LOGARITHMIC CONVEXITY OF THE ONE-PARAMETER MEAN VALUES

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ABSTRACT. In this article, the logarithmic convexity of the one-parameter mean values $J(r)$ and the monotonicity of the product $J(r)J(-r)$ with $r \in \mathbb{R}$ are presented. Some more general results are established. Three open problems are posed.

CONTENTS

1. INTRODUCTION

Define a function $g(r; x, y)$ for $x \neq y$ by

$$g(t) \triangleq g(t; x, y) = \begin{cases} \dfrac{y^t - x^t}{t}, & t \neq 0; \\ \ln y - \ln x, & t = 0. \end{cases} \tag{1}$$

The following integral form of g is presented and applied in [11, 13, 16, 17]:

$$g(t) = \int_x^y u^{t-1}\,\mathrm{d}u, \quad t \in \mathbb{R}, \tag{2}$$

$$g^{(n)}(t) = \int_x^y (\ln u)^n u^{t-1}\,\mathrm{d}u, \quad t \in \mathbb{R}. \tag{3}$$

Straightforward computation results in

$$\left(\frac{g'(t)}{g(t)}\right)' = \frac{g''(t)g(t) - [g'(t)]^2}{g^2(t)}, \tag{4}$$

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$$\left(\frac{g'(t)}{g(t)}\right)'' = \frac{g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2[g'(t)]^3}{g^3(t)}. \tag{5}$$

In [11], Corallary 3 states that, for $y > x > 0$, if $t > 0$, then

$$g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2[g'(t)]^3 < 0; \tag{6}$$

if $t < 0$, inequality (6) reverses.

The function $g(t; x, y)$ and its integral expressions (2) and (3) are very important in the proofs of the logarithmic convexity [11, 13] and Schur-convexity [12, 13, 15] of the extended mean values $E(r, s; x, y)$, which is a generalization of the one-parameter mean values $J(r)$ with $E(r, r + 1; x, y) = J(r; x, y)$. The monotonicity and comparison of $E(r, s; x, y)$ were studied in [6, 7, 8, 13]. The concepts of mean values are generalized in [9, 10, 13, 19]. For more information about the extended mean values $E(r, s; x, y)$, please refer to the expository article [13] and the references therein.

The one-parameter mean values $J(r; x, y)$ for $x \neq y$ are defined in [1, 20] and introduced in [5, p. 44] by

$$J(r) \triangleq J(r; x, y) = \begin{cases} \dfrac{r(x^{r+1} - y^{r+1})}{(r + 1)(x^r - y^r)}, & r \neq 0, -1; \\ \dfrac{x - y}{\ln x - \ln y}, & r = 0; \\ \dfrac{xy(\ln x - \ln y)}{x - y}, & r = -1. \end{cases} \tag{7}$$

In [4, p. 49], the following results in [2, 3] by Alzer are mentioned:

(1) When $r \neq 0$, we have

$$G(x, y) < \sqrt{J(r; x, y)J(-r; x, y)} < L < \frac{J(r; x, y) + J(-r; x, y)}{2} < A(x, y). \tag{8}$$

(2) For $x_1 > 0$, $x_2 > 0$, $y_1 > 0$ and $y_2 > 0$, if $r \geq 1$, then

$$J(r; x_1 + y_1, x_2 + y_2) \leq J(r; x_1, x_2) + J(r; y_1, y_2); \tag{9}$$

if $r \leq 1$, inequality (9) is reversed.

(3) If (x_1, x_2) and (y_1, y_2) are similarly or oppositely ordered, then, if $r < -\frac{1}{2}$, we have

$$J(r; x_1y_1 + x_2y_2) \geq J(r; x_1, x_2)J(r; y_1, y_2); \tag{10}$$

if $r \geq -\frac{1}{2}$, then inequality (10) is reversed.

(4) For $x > 0$ and $y > 0$, if $r < s < t \leq -\frac{1}{2}$, then

$$[J(s; x, y)]^{t-r} \leq [J(r; x, y)]^{t-s}[J(t; x, y)]^{s-r}; \tag{11}$$

if $-\frac{1}{2} \leq r < s < t$, inequality (11) is reversed.

Moreover, H. Alzer in [3] raised a question about the convexity of $r \ln J(r; x, y)$ and proved that $(r + 1)J(r; x, y)$ is convex.

In April of 2004, Witkowski looked for the reference to the inequality

$$J(r; x, y)J(-r; x, y) \leq [J(0; x, y)]^2 = L^2(x, y), \tag{12}$$

which is contained in (8), through S. S. Dragomir by an e-mail which was forwarded to all members of the Research Group in Mathematical Inequalities and Applications at `http://rgmia.vu.edu.au`.

The main purpose of this paper is to prove the logarithmic convexity of the one-parameter mean values $J(r;x,y)$ and the monotonicity of $J(-r)J(r)$ for $r\in\mathbb{R}$.

Our main results are as follows.

Theorem 1. *For fixed positive numbers x and y with $x\ne y$, we have*

(i) *The one-parameter mean values $J(r)$ defined by (7) are strictly increasing in $r\in\mathbb{R}$;*

(ii) *The one-parameter mean values $J(r)$ defined by (7) are strictly logarithmically convex in $\left(-\infty,-\frac12\right)$ and strictly logarithmically concave in $\left(-\frac12,\infty\right)$.*

Remark 1. Though the monotonicity property of $J(r;x,y)$ with $r\in\mathbb{R}$ is well known, as a by-product of Theorem 1 and for completeness, we give it other two proofs below. However, we cannot affirm whether they are new proofs or not.

Theorem 2. *Let $\mathcal{J}(r)=J(r)J(-r)$ with $r\in\mathbb{R}$ for fixed positive numbers x and y with $x\ne y$. Then the function $\mathcal{J}(r)$ is strictly increasing in $(-\infty,0)$ and strictly decreasing in $(0,\infty)$.*

Remark 2. Inequality (12) is clearly a direct consequence of Theorem 2.

2. Proofs of theorems

Proof of Theorem 1. (i) Formula (6) implies that, for $y>x>0$,

$$\left(\frac{g'(t)}{g(t)}\right)''\begin{cases}>0, & t<0,\\ =0, & t=0,\\ <0, & t>0.\end{cases}\tag{13}$$

From this, we obtain that the function $\left(\frac{g'(t)}{g(t)}\right)'$ is strictly increasing in $(-\infty,0)$ and strictly decreasing in $(0,\infty)$.

In [14, 18], by using the Cauchy-Schwartz integral inequality or the Tchebycheff integral inequality, it is obtained that

$$\left(\frac{g'(t)}{g(t)}\right)'>0\tag{14}$$

for $t\in\mathbb{R}$. Then the function $\frac{g'(t)}{g(t)}$ is strictly increasing in $(-\infty,\infty)$.

The one-parameter mean values $J(r)$ can be rewritten in terms of g as

$$J(r)=\frac{g(r+1)}{g(r)}\tag{15}$$

with $r\in\mathbb{R}$ for $y>x>0$. Taking logarithm of $J(r)$ yields

$$\ln J(r)=\ln g(r+1)-\ln g(r)=\int_r^{r+1}\frac{g'(u)}{g(u)}\,\mathrm{d}u=\int_0^1\frac{g'(u+r)}{g(u+r)}\,\mathrm{d}u\tag{16}$$

and

$$\bigl[\ln J(r)\bigr]'=\frac{g'(r+1)}{g(r+1)}-\frac{g'(r)}{g(r)}>0.\tag{17}$$

Hence the functions $\ln J(r)$ and $J(r)$ are strictly increasing in $r\in(-\infty,\infty)$. This proves (i).

(ii) If $r < -1$, then $r < r+1 < 0$ and

$$\big[\ln J(r)\big]'' = \left(\frac{g'(r+1)}{g(r+1)}\right)' - \left(\frac{g'(r)}{g(r)}\right)' > 0 \tag{18}$$

which follows from the strictly increasing property of $\left(\frac{g'(r)}{g(r)}\right)'$ in $(-\infty,0)$.

If $r > 0$, then from the strictly decreasing property of $\left(\frac{g'(r)}{g(r)}\right)'$ in $(0,\infty)$, we have $\big[\ln J(r)\big]'' < 0$.

If $-1 < r < 0$, then $r < 0 < r+1$, and we have

$$\begin{aligned}\big[\ln J(r)\big]'' &= \left(\frac{g'(r+1)}{g(r+1)}\right)' - \left(\frac{g'(r)}{g(r)}\right)' \\ &= \frac{g''(r+1)g(r+1)-[g'(r+1)]^2}{g^2(r+1)} - \frac{g''(r)g(r)-[g'(r)]^2}{g^2(r)} \\ &= \frac{g''(u)g(u)-[g'(u)]^2}{g^2(u)} - \frac{g''(-r)g(-r)-[g'(-r)]^2}{g^2(-r)} \\ &= \frac{g''(u)g(u)-[g'(u)]^2}{g^2(u)} - \frac{g''(v)g(v)-[g'(v)]^2}{g^2(v)} \\ &= \left(\frac{g'(u)}{g(u)}\right)' - \left(\frac{g'(v)}{g(v)}\right)',\end{aligned} \tag{19}$$

where $u = r+1 > 0$ and $v = -r > 0$. Thus, $\big[\ln J(r)\big]'' < 0$ for $-1 < r < 0$ and $r+1 > -r$. This means that $\big[\ln J(r)\big]'' < 0$ for $r \in \left(-\frac12, 0\right)$.

Similarly as above, $\big[\ln J(r)\big]'' > 0$ for $-1 < r < 0$ and $-r > r+1$. This means that $\big[\ln J(r)\big]'' > 0$ for $r \in \left(-1, -\frac12\right)$. This proves (ii).

The proof of Theorem 1 is completed. □

Remark 3. From (16), (13) and by direct calculation, we have

$$[\ln J(r)]'' = \int_0^1 \frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{g'(u+r)}{g(u+r)}\right)\mathrm{d}u < 0 \tag{20}$$

for $r \in (0,\infty)$. This means that $J(r;x,y)$ is strictly logarithmically concave in $r \in (0,\infty)$, whether $x > y$ or $x < y$, since $J(r;x,y) = J(r;y,x)$ holds.

By straightforward computation, we have

$$J(r) = \frac{xy}{J(-r-1)} \tag{21}$$

for $r \in \mathbb{R}$. Hence, if $r \in (-\infty,-1)$, from (6), (20) and (13), it follows that

$$[\ln J(r)]'' = -[\ln J(-r-1)]'' = -\int_0^1 \frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{g'(u-r-1)}{g(u-r-1)}\right)\mathrm{d}u > 0. \tag{22}$$

This tells us that the one-parameter mean values $J(r;x,y)$ are strictly logarithmically convex in $r \in (-\infty,-1)$, whether $x > y$ or $x < y$, since $J(r;x,y) = J(r;y,x)$.

Proof of Theorem 2. By standard argument, we obtain

$$\mathcal{J}(r) = \frac{xyJ(r)}{J(r-1)} \tag{23}$$

for $r \in \mathbb{R}$. Then

$$\ln \mathcal{J}(r) = \ln(xy) + \ln J(r) - \ln J(r-1), \tag{24}$$

$$[\ln \mathcal{J}(r)]' = \frac{J'(r)}{J(r)} - \frac{J'(r-1)}{J(r-1)}. \tag{25}$$

Theorem 1 states that the function $J(r)$ is strictly logarithmically convex in $\left(-\infty, -\frac{1}{2}\right)$. Thus, being the derivative of $\ln J(r)$, $\frac{J'(r)}{J(r)}$ is strictly increasing in $\left(-\infty, -\frac{1}{2}\right)$, that is

$$\frac{J'(r)}{J(r)} > \frac{J'(r-1)}{J(r-1)}, \tag{26}$$

or, equivalently, $[\ln \mathcal{J}(r)]' > 0$ for $r \in \left(-\infty, -\frac{1}{2}\right)$, thus $\ln \mathcal{J}(r)$ and $\mathcal{J}(r)$ are strictly increasing in $\left(-\infty, -\frac{1}{2}\right)$.

From (21), it follows that

$$\ln J(r) = \ln(xy) - \ln J(-r-1), \tag{27}$$

$$\frac{J'(r)}{J(r)} = \frac{J'(-r-1)}{J(-r-1)}. \tag{28}$$

Then, from (25), we have

$$[\ln \mathcal{J}(r)]' = \frac{J'(-r-1)}{J(-r-1)} - \frac{J'(r-1)}{J(r-1)}. \tag{29}$$

For $r \in \left(-\frac{1}{2}, 0\right)$, we have $-\frac{3}{2} < r-1 < -1$ and $-1 < -r-1 < -\frac{1}{2}$. Since $\frac{J'(r)}{J(r)}$ is strictly increasing in $\left(-\infty, -\frac{1}{2}\right)$, $[\ln \mathcal{J}(r)]' > 0$ for $r \in \left(-\frac{1}{2}, 0\right)$, therefore $\ln \mathcal{J}(r)$ and $\mathcal{J}(r)$ are also strictly increasing in $\left(-\frac{1}{2}, 0\right)$.

It is clear that the function $\mathcal{J}(r)$ is even in $(-\infty, \infty)$. So, it is easy to see that $\mathcal{J}(r)$ is strictly decreasing in $(0, \infty)$. The proof of Theorem 2 is completed. □

3. Some related results

For $x \neq y$ and $\alpha > 0$, define

$$J_\alpha(r) \triangleq J_\alpha(r; x, y) = \begin{cases} \dfrac{r(x^{r+\alpha} - y^{r+\alpha})}{(r+\alpha)(x^r - y^r)}, & r \neq 0, -\alpha; \\ \dfrac{x^\alpha - y^\alpha}{\alpha(\ln x - \ln y)}, & r = 0; \\ \dfrac{\alpha x^\alpha y^\alpha (\ln x - \ln y)}{x^\alpha - y^\alpha}, & r = -\alpha. \end{cases} \tag{30}$$

We call $J_\alpha(r; x, y)$ the generalized one-parameter mean values for two positive numbers x and y in the interval $(-\infty, \infty)$.

It is clear that $J_1(r; x, y) = J(r; x, y)$ and $J_\alpha(r; x, y) = \frac{g(r+\alpha)}{g(r)}$.

By the same arguments as in the proofs of Theorems 1 and 2, we can obtain the following

Theorem 3. *For positive numbers x and y with $x \neq y$, we have*

(1) *The generalized one-parameter mean values $J_\alpha(r)$ defined by (30) are strictly increasing in $r \in \mathbb{R}$;*

(2) *The generalized one-parameter mean values* $J_\alpha(r)$ *defined by* (30) *are strictly logarithmically convex in* $\left(-\infty,-\frac{\alpha}{2}\right)$ *and strictly logarithmically concave in* $\left(-\frac{\alpha}{2},\infty\right)$.

(3) *Let* $\mathcal{J}_\alpha(r)=J_\alpha(r)J_\alpha(-r)$ *with* $r\in\mathbb{R}$ *for positive numbers* x *and* y *with* $x\ne y$. *Then the function* $\mathcal{J}_\alpha(r)$ *is strictly increasing in* $(-\infty,0)$ *and strictly decreasing in* $(0,\infty)$.

Proof. These follow from combining the identity

$$J_\alpha(r;x,y)=J\left(\frac{r}{\alpha};x^\alpha,y^\alpha\right) \tag{31}$$

with Theorems 1 and 2. □

Theorem 4. *The function* $(r+\alpha)J_\alpha(r)$ *is strictly increasing and strictly convex in* $(-\infty,\infty)$, *and is strictly logarithmically concave for* $r>-\frac{\alpha}{2}$.

Proof. Direct computation gives

$$(r+\alpha)J_\alpha(r;x,y)=\alpha\left(\frac{r}{\alpha}+1\right)J\left(\frac{r}{\alpha};x^\alpha,y^\alpha\right), \tag{32}$$

$$\big\{\ln[(r+\alpha)J_\alpha(r)]\big\}''=-\frac{1}{(r+\alpha)^2}+[\ln J_\alpha(r)]''. \tag{33}$$

From the result by Alzer in [3] that the function $(r+1)J(r;x,y)$ is strictly convex, it is not difficult to obtain that the function $(r+\alpha)J_\alpha(r;x,y)$ is also strictly convex in $(-\infty,\infty)$ by using (32).

By standard argument, we have

$$\lim_{r\to-\infty}[(r+\alpha)J'_\alpha(r)]=\lim_{r\to-\infty}\frac{\alpha(z^{r+\alpha}-1)}{(r+\alpha)(z^r-1)}-\lim_{r\to-\infty}\frac{rz^r(z^\alpha-1)\ln z}{(z^r-1)^2}=0 \tag{34}$$

and

$$\lim_{r\to-\infty}J_\alpha(r)=\min\{x^\alpha,y^\alpha\}, \tag{35}$$

where $z=\frac{y}{x}\ne1$. This leads to

$$\lim_{r\to-\infty}[(r+\alpha)J_\alpha(r)]'=\lim_{r\to-\infty}J_\alpha(r)+\lim_{r\to-\infty}[(r+\alpha)J'_\alpha(r)]=\min\{x^\alpha,y^\alpha\}>0. \tag{36}$$

The convexity of $(r+\alpha)J_\alpha(r)$ means that $[(r+\alpha)J_\alpha(r)]'$ is strictly increasing, in view of (36), $[(r+\alpha)J_\alpha(r)]'>0$, and so $(r+\alpha)J_\alpha(r)$ is strictly increasing in $(-\infty,\infty)$.

Since $J_\alpha(r)$ is strictly logarithmically concave in $\left(-\frac{\alpha}{2},\infty\right)$, we have $[\ln J_\alpha(r)]''<0$, then $\big\{\ln[(r+\alpha)J_\alpha(r)]\big\}''<0$ by (33). This means that the function $(r+\alpha)J_\alpha(r)$ is strictly logarithmically concave in $\left(-\frac{\alpha}{2},\infty\right)$. □

Corollary 1. *If* $r<-\alpha$, *then*

$$0<\frac{J'_\alpha(r)}{J_\alpha(r)}=\frac{J'_\alpha(-r-\alpha)}{J_\alpha(-r-\alpha)}<-\frac{1}{r+\alpha}, \tag{37}$$

$$0<\frac{J''_\alpha(r)}{J'_\alpha(r)}<-\frac{2}{r+\alpha}. \tag{38}$$

Proof. From the monotonicity and convexity of $(r+\alpha)J_\alpha(r)$, we have

$$[(r+\alpha)J_\alpha(r)]'=J_\alpha(r)+(r+\alpha)J'_\alpha(r)>0, \tag{39}$$

$$[(r+\alpha)J_\alpha(r)]''=2J'_\alpha(r)+(r+\alpha)J''_\alpha(r)>0. \tag{40}$$

Inequality (37) follows from the combination of (39) and

$$J_\alpha(r) = \frac{xy}{J_\alpha(-r-\alpha)}. \tag{41}$$

Inequality (38) is a direct consequence of (40). □

Theorem 5. *The function $r\ln J_\alpha(r)$ is strictly convex in $(-\frac{\alpha}{2},0)$.*

Proof. Direct calculation yields

$$[r\ln J_\alpha(r)]'' = 2[\ln J_\alpha(r)]' + r[\ln J_\alpha(r)]''. \tag{42}$$

Since $J_\alpha(r)$ is strictly increasing in $(-\infty,\infty)$ and strictly logarithmically concave in $\left(-\frac{\alpha}{2},\infty\right)$, it follows that $[\ln J_\alpha(r)]' > 0$ and $[\ln J_\alpha(r)]'' < 0$ in $\left(-\frac{\alpha}{2},\infty\right)$. Therefore, $[r\ln J_\alpha(r)]'' > 0$ and $r\ln J_\alpha(r)$ is strictly convex in $\left(-\frac{\alpha}{2},0\right)$. □

Remark 4. If $\alpha = 1$ and $\beta = 0$, then $r\ln J(r)$ is strictly convex in $\left(-\frac{1}{2},0\right)$. This answers partially the question raised by Alzer in [3].

4. Open problems

Finally, we pose the following

Open Problem 1. *The generalized one-parameter mean values $J_\alpha(r)$ defined by (30) are strictly concave in $\left(-\frac{\alpha}{2},\infty\right)$.*

Open Problem 2. *The function $\mathcal{J}_\alpha(t) = J_\alpha(t)J_\alpha(-t)$ is strictly logarithmically convex for $t \notin [-\frac{\alpha}{2},\frac{\alpha}{2}]$ and strictly concave and strictly logarithmically concave for $t \in (-\frac{\alpha}{2},\frac{\alpha}{2})$.*

Open Problem 3. *The function $J_\alpha(r)+J_\alpha(-r)$ is strictly decreasing in $(-\infty,0)$, strictly increasing in $(0,\infty)$, strictly convex in $(-r_\alpha,r_\alpha)$, and strictly concave for $r \notin [-r_\alpha,r_\alpha]$, where $r_\alpha > 0$ is a constant dependent on α.*

Remark 5. The following conclusions are well known.

(1) Although a logarithmically convex function is also convex, a convex function may be not logarithmically convex.
(2) A logarithmically concave function may be not concave.
(3) A concave function may be not logarithmically concave.

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