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Periodic solutions for p-Laplacian Rayleigh equations

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Abstract

By employing Mawhin's continuation theorem, the existence of the p-Laplacian delay Rayleigh equation

$$(\varphi_p(x'(t)))' + f(x'(t)) + \beta g(x(t - \tau(t))) = e(t)$$

under various assumptions is obtained.

Keywords: Periodic solution; Mawhin's continuation theorem; p-Laplacian; Rayleigh equation

1. Introduction

Consider the p-Laplacian Rayleigh differential equation

$$(\varphi_p(x'(t)))' + f(x'(t)) + \beta g(x(t - \tau(t))) = e(t), \tag{1.1}$$

where $p > 1$ is a constant, $\varphi_p : \mathbb{R} \to \mathbb{R}, \varphi_p(u) = |u|^{p-2}u$ is a one-dimensional p-Laplacian; $\beta > 0$ is a constant, $f, g, e, \tau \in C(\mathbb{R}, \mathbb{R})$, e, τ are periodic with period T, $e(t) \not\equiv 0$, $\int_0^T e(s)\mathrm{d}s = 0$, and $\tau(t) \geq 0$ for $t \in [0, T]$.

While there are plenty of results on the existence of periodic solutions for various types of delay differential equation (see [1,8,10–12] and references therein), studies of delay Rayleigh equations is relatively infrequent. The main difficulty lies in the middle term $f(x'(t))$ of (1.1), the existence of which obstructs the usual method of finding *a priori bounds* for delay Duffing or

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Liénard equations from working. In [9], Wang and Cheng discussed a delay Rayleigh equation of the form

$$x''(t) + f(x'(t)) + g(x(t - \tau(t))) = e(t), \tag{1.2}$$

and estimated the existence of periodic solutions to the equation by using Mawhin's continuation theorem. Recently, S.P. Lu, W.G. Ge and Z.X. Zheng (see [6,7]) continued to study (1.2) and improved the result of Wang and Cheng. Now as the p-Laplacian $(\varphi_p(x'(t)))'$ of a function $x(t)$ frequently comes into play in many practical situations (for example, it is used to describe fluid mechanical and nonlinear elastic mechanical phenomena), and in fact, since $x''(t) = \varphi_2(x'(t))'$, the p-Laplacian is a natural generalization of the usual Laplacian, it is natural to try to consider the existence of solutions of p-Laplacian equations, that is, differential equations with the leading term a p-Laplacian. So for example, it should be interesting to consider the aforesaid Eq. (1.2) with $x''(t)$ replaced by $(\varphi_p(x'(t)))'$. The obvious difficulty for the study of general p-Laplacian equations, with $p \neq 2$, is that $(\varphi_p(x'(t)))'$ is no longer linear and so Mawhin's continuation theorem does not apply directly. As far as we are aware, the problem of the existence of solutions for p-Laplacian delay equations had not been studied until, very recently, Cheung and Ren [2,3] investigated p-Laplacian Liénard equations and established sufficient conditions for the existence of periodic solutions of these equations. In this paper, following the line of Cheung and Ren in [2,3], we consider the p-Laplacian differential equation (1.1) and obtain sufficient conditions for the existence of periodic solutions of (1.1).

2. Preparation

We first recall Mawhin's continuation theorem which our study is based upon.

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero; here $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider the supplementary subspaces X_1, Y_1, of X, Y respectively, such that $X = \operatorname{Ker} L \oplus X_1$, $Y = \operatorname{Im} L \oplus Y_1$, and let $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y_1$ be the natural projections. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$; thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Denote by K the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \phi$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1 (*Gaines and Mawhin [4]*). *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If:*

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0, 1)$;
(2) $Nx \notin \operatorname{Im} L, \forall x \in \partial\Omega \cap \operatorname{Ker} L$; *and*
(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism,*

then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \bigcap D(L)$.

The next result is useful in obtaining *a priori bounds* of periodic solutions.

Lemma 2.2 (*[5]*). *Let $s \in C(\mathbb{R}, \mathbb{R})$ be periodic with period T and $\max_{t\in[0,T]} |s(t)| \leq \alpha$ for some constant $\alpha \in [0, T]$. Then for any $u \in C^1(\mathbb{R}, \mathbb{R})$ which is periodic with period T, we have*

$$\int_0^T |u(t) - u(t - s(t))|^2 \,\mathrm{d}t \leq 2\alpha^2 \int_0^T |u'(t)|^2 \,\mathrm{d}t.$$

In order to apply Mawhin's continuation theorem to study the existence of T-periodic solutions for (1.1), we rewrite (1.1) in the following form:

$$\begin{cases} x_1'(t) = \varphi_q(x_2(t)) = |x_2(t)|^{q-2}x_2(t) \\ x_2'(t) = -f(\varphi_q(x_2(t))) - \beta g(x_1(t-\tau(t))) + e(t), \end{cases} \tag{2.1}$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (x_1(t), x_2(t))^\top$ is a T-periodic solution to (2.1), then $x_1(t)$ must be a T-periodic solution to (1.1). Thus, the problem of finding a T-periodic solution for (1.1) reduces to finding one for (2.1).

Now, we set $C_T = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t+T) \equiv \phi(t)\}$ with norm $|\phi|_0 = \max_{t\in[0,T]} |\phi(t)|$, $X = Y = \{x = (x_1(\cdot), x_2(\cdot)) \in C(\mathbb{R}, \mathbb{R}^2) : x(t) \equiv x(t+T)\}$ with norm $\|x\| = \max\{|x_1|_0, |x_2|_0\}$. Clearly, X and Y are Banach spaces. Meanwhile, let

$$L : D(L) \subset X \to Y, \qquad Lx = x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$N : X \to Y, \qquad Nx = \begin{pmatrix} \varphi_q(x_2) \\ -f(\varphi_q(x_2(t))) - \beta g(x_1(t-\tau(t))) + e(t) \end{pmatrix}.$$

It is easy to see that $\operatorname{Ker} L = \mathbb{R}^2$ and $\operatorname{Im} L = \{y \in Y : \int_0^T y(s)\,\mathrm{d}s = 0\}$. So L is a Fredholm operator with index zero. Let $P : X \to \operatorname{Ker} L$ and $Q : Y \to \operatorname{Im} Q \subset \mathbb{R}^2$ be defined by

$$Px = \frac{1}{T}\int_0^T x(s)\,\mathrm{d}s; \qquad Qy = \frac{1}{T}\int_0^T y(s)\,\mathrm{d}s,$$

and let K denote the inverse of $L|_{\operatorname{Ker} P\cap D(L)}$. Obviously, $\operatorname{Ker} L = \operatorname{Im} Q = \mathbb{R}^2$ and

$$[Ky](t) = \int_0^T G(t,s)y(s)\,\mathrm{d}s \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \dfrac{s}{T}, & 0 \le s < t \le T. \\ \dfrac{s-T}{T}, & 0 \le t \le s \le T. \end{cases}$$

By (2.2), N is L-compact on $\overline{\Omega}$, where Ω is an open, bounded subset of X.

For the sake of convenience, denote by $|x|_w := (\int_0^T |x(s)|^w \mathrm{d}s)^{1/w}$ for $w \ge 1$; we list the following assumptions which will be useful in the study of the existence of T-periodic solutions to Eq. (1.1) in Section 3.

[H_1] There exist constants $\sigma > 0$ and $n \in \mathbb{N}$ such that $f(x)x \ge \sigma|x|^{n+1}$ $\forall x \in \mathbb{R}$ (or $f(x)x \le -\sigma|x|^{n+1}$ $\forall x \in \mathbb{R}$).

[H_2] There exist constants $\sigma > 0$ and $n \in \mathbb{N}$ such that $f(x) \ge \sigma|x|^n$ $\forall x \in \mathbb{R}$ (or $f(x) \le -\sigma|x|^n$ $\forall x \in \mathbb{R}$).

[H_3] There exists a constant $d > 0$ such that $xg(x) > 0$ (or $xg(x) < 0$) and $|g(x)| > \frac{|e|_0}{\beta}$ for $|x| > d$.

[H_4] There exists a constant $l > 0$ such that $|g(u_1) - g(u_2)| \le l|u_1 - u_2|$ $\forall u_1, u_2 \in \mathbb{R}$.

[H_5] There exist constants $r \in [0, +\infty)$ and $m \in \mathbb{N}$ such that $\lim_{|u|\to+\infty} \frac{|g(u)|}{|u|^m} = r$.

3. Main results

Theorem 3.1. *Suppose* $f(0) = 0$ *and* $[H_1]$, $[H_3]$, $[H_4]$ *hold. Suppose further that there exist* $k \in \mathbb{Z}$ *and* $\delta \geq 0$ *such that* $\max_{t\in[0,T]} |\tau(t) - kT| \leq \delta$. *If:*

(1) $n = 1$ *and* $\sigma > \sqrt{2}l\delta\beta$,

or

(2) $n > 1$,

then Eq. (1.1) *has at least one* T*-periodic solution.*

Proof. Consider the operator equation

$$Lx = \lambda Nx, \qquad \lambda \in (0, 1). \tag{3.1}$$

Let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in (0, 1)\}$. If $x(t) = \binom{x_1(t)}{x_2(t)} \in \Omega_1$, then from (3.1) we have

$$\begin{cases} x_1'(t) = \lambda\varphi_q(x_2(t)) = \lambda|x_2(t)|^{q-2}x_2(t) \\ x_2'(t) = -\lambda f(\varphi_q(x_2(t))) - \lambda\beta g(x_1(t - \tau(t))) + \lambda e(t). \end{cases} \tag{3.2}$$

From the first equation of (3.2), we have

$$x_2(t) = \varphi_p\left(\frac{1}{\lambda}x_1'(t)\right). \tag{3.3}$$

Let $\bar{t}$ and $\underline{t}$ be, respectively, the global maximum point and global minimum point of $x_1(t)$ on $[0, T]$; then $x_1'(\bar{t}) = 0$ and there exists $\varepsilon > 0$ such that $x_1'(t)$ is decreasing for $t \in (\bar{t} - \varepsilon, \bar{t} + \varepsilon)$. By (3.3), $x_2(\bar{t}) = 0$ and $x_2(t)$ is also decreasing for $t \in (\bar{t} - \varepsilon, \bar{t} + \varepsilon)$ which yields $x_2'(\bar{t}) \leq 0$. From $f(0) = 0$ and the second equation of (3.2), we have

$$-\beta g(x_1(\bar{t} - \tau(\bar{t}))) + e(\bar{t}) \leq 0,$$

i.e.,

$$g(x_1(\bar{t} - \tau(\bar{t}))) \geq \frac{e(\bar{t})}{\beta} \geq -\frac{|e|_0}{\beta}. \tag{3.4}$$

Similarly, we get

$$g(x_1(\underline{t} - \tau(\underline{t}))) \leq \frac{e(\underline{t})}{\beta} \leq \frac{|e|_0}{\beta}. \tag{3.5}$$

If $g(x_1(\bar{t}-\tau(\bar{t}))) \geq \frac{|e|_0}{\beta}$, then from (3.5) there is a point $\eta \in [0, T]$ such that $g(x_1(\eta-\tau(\eta))) = \frac{|e|_0}{\beta}$. So by assumption $[H_3]$ we find

$$|x_1(\eta - \tau(\eta))| \leq d. \tag{3.6}$$

On the other hand, if $g(x_1(\bar{t} - \tau(\bar{t}))) < \frac{|e|_0}{\beta}$, Eq. (3.4) implies that $|g(x_1(\bar{t} - \tau(\bar{t})))| < \frac{|e|_0}{\beta}$ which together with assumption $[H_3]$ guarantees that

$$|x_1(\bar{t} - \tau(\bar{t}))| \leq d. \tag{3.7}$$

Combining the above, we see that there is always a point $\xi \in [0, T]$ such that

$$|x_1(\xi - \tau(\xi))| \leq d.$$

Write $\xi-\tau(\xi)=kT+t^*$ when $k\in\mathbb{Z}$ and $t^*\in[0,T]$. Then by $|x_1(t)|\le|\int_{t^*}^t|x_1'(s)|\mathrm{d}s|+|x_1(t^*)|$ for all $t\in[0,T]$, we have

$$|x_1|_0\le|x_1(t^*)|+\int_0^T|x_1'(s)|\,\mathrm{d}s\le d+|x_1'|_1. \tag{3.8}$$

On the other hand, substituting (3.3) into the second equation of (3.2), we obtain

$$\left[\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right]'+\lambda f\left(\varphi_q\left(\varphi_p\left(\frac{1}{\lambda}x_1'(t)\right)\right)\right)+\lambda\beta g(x_1(t-\tau(t)))=\lambda e(t),$$

or

$$[\varphi_p(x_1'(t))]'+\lambda^p f\left(\frac{1}{\lambda}x_1'(t)\right)+\lambda^p\beta g(x_1(t-\tau(t)))=\lambda^p e(t). \tag{3.9}$$

Multiplying both sides of Eq. (3.9) by $x_1'(t)$ and integrating over $[0,T]$, we have

$$\begin{aligned}&\int_0^T[\varphi_p(x_1'(t))]'x_1'(t)\,\mathrm{d}t+\lambda^p\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)x_1'(t)\,\mathrm{d}t\\&\quad+\lambda^p\int_0^T\beta g(x_1(t-\tau(t)))x_1'(t)\,\mathrm{d}t=\lambda^p\int_0^T e(t)x_1'(t)\,\mathrm{d}t.\end{aligned} \tag{3.10}$$

If we write $w(t)=\varphi_p(x_1'(t))$, then $\int_0^T[\varphi_p(x_1'(t))]'x_1'(t)\,\mathrm{d}t=\int_0^T\varphi_q(w(t))\,\mathrm{d}w(t)=0$. Hence (3.10) reduces to

$$\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)x_1'(t)\,\mathrm{d}t=-\int_0^T\beta g(x_1(t-\tau(t)))x_1'(t)\mathrm{d}t+\int_0^T e(t)x_1'(t)\,\mathrm{d}t. \tag{3.11}$$

From $[H_1]$, we have

$$\begin{aligned}\left|\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)x_1'(t)\,\mathrm{d}t\right|&=\int_0^T\left|f\left(\frac{1}{\lambda}x_1'(t)\right)x_1'(t)\right|\,\mathrm{d}t\ge\lambda\sigma\int_0^T\left|\frac{1}{\lambda}x_1'(t)\right|^{n+1}\,\mathrm{d}t\\&=\frac{\sigma}{\lambda^n}\int_0^T|x_1'(t)|^{n+1}\,\mathrm{d}t,\end{aligned}$$

which together with (3.11), $[H_4]$, and Hölder's inequality implies

$$\begin{aligned}&\sigma\int_0^T|x_1'(t)|^{n+1}\,\mathrm{d}t\\&\quad\le\lambda^n\left|\int_0^T\beta g(x_1(t-\tau(t)))x_1'(t)\,\mathrm{d}t\right|+\lambda^n\left|\int_0^T e(t)x_1'(t)\,\mathrm{d}t\right|\\&\quad\le\beta\left|\int_0^T g(x_1(t))x_1'(t)\,\mathrm{d}t+\int_0^T[g(x_1(t-\tau(t)))-g(x_1(t))]x_1'(t)\,\mathrm{d}t\right|\\&\qquad+\left|\int_0^T e(t)x_1'(t)\,\mathrm{d}t\right|\\&\quad\le l\beta\int_0^T|x_1(t-\tau(t))-x(t)||x_1'(t)|\,\mathrm{d}t+\left|\int_0^T e(t)x_1'(t)\,\mathrm{d}t\right|\\&\quad\le l\beta|x_1'|_2\left(\int_0^T|x_1(t)-x_1(t-\tau(t))|^2\,\mathrm{d}t\right)^{\frac{1}{2}}+|e|_2|x_1'|_2.\end{aligned} \tag{3.12}$$

Meanwhile, by assumption, there exist $k \in \mathbb{Z}$ and $\delta > 0$ such that $\max_{t\in[0,T]} |\tau(t) - kT| \leq \delta$. Hence Lemma 2.2 gives

$$\left(\int_0^T |x_1(t) - x_1(t-\tau(t))|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} = \left(\int_0^T |x_1(t) - x_1(t-\tau(t)+kT)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \leq \sqrt{2}\delta|x_1'|_2. \tag{3.13}$$

Substituting (3.13) into (3.12), and using Hölder's inequality, we have

$$\begin{aligned} &\sigma \int_0^T |x_1'(t)|^{n+1} \,\mathrm{d}t \\ &\leq \sqrt{2}l\delta\beta|x_1'|_2^2 + |e|_2|x_1'|_2 \\ &\leq \sqrt{2}l\delta\beta T^{(n-1)/(n+1)} \left(\int_0^T |x_1'(t)|^{n+1} \,\mathrm{d}t\right)^{2/(n+1)} \\ &\quad + |e|_2 T^{(n-1)/(2(n+1))} \left(\int_0^T |x_1'(t)|^{n+1} \,\mathrm{d}t\right)^{1/(n+1)}. \end{aligned} \tag{3.14}$$

Case 1. If $n = 1$ and $\sigma > \sqrt{2}l\delta\beta$, it follows from (3.14) that

$$|x_1'|_2 \leq \frac{|e|_2}{\sigma - \sqrt{2}l\delta\beta} := M. \tag{3.15}$$

So by (3.8) and Hölder's inequality, we have

$$|x_1|_0 \leq d + |x_1'|_1 \leq d + T^{1/2}|x_1'|_2 \leq d + T^{1/2}M. \tag{3.16}$$

Case 2. For $n > 1$, as $1/(n+1) < 2/(n+1) < 1$, it follows from (3.14) that there must be a constant $M^* > 0$ such that $\int_0^T |x_1'(t)|^{n+1} \,\mathrm{d}t \leq M^*$. Thus by (3.8) and Hölder's inequality,

$$\begin{aligned} |x_1|_0 &\leq d + |x_1'|_1 \leq d + T^{n/(n+1)} \left(\int_0^T |x_1'(t)|^{n+1} \,\mathrm{d}t\right)^{1/(n+1)} \\ &\leq d + T^{n/(n+1)}(M^*)^{1/(n+1)}. \end{aligned} \tag{3.17}$$

From (3.16) and (3.17), we see that in both cases there exists a constant M_1 such that

$$|x_1|_0 \leq M_1. \tag{3.18}$$

On the other hand, by the first equation of (3.2), we have

$$\int_0^T |x_2(s)|^{q-2}x_2(s) \,\mathrm{d}s = 0, \tag{3.19}$$

which implies that there is a constant $t_2 \in [0, T]$ such that $x_2(t_2) = 0$. So

$$|x_2|_0 \leq \int_0^T |x_2'(s)| \,\mathrm{d}s. \tag{3.20}$$

Now multiplying by $x_2'(t)$ on both sides of the second equation of (3.2) and integrating over $[0, T]$, we obtain

$$\begin{aligned}
|x_2'|_2^2 &= -\lambda \int_0^T f(\varphi_q(x_2(t)))x_2'(t)\,\mathrm{d}t - \lambda \int_0^T \beta g(x_1(t-\tau(t)))x_2'(t)\,\mathrm{d}t \\
&\quad + \lambda \int_0^T e(t)x_2'(t)\,\mathrm{d}t \\
&= -\lambda \int_0^T \beta g(x_1(t-\tau(t)))x_2'(t)\,\mathrm{d}t + \lambda \int_0^T e(t)x_2'(t)\,\mathrm{d}t \\
&\le \left| \int_0^T \beta g(x_1(t-\tau(t)))x_2'(t)\,\mathrm{d}t \right| + \left| \int_0^T e(t)x_2'(t)\,\mathrm{d}t \right| \\
&\le \beta g_{M_1} \cdot |x_2'|_1 + |e|_2 |x_2'|_2 \\
&\le \beta g_{M_1} T^{\frac{1}{2}} |x_2'|_2 + |e|_2 |x_2'|_2
\end{aligned} \tag{3.21}$$

where $g_{M_1} := \max_{|u|\le M_1} |g(u)|$. Hence

$$|x_2'|_2 \le \beta g_{M_1} T^{\frac{1}{2}} + |e|_2 := M^{**}, \tag{3.22}$$

which, together with (3.20), yields

$$|x_2|_0 \le \int_0^T |x_2'(s)|\,\mathrm{d}s \le T^{\frac{1}{2}} |x_2'|_2 \le T^{\frac{1}{2}} M^{**} := M_2. \tag{3.23}$$

Let $\Omega_2 = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}$. If $x \in \Omega_2$, then $x \in \operatorname{Ker} L$ and $QNx = 0$. Obviously $|x_2|^{q-2}x_2 = 0$; then by the assumption on e, we see that $g(x_1) = 0$. So

$$|x_1| \le d \le M_1, \qquad x_2 = 0 \le M_2. \tag{3.24}$$

Let $\Omega = \{x = (x_1, x_2)^\top \in X : |x_1|_0 < N_1, |x_2|_0 < N_2\}$, where N_1 and N_2 are constants with $N_1 > M_1$, $N_2 > M_2$ and $(N_2)^q > d\beta g_d$, where $g_d = \max_{|u|\le d} |g(u)|$. Then $\overline{\Omega}_1 \subset \Omega$, $\overline{\Omega}_2 \subset \Omega$. From (3.18), (3.23) and (3.24), it is easy to see that conditions (1) and (2) of Lemma 2.1 are satisfied.

Next, we claim that condition (3) of Lemma 2.1 is also satisfied. For this, define the isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ by $J(x_1, x_2) = (x_2, -x_1)$ and let $H(v, \mu) := \mu v + (1-\mu)JQNv$, $(v, \mu) \in \Omega \times [0, 1]$. By simple calculations, we obtain, for $(x, \mu) \in \partial(\Omega \cap \operatorname{Ker} L) \times [0, 1]$,

$$x^\top H(x, \mu) = \mu(x_1^2 + x_2^2) + (1-\mu)(\beta x_1 g(x_1) + |x_2|^q) > 0. \tag{3.25}$$

Hence

$$\begin{aligned}
&\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{H(x, 0), \Omega \cap \operatorname{Ker} L, 0\} \\
&\quad = \deg\{H(x, 1), \Omega \cap \operatorname{Ker} L, 0\} = \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \\
&\quad \ne 0,
\end{aligned} \tag{3.26}$$

and so condition (3) of Lemma 2.1 is also satisfied.

Therefore, by Lemma 2.1, we conclude that equation

$$Lx = Nx \tag{3.27}$$

has a solution $x(t) = (x_1(t), x_2(t))^\top$ on $\overline{\Omega}$, i.e., Eq. (1.1) has a T-periodic solution $x_1(t)$ with $|x_1|_0 \le M_1$. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Suppose $f(0) = 0$ and $[H_2]$, $[H_3]$, $[H_5]$ hold. If:*

(1) *$n = m$ and $\sigma > r\beta T^n$,*

or

(2) *$n > m$,*

then Eq. (1.1) has at least one T-periodic solution.

Proof. As proved in Theorem 3.1, let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in (0, 1)\}$. If $x(t) = \binom{x_1(t)}{x_2(t)} \in \Omega_1$, then we have

$$[\varphi_p(x_1'(t))]' + \lambda^p f\left(\frac{1}{\lambda}x_1'(t)\right) + \lambda^p \beta g(x_1(t - \tau(t))) = \lambda^p e(t) \tag{3.28}$$

and

$$|x_1|_0 \le |x_1(t^*)| + \int_0^T |x_1'(s)|\,\mathrm{d}s \le d + |x_1'|_1. \tag{3.29}$$

Integrating both sides of (3.28) over $[0, T]$, we get

$$\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)\mathrm{d}t + \int_0^T \beta g(x_1(t - \tau(t)))\,\mathrm{d}t = 0. \tag{3.30}$$

It follows from $[H_2]$ that

$$\left|\int_0^T f\left(\frac{1}{\lambda}x_1'(t)\right)\mathrm{d}t\right| = \int_0^T \left|f\left(\frac{1}{\lambda}x_1'(t)\right)\right|\mathrm{d}t \ge \frac{\sigma}{\lambda^n}\int_0^T |x_1'(t)|^n\,\mathrm{d}t \tag{3.31}$$

and so

$$\sigma\int_0^T |x_1'(t)|^n\,\mathrm{d}t \le \lambda^n\left|\int_0^T \beta g(x_1(t - \tau(t)))\,\mathrm{d}t\right| \le \beta\int_0^T |g(x_1(t - \tau(t)))|\,\mathrm{d}t. \tag{3.32}$$

For an arbitrary constant $\varepsilon > 0$, we have from $[H_5]$ that there is a constant $\rho > d$ (independent of λ) such that $|g(u)| \le (r + \varepsilon)|u|^m$ for $|u| > \rho$. Let $E_1 = \{t \in [0, T] : |x_1(t - \tau(t))| > \rho\}$, $E_2 = \{t \in [0, T] : |x_1(t - \tau(t))| \le \rho\}$ and $g_\rho = \max_{|u|\le\rho} |g(u)|$. By (3.32), we have

$$\begin{aligned}\sigma\int_0^T |x_1'(t)|^n\,\mathrm{d}t &\le \beta\int_{E_1} |g(x_1(t - \tau(t)))|\,\mathrm{d}t + \beta\int_{E_2} |g(x_1(t - \tau(t)))|\,\mathrm{d}t \\ &\le (r + \varepsilon)\beta T|x_1|_0^m + \beta g_\rho T \\ &\le (r + \varepsilon)\beta T(d + |x_1'|_1)^m + \beta g_\rho T.\end{aligned} \tag{3.33}$$

We claim that there exists a constant $M_1 > 0$ such that

$$|x_1|_0 \le M_1. \tag{3.34}$$

If $|x_1'|_1 = 0$, then by (3.29), $|x_1|_0 \le d$. So suppose $|x_1'|_1 > 0$; then

$$\left[d + \int_0^T |x_1'(s)|\,\mathrm{d}s\right]^m = \left(\int_0^T |x_1'(s)|\,\mathrm{d}s\right)^m \left[1 + \frac{d}{\int_0^T |x_1'(s)|\,\mathrm{d}s}\right]^m. \tag{3.35}$$

From elementary analysis, there is a constant $h > 0$ (independent of λ) such that

$$(1 + x)^m < 1 + (1 + m)x, \quad \forall x \in (0, h]. \tag{3.36}$$

If $\frac{d}{|x_1'|_1} \geq h$, then

$$|x_1'|_1 \leq d/h,$$

and (3.29) implies that

$$|x_1|_0 \leq d + d/h. \tag{3.37}$$

If $\frac{d}{|x_1'|_1} < h$, then from (3.36) we have

$$\begin{aligned}
&\left[d + |x_1'|_1\right]^m \\
&\quad \leq \left(\int_0^T |x_1'(s)|\,\mathrm{d}s\right)^m \left[1 + \frac{(m+1)d}{\int_0^T |x_1'(s)|\,\mathrm{d}s}\right] \\
&\quad = \left(\int_0^T |x_1'(s)|\,\mathrm{d}s\right)^m + (m+1)d\left(\int_0^T |x_1'(s)|\,\mathrm{d}s\right)^{m-1} \\
&\quad \leq T^{m(n-1)/n}\left(\int_0^T |x_1'(s)|^n\,\mathrm{d}s\right)^{\frac{m}{n}} \\
&\qquad + (m+1)\,dT^{(n-1)(m-1)/n}\left(\int_0^T |x_1'(s)|^n\,\mathrm{d}s\right)^{(m-1)/n}.
\end{aligned} \tag{3.38}$$

Substituting (3.38) into (3.33), we obtain

$$\begin{aligned}
&\sigma\int_0^T |x_1'(t)|^n\,\mathrm{d}t \\
&\quad \leq (r+\varepsilon)\beta T^{1+m-\frac{m}{n}}\left(\int_0^T |x_1'(s)|^n\,\mathrm{d}s\right)^{\frac{m}{n}} \\
&\qquad + (r+\varepsilon)(m+1)\beta\,dT^{m+\frac{1}{n}-\frac{m}{n}}\left(\int_0^T |x_1'(s)|^n\,\mathrm{d}s\right)^{\frac{m-1}{n}} + \beta g_\rho T.
\end{aligned} \tag{3.39}$$

Case 1. If $n = m$ and $\sigma > r\beta T^n$, we can choose $\varepsilon < \frac{\sigma}{\beta T^n} - r$. Then $\sigma > (r+\varepsilon)\beta T^n$. So by (3.39), we see that $\int_0^T |x_1'(t)|^n \mathrm{d}t$ is bounded.

Case 2. For $n > m$, as $\frac{m-1}{m} < \frac{m}{n} < 1$, it is also easy to see that $\int_0^T |x_1'(t)|^n \mathrm{d}t$ is bounded.

Hence we know that in both cases there is a constant $M > 0$ such that

$$\int_0^T |x_1'(t)|^n\,\mathrm{d}t \leq M$$

which together with (3.29) yields that

$$|x_1|_0 \leq d + T^{(n-1)/n}(M)^{1/n} := M_1. \tag{3.40}$$

This proves the claim and the rest of the proof of the theorem is identical to that of Theorem 3.1.

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