mathfrak{l}''_{S,T,d,V} = \mathrm{Ad}^{-1}_{(\dot{x}_S,e)} \mathfrak{l}_{-w_0(S),T,\, d x_S^{-1}, \mathrm{Ad}_{(\dot{x}_S,e)} V}, \tag{4.6}$$

where $\dot{x}_T$ and $\dot{x}_S$ are representatives in G of x_T and x_S respectively.

Denote also by θ_d the (unique) group isomorphism $M_S/Z_S \to M_T/Z_T$ induced by $\theta_d : \bar{\mathfrak{m}}_S \to \bar{\mathfrak{m}}_T$. Corresponding to the subalgebras in (4.2)-(4.4), we define

$$R_{S,T,d} = \{(p_1,p_2) \in P_S \times P_T \mid \theta_d(\chi_S(p_1)) = \chi_T(p_2)\} \subset P_S \times P_T, \tag{4.7}$$

$$R'_{S,T,d} = \{(p_1,p_2) \in P_S \times P_T^- \mid \theta_d(\chi_S(p_1)) = \chi_T(p_2)\} \subset P_S \times P_T^-, \tag{4.8}$$

$$R''_{S,T,d} = \{(p_1,p_2) \in P_S^- \times P_T \mid \theta_d(\chi_S(p_1)) = \chi_T(p_2)\} \subset P_S^- \times P_T. \tag{4.9}$$

One knows that $R_{S,T,d}$, $R'_{S,T,d}$, and $R''_{S,T,d}$ are all connected [8, Lemma 2.19]. Corresponding to (4.5) and (4.6), we have

$$R'_{S,T,d} = \mathrm{Ad}^{-1}_{(e,\dot{x}_T)} R_{S,-w_0(T),\, x_T d}, \tag{4.10}$$

$$R''_{S,T,d} = \mathrm{Ad}^{-1}_{(\dot{x}_S,e)} R_{-w_0(S),T,\, d x_S^{-1}}. \tag{4.11}$$

The Lie algebras of $R_{S,T,d}, R'_{S,T,d}$, and $R''_{S,T,d}$ will be denoted by $\mathfrak{r}_{S,T,d}, \mathfrak{r}'_{S,T,d}$, and $\mathfrak{r}''_{S,T,d}$ respectively.

Proposition 4.1. [8] *Every $(G \times G)$-orbit in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ passes through an $\mathfrak{l}_{S,T,d,V}$ for a unique generalized Belavin–Drinfeld triple (S,T,d) and a unique $V \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$. The normalizer subgroup of $\mathfrak{l}_{S,T,d,V}$ in $G \times G$ is $R_{S,T,d}$.*

Definition 4.2. For generalized Belavin-Drinfeld triples (S_i, T_i, d_i), $i = 1,2$, let

$$S_2^{d_1^{-1}d_2} = \{\alpha \in S_2 \mid \ (d_1^{-1}d_2)^n \alpha \text{ is defined and is in } S_2 \text{ for } n = 1,2,\cdots\}.$$

A **generalized Belavin–Drinfeld system** is a pair of quadruples (S_1, T_1, d_1, V_1) and (S_2, T_2, d_2, V_2), where for $i = 1,2$, (S_i, T_i, d_i) is a generalized Belavin–Drinfeld triple and $V_i \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_{S_i} \oplus \mathfrak{z}_{T_i})$, such that

1) $S_2^{d_1^{-1}d_2} = \emptyset$;
2) $\mathfrak{h}_1 \cap \mathfrak{h}_2 = \{0\}$, where $\mathfrak{h}_i = V_i + \{(x, \theta_d(x)) \mid x \in \mathfrak{h}_{S_i}\} \subset \mathfrak{h} \oplus \mathfrak{h}$ for $i = 1,2$.

Theorem 4.3. [4, Delorme] *Every Lagrangian splitting of $\mathfrak{g} \oplus \mathfrak{g}$ is conjugate by an element in $G \times G$ to one of the form $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$, where*

$$\mathfrak{l}_1 = \mathfrak{l}'_{S_1,T_1,d_1,V_1} \quad \textit{and} \quad \mathfrak{l}_2 = \mathfrak{l}''_{S_2,T_2,d_2,V_2} \tag{4.12}$$

for a generalized Belavin–Drinfeld system (S_1, T_1, d_1, V_1), (S_2, T_2, d_2, V_2).

Let $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ be a Lagrangian splitting with $\mathfrak{l}_1$ and $\mathfrak{l}_2$ given in (4.12). By Theorem 3.2, any non-empty intersection of an $N(\mathfrak{l}_1)$ and an $N(\mathfrak{l}_2)$-orbit in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ is a regular Poisson subvariety for the Poisson structure $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$. A classification of $N(\mathfrak{l}_1)$ and $N(\mathfrak{l}_2)$-orbits will be given in § 4.3. We now prove that every non-empty intersection of an $N(\mathfrak{l}_1)$-orbit and an $N(\mathfrak{l}_2)$-orbit in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ is smooth and irreducible. Let H be the connected subgroup of G with Lie algebra $\mathfrak{h}$.

Proposition 4.4. *For a Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ with $\mathfrak{l}_1$ and $\mathfrak{l}_2$ given by* (4.12)*, $N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2)$ is a subtorus of $H \times H$ of dimension* $\dim \mathfrak{z}_{S_1} + \dim \mathfrak{z}_{S_2}$. *In particular, $N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2)$ is connected.*

Proof. It follows from Proposition 4.1 that

$$N(\mathfrak{l}_1) = R'_{S_1,T_1,d_1} \quad \text{and} \quad N(\mathfrak{l}_2) = R''_{S_2,T_2,d_2}. \tag{4.13}$$

For notational simplicity, let $x_1 = x_{T_1}, x_2 = x_{S_2}$. By (4.10) and (4.11),

$$N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2) = \left(\mathrm{Ad}^{-1}_{(e,\dot{x}_1)} R_{S_1,-w_0(T_1),x_1d_1}\right) \cap \left(\mathrm{Ad}^{-1}_{(\dot{x}_2,e)} R_{-w_0(S_2),\,T_2,\,d_2x_2^{-1}}\right).$$

Since $x_2^{-1} \in W^{-w_0(S_2)}$ and $x_1 \in {}^{-w_0(T_1)}W$, we can use [14, Theorem 2.5] to determine $N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2)$. Set $N = N_\emptyset$. Since $S_2^{d_1^{-1}d_2} = \emptyset$, [14, Theorem 2.5] implies that $N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2) \subset B \times B$ and $N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2) = (N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2))^{\mathrm{red}} (N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2))^{\mathrm{uni}}$, where

$$(N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2))^{\mathrm{red}} = N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2) \cap (H \times H),$$
$$(N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2))^{\mathrm{uni}} = N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2) \cap (N \times N).$$

Moreover, [14, Theorem 2.5] also tells us that

$$(N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2))^{\mathrm{uni}} \cong (N \cap \mathrm{Ad}^{-1}_{\dot{x}_2}(N_{-w_0(S_2)})) \times (N \cap \mathrm{Ad}^{-1}_{\dot{x}_1}(N_{-w_0(T_1)})).$$

It is easy to see that $(N \cap \mathrm{Ad}^{-1}_{\dot{x}_2}(N_{-w_0(S_2)})) \times (N \cap \mathrm{Ad}^{-1}_{\dot{x}_1}(N_{-w_0(T_1)}))$ is the trivial group. Thus $N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2) = N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2) \cap (H \times H)$ consists of all $(h_1, h_2) \in H \times H$ such that

$$\begin{cases} \theta_{d_1}\chi_{S_1}(h_1) = \chi_{T_1}(h_2), \\ \theta_{d_2}\chi_{S_2}(h_1) = \chi_{T_2}(h_2), \end{cases}$$

which are equivalent to

$$h_1^\alpha = h_2^{d_1\alpha}, \ \ \forall \alpha \in S_1 \quad \text{and} \quad h_1^\beta = h_2^{d_2\beta}, \ \ \forall \beta \in S_2. \tag{4.14}$$

Let $\Gamma = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ be the set of simple roots of $\mathfrak{g}$. Since G is the adjoint group of $\mathfrak{g}$, we can identify $H \times H$ with the torus $(\mathbb{C}^\times)^{2r}$ by the map

$$H \times H \longrightarrow (\mathbb{C}^\times)^{2r}: \quad (h_1, h_2) \longmapsto (h_1^{\alpha_1},\, h_1^{\alpha_2},\, \cdots,\, h_1^{\alpha_r},\, h_2^{\alpha_1},\, h_2^{\alpha_2},\, \cdots,\, h_2^{\alpha_r}).$$

The conditions in (4.14) imply that the coordinates h_1^α of h_1 for $\alpha \in S_1 \cup S_2$ are expressed in terms of coordinates of h_2, and we have the extra conditions

$$h_2^{d_1\alpha} = h_2^{d_2\alpha}, \quad \alpha \in S_1 \cap S_2 \tag{4.15}$$

for the coordinates of h_2. To understand the conditions in (4.15), recall that $S_2^{d_1^{-1}d_2} = \emptyset$. Thus for every $\alpha \in S_1 \cap S_2$, there is a unique integer $n \geq 1$ and unique elements $\alpha^{(0)} = \alpha$, $\alpha^{(1)}$, $\alpha^{(2)}$, $\cdots$, $\alpha^{(n-1)} \in S_1 \cap S_2$ such that

$$d_2\alpha^{(0)} = d_1\alpha^{(1)}, \;\; d_2\alpha^{(1)} = d_1\alpha^{(2)}, \;\; \cdots \;\; d_2\alpha^{(n-2)} = d_1\alpha^{(n-1)}$$

and either $d_2\alpha^{(n-1)} \notin T_1$ or $d_2\alpha^{(n-1)} = d_1\alpha^{(n)}$ for some $\alpha^{(n)} \in S_1$ but $\alpha^{(n)} \notin S_2$. Then the conditions in (4.15) are equivalent to

$$h_2^{d_1\alpha} = h_2^{d_2\alpha} = h_2^{d_2\alpha^{(1)}} = \cdots = h_2^{d_2\alpha^{(n-1)}}.$$

Since $d_2\alpha^{(n-1)} \notin d_1(S_1 \cap S_2)$, we see that the conditions (4.15) express $h_2^{d_1\alpha}$ for every $\alpha \in S_1 \cap S_2$ in terms of h_2^{β} for some $\beta \notin d_1(S_1 \cap S_2)$. We conclude that the set of $(h_1, h_2) \in H \times H$ satisfying (4.14) is a subtorus of $H \times H$ with dimension equal to

$$2\dim H - |S_1 \cup S_2| - |S_1 \cap S_2| = 2\dim H - |S_1| - |S_2| = \dim \mathfrak{z}_{S_1} + \dim \mathfrak{z}_{S_2}.$$

Q.E.D.

Corollary 4.5. *For any Lagrangian splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$, all $N(\mathfrak{l}_1)$-orbits and $N(\mathfrak{l}_2)$-orbits in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ intersection transversally, and every such non-empty intersection is smooth and irreducible.*

Proof. Clearly $\mathfrak{n}(\mathfrak{l}_1) + \mathfrak{n}(\mathfrak{l}_2) = \mathfrak{g} \oplus \mathfrak{g}$, where $\mathfrak{n}(\mathfrak{l}_i)$ is the Lie algebra of $N(\mathfrak{l}_i)$ for $i = 1, 2$. Since $N(\mathfrak{l}_1) \cap N(\mathfrak{l}_2)$ is connected, Corollary 4.5 follows from [19, Corollary 1.5].

Q.E.D.

4.3. $N(\mathfrak{l}_1)$ **and** $N(\mathfrak{l}_2)$**-orbits in** $\mathcal{L}(\mathfrak{g}\oplus\mathfrak{g})$**.** Assume that $\mathfrak{g}\oplus\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ is a Lagrangian splitting with $\mathfrak{l}_1$ and $\mathfrak{l}_2$ given in (4.12). We now use results in [14] to describe $N(\mathfrak{l}_1)$ and $N(\mathfrak{l}_2)$-orbits in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$.

For $A \subset \Gamma$, let W^A and ${}^AW$ be respectively the sets of minimal length representatives in the cosets in W/W_A and $W_A \backslash W$. For each $w \in W$, we fix a representative $\dot{w}$ of w in the normalizer of H in G.

Proposition 4.6. *1) Every $N(\mathfrak{l}_1) = R'_{S_1,T_1,d_1}$-orbit in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ is of the form*

$$R'_{S_1,T_1,d_1}\mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{l}_{S,T,d,V}$$

for a unique generalized Belavin-Drinfeld triple (S, T, d), a unique $V \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$, a unique pair $(v_1, v_2) \in W^S \times {}^{T_1}W$, and some $m_2 \in M_{T(v_1,v_2)}$ with

$$T(v_1, v_2) = \{\alpha \in T \mid (v_2^{-1}d_1v_1d^{-1})^n\alpha \textit{ is defined and is in } T \textit{ for } n = 1, 2, \cdots\}.$$

2) Every $N(\mathfrak{l}_2) = R''_{S_2,T_2,d_2}$-orbit in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ is of the form

$$R''_{S_2,T_2,d_2}\mathrm{Ad}_{(\dot{w}_1 m_1,\dot{w}_2)}\mathfrak{l}_{S,T,d,V}$$

for a unique generalized Belavin-Drinfeld triple (S, T, d), *a unique* $V \in \mathcal{L}_{\text{space}}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$, *a unique pair* $(w_1, w_2) \in {}^{S_2}W \times W^T$, *and some* $m_1 \in M_{S(w_1,w_2)}$ *with*

$$S(w_1, w_2) = \{\alpha \in S \mid (w_1^{-1}d_2^{-1}w_2 d)^n\alpha \text{ is defined and is in } S \text{ for } n = 1, 2, \cdots\}.$$

Proof. In [14] we gave a description of the $(R_{S_1,T_1,d_1}, R_{S,T,d})$-double cosets in $G \times G$. The first part follows from (4.10), [14, Theorem 2.2], and the fact that ${}^{-w_0(A)}W = x_A {}^AW$, $\forall A \subset \Gamma$, see §4.2 for the definition of x_A.

Let $\sigma : G \times G \to G \times G : (g_1, g_2) \mapsto (g_2, g_1)$. Then using first part and the facts that

$$R''_{S_2,T_2,d_2} = \sigma\left(R'_{T_2,S_2,d_2^{-1}}\right) \quad \text{and} \quad R_{S,T,d} = \sigma(R_{T,S,d^{-1}})$$

we get part 2).

Q.E.D.

Let $\mathcal{O}_1$ be an $N(\mathfrak{l}_1) = R'_{S_1,T_1,d_1}$-orbit and $\mathcal{O}_2$ an $N(\mathfrak{l}_2) = R''_{S_2,T_2,d_2}$-orbit in $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$. By Proposition 4.6, we can assume that

$$\mathcal{O}_1 = R'_{S_1,T_1,d_1}\mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{l}_{S,T,d,V}, \tag{4.16}$$

$$\mathcal{O}_2 = R''_{S_2,T_2,d_2}\mathrm{Ad}_{(\dot{w}_1 m_1,\dot{w}_2)}\mathfrak{l}_{S,T,d,V}, \tag{4.17}$$

where (S, T, d, V), v_1, v_2, w_1, w_2 and m_1 and m_2 are as in Proposition 4.6. Let

$$S_1(v_1, v_2) = d_1^{-1}v_2T(v_1, v_2) = v_1 d^{-1}T(v_1, v_2) \subset S_1 \tag{4.18}$$

$$S_2(w_1, w_2) = w_1S(w_1, w_2) = d_2^{-1}w_2 dS(w_1, w_2) \subset S_2. \tag{4.19}$$

In other words, $S_1(v_1, v_2)$ is the largest subset of S_1 that is invariant under the partial map $v_1d^{-1}v_2^{-1}d_1 : \Gamma \to \Gamma$, and $S_2(w_1, w_2)$ is the largest subset of S_2 that is invariant under the partial map $w_1d^{-1}w_2^{-1}d_2 : \Gamma \to \Gamma$.

In order to compute the rank of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ at an $\mathfrak{l} \in \mathcal{O}_1 \cap \mathcal{O}_2$ using Lemma 2.16, we need to compute the dimensions of various intersections of subalgebras in $\mathfrak{n}(\mathfrak{l}_1)$, $\mathfrak{n}(\mathfrak{l}_2)$ and $\mathfrak{n}(\mathfrak{l})$. Such intersections can be described using the following Proposition 4.8 which is derived from [14, Theorem 2.5].

Recall that for $\mathfrak{q} \subset \mathfrak{g} \oplus \mathfrak{g}$, $\mathfrak{q}^\perp = \{x \in \mathfrak{g} \oplus \mathfrak{g} \mid \langle x, y\rangle = 0, \ \forall y \in \mathfrak{q}\}$. Clearly

$$\mathfrak{r}^\perp_{S,T,d} = \mathfrak{n}_S \oplus \mathfrak{n}_T + \{(x, \theta_d(x)) \mid x \in \bar{\mathfrak{m}}_S\},$$
$$\mathfrak{r}'^{,\perp}_{S_1,T_1,d_1} = \mathfrak{n}_{S_1} \oplus \mathfrak{n}^-_{T_1} + \{(x, \theta_{d_1}(x)) \mid x \in \bar{\mathfrak{m}}_{S_1}\},$$
$$\mathfrak{r}''^{,\perp}_{S_2,T_2,d_2} = \mathfrak{n}^-_{S_2} \oplus \mathfrak{n}_{T_2} + \{(x, \theta_{d_2}(x)) \mid x \in \bar{\mathfrak{m}}_{S_2}\}.$$

Notation 4.7. Let $\mathcal{S}$ (reps. $\mathcal{S}'$, $\mathcal{S}''$) be the set of all subspaces of $\mathfrak{r}_{S,T,d}$ (resp. $\mathfrak{r}'_{S_1,T_1,d_1}$, $\mathfrak{r}''_{S_2,T_2,d_2}$) that contain $\mathfrak{r}^\perp_{S,T,d}$ (resp. $\mathfrak{r}'^{,\perp}_{S_1,T_1,d_1}$, $\mathfrak{r}''^{,\perp}_{S_2,T_2,d_2}$). For $\mathfrak{a} \in \mathcal{S}$, let $V_\mathfrak{a} = \mathfrak{a} \cap (\mathfrak{z}_S \oplus \mathfrak{z}_T)$ and

$$X'_\mathfrak{a} = V_\mathfrak{a} + \{(x, \theta_d(x)) \mid x \in \mathfrak{z}_{d^{-1}T(v_1,v_2)} \cap \mathfrak{h}_S\},$$
$$X''_\mathfrak{a} = V_\mathfrak{a} + \{(x, \theta_d(x)) \mid x \in \mathfrak{z}_{S(w_1,w_2)} \cap \mathfrak{h}_S\}.$$

For $\mathfrak{a}' \in \mathcal{S}'$ and $\mathfrak{a}'' \in \mathcal{S}''$, let

$$V_{\mathfrak{a}'} = \mathfrak{a}' \cap (\mathfrak{z}_{S_1} \oplus \mathfrak{z}_{T_1}) \quad \text{and} \quad Y_{\mathfrak{a}'} = V_{\mathfrak{a}'} + \{(x, \theta_{d_1}(x)) \mid x \in \mathfrak{z}_{S_1(v_1,v_2)} \cap \mathfrak{h}_{S_1}\},$$
$$V_{\mathfrak{a}''} = \mathfrak{a}'' \cap (\mathfrak{z}_{S_2} \oplus \mathfrak{z}_{T_2}) \quad \text{and} \quad Y_{\mathfrak{a}''} = V_{\mathfrak{a}''} + \{(x, \theta_{d_2}(x)) \mid x \in \mathfrak{z}_{S_2(w_1,w_2)} \cap \mathfrak{h}_{S_2}\}.$$

We also set

$$\mathfrak{f}_1 = \mathfrak{r}'^{,\perp}_{S_1,T_1,d_1} \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)} \mathfrak{r}^{\perp}_{S,T,d} \cap (\bar{\mathfrak{m}}_{S_1(v_1,v_2)} \oplus \bar{\mathfrak{m}}_{d_1 S_1(v_1,v_2)}). \tag{4.20}$$

Proposition 4.8. *For any $\mathfrak{a}' \in \mathcal{S}_1$ and $\mathfrak{a} \in \mathcal{S}$, one has the direct sum*

$$\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a} = (\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a})^{\mathrm{red}} + (\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a})^{\mathrm{nil}},$$

where

$$\begin{aligned}(\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a})^{\mathrm{red}} &= \mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a} \cap (\mathfrak{m}_{S_1(v_1,v_2)} \oplus \mathfrak{m}_{d_1 S_1(v_1,v_2)}) \\ &= Y_{\mathfrak{a}'} \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2)} X'_{\mathfrak{a}} + \mathfrak{f}_1 \quad (\textit{direct sum}),\end{aligned}$$

and $(\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a})^{\mathrm{nil}} = \mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a} \cap (\mathfrak{n}_{S_1(v_1,v_2)} \oplus \mathfrak{n}^-_{d_1 S_1(v_1,v_2)})$. *The dimension of the latter is equal to* $l(v_2) + \dim(\mathfrak{n} \cap \mathrm{Ad}_{\dot{v}_1}(\mathfrak{n}_S))$.

Proof. Using (4.5) and [14, Theorem 2.5], one can see that

$$\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a} \subset \mathfrak{p}_{S_1(v_1,v_2)} \oplus \mathfrak{p}^-_{d_1 S_1(v_1,v_2)}$$

and that

$$\begin{aligned}\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a} &= (\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a})^{\mathrm{red}} + (\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a})^{\mathrm{nil}} \\ &= \mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a} \cap (\mathfrak{m}_{S_1(v_1,v_2)} \oplus \mathfrak{m}_{d_1 S_1(v_1,v_2)}) \\ &\quad + \mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a} \cap (\mathfrak{n}_{S_1(v_1,v_2)} \oplus \mathfrak{n}^-_{d_1 S_1(v_1,v_2)}).\end{aligned}$$

The dimension formula for $(\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a})^{\mathrm{nil}}$ also follows from [14, Theorem 2.5]. It now remains to show that $(\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a})^{\mathrm{red}} = Y_{\mathfrak{a}'} \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2)} X'_{\mathfrak{a}} + \mathfrak{f}_1$ as a direct sum. Clearly, $Y_{\mathfrak{a}'} \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2)} X'_{\mathfrak{a}}$ and $\mathfrak{f}_1$ intersect trivially, and their sum is contained in $(\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a})^{\mathrm{red}}$.

Suppose that $(x, y) \in (\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)}\mathfrak{a})^{\mathrm{red}}$. Write $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1 \in \mathfrak{z}_{S_1(v_1,v_2)}, x_2 \in \bar{\mathfrak{m}}_{S_1(v_1,v_2)}, y_1 \in \mathfrak{z}_{d_1 S_1(v_1,v_2)}$ and $y_1 \in \bar{\mathfrak{m}}_{d_1 S_1(v_1,v_2)}$. Because $(x, y) \in \mathfrak{a}' \subset \mathfrak{r}'_{S_1,T_1,d_1}$, we have $\theta_{d_1}\chi_{S_1}(x_1) + \theta_{d_1}(x_2) = \chi_{T_1}(y_1) + y_2$. It follows from the direct sum decomposition $\mathfrak{z}_{d_1 S_1(v_1,v_2)} = \mathfrak{z}_{T_1} + \mathfrak{z}_{d_1 S_1(v_1,v_2)} \cap \mathfrak{h}_{T_1}$ that $\chi_{T_1}(y_1) \in \mathfrak{z}_{S_1(v_1,v_2)} \cap \mathfrak{h}_{T_1}$. Similarly, $\chi_{S_1}(x_1) \in \mathfrak{z}_{S_1(v_1,v_2)} \cap \mathfrak{h}_{S_1}$. Thus $\theta_{d_1}\chi_{S_1}(x_1) = \chi_{T_1}(y_1)$ and $\theta_{d_1}(x_2) = y_2$. Hence $(x_2, y_2) \in \mathfrak{r}'^{,\perp}_{S_1,T_1,d_1} \subset \mathfrak{a}'$, and therefore $(x_1, y_1) \in (\mathfrak{z}_{S_1(v_1,v_2)} \oplus \mathfrak{z}_{d_1 S_1(v_1,v_2)}) \cap \mathfrak{a}' = Y_{\mathfrak{a}'}$. In the same way one shows that $(x_1, y_1) \in \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2)} X'_{\mathfrak{a}}$ and $(x_2, y_2) \in \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)} \mathfrak{r}^{\perp}_{S,T,d}$. Thus $(x_1, y_1) \in Y_{\mathfrak{a}'} \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2)} X'_{\mathfrak{a}}$ and $(x_2, y_2) \in \mathfrak{f}_1$.

Q.E.D.

Corollary 4.9. *For any* $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}$, $\mathfrak{a}', \mathfrak{b}' \in \mathcal{S}'$, *and* $\mathfrak{a}'', \mathfrak{b}'' \in \mathcal{S}''$,

$$\begin{aligned} \dim(\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2 m_2)} \mathfrak{a}) - \dim(\mathfrak{b}' \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2 m_2)} \mathfrak{b}) \\ = \dim(Y_{\mathfrak{a}'} \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2)} X'_{\mathfrak{a}}) - \dim(Y_{\mathfrak{b}'} \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2)} X'_{\mathfrak{b}}), \end{aligned} \tag{4.21}$$

$$\begin{aligned} \dim(\mathfrak{a}'' \cap \mathrm{Ad}_{(\dot{w}_1 m_1, \dot{w}_2)} \mathfrak{a}) - \dim(\mathfrak{b}'' \cap \mathrm{Ad}_{(\dot{w}_1 m_1, \dot{w}_2)} \mathfrak{b}) \\ = \dim(Y_{\mathfrak{a}''} \cap \mathrm{Ad}_{(\dot{w}_1, \dot{w}_2)} X''_{\mathfrak{a}}) - \dim(Y_{\mathfrak{b}''} \cap \mathrm{Ad}_{(\dot{w}_1, \dot{w}_2)} X''_{\mathfrak{b}}). \end{aligned} \tag{4.22}$$

Proof. Let $\mathfrak{f}_1$ be as in(4.20). We know from Proposition 4.8 that

$$(\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2 m_2)} \mathfrak{a})^{\mathrm{red}} = Y_{\mathfrak{a}'} \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2)} X'_{\mathfrak{a}} + \mathfrak{f}_1$$

is a direct sum. Replacing $\mathfrak{a}'$ by $\mathfrak{b}'$ and $\mathfrak{a}$ by $\mathfrak{b}$, we get

$$\begin{aligned} &\dim(\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2 m_2)} \mathfrak{a}) - \dim(\mathfrak{b}' \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2 m_2)} \mathfrak{b}) \\ &\quad = \dim(\mathfrak{a}' \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2 m_2)} \mathfrak{a})^{\mathrm{red}} - \dim(\mathfrak{b}' \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2 m_2)} \mathfrak{b})^{\mathrm{red}} \\ &\quad = \dim(Y_{\mathfrak{a}'} \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2)} X'_{\mathfrak{a}}) - \dim(Y_{\mathfrak{b}'} \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2)} X'_{\mathfrak{b}}). \end{aligned}$$

(4.22) is proved by using (4.21) and the map $\sigma : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g} : (x, y) \mapsto (y, x)$.

Q.E.D.

4.4. The rank of the Poisson structure $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ on $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$.

Theorem 4.10. *Let* $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ *be a Lagrangian splitting as in* (4.12) *and let* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *be respectively an* $N(\mathfrak{l}_1)$ *and an* $N(\mathfrak{l}_2)$*-orbit in* $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ *as in* (4.16) *and* (4.17)*. Then the corank of* $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ *in* $\mathcal{O}_1 \cap \mathcal{O}_2$ *is equal to*

$$\begin{aligned} \dim \mathfrak{z}_{S_1} + \dim \mathfrak{z}_{S_2} + \dim \mathfrak{z}_S - \dim(Y_1 \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2)} X_1) + \dim(Z_1 \cap \mathrm{Ad}_{(\dot{v}_1, \dot{v}_2)} X_1) \\ - \dim(Y_2 \cap \mathrm{Ad}_{(\dot{w}_1, \dot{w}_2)} X_2) + \dim(Z_2 \cap \mathrm{Ad}_{(\dot{w}_1, \dot{w}_2)} \widetilde{X}), \end{aligned}$$

where

$$\begin{aligned} X_1 &= (\mathfrak{z}_{d^{-1}T(v_1, v_2)} \oplus \mathfrak{z}_{T(v_1, v_2)}) \cap r_{S,T,d} \\ &= \mathfrak{z}_S \oplus \mathfrak{z}_T + \{(x, \theta_d(x)) \mid x \in \mathfrak{z}_{d^{-1}T(v_1, v_2)} \cap \mathfrak{h}_S\} \\ X_2 &= (\mathfrak{z}_{S(w_1, w_2)} \oplus \mathfrak{z}_{dS(w_1, w_2)}) \cap r_{S,T,d} \\ &= \mathfrak{z}_S \oplus \mathfrak{z}_T + \{(x, \theta_d(x)) \mid x \in \mathfrak{z}_{S(w_1, w_2)} \cap \mathfrak{h}_S\} \\ Y_1 &= (\mathfrak{z}_{S_1(v_1, v_2)} \oplus \mathfrak{z}_{d_1 S_1(v_1, v_2)}) \cap \mathfrak{r}'_{S_1, T_1, d_1} \\ &= \mathfrak{z}_{S_1} \oplus \mathfrak{z}_{T_1} + \{(x, \theta_{d_1}(x)) \mid x \in \mathfrak{z}_{S_1(v_1, v_2)} \cap \mathfrak{h}_{S_1}\} \\ Y_2 &= (\mathfrak{z}_{S_2(w_1, w_2)} \oplus \mathfrak{z}_{d_2 S_2(w_1, w_2)}) \cap \mathfrak{r}''_{S_2, T_2, d_2} \\ &= \mathfrak{z}_{S_2} \oplus \mathfrak{z}_{T_2} + \{(x, \theta_{d_2}(x)) \mid x \in \mathfrak{z}_{S_2(w_1, w_2)} \cap \mathfrak{h}_{S_2}\} \\ Z_1 &= (\mathfrak{z}_{S_1(v_1, v_2)} \oplus \mathfrak{z}_{d_1 S_1(v_1, v_2)}) \cap \mathfrak{l}'_{S_1, T_1, d_1, V_1} \\ &= V_1 + \{(x, \theta_{d_1}(x)) \mid x \in \mathfrak{z}_{S_1(v_1, v_2)} \cap \mathfrak{h}_{S_1}\} \\ Z_2 &= (\mathfrak{z}_{S_2(w_1, w_2)} \oplus \mathfrak{z}_{d_2 S_2(w_1, w_2)}) \cap \mathfrak{l}''_{S_2, T_2, d_2, V_2} \\ &= V_2 + \{(x, \theta_{d_2}(x)) \mid x \in \mathfrak{z}_{S_2(w_1, w_2)} \cap \mathfrak{h}_{S_2}\}, \end{aligned}$$

and $\tilde{X} = p(X_1 \cap \mathrm{Ad}^{-1}_{(\dot{v}_1,\dot{v}_2)} Z_1) + \{(x, \theta_d(x)) \mid x \in \mathfrak{z}_{S(w_1,w_2)} \cap \mathfrak{h}_S\}$ *with* $p : \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{z}_S \oplus \mathfrak{z}_T$ *being the projection along* $\mathfrak{h}_S \oplus \mathfrak{h}_T$*.*

Proof. Let $\mathfrak{l} \in \mathcal{O}_1 \cap \mathcal{O}_2$ be given by

$$\mathfrak{l} = \mathrm{Ad}_{(r'_1,r'_2)(\dot{v}_1,\dot{v}_2 m_2)} \mathfrak{l}_{S,T,d,V} = \mathrm{Ad}_{(r''_1,r''_2)(\dot{w}_1 m_1,\dot{w}_2)} \mathfrak{l}_{S,T,d,V}, \tag{4.23}$$

where $(r'_1, r'_2) \in N(\mathfrak{l}_1) = R'_{S_1,T_1,d_1}$ and $(r''_1, r''_2) \in N(\mathfrak{l}_2) = R''_{S_2,T_2,d_2}$. The formula for the corank of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ at $\mathfrak{l} \in \mathcal{O}_1 \cap \mathcal{O}_2$ involves the Drinfeld subalgebra $\mathcal{T}(\mathfrak{l})$ of $\mathfrak{g} \oplus \mathfrak{g}$ defined by $\mathcal{T}(\mathfrak{l}) = \mathfrak{n}(\mathfrak{l})^\perp + \mathfrak{l}_1 \cap \mathfrak{n}(\mathfrak{l})$, cf. Proposition 2.5, where again $\mathfrak{n}(\mathfrak{l})$ is the normalizer subalgebra of $\mathfrak{l}$ in $\mathfrak{g} \oplus \mathfrak{g}$. We first compute $\mathcal{T}(\mathfrak{l})$. Since

$$\mathrm{Ad}_{(r'_1,r'_2)(\dot{v}_1,\dot{v}_2 m_2)} \mathfrak{r}^\perp_{S,T,d} = \mathfrak{n}(\mathfrak{l})^\perp \subset \mathcal{T}(\mathfrak{l}) \subset \mathfrak{n}(\mathfrak{l}) = \mathrm{Ad}_{(r'_1,r'_2)(\dot{v}_1,\dot{v}_2 m_2)} \mathfrak{r}_{S,T,d},$$

we know that $\mathcal{T}(\mathfrak{l}) = \mathrm{Ad}_{(r'_1,r'_2)(\dot{v}_1,\dot{v}_2 m_2)} \mathfrak{l}_{S,T,d,\widetilde{V}}$ for some $\widetilde{V} \in \mathcal{L}(\mathfrak{z}_S \oplus \mathfrak{z}_T)$. On the other hand,

$$\mathcal{T}(\mathfrak{l}) = \mathrm{Ad}_{(r'_1,r'_2)(\dot{v}_1,\dot{v}_2 m_2)} \left(\mathfrak{r}^\perp_{S,T,d} + r_{S,T,d} \cap \mathrm{Ad}^{-1}_{(\dot{v}_1,\dot{v}_2 m_2)} \mathfrak{l}_1 \right).$$

Thus $\mathfrak{l}_{S,T,d,\widetilde{V}} = \mathfrak{r}^\perp_{S,T,d} + r_{S,T,d} \cap \mathrm{Ad}^{-1}_{(\dot{v}_1,\dot{v}_2 m_2)} \mathfrak{l}_1 = X_1 \cap \mathrm{Ad}^{-1}_{(\dot{v}_1,\dot{v}_2)} Z_1 + \mathfrak{r}^\perp_{S,T,d}$, where the second identity comes from Proposition 4.8. Hence

$$\widetilde{V} = p(X_1 \cap \mathrm{Ad}^{-1}_{(\dot{v}_1,\dot{v}_2)} Z_1).$$

Now by Lemma 2.16, the corank of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ in $\mathcal{O}_1 \cap \mathcal{O}_2$ at the Lagrangian subalgebra $\mathfrak{l}$ given by (4.23) is

$$\begin{aligned} \mathrm{Corank}_{\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}}(\mathfrak{l}) &= \dim \mathfrak{z}_{S_1} + \dim \mathfrak{z}_{S_2} + \dim \mathfrak{z}_S \\ &\quad - \dim(\mathfrak{r}'_{S_1,T_1,d_1} \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)} \mathfrak{r}_{S,T,d}) + \dim(\mathfrak{l}'_{S_1,T_1,d_1,V_1} \cap \mathrm{Ad}_{(\dot{v}_1,\dot{v}_2 m_2)} \mathfrak{r}_{S,T,d}) \\ &\quad - \dim(\mathfrak{r}''_{S_2,T_2,d_2} \cap \mathrm{Ad}_{(\dot{w}_1 m_1,\dot{w}_2)} \mathfrak{r}_{S,T,d}) + \dim(\mathfrak{l}''_{S_2,T_2,d_2} \cap \mathrm{Ad}_{(\dot{w}_1 m_1,\dot{w}_2)} \mathfrak{l}_{S,T,d,\widetilde{V}}). \end{aligned}$$

Applying Corollary 4.9, we get the desired formula for the corank of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ in $\mathcal{O}_1 \cap \mathcal{O}_2$. This completes the proof of Theorem 4.10.

Q.E.D.

Example 4.11. Let $\mathfrak{g}_{\mathrm{diag}} = \{(x, x) \mid x \in \mathfrak{g}\}$. A Lagrangian splitting of the form $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\mathrm{diag}} + \mathfrak{l}$, where $\mathfrak{l} \in \mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$, is called a Belavin-Drinfeld splitting. Let $G_{\mathrm{diag}} = \{(g, g) \mid g \in G\}$. It is shown in [1] (see also [8, Corollary 3.18]) that every Belavin-Drinfeld splitting of $\mathfrak{g} \oplus \mathfrak{g}$ is conjugate by an element in G_{diag} to a splitting of the form

$$\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\mathrm{diag}} + \mathfrak{l}''_{S_2,T_2,d_2,V_2}, \tag{4.24}$$

where (S_2, T_2, d_2) is a Belavin-Drinfeld triple in the sense that

$$S_2^{d_2} = \{\alpha \in S_2 \mid d_2^n \alpha \text{ is defined and is in } S_2 \text{ for } n = 1, 2, \cdots\} = \emptyset,$$

and $V_2 \in \mathcal{L}_{\mathrm{space}}(\mathfrak{z}_{S_2} \oplus \mathfrak{z}_{T_2})$ is such that $\mathfrak{h}_{\mathrm{diag}} \cap (V_2 + \{(x, \theta_{d_2}(x)) \mid x \in \mathfrak{h}_{S_2}\} = 0$. In other words, (4.24) is the special case of the splitting in (4.12) with $\mathfrak{l}_1 = \mathfrak{g}_{\mathrm{diag}}$.

Keeping the notation as in Theorem 4.10, we have $v_2 = 1$, and the corank of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ in $\mathcal{O}_1 \cap \mathcal{O}_2$ in this special case simplifies to

$$\dim \mathfrak{z}_{S_2} + \dim \mathfrak{z}_S - \dim(Y_2 \cap \mathrm{Ad}_{(\dot{w}_1,\dot{w}_2)} X_2) + \dim(Z_2 \cap \mathrm{Ad}_{(\dot{w}_1,\dot{w}_2)} \widetilde{X}).$$

When $\mathfrak{l}''_{S_2,T_2,d_2,V_2} = \mathfrak{l}_0 := \mathfrak{n}^- \oplus \mathfrak{n} + \mathfrak{h}_{-\mathrm{diag}}$, where $\mathfrak{n}$ and $\mathfrak{n}^-$ are respectively the span by positive and negative root vectors and $\mathfrak{h}_{-\mathrm{diag}} = \{(x,-x) \mid x \in \mathfrak{h}\}$, the splitting $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{g}_{\mathrm{diag}} + \mathfrak{l}_0$ is called the *standard splitting* of $\mathfrak{g} \oplus \mathfrak{g}$ [8]. In this case, $N(\mathfrak{l}_0) = B^- \times B$, where $B^- = P_\emptyset^-$ and $B = P_\emptyset$ are two opposite Borel subgroups, and in the notation of Theorem 4.10, $w_1 \in W$ and $w_2 \in W^T$. The corank of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ in $\mathcal{O}_1 \cap \mathcal{O}_2$ in this special case further simplifies to $\dim(\mathfrak{h}_{-\mathrm{diag}} \cap \mathrm{Ad}_{(\dot{w}_1,\dot{w}_2)} \widetilde{X})$, where

$$\widetilde{X} = \{(\mathrm{Ad}_{\dot{v}_1}^{-1} x, x) \mid x \in \mathfrak{z}_{T(v_1)},\ \theta_d \chi_S(\mathrm{Ad}_{\dot{v}_1}^{-1} x) = \chi_T(x)\} + \{(y, \theta_d(y)) \mid y \in \mathfrak{h}_S\}.$$

This formula has been obtained in [8].

4.5. **The wonderful compactification of G.** Recall [3] that the wonderful compactification $\overline{G}$ of G is the closure of the Lagrangian subalgebra $\mathfrak{g}_{\mathrm{diag}}$ inside $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$. Let $\mathfrak{g} \oplus \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$ be a Lagrangian splitting with $\mathfrak{l}_1$ and $\mathfrak{l}_2$ given by (4.12). Then $\overline{G}$ is a Poisson submanifold of $\mathcal{L}(\mathfrak{g} \oplus \mathfrak{g})$ with respect to the Poisson structure $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ because it is $(G \times G)$-stable. In [15], to each of the above Lagrangian splittings we associated two partition $\mathcal{P}_i$, $i = 1, 2$, of $\overline{G}$ into finitely many smooth irreducible locally closed $N(\mathfrak{l}_i)$-stable subsets. The strata of $\mathcal{P}_i$ are indexed by the Weyl group elements in Proposition 4.6 and are obtained by putting together the $N(\mathfrak{l}_i)$-orbits corresponding to different continuous parameters. When $\mathfrak{l}_1 = \mathfrak{g}_{\mathrm{diag}}$, the subsets in $\mathcal{P}_1$ are the G_{diag}-stable pieces introduced by Lusztig [17, 18]. Each stratum of $\mathcal{P}_1$ and $\mathcal{P}_2$ is a Poisson submanifolds of $(\overline{G}, \Pi_{\mathfrak{l}_1,\mathfrak{l}_2})$. Theorem 4.10 shows that the corank of $\Pi_{\mathfrak{l}_1,\mathfrak{l}_2}$ at an $\mathfrak{l} \in \overline{G}$ in $(N(\mathfrak{l}_1).\mathfrak{l}) \cap (N(\mathfrak{l}_2).\mathfrak{l})$ depends only on the stratum of $\mathcal{P}_1$ (or $\mathcal{P}_2$) to which $\mathfrak{l}$ belongs.

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