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On a non-abelian invariant on complex surfaces of general type

Dedicated to Professor Sheng GONG on the occasion of his 75th birthday

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Abstract In this paper, we give certain homotopy and diffeomorphism versions as a generalization to an earlier result due to W.S. Cheung, Bun Wong and Stephen S. T. Yau concerning a local rigidity problem of the tangent bundle over compact surfaces of general type.

Keywords: Chern numbers, complex surfaces of general type, coholomogy group, local moduli, local deformation space, Miyaoka-Yau inequality, Yau's global rigidity theorem, Yau's uniformization theorem.

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1 Introduction

Let X be a compact complex manifold with a holomorphic tangent bundle $T(X)$. An intrinsic complex analytic invariant attached to X is the cohomology groups $H^i(X, \mathrm{End}T)$, $0 \leqslant i \leqslant n$, where $\mathrm{End}T$ is the endomorphism tangent bundle of X. The 1st cohomology group $H^1(X, \mathrm{End}T)$ is particularly important as it parametrizes the local deformation space of the tangent bundle $T(X)$ subject to constraint of the Kuranishi obstruction. There is a decomposition $H^1(X, \mathrm{End}T) = H^1(\theta) \oplus H^1(X, \mathrm{End}_0 T)$, where θ is the sheaf of germs of complex analytic functions and $\mathrm{End}_0 T$ represents the bundle of trace-free endomorphisms of T. The trace-free endomorphism cohomology group $H^1(X, \mathrm{End}_0 T)$, subject to the Kuranishi obstruction, parametrizes the local deformation space of $T(X)$ with a fixed determinant line bundle $\det T$. This trace-free local moduli space will be denoted by $K_0\big(T(X)\big)$ throughout this paper.

In ref. [1] we are interested in computing $\dim_{\mathbb{C}} K_0\big(T(X)\big)$ and $\dim_{\mathbb{C}} H^1(X, \mathrm{End}_0 T)$ when X is a compact complex two-fold of a general type. Some of the results obtained in ref. [1] will be summarized in sec. 2. In sec. 3 we will prove certain homotopy versions of the corresponding results. The attempt is to point out some observations to support our belief that the local moduli of tangent bundles of complex surfaces of a general type might contain some interesting topological invariances that are worthy of further studies. The claims presented in this paper can only be regarded as a report at a preliminary stage of our project, a deeper investigation along this line is needed in order to gain more concrete results.

2 $K_0(T(M))$ and $H^1(M, \mathrm{End}_0 T)$ on a complex surface of general type

The Hirzebruch's Riemann-Roch formula for the bundle $\mathrm{End}_0 T$ over a compact complex two-fold X, after a rearrangement of the terms, can be written as

$$\dim_{\mathbb{C}} H^1(X, \mathrm{End}_0 T) - \dim_{\mathbb{C}} H^2(X, \mathrm{End}_0 T) = \dim_{\mathbb{C}} H^0(X, \mathrm{End}_0 T) + \frac{5}{4}(3C_2 - C_1^2),$$

where C_1^2 and C_2 are the 1st and 2nd Chern numbers of X.

Theorem 2.1.[1] Let M be a compact complex two-fold of a general type. Then $K_0(T(M))$ is non-trivial unless M is covered holomorphically by the Euclidean ball.

Proof. The expected dimension $d_0(T(M))$ of $K_0(T(M))$ is defined as the number $\dim_{\mathbb{C}} H^1(M, \mathrm{End}_0 T) - \dim_{\mathbb{C}} H^2(M, \mathrm{End}_0 T)$. It is always true that $\dim_{\mathbb{C}} K_0(T(M)) \geqslant d_0(T(M))$. The Miyaoka-Yau inequality of a complex surface of a general type says that $C_1^2 \leqslant 3C_2$, with this one observes that $d_0(T(M)) \geqslant 0$. Assuming $\dim_{\mathbb{C}} K_0(T(M)) = 0$ one obtains the equality $C_1^2 = 3C_2$. This equality implies that the canonical line bundle must be ample. By the Yau's uniformization[2] M is covered holomorphically by the Euclidean ball. This completes the proof.

Theorem 2.2.[1] Let M be a compact complex two-fold of a general type. Then the following statements are true:

(1) If M is not covered by the ball, then $\dim_{\mathbb{C}} H^1(M, \mathrm{End}_0 T) \neq 0$.

(2) If M is covered by the ball with a non-trivial holomorphic one form, then $\dim_{\mathbb{C}} H^1(M, \mathrm{End}_0 T) \neq 0$.

Proof. The proof of statement (1) follows from Theorem 2.1 and the fact that $\dim_{\mathbb{C}} H^1(M, \mathrm{End}_0 T) \geqslant \dim_{\mathbb{C}} K_0(T(M)) \geqslant 0$.

As for statement (2), the argument goes as follows. On a ball quotient the 1st and 2nd Chern numbers satisfy an equality $C_1^2 = 3C_2$. In this case the Hirzebruch formula for $\mathrm{End}_0 T$ has the following simple form:

$$\dim_{\mathbb{C}} H^1(M, \mathrm{End}_0 T) = \dim_{\mathbb{C}} H^0(M, \mathrm{End}_0 T) + \dim_{\mathbb{C}} H^2(M, \mathrm{End}_0 T) .$$

The following facts are true, the proofs of which can be found in ref. [1].

(a) $\dim_{\mathbb{C}} H^0(M, \mathrm{End}_0 T) = 0$ if M is a ball quotient.

(b) $\dim_{\mathbb{C}} H^2(M, \mathrm{End}_0 T) = \dim_{\mathbb{C}} H^0(M, S^2 T^*)$ is ture for any compact complex two-fold M, where $S^2 T^*$ is the symmetric two tensor product of the cotangent bundle T^*. We note that $H^0(M, S^2 T^*)$ is the space of all holomorphic symmetric two tensors on M.

If M admits a non-trivial holomorphic one form ω, then the tensor product $\omega \otimes \omega$ is a non-trivial holomorphic symmetric two tensor on M. Using (a) and (b) one can conclude that $H^1(M, \mathrm{End}_0 T) \neq 0$.

To conclude the discussion of this section we would like to mention that we cannot determine whether $H^1(M, \mathrm{End}_0 T)$ vanishes or not on the ball quotient without the holomorphic one form. An important example is an arithmetic surface constructed by David Mumford which is a ball quotient with the same homology as the complex projective space. Now we are unable to produce a non-trivial holomorphic symmetric two-tensor on the Mumford surface.

3 Local moduli of the tangent bundle over complex surface as a topological invariance

Theorem 3.1. Let M be a compact complex two-fold of a general type and X be another compact complex two-fold. Suppose there exists an oriented homotopy equivalence between M and X, then the following statements are true:

(a) If M is a ball quotient, then $\dim_{\mathbb{C}} K_0\big(T(M)\big) = \dim_{\mathbb{C}} K_0\big(T(X)\big)$.

(b) If M is not a ball quotient, then $\dim_{\mathbb{C}} K_0\big(T(X)\big) > 0$.

Proof. (a) This follows from the Yau's global rigidity Theorem in ref. [2].

(b) Let the Chern numbers of M and X be $\{C_1^2, C_2\}$ and $\{\tilde{C}_1^2, \tilde{C}_2\}$, respectively. The index $\frac{1}{3}(C_1^2 - 2C_2) = \frac{1}{3}(\tilde{C}_1^2 - 2\tilde{C}_2)$ is an invariance under an oriented homotopy equivalence for four manifolds. Since $C_2 = \tilde{C}_2$ one has the equality $3\tilde{C}_2 - \tilde{C}_1^2 = 3C_2 - C_1^2$. $3\tilde{C}_2 - \tilde{C}_1^2$ is a positive number as a consequence of the Miyaoka-Yau inequality and the Yau's uniformization theorem. It follows from the Hirzebruch formula for $\mathrm{End}_0\big(T(X)\big)$ that the expected dimension $d_0\big(T(X)\big)$ for $K_0\big(T(X)\big)$ must be positive. Since $\dim_{\mathbb{C}} K_0\big(T(X)\big)$ is never less than $d_0\big(T(X)\big)$, one can conclude that $\dim_{\mathbb{C}} K_0\big(T(X)\big) > 0$.

The following remark addresses the orientation reversed cases which are ample in existence.

Theorem 3.2. Suppose there exists an orientation reversed homotopy equivalence between two compact complex two-folds M and X. Assume that the Chern number $C_1^2 - C_2$ of M is positive, then $\dim_{\mathbb{C}} K_0\big(T(X)\big) > 0$.

Proof. Using the same notation as in the proof of Theorem 3.1(b) one has $\frac{1}{3}(C_1^2 - 2C_2) = -\frac{1}{3}(\tilde{C}_1^2 - 2\tilde{C}_2)$ as the indices of M and X have different signs if there is an orientation reversed homotopy equivalence between them. Since $C_2 = \tilde{C}_2$ one has $3\tilde{C}_2 - \tilde{C}_1^2 = C_1^2 - C_2$. From the Hirzebruch formula for $\mathrm{End}_0 T(X)$ and the assumption $C_1^2 - C_2 > 0$ we obtain $d_0\big(T(X)\big) > 0$. This proves $\dim_{\mathbb{C}} K_0\big(T(X)\big) > 0$ for the same reason as before.

It is now known from the Donaldson and Seiberg-Witten invariants that the Kodaira dimensions of complex surfaces are a diffeomorphic invariant. The following results are the consequences along our path of discussion.

Theorem 3.3. Let M be a compact complex two-fold of a general type. Suppose X is a compact complex two-fold diffeomorphic to M, then the following statements are true:

(a) If M is a ball quotient and the diffeomorphism preserves the orientation, then

$\dim_{\mathbb{C}} K_0\big(T(M)\big) = \dim_{\mathbb{C}} K_0\big(T(X)\big)$.

(b) If M is not a ball quotient and the diffeomorphism preserves the orientation, then $\dim_{\mathbb{C}} K_0\big(T(X)\big) > 0$.

(c) If the diffeomorphism reverses the orientation, then $\dim_{\mathbb{C}} K_0\big(T(X)\big) > 0$.

Proof. (a) and (b) are contained in Theorem 3.1. We only have to prove statement (c). X is of a general type as it is diffeomorphic to M. If $\dim_{\mathbb{C}} K_0\big(T(X)\big) = 0$, by Theorem 2.1, X must be a ball quotient, hence its Chern numbers satisfy $3\tilde{C}_2 - \tilde{C}_1^2 = 0$. The orientation of the diffeomorphism is reversed, thus $\frac{1}{3}(C_1^2 - 2C_2) = -\frac{1}{3}(\tilde{C}_1^2 - 2\tilde{C}_2)$. Since $C_2 = \tilde{C}_2$, this implies $C_1^2 - C_2 = 3\tilde{C}_2 - \tilde{C}_1^2$, which is equal to zero. Nonetheless, by a result in ref. [3] the index $\frac{1}{3}(C_1^2 - 2C_2)$ of M must vanish if such an unoriented diffeomorphism exists. This would imply $C_1 = 0$ and $C_2 = 0$. This is a contradiction because C_2 is always positive on a surface of a general type (in fact M and X are very restrictive under the assumption that they are reversedly diffeomorphic[3,4]). Therefore $\dim_{\mathbb{C}} K_0\big(T(X)\big)$ has to be positive.

In the statements of Theorem 3.1(a),(b), Theorem 3.2, and Theorem 3.3(a),(b),(c), one can make the same conclusions for the expected dimension $d_0\big(T(X)\big)$ and $\dim_{\mathbb{C}} H^1(X, \mathrm{End}_0 T)$ using more or less the same arguments. They are all bounded below by the Chern number $\frac{5}{4}(3C_2 - C_1^2)$ in the oriented cases and by $\frac{5}{4}(C_1^2 - C_2)$ in the unoriented cases.

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