

A Fast Algorithm for the Spread of HIV in a System of Prisons

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Abstract

In this paper, we propose a continuous time model for modeling the spread of HIV in a network of prisons. We give some sufficient conditions for the equilibrium points of the system to be stable. We also develop an efficient algorithm based on Newton's method and the Sherman-Morrison-Woodbury Formula for computing the equilibrium values of the infectives in each prison.

Key Words: HIV, Prison System, Epidemic Model, Equilibrium Point, Newton's Method, Sherman-Morrison-Woodbury Formula.

1 Introduction

The spread of HIV has captured the attention of a lot of researchers. There have been many research papers on the modeling of the spread of HIV in the community. Greenhalgh and Lewis [6] develop a model for the spread of HIV amongst a population of injecting drug users. The model they discuss focuses on the transmission of HIV through the sharing of contaminated drug injection equipment and in particular they examine the mixing of addicts and needles when the AIDS incubation period is divided into distinct infectious stages. Huang and Villasans [10] extended the Kermack-Mckendrick model for the spread of HIV. Their model is an SI type epidemic model which takes preventive education into account. Global behavior of an SEIRS epidemic model and SIR model with time delays have been studied by Wang [21] and Ma et al. [14] respectively. A discrete branching process model was proposed by Knolle [13] for modeling the spread of HIV via steady sexual partnerships.

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A serious problem of prison life is the spread of HIV by both sexual contacts and needle sharing activities among the prisoners. In [4], it is mentioned that a special report in the New York Times notes that the number of HIV infectives in Argentine federal prisons is of the order 30% of all inmates and suggests that local and rural prisons have similar prevalence, as do prisons in other Latin American countries, including Brazil. Motivated by the report, Gani et al. [4] proposed deterministic epidemic models in both continuous time and discrete time and also stochastic models in discrete time for modeling the spread of HIV in a prison system. Recently, due to the new results in complex networks, we understand better our social networks and the Internet [15]. It is mentioned in [3] that by 2030, more than 60% people in the world will live in cities. They are connected by different transportation methods. Dangerous diseases such as SARS can be spread very fast [1, 17]. Researchers have paid more attention to the spread of diseases and computer viruses through a network [3]. With these motivations, we extend the deterministic continuous time model of [4] to the case of a network of prisons.

We first review a continuous deterministic model for one prison system in [4]. We then give the outline of the paper. The model in [4] considers a prison of N prisoners in continuous time. At time point t , there are simultaneous inflows and outflows of n prisoners where $n < N$. After that there are $y(t)$ prisoners who are HIV⁺ and $(N - y(t))$ susceptibles. We then assume that homogeneous mixing occurs in the prison. The model further assumes that the new infectives produced is proportional to

$$y(t)(N - y(t))$$

with a proportioned constant β , for instance [2, p. 9]. This means the number of new infectives produced is

$$\beta \cdot y(t)(N - y(t)).$$

They also assume an inflow of n new prisoners joining the prison from the outside world, where the proportion of infectives in the outside world is μ ($0 < \mu < 1$). Thus there are $n\mu$ infectives added to the prison. At the same time, after mixing with the prisoners, an outflow of n prisoners leaves the prison of which

$$n \left(\frac{y(t)}{N} \right)$$

are infectives. Thus the differential equation governing the infectives in the system is given by

$$\frac{dy}{dt} = n\mu - \frac{n}{N}y(t) + \beta y(t)(N - y(t)) = \beta(\eta_1 - y(t))(\eta_2 + y(t))$$

where

$$\eta_1 = \frac{1}{2} \left\{ A + \sqrt{A^2 + \frac{4n\mu}{\beta}} \right\}, \quad \eta_2 = \frac{1}{2} \left\{ A - \sqrt{A^2 + \frac{4n\mu}{\beta}} \right\}$$

and

$$A = N - \frac{n}{\beta N}.$$

Solving the ordinary differential equation, we get

$$y(t) = \frac{\eta_1 K e^{\beta(\eta_1 + \eta_2)t} - \eta_2}{1 + K e^{\beta(\eta_1 + \eta_2)t}} \quad \text{where} \quad K = \frac{\eta_2 + y(0)}{\eta_1 - y(0)}$$

and

$$\lim_{t \rightarrow \infty} y(t) = \eta_1. \quad (1)$$

The remainder of the paper is organized as follows. In Section 2, we consider a continuous time two-prison system. In Section 3, we extend our modeling to a general network of prisons. We develop a fast algorithm based on Newton's method and the Sherman-Morrison-Woodbury formula for computing the limiting values of the infectives in each prison. Numerical examples are given in Section 4. Finally, there are some concluding remarks in Section 5.

2 The Two-Prison Case

In this section, before we present a general network of prisons, let us first consider a two-prison system. A discrete time model for a two-prison system has been proposed in [4]. Here we consider a continuous time model for a two-prison system. Both prisons are allowed to have interactions with the outside world. In the model, we assume that the death rate is relatively insignificant. We define the following notations for our model.

1. N_i , the number of prisoners in Prison i , $i = 1, 2$.
2. $y_i(t)$, the number of infectives in Prison i ($i = 1, 2$) at time t .
3. β_i , the infection rate in Prison i ($i = 1, 2$).
4. μ , the mean proportion of infectives in the outside world.
5. n_i , the number of prisoners moving in and out from the outside world to Prison i ($i = 1, 2$).
6. m , the number of prisoners exchanged between the two prisons.

With the above notations, the system of ordinary differential equations governing the number of infectives in each prison is given by

$$\begin{cases} \frac{dy_1(t)}{dt} = n_1\mu + \beta_1 y_1(t)(N_1 - y_1(t)) - \frac{n_1 y_1(t)}{N_1} - \frac{m y_1(t)}{N_1} + \frac{m y_2(t)}{N_2} \\ \frac{dy_2(t)}{dt} = n_2\mu + \beta_2 y_2(t)(N_2 - y_2(t)) - \frac{n_2 y_2(t)}{N_2} - \frac{m y_2(t)}{N_2} + \frac{m y_1(t)}{N_1}. \end{cases} \quad (2)$$

Re-arranging the terms we have

$$\begin{cases} \frac{dy_1(t)}{dt} = n_1\mu + (\beta_1 N_1 - \frac{n_1 + m}{N_1})y_1(t) - \beta_1 y_1^2(t) + \frac{m y_2(t)}{N_2} \\ \frac{dy_2(t)}{dt} = n_2\mu + (\beta_2 N_2 - \frac{n_2 + m}{N_2})y_2(t) - \beta_2 y_2^2(t) + \frac{m y_1(t)}{N_1}. \end{cases} \quad (3)$$

In equilibrium, we have

$$\begin{cases} 0 = F_1(y_1^*, y_2^*) = n_1\mu + (\beta_1 N_1 - \frac{n_1 + m}{N_1})y_1^* - \beta_1 y_1^{*2} + \frac{m y_2^*}{N_2} \\ 0 = F_2(y_1^*, y_2^*) = n_2\mu + (\beta_2 N_2 - \frac{n_2 + m}{N_2})y_2^* - \beta_2 y_2^{*2} + \frac{m y_1^*}{N_1}. \end{cases} \quad (4)$$

To solve for the equilibrium points, one may apply Newton's method [12, p. 586]. Let $y_1^*(0)$ and $y_2^*(0)$ be the initial guesses, then the iterative scheme reads:

$$\begin{pmatrix} y_1^*(n+1) \\ y_2^*(n+1) \end{pmatrix} = \begin{pmatrix} y_1^*(n) \\ y_2^*(n) \end{pmatrix} - \begin{pmatrix} -2\beta_1 y_1^*(n) + \beta_1 N_1 - \frac{n_1 + m}{N_1} & \frac{m}{N_2} \\ \frac{m}{N_1} & -2\beta_2 y_2^*(n) + \beta_2 N_2 - \frac{n_2 + m}{N_2} \end{pmatrix}^{-1} \begin{pmatrix} F_1(y_1^*(n), y_2^*(n)) \\ F_2(y_1^*(n), y_2^*(n)) \end{pmatrix}.$$

To classify the equilibrium points, one may follow the analysis in [19, p. 261] and [8, 92].

Proposition 1 *Let (y_1^*, y_2^*) be the non-negative equilibrium point, if $N_1 = N_2$ and*

$$\beta_i < \frac{n_i}{N_i^2}$$

then the equilibrium point is a stable one.

Proof: We consider the matrix

$$\begin{pmatrix} -2\beta_1 y_1^* + \beta_1 N_1 - \frac{n_1 + m}{N_1} & \frac{m}{N_2} \\ \frac{m}{N_1} & -2\beta_2 y_2^* + \beta_2 N_2 - \frac{n_2 + m}{N_2} \end{pmatrix}.$$

By applying the Gershgorin disc theorem [5, p. 341] to the matrix above, the real part of its eigenvalues lie in the union of the two intervals:

$$\left(-2\beta_1 y_1^* + \beta_1 N_1 - \frac{n_1 + m}{N_1} - \frac{m}{N_2}, -2\beta_1 y_1^* + \beta_1 N_1 - \frac{n_1 + m}{N_1} + \frac{m}{N_2} \right) \subset (-\infty, -2\beta_1 y_1^*)$$

and

$$\left(-2\beta_2 y_2^* + \beta_2 N_2 - \frac{n_2 + m}{N_2} - \frac{m}{N_1}, -2\beta_2 y_2^* + \beta_2 N_2 - \frac{n_2 + m}{N_2} + \frac{m}{N_1} \right) \subset (-\infty, -2\beta_2 y_2^*).$$

Therefore the real part of the eigenvalues are negative and the equilibrium point is stable [8].

We remark that one can derive similar results for the case $N_1 \neq N_2$. \square

Example 1 Consider the following example with parameters

$$\begin{cases} N_1 = 500, & N_2 = 200, & n_1 = 50, & n_2 = 20, & m = 10, \\ \mu = 0.1, & \beta_1 = 0.00002 & \beta_2 = 0.0003, \\ y_1(0) = 250, & y_2(0) = 100. \end{cases}$$

Newton's method converges in a few steps and the equilibrium point is

$$(y_1^*, y_2^*) = (59.3593, 31.9999) \approx (59, 32).$$

Moreover, we have tried a number of different initial guesses, and Newton's method still converges very fast to the same solution.

3 A Network of Prisons

In this section, we consider a network of prisons. A simple epidemic model for interacting groups has been studied by Rushton and Mautner [18], see also [2, pp. 23-27]. We assume there are interactions among the prisons and also with the outside world. Let us give the notations of our model as follows:

1. N_i , the number of prisoners in Prison i ($i = 1, 2, \dots, s$).
2. $y_i(t)$, the number of infectives in Prison i ($i = 1, 2, \dots, s$) at time t .
3. β_i , the infection rate in Prison i ($i = 1, 2, \dots, s$).
4. μ , the mean proportion of infectives in the outside world.
5. n_i , the number of prisoners moving in and out from the outside world to Prison i ($i = 1, 2, \dots, s$).

For simplicity of discussion, we assume that the prisoners are exchanged among the s prisons so that the number m of prisoners moving out to other prisons is equal to the number of prisoners received from other prisons. We assume that there is a mechanism to mix up the prisoners uniformly and we have for $i = 1, 2, \dots, s$

$$\frac{dy_i(t)}{dt} = n_i\mu + \beta_i y_i(t)(N_i - y_i(t)) - \frac{n_i y_i(t)}{N_i} - \frac{m(s-1)y_i(t)}{N_i} + \sum_{k=1, k \neq i}^s \frac{y_k(t)m}{N_k}. \quad (5)$$

In equilibrium, we have

$$0 = F_i(y_1^*, y_2^*, \dots, y_s^*) = n_i\mu + (\beta_i N_i - \frac{n_i + m(s-1)}{N_i})y_i^* - \beta_i y_i^{*2} + \sum_{k=1, k \neq i}^s \frac{y_k^* m}{N_k}. \quad (6)$$

We remark that by Bezout's theorem [7, p. 87], there are at most $2s$ equilibrium points. Newton's method is a popular and effective method for solving non-linear system of equations [11, 16]. Other methods such as inexact Newton's method and Broyden's method can be found in [11].

Since for our problem, Newton's method works very well, we will only focus on Newton's method. When Newton's method is applied to solve equations (6), the matrix system to be solved is given by $(D + E)$ where D is a diagonal matrix

$$D_{ii} = -2\beta_i y_i^* + \beta_i N_i - \frac{n_i + sm}{N_i}. \quad (7)$$

We remark that if $\beta_i < n_i/N_i^2$ then the matrix D is invertible. The matrix E is a rank one matrix given by

$$(1, 1, \dots, 1)^t \left(\frac{m}{N_1}, \frac{m}{N_2}, \dots, \frac{m}{N_s} \right).$$

Since E is a rank one matrix, the linear system can be solved by using the Sherman-Morrison-Woodbury Formula.

Proposition 2 (*Sherman-Morrison-Woodbury Formula*) [5, p. 51] *Let M be an non-singular $r \times r$ matrix, u and v be two $r \times l$ ($l \leq r$) matrices such that the matrix $(I_l + v^t M u)$ is non-singular. Then we have*

$$(M + uv^t)^{-1} = M^{-1} - M^{-1}u (I_l + v^t M^{-1}u)^{-1} v^t M^{-1}.$$

Thus by using Proposition 2, we have

$$(D + E)^{-1} = D^{-1} - D^{-1}(1, 1, \dots, 1)^t \left(1 + \sum_{i=1}^s \frac{m_i}{N_i D_{ii}} \right)^{-1} \left(\frac{m}{N_1}, \frac{m}{N_2}, \dots, \frac{m}{N_s} \right) D^{-1}$$

or

$$(D + E)^{-1} = D^{-1} - \left(1 + \sum_{i=1}^s \frac{m}{N_i D_{ii}} \right)^{-1} \left(\frac{1}{D_{11}}, \dots, \frac{1}{D_{ss}} \right)^t \left(\frac{m}{N_1 D_{11}}, \dots, \frac{m}{N_s D_{ss}} \right). \quad (8)$$

For Newton's method, we have the following convergence theorem.

Proposition 3 (*Kantorovich Theorem* [9, p. 249]) *Let \mathbf{a} be a point in \mathbb{R}^K , U be an open neighborhood of \mathbf{a} in \mathbb{R}^K and $F : U \rightarrow \mathbb{R}^K$ be a differential mapping with its derivative $[DF(\mathbf{a})]$ being invertible. Define*

$$\mathbf{h} = -[DF(\mathbf{a})]^{-1}F(\mathbf{a}), \quad \mathbf{a}_1 = \mathbf{a} + \mathbf{h} \quad \text{and} \quad U_0 = B_{|\mathbf{h}|}(\mathbf{a}_1).$$

If $U_0 \subset U$ and the derivative $[DF(\mathbf{x})]$ satisfies the condition

$$\|DF(\mathbf{u}_1) - DF(\mathbf{u}_2)\| \leq M|\mathbf{u}_1 - \mathbf{u}_2|$$

for all points \mathbf{u}_1 and \mathbf{u}_2 and if the inequality

$$\|F(\mathbf{a})\| \|DF(\mathbf{a})\|^{-2} M \leq \frac{1}{2}$$

is satisfied, then the equation $F(\mathbf{x}) = \mathbf{0}$ has a unique solution in U_0 and Newton's method converges with an initial guess \mathbf{a} .

Remark 1 *If we set $\mathbf{a} = \frac{1}{2}(N_1, N_2, \dots, N_s)^t$ as the initial guess then it can be shown that we*

can take (see [9, p. 246])

$$M^2 = 4 \sum_{i=1}^s \beta_i^2.$$

Moreover, we have

$$|F(\mathbf{a})|^2 = \sum_{i=1}^s \left(n_i \left(\mu - \frac{1}{2} \right) + \frac{\beta_i N_i^2}{4} \right)^2.$$

By applying the Gershgorin disc theorem to (7), we have

$$|DF(\mathbf{a})|^{-2} \leq \prod_{i=1}^s \left(\frac{N_i}{n_i} \right)^2.$$

Thus by Proposition 3, a sufficient condition for Newton's method to be convergent with the initial guess \mathbf{a} is

$$\left(\sum_{i=1}^s \beta_i^2 \right) \left(\sum_{i=1}^s \left(n_i \left(\mu - \frac{1}{2} \right) + \frac{\beta_i N_i^2}{4} \right)^2 \right) \prod_{i=1}^s \left(\frac{N_i}{n_i} \right)^4 \leq \frac{1}{16}.$$

In view of Proposition 1, it is straightforward to show that

Proposition 4 *Let $(y_1^*, y_2^*, \dots, y_s^*)$ be the non-negative equilibrium point, if*

$$N_1 = N_2 = \dots = N_s$$

and

$$\beta_i < \frac{n_i}{N_i^2}$$

then the equilibrium point is a stable one.

4 Numerical Examples

In this section, we present some numerical examples on the equilibrium points for different system parameters. In all the numerical tests, we assume each $N_i = 500$ and

$$n_i = i \times 10 \quad \text{for } i = 1, 2, \dots, 8.$$

The initial guess is taken to be $\frac{1}{2}(500, 500, \dots, 500)$ for all cases. The stopping criterion for all the numerical results is the following:

$$\|F(y_1^*, y_2^*, \dots, y_s^*)\|_2 \leq 10^{-6}.$$

We remark that Newton's method converges in a few steps in all the tested cases.

In the first three tables, $\beta_i = 0.00002$ and for each value of $\mu = 0.01, 0.05$ and 0.1 , we solve for the following 9 cases of equilibrium points

$$s = 2, 4, 8 \quad \text{and} \quad m = 10, 20, 40$$

Table 1: The Equilibrium Points of the Infectives when $\mu = 0.01$

| | $s = 2$ | $s = 4$ | $s = 8$ |
|----------|-----------|-------------------|-----------------------------------|
| $m = 10$ | (8.1,7.2) | (6.8,6.4,6.2,6.1) | (5.9,5.8,5.8,5.7,5.6,5.6,5.6,5.5) |
| $m = 20$ | (7.8,7.3) | (6.5,6.4,6.2,6.1) | (5.8,5.7,5.7,5.7,5.6,5.6,5.6,5.6) |
| $m = 40$ | (7.7,7.4) | (6.4,6.3,6.2,6.2) | (5.7,5.7,5.7,5.6,5.6,5.6,5.6,5.6) |

Table 2: The Equilibrium Points of the Infectives when $\mu = 0.05$

| | $s = 2$ | $s = 4$ | $s = 8$ |
|----------|-------------|-----------------------|---|
| $m = 10$ | (39.1,35.1) | (33.3,31.8,30.7,30.0) | (29.5,29.0,28.6,28.3,28.0,27.8,27.6,27.5) |
| $m = 20$ | (37.9,35.6) | (32.1,31.4,30.8,30.3) | (28.7,28.5,28.3,28.2,28.0,27.9,27.8,27.6) |
| $m = 40$ | (37.1,35.9) | (31.5,31.1,30.8,30.5) | (28.4,28.3,28.2,28.1,28.0,27.9,27.8,27.8) |

the results are reported in Tables 1, 2 and 3. We observe that when μ increases, all the numbers of infectives in each of the prisons in equilibrium are also increased. When we consider the connection with the outside world, μ increases means the inflow of infectives by exchanging with the outside world is also increased. For each fixed s , when m is increased, the variation of the infectives in equilibrium among the prisons is decreased. When m is increased, the interactions between prisons are more frequent and we expect this will decrease the variation of the infectives in equilibrium.

For the numerical tests in Table 4, we assume $s = 4, \mu = 0.1$ and we calculate the equilibrium points for the cases where

$$\beta_i = 0.0000003, \quad 0.000003 \quad \text{and} \quad 0.00003$$

for $m = 10, 20, 40$. We observe that for each fixed β_i , when m is increased, the variation of the infectives in equilibrium among the prisons will be decreased. Similar to the previous argument, an increase in m enhance the exchanges between the prisons. Moreover, there is a significant jump in the number of infectives when β_i increases from 0.000003 to 0.00003.

One may further consider the case that β_i to be different from one each other. To simplify this case, for each m , we calculate the equilibrium points for the following three cases (three rows): β_1 changes from 0.0000003, 0.000003, 0.00003, while $\beta_i = 0.0000003$ for $i \neq 1$. Then we get Tables 5, 6 and 7 for $\mu = 0.01, 0.05$ and 0.1 respectively. We observe the following phenomenon. For each fixed β_i , when m is increased, the less variation of the infectives in equilibrium among the prisons is observed. The reason for this is when m is increased, the interactions between prisons are more frequent. Moreover, the infectives in equilibrium increases when μ increases and this is expected.

5 Concluding Remarks

In this paper, we have proposed a continuous time model for modeling the spread of HIV in a network of prisons. We give some sufficient conditions for the equilibrium points of the system

Table 3: The Equilibrium Points of the Infectives when $\mu = 0.1$

| | $s = 2$ | $s = 4$ | $s = 8$ |
|----------|-------------|-----------------------|---|
| $m = 10$ | (75.0,68.1) | (65.1,62.4,60.6,59.2) | (58.3,57.5,56.8,56.2,55.7,55.3,54.9,54.6) |
| $m = 20$ | (73.0,68.9) | (63.1,61.8,60.7,59.7) | (57.0,56.6,56.2,55.9,55.6,55.4,55.1,54.9) |
| $m = 40$ | (71.7,69.5) | (62.0,61.3,60.7,60.1) | (56.3,56.1,55.9,55.7,55.6,55.4,55.3,55.2) |

Table 4: The Equilibrium Points of the Infectives when $s = 4$

| | $\beta_i = 0.000003$ | $\beta_i = 0.000003$ | $\beta_i = 0.00003$ |
|----------|-----------------------|-----------------------|-----------------------|
| $m = 10$ | (50.2,50.2,50.1,50.1) | (51.9,51.6,51.4,51.2) | (75.3,70.7,67.5,65.1) |
| $m = 20$ | (50.2,50.2,50.1,50.1) | (51.7,51.5,51.4,51.3) | (71.8,69.5,67.6,66.1) |
| $m = 40$ | (50.2,50.1,50.1,50.1) | (51.5,51.5,51.4,51.3) | (69.7,68.6,67.6,66.7) |

Table 5: The Equilibrium Point of the Infectives when $\mu=0.01$

| | $s = 2$ | $s = 4$ | $s = 8$ |
|----------|------------|----------------------|--|
| $m = 10$ | (5.0, 5.0) | (5.0, 5.0, 5.0, 5.0) | (5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0) |
| | (5.2, 5.1) | (5.1, 5.0, 5.0, 5.0) | (5.1, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0) |
| | (9.0, 6.3) | (6.5, 5.5, 5.4, 5.3) | (5.6, 5.1, 5.1, 5.1, 5.1, 5.1, 5.1, 5.1) |
| $m = 20$ | (5.0, 5.0) | (5.0, 5.0, 5.0, 5.0) | (5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0) |
| | (5.2, 5.1) | (5.1, 5.0, 5.0, 5.0) | (5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0) |
| | (7.9, 6.5) | (6.0, 5.4, 5.4, 5.4) | (5.4, 5.1, 5.1, 5.1, 5.1, 5.1, 5.1, 5.1) |
| $m = 40$ | (5.0, 5.0) | (5.0, 5.0, 5.0, 5.0) | (5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0) |
| | (5.2, 5.1) | (5.1, 5.0, 5.0, 5.0) | (5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0, 5.0) |
| | (7.3, 6.6) | (5.7, 5.4, 5.4, 5.4) | (5.2, 5.1, 5.1, 5.1, 5.1, 5.1, 5.1, 5.1) |

Table 6: The Equilibrium Point of the Infectives when $\mu=0.05$

| | $s = 2$ | $s = 4$ | $s = 8$ |
|----------|--------------|--------------------------|--|
| $m = 10$ | (25.1 ,25.1) | (25.1, 25.1, 25.1, 25.1) | (25.1, 25.1, 25.0, 25.0, 25.0, 25.0, 25.0, 25.0) |
| | (26.2 ,25.4) | (25.6, 25.2, 25.2, 25.2) | (25.3, 25.1, 25.1, 25.1, 25.1, 25.1, 25.1, 25.1) |
| | (42.6 ,31.0) | (32.0, 27.1, 26.8, 26.6) | (26.0, 25.8, 25.7, 25.6, 25.6, 25.5, 25.5, 25.4) |
| $m = 20$ | (25.1 ,25.1) | (25.1, 25.1, 25.1, 25.1) | (25.1, 25.0, 25.0, 25.0, 25.0, 25.0, 25.0, 25.0) |
| | (26.0 ,25.5) | (25.4, 25.2, 25.2, 25.2) | (25.2, 25.1, 25.1, 25.1, 25.1, 25.1, 25.1, 25.1) |
| | (38.3 ,31.7) | (29.6, 27.1, 26.9, 26.7) | (26.8, 25.6, 25.6, 25.6, 25.5, 25.5, 25.5, 25.5) |
| $m = 40$ | (25.1 ,25.1) | (25.1, 25.1, 25.1, 25.1) | (25.0, 25.0, 25.0, 25.0, 25.0, 25.0, 25.0, 25.0) |
| | (25.8 ,25.6) | (25.3, 25.2, 25.2, 25.2) | (25.1, 25.1, 25.1, 25.1, 25.1, 25.1, 25.1, 25.1) |
| | (35.7 ,32.2) | (28.3, 27.0, 26.9, 26.8) | (26.2, 25.6, 25.6, 25.6, 25.5, 25.5, 25.5, 25.5) |

Table 7: The Equilibrium Point of the Infectives when $\mu=0.1$

| | $s = 2$ | $s = 4$ | $s = 8$ |
|----------|--------------|--------------------------|--|
| $m = 10$ | (50.3, 50.2) | (50.2, 50.2, 50.1, 50.1) | (50.1, 50.1, 50.1, 50.1, 50.1, 50.1, 50.1, 50.1) |
| | (52.2, 50.8) | (51.1, 50.4, 50.4, 50.3) | (50.6, 50.2, 50.2, 50.2, 50.2, 50.1, 50.1, 50.1) |
| | (80.5, 60.3) | (62.8, 53.9, 53.3, 52.9) | (55.5, 51.3, 51.2, 51.1, 51.0, 50.9, 50.9, 50.9) |
| $m = 20$ | (50.3, 50.2) | (50.2, 50.1, 50.1, 50.1) | (50.1, 50.1, 50.1, 50.1, 50.1, 50.1, 50.1, 50.1) |
| | (51.8, 51.0) | (50.8, 50.4, 50.4, 50.4) | (50.4, 50.2, 50.2, 50.2, 50.2, 50.1, 50.1, 50.1) |
| | (73.7, 61.9) | (58.6, 53.9, 53.5, 53.2) | (53.3, 51.2, 51.1, 51.1, 51.0, 51.0, 51.1, 50.1) |
| $m = 40$ | (50.2, 50.2) | (50.2, 50.1, 50.1, 50.1) | (50.1, 50.1, 50.1, 50.1, 50.1, 50.1, 50.1, 50.1) |
| | (51.6, 51.1) | (50.7, 50.4, 50.4, 50.4) | (50.3, 50.2, 50.2, 50.2, 50.2, 50.2, 50.2, 50.1) |
| | (69.3, 62.9) | (56.2, 53.8, 53.6, 53.4) | (52.2, 51.1, 51.1, 51.1, 51.0, 51.0, 51.0, 51.0) |

to be stable. Newton's method with the Sherman-Morrison-Woodbury Formula are used to compute the equilibrium values of the infectives. Numerical results indicate that the method is efficient.

From the tested numerical examples, we observe that the model has a unique positive equilibrium point. We seek a mathematical proof for the existence and uniqueness of a positive equilibrium point in the network. The model proposed here is of the SI type; it is therefore interesting to extend it to the SIR type epidemic model [2] and also the case of quarantining infected prisoners. Moreover, the setting of the problem can be a general urban social network [3]. In our model, we assume that the number of prisoners exchanged among the prisons are the same. One may further extend the result to the case when this restriction is removed.

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