POISSON GEOMETRY OF THE GROTHENDIECK RESOLUTION OF A COMPLEX SEMISIMPLE GROUP

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ABSTRACT. Let G be a complex semi-simple Lie group with a fixed pair of opposite Borel subgroups (B, B_-). We study a Poisson structure π on G and a Poisson structure Π on the Grothendieck resolution X of G such that the Grothendieck map $\mu : (X, \Pi) \to (G, \pi)$ is Poisson. We show that the orbits of symplectic leaves of π in G under the conjugation action by the Cartan subgroup $H = B \cap B_-$ are intersections of conjugacy classes and Bruhat cells BwB_-, while the H-orbits of symplectic leaves of Π on X give desingularizations of intersections of Steinberg fibers and Bruhat cells in G. We also give birational Poisson isomorphisms from quotients by $H \times H$ of products of double Bruhat cells in G to intersections of Steinberg fibers and Bruhat cells.

Dedicated to Victor Ginzburg for his 50th birthday

CONTENTS

1. Introduction

Let G be a complex connected semi-simple Lie group. It is well-known that the choice of a pair (B, B_-) of opposite Borel subgroups of G leads to a standard Poisson structure π_G on G making (G, π_G) into a Poisson Lie group [26]. The Poisson Lie group (G, π_G) is the semi-classical limit of the quantum group $\mathbb{C}_q(G)$, the dual (as Hopf algebras) of the much studied quantized universal enveloping algebra $U_q\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. The geometry of the Poisson structure π_G is closely related to the representation theory of $\mathbb{C}_q(G)$ and $U_q\mathfrak{g}$, as seen in the fascinating work of Lusztig, Fomin, Zelevinsky and others on total positivity, canonical bases, and cluster algebras (see, for example, [4, 12, 13, 14, 30]). In particular, let $H = B \cap B_-$, a Cartan subgroup of G. Then the H-orbits of symplectic leaves of π_G in G are [19, 25] the so-called double Bruhat cells defined as

$$G^{u,v} = BuB \cap B_-vB_-, \qquad u, v \in W,$$

where W is the Weyl group of G with respect to H. Double Bruhat cells are the main motivating examples for cluster algebras [12, 13, 14].

Because of the connections between (G, π_G) and the quantum groups and cluster algebras, it is natural to study the dual Poisson Lie group (G^*, π_{G^*}) of (G, π_G), which is the semi-classical limit of $U_q\mathfrak{g}$. Let U and U_- be the unipotent radicals of B and B_- respectively. Then G^* is the subgroup of $B \times B_-$ given by

$$G^* = \{(uh, h^{-1}u_-) \mid u \in U, u_- \in U_-, h \in H\} \subset B \times B_-.$$

The map $\eta : G^* \to G : \ (b, b_-) \mapsto bb_-^{-1}$ is a local diffeomorphism and $\eta(G^*) = BB_-$ is open in G. It turns out that there is a unique Poisson structure π on G such that $\eta : (G^*, \pi_{G^*}) \to (G, -\pi)$ is a Poisson map. Thus we may study the Poisson geometry of (G^*, π_{G^*}) by studying $(G, -\pi)$. The first part of the paper, §2, is devoted to the Poisson structure π on G. (When G is of adjoint type, the Poisson structure π was considered in our previous papers [10] and [11], where its extension to the wonderful compactification of G is also discussed).

Recall that a *Poisson action* of the Poisson Lie group (G, π_G) is an action of G on a Poisson manifold (P, π_P) such that the action map $(G, \pi_G) \times (P, \pi_P) \to (P, \pi_P)$ is Poisson. We show in Proposition 2.10 and Proposition 2.17 that

1) the conjugation action of (G, π_G) on (G, π) is Poisson. In particular, π is invariant under conjugation by H;

2) the H-orbits of symplectic leaves of π in G are the nonempty intersections of conjugacy classes of G and Bruhat cells BwB_- for $w \in W$.

Since conjugacy classes and Bruhat cells in G do not always intersect (see Remark 2.14), we study Steinberg fibers in place of conjugacy classes. Intersections of Steinberg fibers and Bruhat cells in G are studied in §3. Recall [21] that a Steinberg fiber in G is the closure of a regular conjugacy class in G and is a finite union of conjugacy classes. It is shown in Proposition 3.3 that the intersection of a Steinberg fiber and a Bruhat cell is always nonempty. For $t \in H$, let F_t be the Steinberg fiber through t. Then $F_t = F_{t'}$ if and only if $t' = w.t$ for some $w \in W$, and

$$G = \bigsqcup_{t \in T/W, w \in W} (F_t \cap BwB_-) \quad \text{(disjoint union)}.$$

For $t \in H$ and $w \in W$, set $F_{t,w} = F_t \cap BwB_-$. Since F_t is a finite union of conjugacy classes, $F_{t,w}$ is a finite union of H-orbits of symplectic leaves of π. The subvariety $F_{t,w}$ of G is irreducible (Proposition 3.3) but may be singular (see Example 3.6).

Aside from our Poisson geometric motivation, the geometry of Steinberg fibers is quite important and subtle. In particular, the unipotent variety is a special case of a Steinberg fiber, and plays a key role in the Springer correspondence and the study of subregular singularities [21, 37]. Further, we believe that intersections of Steinberg fibers and Bruhat cells in G are important, and that it is worthwhile to study their algebro-geometric properties as well as the Poisson structure π on them. To do this, we consider in §4 the Grothendieck resolution $X = G \times_B B$ of G with the resolution map

$$\mu : \; X \longrightarrow G : \quad [g, b] \longmapsto gbg^{-1}, \qquad g \in G, b \in B,$$

where $[gb_1, b_1^{-1}bb_1] = [g, b]$ for $g \in G$ and $b, b_1 \in B$. Note that μ is G-equivariant, where G acts on G by conjugation and on X by

$$\sigma : \; G \times X \longrightarrow X : \quad g_1.[g, b] = [g_1 g, \, b], \;\; g_1, g \in G, b \in B.$$

We introduce a Poisson structure Π on X and show in Proposition 3.12, Proposition 4.5, and Theorem 4.8 that Π has the following properties:

1) the morphism $\mu : (X, \Pi) \to (G, \pi)$ is Poisson;

2) the action σ of (G, π_G) on (X, Π) is Poisson. In particular, Π is H-invariant;

3) the H-orbits of symplectic leaves of Π in X are the smooth and irreducible subvarieties $X_{t,w}$ of X, where $t \in H, w \in W$, and

$$X_{t,w} = (G \times_B tU) \cap \mu^{-1}(BwB_-) \subset X.$$

Assume that G is simply connected. For $t \in H$, let $X_t = G \times_B tU \subset X$. It is well-known [37, 40] that $\mu : X_t \to F_t$ is a resolution of singularities of F_t. We prove in Corollary 3.10 that for every $t \in H$ and $w \in W$,

$$\mu : \;\; X_{t,w} \longrightarrow F_{t,w}$$

is a resolution of singularities of $F_{t,w}$. Note that each $F_{t,w}$ is a *finite union* of H-orbits of symplectic leaves of π in G, while $X_{t,w}$ is a *single* H-orbit of symplectic leaves of Π in X. Thus, for any $t \in H$ and $w \in W$, the Poisson morphism

$$\mu : (X_{t,w}, \Pi) \longrightarrow (F_{t,w}, \pi)$$

may be used to better understand the singular Poisson structure π on $F_{t,w}$, and we call it an H-equivariant *Poisson desingularization* (see §4.4).

In §5, the desingularization $\mu : X_{t,w} \to F_{t,w}$ is used to obtain rational parametrizations of $F_{t,w}$. More precisely, let w_0 be the longest element in W. We construct an explicit biregular Poisson isomorphism between a Zariski open subset of $X_{t,w}$ and the Poisson variety $(G^{1,w^{-1}w_0}, \pi_G) \times_H (G^{1,w_0}, \pi_G)/H$, where $H \times H$ acts on $G^{1,w^{-1}w_0} \times G^{1,w_0}$ from the right by

$$(g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, \; h_1^{-1} g_2 h_2), \qquad g_1, g_2 \in G, \, h_1, h_2 \in H.$$

In future work, we will use the birational isomorphism to study log-canonical coordinates for $(X_{t,w}, \Pi)$ and $(F_{t,w}, \pi)$ and investigate the combinatorial consequences and relations to work of Fomin, Kogan, and Zelevinsky in [12, 25].

Our Poisson geometric interpretation of the Grothendieck resolution $\mu : G \times_B B \to G$ is heavily influenced by ideas of Victor Ginzburg, who emphasized the symplectic nature of the Grothendieck resolution

$$\mu_0 : \quad G \times_B \mathfrak{b} \to \mathfrak{g} : \quad (g, x) \longmapsto \mathrm{Ad}_g(x), \qquad g \in G, \, x \in \mathfrak{g},$$

in geometry and representation theory [6], where $\mathfrak{g}$ and $\mathfrak{b}$ are the Lie algebras of G and B respectively. Note that the verctor space $\mathfrak{g}$ can be given the linear Kostant-Kirillov Poisson structure π_0, while $G \times_B \mathfrak{b}$ has the Poisson structure Π_0 whose symplectic leaves are $G \times_B (y + \mathfrak{n})$ with the twisted cotangent bundle symplectic structures, where $y \in \mathfrak{h}$, and $\mathfrak{h}$ and $\mathfrak{n}$ are the Lie algebras of H and U respectively. It is easy to see that the linearization of π at the identity element $e \in G$ is $(\mathfrak{g}, \pi_0)$, so (G, π) can be regarded as a deformation of $(\mathfrak{g}, \pi_0)$. In future work, we hope to study in more detail the relation between the Poisson morphisms $\mu : (G \times_B B, \, \Pi) \to (G, \pi)$ and $\mu_0 : (G \times_B \mathfrak{b}, \Pi_0) \to (\mathfrak{g}, \pi_0)$.

In §6, the Appendix, we collect some results on Poisson Lie groups and coisotropic reduction that are needed in constructing the Poisson structures π on G and Π on $G \times_B B$. We also prove that for a Steinberg fiber F and $w \in W$, $F \cap \overline{BwB_-}$ is normal and Cohen-Macaulay. We compute its dimension and describe its singular set.

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1.1. **Notation.** If G is a Lie group with Lie algebra $\mathfrak{g}$, and if $X \in \wedge^k \mathfrak{g}$, X^L and X^R will denote respectively the left and right invariant k-vector fields on G with $X^L(e) = X^R(e) = X$, where e is the identity element of G. If π is a bi-vector field on a manifold P, $\tilde{\pi}$ denotes the bundle map

$$\tilde{\pi} : \quad T^*P \longrightarrow TP : \quad (\tilde{\pi}(\alpha), \, \beta) = \pi(\alpha, \beta), \qquad \alpha, \beta \in \Omega^1(P). \tag{1.1}$$

By a variety, we mean a complex quasi-projective variety, and a subvariety is a locally closed subset of a variety. For a morphism $f : X \to Y$ between varieties, we also use $f : TX \to TY$ for its differential. When a group G acts on a space X, $g.x$ will stand for the action of $g \in G$ on $x \in X$.

2. The Poisson structure π on G

Let G be a connected complex semi-simple Lie group. In this section, we recall the definition of two Poisson structures π_G and π on G. The Poisson structure π_G on G is *multiplicative* [28] and (G, π_G) is a Poisson Lie group [26]. The Poisson structure π has the property that the conjugation action

$$(G, \pi_G) \times (G, \pi) \longrightarrow (G, \pi) : \quad (g, h) \longmapsto ghg^{-1}, \qquad g, h \in G$$

is Poisson. One obtains π as a special case of a general construction in the theory of Poisson Lie groups, which is reviewed in §6.1 in the Appendix.

2.1. **The Poisson Lie group (G, π_G) and its dual group (G^*, π_{G^*}).** Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ be the direct product Lie algebra. Let $\ll \, , \, \gg$ be a fixed non-zero scalar multiple of the Killing form of $\mathfrak{g}$, and let $\langle \, , \, \rangle$ be the symmetric ad-invariant nondegenerate bilinear form on $\mathfrak{d}$ given by

$$\langle x_1 + y_1, \; x_2 + y_2 \rangle = \ll x_1, x_2 \gg - \ll y_1, y_2 \gg, \qquad x_1, x_2, y_1, y_2 \in \mathfrak{g}.$$

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a choice Φ^+ of positive roots in the set Φ of roots for $(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}^\alpha$ be the corresponding root decomposition, and let

$$\mathfrak{n} = \sum_{\alpha \in \Phi^+} \mathfrak{g}^\alpha, \qquad \mathfrak{n}_- = \sum_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha}.$$

The so-called *standard Manin triple* associated to $\mathfrak{g}$ (see [26]) is the quadruple $(\mathfrak{d}, \mathfrak{g}_\Delta, \mathfrak{g}^*_{\rm st}, \langle \, , \, \rangle)$, where $\mathfrak{g}_\Delta = \{(x, x) : x \in \mathfrak{g}\}$ is the diagonal of $\mathfrak{d}$, and

$$\mathfrak{g}^*_{\rm st} = \mathfrak{h}_{-\Delta} + (\mathfrak{n} \oplus \mathfrak{n}_-) = \{(x + y, -y + x_-) : x \in \mathfrak{n}, x_- \in \mathfrak{n}_-, y \in \mathfrak{h}\}. \tag{2.1}$$

In particular, both $\mathfrak{g}_\Delta$ and $\mathfrak{g}^*_{\rm st}$ are maximal isotropic with respect to $\langle \, , \, \rangle$, and $\langle \, , \, \rangle$ gives rise to a non-degenerate pairing between $\mathfrak{g}_\Delta$ and $\mathfrak{g}^*_{\rm st}$.

Let $G_\Delta = \{(g, g) : g \in G\} \subset G \times G$, and let G^* be the connected subgroup of $G \times G$ with Lie algebra $\mathfrak{g}^*_{\rm st}$. The splitting $\mathfrak{d} = \mathfrak{g}_\Delta + \mathfrak{g}^*_{\rm st}$ gives rise to multiplicative Poisson structures π_G on $G \cong G_\Delta$ and π_{G^*} on G^* making them into a pair of dual Poisson Lie groups [26] (see also the Appendix). If U, U_-, and H are the connected subgroups of G with Lie algebras $\mathfrak{n}, \mathfrak{n}_-$, and $\mathfrak{h}$ respectively, then

$$G^* = \{(nh, h^{-1}n_-) : n \in U, n_- \in U_-, h \in H\} \subset G \times G. \tag{2.2}$$

We will refer to π_G as the *standard multiplicative Poisson structure* on G.

Notation 2.1. Throughout the paper, for each $\alpha \in \Phi^+$, we fix root vectors $E_\alpha \in \mathfrak{g}^\alpha$ and $E_{-\alpha} \in \mathfrak{g}^{-\alpha}$ such that $\ll E_\alpha, E_{-\alpha} \gg = 1$.

The following fact on π_G is well-known [26].

Proposition 2.2. *The Poisson structure π_G is given by $\pi_G = \Lambda_0^R - \Lambda_0^L$, where $\Lambda_0 = \frac{1}{2} \sum_{\alpha \in \Phi^+} E_\alpha \wedge E_{-\alpha} \in \wedge^2 \mathfrak{g}$. (See notation in §1.1).*

Example 2.3. Let $G = SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \, a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}$, and let $\ll x, y \gg = \frac{1}{2\lambda} \operatorname{tr}(xy)$ for $x, y \in \mathfrak{sl}(2, \mathbb{C})$, where $\lambda \in \mathbb{C}, \lambda \neq 0$. Let $\mathfrak{h}$ be the Cartan subalgebra consisting of diagonal matrices in $\mathfrak{sl}(2, \mathbb{C})$, and take the standard choices of positive and negative roots. Then $\Lambda_0 = \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and the Poisson brackets with respect to π_G between the functions a, b, c and d are:

$$\{a, b\}_\lambda = \lambda ab, \quad \{a, c\}_\lambda = \lambda ac, \quad \{b, d\}_\lambda = \lambda bd,$$
$$\{c, d\}_\lambda = \lambda cd, \quad \{a, d\}_\lambda = 2\lambda bc, \quad \{b, c\}_\lambda = 0.$$

We will denote this Poisson structure by $\pi^\lambda_{SL(2,\mathbb{C})}$.

Notation 2.4. For $\alpha \in \Phi^+$, let $H_\alpha \in \mathfrak{h}$ be such that $\ll H_\alpha, x \gg = \alpha(x)$ for all $x \in \mathfrak{h}$, and let $\ll \alpha, \alpha \gg = \ll H_\alpha, H_\alpha \gg$. Set

$$h_\alpha = \frac{2}{\ll \alpha, \alpha \gg} H_\alpha, \qquad e_\alpha = \sqrt{\frac{2}{\ll \alpha, \alpha \gg}} E_\alpha, \qquad e_{-\alpha} = \sqrt{\frac{2}{\ll \alpha, \alpha \gg}} E_{-\alpha}.$$

Then the linear map $\phi_\alpha : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ given by

$$\phi_\alpha : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longmapsto h_\alpha, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto e_\alpha, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longmapsto e_{-\alpha}$$

is a Lie algebra homomorphism. The corresponding Lie group homomorphism from $SL(2,\mathbb{C})$ to G will also be denoted by ϕ_α.

The following fact is also well-known [26].

Proposition 2.5. *Let α be any simple root, and let $\lambda_\alpha = \frac{\ll\alpha,\alpha\gg}{4}$. Then the map*

$$\phi_\alpha : \quad \left(SL(2,\mathbb{C}), \ \pi^{\lambda_\alpha}_{SL(2,\mathbb{C})}\right) \longrightarrow (G, \pi_G)$$

is Poisson, where $\pi^{\lambda_\alpha}_{SL(2,\mathbb{C})}$ is the Poisson structure on $SL(2,\mathbb{C})$ in Example 2.3.

Since π_G vanishes at points in H, π_G is invariant under left translation by elements in H. By an H-orbit of symplectic leaves of π_G, we mean a set of the form $\cup_{h\in H} h\Sigma$, where Σ is a symplectic leaf of π_G. For $u, v \in W$, let $G^{u,v} \subset G$ be the double Bruhat cell given by

$$G^{u,v} = BuB \cap B_-vB_-. \tag{2.3}$$

By [12, Theorem 1.1], $\dim(G^{u,v}) = l(u)+l(v)+\dim(H)$, where l is the length function on W.

Lemma 2.6. [19, 20, 25, 34] *The H-orbits of symplectic leaves of π_G in G are precisely the double Bruhat cells $G^{u,v}$ for $u, v \in W$.*

2.2. **Definition of the Poisson structure π on G.** Let $n = \dim \mathfrak{g}$. Let $\{x_j\}_{j=1}^n$ be a basis of $\mathfrak{g}_\Delta$ and let $\{\xi_j\}_{j=1}^n$ be the basis of $\mathfrak{g}^*_{\rm st}$ such that $\langle x_j, \xi_k\rangle = \delta_{jk}$ for $1 \leq j, k \leq n$. Let

$$\Lambda = \frac{1}{2}\sum_{j=1}^n (\xi_j \wedge x_j) \in \wedge^2(\mathfrak{g}\oplus\mathfrak{g}). \tag{2.4}$$

Let $D = G \times G$. Since $(\mathfrak{g}\oplus\mathfrak{g}, \mathfrak{g}_\Delta, \mathfrak{g}^*_{\rm st})$ is a Manin triple, the bi-vector field

$$\pi_D^+ = \Lambda^R + \Lambda^L \tag{2.5}$$

on D is Poisson (see §6.1 in the Appendix). Let $p : D \to D/G_\Delta$ be the natural projection. Then (see Proposition 6.2 in the Appendix), $p(\pi_D^+)$ is a Poisson structure on D/G_Δ. Since $p(\Lambda^L) = 0$, we also have $p(\pi_D^+) = p(\Lambda^R)$.

Notation 2.7. Identify $D/G_\Delta \cong G$ via the map $(g_1, g_2)G_\Delta \mapsto g_1g_2^{-1}$, and let

$$\eta : \quad D = G\times G \longrightarrow G : \quad (g_1, g_2) \longmapsto g_1g_2^{-1}. \tag{2.6}$$

Througout this paper, π will denote the Poisson structure $\eta(\pi_D^+)$ on G.

Remark 2.8. By Lemma 6.3 in the Appendix, the local diffeomorphism

$$\eta|_{G^*} : \quad (G^*, \ -\pi_{G^*}) \longrightarrow (G, \pi) : \quad (b, b_-) \longmapsto bb_-^{-1}$$

is Poisson.

Let $\{y_i\}_{i=1}^r$ be a basis of $\mathfrak{h}$ such that $2 \ll y_i, y_j \gg = \delta_{ij}$ for $1 \leq i, j \leq r = \dim \mathfrak{h}$. As bases of $\mathfrak{g}_\Delta$ and $\mathfrak{g}_{\mathrm{st}}^*$, let

$$\{x_i\} = \{(y_1, y_1),\ (y_2, y_2), \ldots, (y_r, y_r),\ (E_\alpha, E_\alpha),\ (E_{-\alpha}, E_{-\alpha}) : \alpha \in \Phi^+\} \tag{2.7}$$

$$\{\xi_i\} = \{(y_1, -y_1),\ (y_2, -y_2), \ldots, (y_r, -y_r),\ (0, -E_{-\alpha}),\ (E_\alpha, 0) : \alpha \in \Phi^+\}. \tag{2.8}$$

The element $\Lambda \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g})$ in (2.4) is then given by

$$\begin{aligned} \Lambda = & \frac{1}{2}\sum_{i=1}^r (y_i, -y_i) \wedge (y_i, y_i) \\ & + \frac{1}{2}\sum_{\alpha \in \Phi^+} \left((E_\alpha, 0) \wedge (E_{-\alpha}, E_{-\alpha}) + (0, -E_{-\alpha}) \wedge (E_\alpha, E_\alpha)\right). \end{aligned} \tag{2.9}$$

Since $\eta((x,0)^R) = x^R$ and $\eta((0,x)^R) = -x^L$ for $x \in \mathfrak{g}$,

$$\pi = \sum_{i=1}^r y_i^L \wedge y_i^R + \frac{1}{2}\sum_{\alpha \in \Phi^+} (E_\alpha^R \wedge E_{-\alpha}^R + E_\alpha^L \wedge E_{-\alpha}^L) + \sum_{\alpha \in \Phi^+} E_{-\alpha}^L \wedge E_\alpha^R. \tag{2.10}$$

Example 2.9. Let $G = SL(2, \mathbb{C})$. Using $\ll x, y \gg = \mathrm{tr}(xy)$ for $x, y \in \mathfrak{sl}(2, \mathbb{C})$, we can compute the Poisson structure π on $SL(2, \mathbb{C})$ directly from (2.10) to get

$$\begin{aligned} &\{a, b\}_\pi = bd, \quad \{a, c\}_\pi = -cd, \quad \{a, d\}_\pi = 0, \\ &\{b, c\}_\pi = ad - d^2, \quad \{b, d\}_\pi = bd, \quad \{c, d\}_\pi = -cd. \end{aligned}$$

It is easy to see that $a + d$ is a Casimir function.

The following Proposition 2.10 is a direct consequence of Proposition 6.2 in the Appendix and the fact that π_G vanishes at every point in H.

Proposition 2.10. *1) The following group actions are Poisson :*

$$(G, \pi_G) \times (G, \pi) \longrightarrow (G, \pi) : \quad (g_1, g) \longmapsto g_1 g g_1^{-1}, \tag{2.11}$$

$$(G^*, -\pi_{G^*}) \times (G, \pi) \longrightarrow (G, \pi) : \quad ((b, b_-), g) \longmapsto b g b_-^{-1}; \tag{2.12}$$

2) The Poisson structure π on G is invariant under conjugation by H.

Recall the definition of a coisotropic submanifold of a Poisson manifold from §6.2 in the Appendix. The following property of π will be used in §4.1.

Lemma 2.11. *For every $t \in H$, tU is a coisotropic submanifold of (G, π).*

Proof. By Remark 6.5 in the Appendix, it suffices to show that $\pi(b) \in T_b(tU) \wedge T_b G$ for every $b = tn \in tU$ with $n \in U$. First note that for $1 \leq i \leq r$,

$$y_i^L(b) \wedge y_i^R(b) = y_i^L(b) \wedge (y_i^R - y_i^L)(b) \in T_b(tU) \wedge T_b G.$$

Now since $E_\alpha^R(b) \in T_b(tU)$ and $E_\alpha^L(b) \in T_b(tU)$ for any $\alpha \in \Phi^+$, it follows from (2.10) that $\pi(b) \in T_b(tU) \wedge T_b G$. □

2.3. **Symplectic leaves of π in G.** The following Proposition 2.12 is a direct consequence of Proposition 6.2 of the Appendix. See also [1, 29].

Proposition 2.12. *The symplectic leaves of π in G are precisely the connected components of intersections of conjugacy classes in G and the G^*-orbits in G, where G^* acts on G by (2.12).*

The G^*-orbits in G can be determined by the Bruhat decomposition of G. Indeed, let W be the Weyl group of (G, H). For each $w \in W$, fix a representative $\dot{w}$ of w in the normalizer $N_G(H)$ of H in G, and let

$$U^w = U \cap \dot{w} U \dot{w}^{-1} \subset U \quad \text{and} \quad H_w = \{hw(h) : h \in H\} \subset H. \tag{2.13}$$

Lemma 2.13. *1) Every G^*-orbit in G for the action in (2.12) is of the form $G^*.(h\dot{w})$ for a unique $w \in W$ and for some $h \in H$. Moreover, for any $w \in W$ and $h_1, h_2 \in H$, $G^*.(h_1\dot{w}) = G^*.(h_2\dot{w})$ if and only if $h_1 h_2^{-1} \in H_w$;*

2) For any $w \in W$ and $h \in H$, the map

$$\phi_{w,h} : \ U^w \times H_w \times U_- \longrightarrow G^*.(h\dot{w}) : \ (n, h_1, n_-) \longmapsto n h h_1 \dot{w} n_-^{-1} \tag{2.14}$$

is a biregular isomorphism.

Proof. By the Bruhat decomposition, $G = \sqcup_{w \in W} BwB_-$ as a disjoint union. Clearly each BwB_- is a union of G^*-orbits. Moreover, for each $w \in W$,

$$\phi_w : \ U^w \times H \times U_- \longrightarrow BwB_- : \ (n, h, n_-) \longmapsto n h \dot{w} n_- \tag{2.15}$$

is a biregular isomorphism. Since $U \times U_- \subset G^*$, every G^*-orbit in BwB_- is of the form $G^*.(h\dot{w})$ for some $h \in H$. Now let $h_1, h_2 \in H$. Then $G^*.(h_1\dot{w}) = G^*.(h_2\dot{w})$ if and only if there exist $n \in U, n_- \in U_-$ and $h \in H$ such that $h_1\dot{w} = nhh_2\dot{w}hn_-$ which is equivalent to $h_1 h_2^{-1} \in H_w$. 2) follows from the fact that ϕ_w is a biregular isomorphism. □

Remark 2.14. A conjugacy class in G may not intersect every G^*-orbit in G. For example, the conjugacy class of the identity element of G intersects $G^*.(h\dot{w})$ if and only if $w = 1$. Moreover, the intersection of a conjugacy class and a G^*-orbit in G may not be connected. As an example, consider $G = SL(3, \mathbb{C})$, and let C be the conjugacy class of subregular unipotent elements, i.e.,

$$C = \{g \in SL(3, \mathbb{C}) : \ (g - I_3)^2 = 0, \ g \neq I_3\},$$

where I_3 is the 3×3 identity matrix. Let B and B_- be the subgroups of all upper and lower triangular matrices in $SL(3, \mathbb{C})$ respectively, and let $H = B \cap B_-$. Let w_0 be the longest element in the Weyl group $W \cong S_3$ and let

$$\dot{w}_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Fix $h = \mathrm{diag}(h_1, h_2, h_2) \in H$. It is easy to see that

$$G^*.(h\dot{w}_0) = \left\{ g = \begin{pmatrix} a & b & h_1 x \\ c & h_2 x^{-2} & 0 \\ -h_3 x & 0 & 0 \end{pmatrix} : \ a, b, c, x \in \mathbb{C}, x \neq 0 \right\}.$$

One then checks that an element $g \in G^*.(h\dot{w}_0)$ lies in C if and only if $a = 2, b = c = 0$ and $x^2 = h_2$. Thus the intersection $C \cap (G^*.(h\dot{w}_0))$ consists of exactly two points.

2.4. **Intersections of conjugacy classes and Bruhat cells.** In contrast to the situation for G^*-orbits in G, the intersection of a conjugacy class and a Bruhat cell BwB_- is always connected, as stated in the following Proposition 2.15.

Proposition 2.15. *Let C be a conjugacy class in G and let $w \in W$. Assume that $C \cap (BwB_-)$ is nonempty. Then $C \cap (BwB_-)$ is smooth and irreducible, and* $\dim(C \cap (BwB_-)) = \dim(C) - l(w)$.

Proof. This follows from [35, Corollary 1.5] because $G_\Delta \cap (B \times B_-)$ is connected. See also [11, Proposition 4.10]. □

Recall [21] that a conjugacy class C in G is said to be regular if $\dim C = \dim G - \dim H$.

Proposition 2.16. *If C is a regular conjugacy class in G, then $C \cap (BwB_-) \neq \emptyset$ for every $w \in W$.*

Proof. By [9, Proposition 5.1], $C \cap (BwB) \neq \emptyset$. It follows that $C \cap (Bw) \neq \emptyset$, which implies the result. □

2.5. **H-orbits of symplectic leaves of π.** Since the Poisson structure π is invariant under conjugation by elements in H, if Σ is a symplectic leaf of π, so is $h\Sigma h^{-1}$ for every $h \in H$. Let Σ be a symplectic leaf of π. The set

$$H.\Sigma := \{h\Sigma h^{-1} : h \in H\}$$

will be called an *H-orbit of symplectic leaves* of π in G. By Proposition 2.12 and Lemma 2.13, every H-orbit of symplectic leaves of π in G is contained in $C \cap (BwB_-)$ for some conjugacy class C in G and some $w \in W$. When G has trivial center, the following Proposition 2.17 is a special case of [11, Corollary 4.7 and Theorem 4.14].

Proposition 2.17. *Let C be a conjugacy class in G and let $w \in W$ be such that $C \cap (BwB_-) \neq \emptyset$. Then*

1) every symplectic leaf of π in $C \cap (BwB_-)$ has dimension equal to

$$\dim(C \cap (BwB_-)) - \dim(H/H_w) = \dim C - l(w) - \dim(H/H_w);$$

2) $C \cap (BwB_-)$ is a single H-orbit of symplectic leaves of π in G.

Proof. 1) Let Σ be a symplectic leaf of π in $C \cap (BwB_-)$. By Proposition 2.12, Σ is a connected component of $C \cap (G^*.(h\dot{w}))$ for some $h \in H$. Since C and $G^*.(h\dot{w})$ intersect transversally, using Lemma 2.13 and Proposition 2.15, one gets

$$\begin{aligned}\dim(\Sigma) &= \dim(C \cap (G^*.(h\dot{w})) = \dim(C) + \dim(G^*.(h\dot{w})) - \dim(G)\\ &= \dim(C) - l(w) - \dim(H/H_w) = \dim(C \cap (BwB_-)) - \dim(H/H_w).\end{aligned}$$

2) It is clear that if Σ and Σ' are two symplectic leaves of π in $C \cap (BwB_-)$, then either $H.\Sigma = H.\Sigma'$ or $(H.\Sigma) \cap (H.\Sigma') = \emptyset$. Since $C \cap (BwB_-)$ is connected by Proposition 2.15, to prove 2), it suffices to show that $H.\Sigma$ is open in $C \cap (BwB_-)$ for every symplectic leaf Σ in $C \cap (BwB_-)$. To this end, we show that the map $H \times \Sigma \to C \cap (BwB_-)$ for the conjugation action by H is a submersion.

Consider the action map $\alpha : H \times BwB_- \to BwB_-$ of H on BwB_- by conjugation. For $g \in BwB_-$, let $\alpha_* : \mathfrak{h} \times T_g(BwB_-) \to T_g(BwB-)$ be the differential of α at

$(e,g) \in H \times BwB_-$, and let $H.g$ be the H-orbit in BwB_- through g. Then for any submanifold S of BwB_- and $g \in S$, $\alpha_*(\mathfrak{h} \times T_gS) = T_g(H.g) + T_gS \subset T_g(BwB_-)$, so

$$\dim \alpha_*(\mathfrak{h} \times T_gS) = \dim(T_g(H.g)) + \dim(T_gS) - \dim(T_g(H.g) \cap T_gS). \tag{2.16}$$

First consider the case when $S = G^*.(h\dot{w})$ for some $h \in H$. By using the isomorphism $\phi_w : U^w \times H \times U_- \to BwB_-$ in (2.15), the map $\alpha : H \times BwB_- \to BwB_-$ becomes the map $\beta : H \times (U^w \times H \times U_-) \to U^w \times H \times U_-$ given by

$$\beta(h_1, (n, h_2, n_-)) = (h_1 n h_1^{-1},\ h_2 h_1 w(h_1^{-1}),\ h_1 n_- h_1^{-1}), \tag{2.17}$$

where $h_1, h_2 \in H, n \in U^w$ and $n_- \in U_-$. Let $\mathfrak{h}_w = \{x + w(x) : x \in \mathfrak{g}\}$ be the Lie algebra of H_w and let $\mathfrak{h}_w^- = \{x - w(x) : x \in \mathfrak{h}\}$. By using the isomorphism $\phi_{w,h} : U^w \times H_w \times U_- \to G^*.(h\dot{w})$ and the fact that $\mathfrak{h} = \mathfrak{h}_w + \mathfrak{h}_w^-$, one sees from (2.17) that $\alpha_*(\mathfrak{h} \times T_g(G^*.(h\dot{w}))) = T_g(BwB_-)$ for every $g \in G^*.(h\dot{w})$. It follows from (2.16) that for every $g \in G^*.(h\dot{w})$,

$$\begin{aligned}\dim(T_g(H.g) \cap T_g(G^*.(h\dot{w}))) &= \dim(T_g(H.g)) + \dim(T_g(G^*.(h\dot{w}))) - \dim BwB_- \\ &= \dim(T_g(H.g)) - \dim(H/H_w).\end{aligned}$$

Now let $S = \Sigma$, a connected component of $C \cap (G^*.(h\dot{w}))$. Then by (2.16),

$$\begin{aligned}\dim \alpha_*(\mathfrak{h} \times T_g\Sigma) &\geq \dim(T_g(H.g)) + \dim(T_g\Sigma) - \dim(T_g(H.g) \cap T_g(G^*.(h\dot{w}))) \\ &= \dim(T_g\Sigma) + \dim(H/H_w) = \dim(C \cap (BwB_-))\end{aligned}$$

for every $g \in \Sigma$, where in the last step we used 1). Since $\alpha(H \times \Sigma) \subset C \cap (BwB_-)$, we obtain $\dim \alpha_*(\mathfrak{h} \times T_g\Sigma) = \dim(C \cap (BwB_-))$. Thus, the map $\alpha|_{H\times\Sigma} : H \times \Sigma \to C \cap (BwB_-)$ has surjective differential at (e,g) for every $g \in \Sigma$. It follows that the differential of $\alpha|_{H\times\Sigma}$ is surjective everywhere in $H \times \Sigma$, so $H.\Sigma$ is open in $C \cap (BwB_-)$. This finishes the proof of 2).

□

Corollary 2.18. *The intersection of a regular conjugacy class and a G^*-orbit in G is nonempty.*

Proof. Let C be a regular conjugacy class and let $G^*.(h\dot{w})$ be a G^*-orbit in G, where $h \in H$ and $w \in W$. By Proposition 2.16, $C \cap (BwB_-) \neq \emptyset$. By Lemma 2.13 and Equation (2.17) in the proof of Proposition 2.17, $H.(G^*.(h\dot{w})) = BwB_-$. It follows that $C \cap (G^*.(h\dot{w})) \neq \emptyset$. □

3. Intersections of Steinberg fibers and Bruhat cells

We now apply our results on conjugacy classes to the study of Steinberg fibers. We will see in this section that the intersection $F \cap (BwB_-)$ of a Steinberg fiber F and a Bruhat cell BwB_- is always a nonempty irreducible Poisson subvariety of (G, π). The remainder of the paper will be devoted to the study of the algebro-geometric and Poisson geometric properties of the Poisson varieties $(F \cap (BwB_-), \pi)$. We do so by constructing in §4 an H-equivariant Poisson desingularization of $(F \cap (BwB_-), \pi)$ and in §5 a birational Poisson isomorphism between $(F \cap (BwB_-), \pi)$ and the $(H \times H)$-quotient of a product of double Bruhat cells in G.

3.1. **Steinberg fibers.** Recall that a regular class function on G is a regular function on G that is invariant under conjugation. Two elements $g_1, g_2 \in G$ are said to be in the same Steinberg fiber [38, Section 6] if $f(g_1) = f(g_2)$ for every regular class function f on G.

Proposition 3.1. [38, 6.11 and 6.15] *Let F be a Steinberg fiber in G. Then*
1) F is a closed irreducible subvariety of G with codimension $r = \dim H$;
2) F is a finite union of conjugacy classes;
3) F contains a single regular conjugacy class which is dense and open in F.

We now state a lemma that will be used several times in the paper.

Lemma 3.2. *Let Y be an algebraic variety. Assume that $k \geq 0$ is an integer such that 1) each irreducible component of Y has dimension at least k, and 2) $Y = Y_1 \cup Y_2 ... \cup Y_n$ is a disjoint union of subvarieties, where Y_1 is irreducible and $\dim Y_i < k$ for $i \geq 2$. Then $Y = \overline{Y_1}$ is irreducible.*

Proposition 3.3. *For any Steinberg fiber F in G and $w \in W$,*
1) $F \cap (BwB_-)$ is a nonempty irreducible subvariety of G with dimension equal to $\dim G - \dim H - l(w)$;
2) $F \cap (BwB_-)$ is a finite union of H-orbits of symplectic leaves of π in G.

Proof. By Proposition 2.16 and 3) of Proposition 3.1, $F \cap (BwB_-)$ is nonempty. Since $\dim BwB_- = \dim G - l(w)$, each irreducible component of $F \cap BwB_-$ has dimension no less than $\dim G - \dim H - l(w)$ using Proposition 3.1. Decompose $F \cap BwB_- = \cup_{i=1}^n (C_i \cap BwB_-)$, where $C_1, \ldots, C_n$ are the conjugacy classes in F. Part 1) follows by Proposition 2.15 and Lemma 3.2. Part 2) follows from Proposition 2.17, since F is a finite union of conjugacy classes. □

3.2. **The varieties $F_{t,w}$ and $X_{t,w}$.** For $t \in H$, let F_t be the Steinberg fiber containing t. By the Jordan decomposition of elements in G, every Steinberg fiber is of the form F_t for some $t \in H$, and $F_{w(t)} = F_t$ for $t \in H$ and $w \in W$.

Let B act on $G \times B$ from the right by

$$(G \times B) \times B \longrightarrow G \times B : \quad ((g, b), b_1) \longmapsto (gb_1, \, b_1^{-1} b b_1). \tag{3.1}$$

Denote the B-orbit through the point $(g, b) \in G \times B$ by $[g, b]$. The map

$$\mu : \quad G \times_B B \longrightarrow G : \quad [g, b] \longmapsto gbg^{-1}. \tag{3.2}$$

is called the *Grothendieck (simultaneous) resolution* of G. Since G is the union of all Borel subgroups of G [39, Page 69], μ is surjective. Note that μ is G-equivariant, where G acts on G by conjugation and on $G \times_B B$ by

$$\sigma : \; G \times (G \times_B B) \longrightarrow G \times_B B : \quad g_1.[g, b]) = [g_1 g, b], \qquad g, g_1 \in G, b \in B. \tag{3.3}$$

For $t \in H$, since tU is invariant under conjugation by B, $G \times_B tU$ is a smooth subvariety of $G \times_B B$. Set

$$X_t := G \times_B tU.$$

Then $\mu|_{X_t} : X_t \to F_t$ is onto. Clearly, $G \times_B B = \bigcup_{t \in H} X_t$ is a disjoint union.

Notation 3.4. For $t \in H$ and $w \in W$, let

$$F_{t,w} = F_t \cap BwB_- \subset G \quad \text{and} \quad X_{t,w} = X_t \cap \mu^{-1}(BwB_-) \subset G \times_B B.$$

Note that for any $t \in H$ and $w \in W$, $X_{t,w} \neq \emptyset$ because $F_{t,w} \neq \emptyset$.

3.3. **The singularities of $F_{t,w}$.** In this subsection, we assume that G is simply connected.

A Steinberg fiber F_t, where $t \in H$, may be singular, and it is shown in [21, 4.24] that the set of smooth points in F_t is the unique regular conjugacy class contained in F_t. The following Proposition 3.5 concerning the singularities of $F_{t,w} = F_t \cap (BwB_-)$ follows from Theorem 6.12, Proposition 6.13 and Theorem 6.14 in the Appendix.

Proposition 3.5. *For any $t \in H$ and $w \in W$, the set of smooth points of $F_t \cap BwB_-$ is $R_t \cap (BwB_-)$, where R_t is the unique regular conjugacy class in F_t. Moreover, $F_t \cap (BwB_-)$ is normal and is a complete intersection in BwB_-.*

Example 3.6. Consider again $G = SL(3, \mathbb{C})$ with B and B_- being the subgroups of upper and lower triangular matrices respectively. Let w_0 be the longest element in the Weyl group W. Then

$$Bw_0B_- = Bw_0 = \left\{ \begin{pmatrix} a & b & y \\ c & x & 0 \\ -\frac{1}{xy} & 0 & 0 \end{pmatrix} : \ a, b, c, x, y \in \mathbb{C}, x \neq 0, y \neq 0 \right\}.$$

Let $\mathcal{U}$ be the unipotent subvariety of G, so $\mathcal{U}$ is the Steinberg fiber containing the identity. Let C_r and $C_{\rm sr}$ be the regular and the sub-regular conjugacy class in $\mathcal{U}$ respectively. Then $\mathcal{U} \cap (Bw_0) = (C_r \cap (Bw_0)) \cup (C_{\rm sr} \cap (Bw_0))$ and $\mathcal{U} \cap (Bw_0)$ can be identified with the subset of $\mathbb{C}^5$ with coordinates (a, b, c, x, y) given by

$$\begin{cases} a + x = 3, \\ ax - bc + \frac{1}{x} = 3, \\ x \neq 0, y \neq 0. \end{cases}$$

Using the Jacobian criterion, one sees that $\mathcal{U} \cap (Bw_0)$ is singular exactly at the subregular elements

$$C_{\rm sr} \cap (Bw_0) = \left\{ \begin{pmatrix} 2 & 0 & y \\ 0 & 1 & 0 \\ -\frac{1}{y} & 0 & 0 \end{pmatrix} : \quad y \neq 0 \right\}.$$

3.4. **A desingularization of $F_{t,w}$.** Recall that if V is an irreducible variety and V_s its set of smooth points, a *desingularization* of V is a proper morphism $\xi : X \to V$, where X is an irreducible smooth variety and ξ maps $\xi^{-1}(V_s)$ isomorphically to V_s.

We again assume that G is simply connected. We show that for any $t \in H$ and $w \in W$, $X_{t,w}$ is a desingularization of $F_{t,w}$.

Proposition 3.7. *(Grothendieck, see* [37, Theorem 4.4] *and* [40, Corollary 6.4]*) For $t \in H$, $\mu : X_t \to F_t$ is a desingularization (called the Springer resolution) of F_t.*

Theorem 3.8. *For any $t \in H$ and $w \in W$, $X_{t,w}$ is smooth and irreducible, and* $\dim X_{t,w} = \dim G - \dim H - l(w)$.

To prove Theorem 3.8, we need a standard result from algebraic geometry.

Lemma 3.9. *Let $f : X \to Z$ be a morphism of smooth algebraic varieties, and let $Y \subset Z$ be a smooth irreducible subvariety. Suppose $f(T_x(X)) + T_{f(x)}(Y) = T_{f(x)}(Z)$ for all $x \in f^{-1}(Y)$. Then $f^{-1}(Y)$ is smooth and each connected component of $f^{-1}(Y)$ has dimension* $\dim X + \dim Y - \dim Z$.

Proof of Theorem 3.8. Consider $\mu_t := \mu|_{X_t} : X_t \to G$. Let $x \in X_{t,w}$, let $y = \mu(x)$. and let C_y be the conjugacy class of y in G. Then $\mu_t(T_x X_t) \supset T_y C_y$. Since C_y and BwB_- intersect transversally at y, one has

$$\mu_t(T_x(X_t)) + T_y(BwB_-) \supset T_y C_y + T_y(BwB_-) = T_y(G).$$

Thus $\mu_t(T_x(X_t)) + T_y(BwB_-) = T_y G$. It follows from Lemma 3.9 that $X_{t,w}$ is smooth with every irreducible component having dimension $\dim X_t - l(w)$. It remains to show that $X_{t,w}$ is irreducible. Let R_t be the regular conjugacy class in F_t. Since $\mu_t : \mu_t^{-1}(R_t) \to R_t$ is bijective, $\mu_t^{-1}(R_t) \cap X_{t,w}$ is irreducible by Proposition 2.15. For any conjugacy class $C \subset F_t \backslash R_t$, since $\dim \mu_t^{-1}(C) < \dim X_t$, by dividing $\mu_t^{-1}(C)$ into finitely many locally closed G-invariant smooth subvarieties of X_t, it follows from Lemma 3.9 that $\dim(\mu_t^{-1}(C) \cap \mu^{-1}(BwB_-)) < \dim X_t - l(w)$. By Lemma 3.2, $X_{t,w}$ is irreducible. Since $\dim X_t = \dim G - \dim H$, the dimension assertion is clear.

□

Corollary 3.10. *For any $t \in H$ and $w \in W$, $\mu : X_{t,w} \to F_{t,w}$ is a desingularization.*

Proof. This follows directly from Theorem 3.8 and the definition of a desingularization. □

3.5. **Irreducibility of $X_{t,w}$ for general G.** In this subsection, we assume G is connected and semisimple but not necessarily simply connected.

Let $p : \tilde{G} \to G$ be a simply connected covering of G with kernel Z. Let $\tilde{B} \supset \tilde{H}$ be a Borel subgroup and maximal torus of $\tilde{G}$ and assume $p(\tilde{B}) = B$ and $p(\tilde{H}) = H$. Let $\tilde{B}_-$ be the opposite Borel subgroup such that $p(\tilde{B}_-) = B_-$, and note that $p : \tilde{U} \to U$ is an isomorphism, where $\tilde{U}$ is the unipotent radical of $\tilde{B}$. For $\tilde{t} \in \tilde{H}$, let $\tilde{X}_{\tilde{t}} = \tilde{G} \times_{\tilde{B}} \tilde{t}\tilde{U}$ with $\tilde{\mu} : \tilde{X}_{\tilde{t}} \to \tilde{G}$ defined as in (3.2). For $w \in W$, let $\tilde{X}_{\tilde{t},w} = \tilde{X}_{\tilde{t}} \cap \tilde{\mu}^{-1}(\tilde{B}w\tilde{B}_-)$. For $t = p(\tilde{t}) \in H$, define

$$\widetilde{P} : \tilde{X}_{\tilde{t}} \to X_t : \quad \widetilde{P}[\tilde{g}, \tilde{t}\tilde{u}] = [p(\tilde{g}), p(\tilde{t})p(\tilde{u})].$$

Lemma 3.11. *For any $\tilde{t} \in \tilde{H}$, $t = p(\tilde{t}) \in H$, and $w \in W$,*
1) $\widetilde{P} : \tilde{X}_{\tilde{t}} \to X_t$ is an isomorphism of varieties;
2) $\widetilde{P}(\tilde{X}_{\tilde{t},w}) = X_{t,w}$.

Proof. For $[p(\tilde{g}), p(\tilde{t}\tilde{u})] \in G \times_B B$, $\widetilde{P}^{-1}([p(\tilde{g}), p(\tilde{t}\tilde{u})]) = \{[\tilde{g}, \tilde{t}z\tilde{u}] : z \in Z\}$. Since $[\tilde{g}, \tilde{t}z\tilde{u}] \in \tilde{X}_{\tilde{t}}$ if and only if z is the identity, $\widetilde{P}$ is a bijection, so 1) follows since X_t is smooth. Part 2) is a consequence of the easily verified assertion

$$\tilde{\mu}([\tilde{g}, \tilde{t}\tilde{u}]) \in \tilde{B}w\tilde{B}_- \iff \mu(\widetilde{P}[\tilde{g}, \tilde{t}\tilde{u}]) \in BwB_-.$$

□

Lemma 3.11 and Theorem 3.8 immediately give

Proposition 3.12. *For a connected complex semisimple Lie group G and for any $t \in H$ and $w \in W$, $X_{t,w}$ is smooth and irreducible and* $\dim X_{t,w} = \dim G - \dim H - l(w)$.

4. The Poisson structure Π on $G \times_B B$

In this section, G is assumed to be connected and semisimple but not necessarily simply connected. We will construct and study a Poisson structure Π on $G \times_B B$ with the following properties:

1) The Grothendick map $\mu : (G \times_B B,\ \Pi) \to (G, \pi)$ is a Poisson map;
2) Π is H-invariant, where H acts on $G \times_B B$ by (3.3).

Moreover, the H-orbits of symplectic leaves of Π in $G \times_B B$ are precisely the $X_{t,w}$'s for $t \in H$ and $w \in W$, as defined in Notation 3.4.

4.1. **Definition of the Poisson structure Π on $G \times_B B$.** Recall that π_D^+ is the Poisson structure on $D = G \times G$ given in (2.5). Let $B_\Delta = \{(b, b) : b \in B\}$, and let ϕ be the projection

$$\phi : \ \ G \times G \longrightarrow (G \times G)/B_\Delta.$$

Recall that π_G is the multiplicative Poisson structure on G defined in §2.1.

Proposition 4.1. *$\phi(\pi_D^+)$ is a well-defined Poisson structure on $(G \times G)/B_\Delta$, and the action of $(G_\Delta,\ \pi_G)$ on $((G \times G)/B_\Delta,\ \phi(\pi_D^+))$ by left multiplication is Poisson.*

Proof. It is well-known [18, 26] that B_Δ is a Poisson subgroup of $(G_\Delta,\ \pi_G)$. By Lemma 6.1 in the Appendix, the action of $(B_\Delta, -\pi_G)$ on $(G \times G, \pi_D^+)$ by right multiplication is Poisson. Thus $\phi(\pi_D^+)$ is a well-defined Poisson structure on $(G \times G)/B_\Delta$ by [28]. Again by Lemma 6.1 in the Appendix, the action by left multiplication of (G_Δ, π_G) on $((G \times G)/B_\Delta,\ \phi(\pi_D^+))$ is Poisson. □

Consider the embedding

$$\psi : \ \ G \times_B B \longrightarrow (G \times G)/B_\Delta : \quad [g,\ b] \longmapsto (gb,\ g).B_\Delta, \ \ g \in G, b \in B. \tag{4.1}$$

Note that $\psi(G \times_B B) = Q/B_\Delta$, where

$$Q = \{(gb,\ g) : g \in G, b \in B\} \subset G \times G.$$

We show that Q/B_Δ is a Poisson submanifold of $((G \times G)/B_\Delta,\ \phi(\pi_D^+))$ and thus obtain a Poisson structure on $G \times_B B$.

For $t \in H$, let $Q_t = \{(gtn,\ g) :\ g \in G, n \in U\}$. Then Q_t is B_Δ-invariant, and $\psi(X_t) = Q_t/B_\Delta$.

Proposition 4.2. *For every $t \in H$, Q_t/B_Δ is a Poisson submanifold of $(G \times G)/B_\Delta$ with respect to $\phi(\pi_D^+)$.*

Proof. By Proposition 6.7 in the Appendix, it suffices to show that Q_t is coisotropic in $G \times G$ with respect to π_D^+ and that the characteristic distribution (see below) of π_D^+ on Q_t coincides with the distribution defined by the tangent spaces to the B_Δ-orbits in Q_t.

By the definition of π in Notation 2.7,

$$\eta : \quad (G \times G,\ \pi_D^+) \longrightarrow (G,\ \pi) : \quad (g_1, g_2) \longmapsto g_1 g_2^{-1}$$

is Poisson. Let $\tau : G \times G \to G \times G : (g_1, g_2) \mapsto (g_1^{-1}, g_2^{-1})$. It is clear from the definition of π_D^+ that τ preserves π_D^+. Thus

$$\eta_1 := \eta\tau : \quad (G \times G,\ \pi_D^+) \longrightarrow (G,\ \pi) : \quad (g_1, g_2) \longmapsto g_1^{-1} g_2$$

is Poisson. Let $t \in H$. By Lemma 2.11, $Ut^{-1} = t^{-1}U$ is coisotropic in G with respect to π. Since η_1 is a submersion, it follows from [41] that $Q_t = \eta_1^{-1}(Ut^{-1})$ is coisotropic in D with respect to π_D^+.

Recall from §6.2 in the Appendix that the characteristic distribution of π_D^+ on Q_t is by definition the image of the bundle map

$$\tilde{\pi}_D^+ : \quad (TQ_t)^0 \longrightarrow TQ_t,$$

where $(TQ_t)^0$ is the conormal bundle of Q_t in $G \times G$, and $\tilde{\pi}_D^+ : T^*(G\times G) \to T(G \times G)$ is defined as in (1.1). Fix $g \in G$, $n \in U$, and let $d = (gtn,\, g) \in Q_t$. We now use formula (6.3) in the proof of Lemma 6.4 in the Appendix to compute $\tilde{\pi}_D^+((T_dQ_t)^0)$. First,

$$T_dQ_t = r_d\mathfrak{g}_\Delta + l_d(\mathfrak{n} \oplus 0) = r_d\left(\mathrm{Ad}_{(g,g)}(\mathfrak{g}_\Delta + (\mathfrak{n} \oplus 0))\right).$$

Identify $\mathfrak{d}$ with $\mathfrak{d}^*$ via the bilinear form $\langle\ ,\ \rangle$, and for $(x, y) \in \mathfrak{d}$, let $\alpha_{(x,y)}$ be the right invariant 1-form on $G \times G$ whose value at the identity element is (x, y). Then

$$(T_dQ_t)^0 = \{\alpha_{(x,x)}(d) : \quad x \in \mathrm{Ad}_g\mathfrak{b}\}.$$

Let $p_1 : \mathfrak{d} \to \mathfrak{g}_\Delta$ be the projection with respect to the decomposition $\mathfrak{d} = \mathfrak{g}_\Delta + \mathfrak{g}_{\mathrm{st}}^*$. Let $x \in \mathrm{Ad}_g\mathfrak{b}$. By formula (6.3) in the proof of Lemma 6.4 in the Appendix,

$$\tilde{\pi}_D^+(\alpha_{(x,x)}(d)) = l_dp_1\mathrm{Ad}_d^{-1}(x,x) = l_dp_1(\mathrm{Ad}_{tn}^{-1}\mathrm{Ad}_g^{-1}x,\ \mathrm{Ad}_g^{-1}x) = l_d(\mathrm{Ad}_g^{-1}x,\ \mathrm{Ad}_g^{-1}x),$$

which is exactly the infinitesimal generator of the B_Δ-action on Q_t in the direction of $(\mathrm{Ad}_g^{-1}x,\ \mathrm{Ad}_g^{-1}x)$. This finishes the proof of Proposition 4.2. □

Corollary 4.3. *Q/B_Δ is a Poisson submanifold of $(G \times G)/B_\Delta$ with respect to the Poisson structure $\phi(\pi_D^+)$ in Proposition 4.1.*

Proof. Corollary 4.3 follows immediately from Proposition 4.2 and the fact that $Q/B_\Delta = \bigcup_{t\in H}(Q_t/B_\Delta)$. □

Recall the isomorphism $\psi : G \times_B B \to Q/B_\Delta$ in (4.1).

Notation 4.4. The Poisson structure $\psi^{-1}(\phi(\pi_D^+))$ on $G \times_B B$ will be denoted by Π.

We summarize some of the properties of the Poisson structure Π on $G \times_B B$. Recall that σ is the left action of G on $G \times_B B$ given in (3.3).

Proposition 4.5. *1) For each $t \in H$, X_t is a Poisson submanifold of $(G \times_B B, \Pi)$;*

2) The Grothendieck map $\mu : (G \times_B B, \Pi) \to (G, \pi)$ is Poisson;

3) With the Poisson structure Π on $G \times_B B$, σ is a Poisson action by the Poisson Lie group (G, π_G);

4) Π is H-invariant for the action σ of H on $G \times_B B$.

Proof. Part 1) follows from the definition of Π and Proposition 4.2, and 2) is a consequence of the definitions of Π and π and Corollary 4.3. Part 3) follows from the definition of Π and Proposition 4.1, and 4) is then clear since π_G vanishes at points in H. □

4.2. **Symplectic leaves of Π in $G \times_B B$.** We use Proposition 6.7 in the Appendix to determine the symplectic leaves of Π in $G \times_B B$.

Lemma 4.6. *The symplectic leaves of π_D^+ are the connected components of the nonempty intersections $G^*(h\dot{w}, e)G_\triangle \cap G_\triangle(e, h'\dot{u})G^*$, where $h, h' \in H$ and $w, u \in W$.*

Proof. By Lemma 6.4 in the Appendix, symplectic leaves of π_D^+ are the connected components of nonempty intersection of $(G^*, G_\triangle)$ and $(G_\triangle, G^*)$-double cosets in $G \times G$. By Lemma 2.13, $(G^*, G_\triangle)$ and $(G_\triangle, G^*)$ double cosets in $G \times G$ are respectively of the form $G^*(h\dot{w}, e)G_\triangle$ and $G_\triangle(e, h'\dot{u})G^*$, where $h, h' \in H$ and $w, u \in W$. □

Recall that for $h \in H$ and $w \in W$, $G^*.(h\dot{w})$ is the G^*-orbit in G through $h\dot{w}$, where G^* acts on G by (2.12).

Proposition 4.7. *1) For any $t, h \in H$ and $w \in W$, $X_t \cap \mu^{-1}(G^*.(h\dot{w}))$ is nonempty, smooth, and all of its connected components have dimension*

$$\dim G - \dim H - l(w) - \dim(H/H_w);$$

2) The symplectic leaves of Π in $G \times_B B$ are precisely the connected components of $X_t \cap \mu^{-1}(G^*.(h\dot{w}))$, where $t, h \in H, w \in W$.

Proof. 1) Let R_t be the unique regular conjugacy class in F_t. By Corollary 2.18, $R_t \cap (G^*.(h\dot{w})) \neq \emptyset$. Since $\mu|_{X_t} : X_t \to F_t$ is onto, $X_t \cap \mu^{-1}(G^*.(h\dot{w})) \neq \emptyset$. An argument similar to the proof of the smoothness of $X_{t,w}$ in Theorem 3.8 shows that $X_t \cap \mu^{-1}(G^*.(h\dot{w}))$ is smooth. The dimension assertion follows from Lemma 3.9.

2) Let $t \in H$. Let $t_1 \in H$ be such that $t_1^2 = t$. Then for any $g \in G$ and $n \in U$, $(gtn, g) = (gt_1, gt_1)(t_1 n, t_1^{-1}) \in G_\triangle G^*$. Thus $Q_t \subset G_\triangle G^*$. By Proposition 6.7 in the Appendix, symplectic leaves of $\phi(\pi_D^+)$ in $Q_t/B_\triangle$ are the connected components of $(Q_t \cap (G^*(h\dot{w}, e)G_\triangle))/B_\triangle$, where $w \in W$ and $h \in H$. It is routine to check that $\mu^{-1}(G^*.(h\dot{w})) = \psi^{-1}((G^*(h\dot{w}, e)G_\triangle))/B_\triangle))$. Now 2) follows from the isomorphism $\psi : G \times_B B \to Q/B_\triangle$ and the fact the X_t's are Poisson submanifolds of $(G \times_B B, \Pi)$. □

4.3. **H-orbits of symplectic leaves of Π in $G \times_B B$.** By 4) of Proposition 4.5, the Poisson structure Π on $G \times_B B$ is H-invariant for the H-action in (3.3).

Theorem 4.8. *The H-orbits of symplectic leaves of Π in $G \times_B B$ are precisely the irreducible smooth subvarieties $X_{t,w}$, where $t \in H$ and $w \in W$. The dimension of every symplectic leaf in $X_{t,w}$ is $\dim G - \dim H - l(w) - \dim(H/H_w)$.*

Proof. By Proposition 4.7 and the fact that each $X_{t,w}$ is H-invariant, every H-orbit of symplectic leaves of Π in $G \times_B B$ is contained in $X_{t,w}$ for some $t \in H$ and $w \in W$. Fix $t \in H$ and $w \in W$. Then $X_{t,w}$ is nonempty, smooth and connected by Proposition 3.12. To prove Theorem 4.8, it suffices to show that for any connected component S in $X_t \cap \mu^{-1}(G^*.(h\dot{w}))$, where $h \in H$, the map

$$\sigma_1 : \ H \times S \longrightarrow X_{t,w} : \quad (h, [g, b]) \longmapsto [hg, b], \qquad h \in H, [g, b] \in S \tag{4.2}$$

has surjective differential everywhere. Let

$$Q_{t,w} = Q_t \cap ((B \times B_-)(h\dot{w}, e)G_\triangle).$$

Then $Q_{t,w}$ is smooth and irreducible since it is a principal B-bundle over the smooth irreducible variety $X_{t,w}$. An argument similar to an argument in the proof of Proposition 2.17, Part 2), shows that for any open submanifold S' of $Q_t \cap G^*(h\dot{w}, e)G_\Delta$, the map

$$H \times S' \longrightarrow Q_{t,w}: \quad (h, (g_1, g_2)) \longmapsto (hg_1, hg_2), \qquad h \in H, (g_1, g_2) \in S'$$

has surjective differential everywhere. It follows that α_1 in (4.2) has surjective differential everywhere. □

4.4. ***H*-equivariant Poisson desingularization.** An irreducible Poisson variety (X, π) is said to be regular if π has constant rank on X.

Definition 4.9. Let (X, π_X) and (Y, π_Y) be irreducible Poisson varieties, with Y normal. A desingularization $f : X \to Y$ is called a *Poisson desingularization* if π_X is a regular Poisson structure on X and if $f : (X, \pi_X) \to (Y, \pi_Y)$ is Poisson. If, in addition, a torus H acts by Poisson isomorphisms on both (X, π_X) and (Y, π_Y) such that f is H-equivariant and H acts transitively on the set of symplectic leaves of π_X in X, we call $f : (X, \pi_X) \to (Y, \pi_Y)$ an *H-equivariant Poisson desingularization.*

Remark 4.10. If π_Y is generically nondegenerate, $f : X \to Y$ is a symplectic resolution in the sense of [23] (see also [2, 15, 17]). Our notion of Poisson desingularization is different from that of Poisson resolution by Fu in [16]. It would be interesting to relate ideas from this paper to the recent paper [27] by Laurent-Gengoux.

Corollary 4.11. *For any $t \in H$ and $w \in W$, $\mu : (X_{t,w}, \Pi) \to (F_{t,w}, \pi)$ is an H-equivariant Poisson desingularization.*

Example 4.12. For $G = SL(2, \mathbb{C})$, $X = G \times_B B$ is the union of two open charts X' and X'', where

$$X' = \left\{ \left[\begin{pmatrix} x_1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} h_1 & y_1 \\ 0 & \frac{1}{h_1} \end{pmatrix} \right] : \ x_1, y_1, h_1 \in \mathbb{C}, h_1 \neq 0 \right\}$$
$$X'' = \left\{ \left[\begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix}, \begin{pmatrix} h_2 & y_2 \\ 0 & \frac{1}{h_2} \end{pmatrix} \right] : \ x_2, y_2, h_2 \in \mathbb{C}, h_2 \neq 0 \right\},$$

and $h_1 = h_2, x_2 = \frac{1}{x_1}$ and $y_2 = x_1(h_1 - \frac{1}{h_1} + x_1 y_1)$ on $X' \cap X''$. Using the formula for π on G in Example 2.9 and the fact that $\mu : (X, \Pi) \to (G, \pi)$ is Poisson, one can compute the Poisson structure Π on X and get

$$\{h_1, x_1\}_\Pi = \{h_1, y_1\}_\Pi = 0, \quad \{x_1, y_1\}_\Pi = h_1 + x_1 y_1 \quad \text{on } X', \quad \text{or} \tag{4.3}$$

$$\{h_2, x_2\}_\Pi = \{h_2, y_2\}_\Pi = 0, \quad \{x_2, y_2\}_\Pi = -\left(\frac{1}{h_2} + x_2 y_2\right) \quad \text{on } X''. \tag{4.4}$$

It follows that each X_t is a Poisson variety of (X, Π). Moreover, for $t = \mathrm{diag}(h, \frac{1}{h})$, note that $X_{t,w_0} \subset X' \cap X''$ and is given by $h + x_1 y_1 = 0$ or $\frac{1}{h} + x_2 y_2 = 0$, so it is clear from (4.3) and (4.4) that Π vanishes on X_{t,w_0} and is symplectic on $X_{t,w=1}$. On the other hand, $\mu : X_{t,w_0} \to F_{t,w_0}$ is an isomorphism for any t, and so is $\mu : X_{t,w=1} \to F_{t,w=1}$ unless $h = \pm 1$, in which case $F_{t,w=1}$ is singular at t and π has two H-orbits of symplectic leaves in $F_{t,w=1}$.

4.5. **Remarks on the Poisson structure $\phi(\pi_D^+)$ on $(G\times G)/B_\Delta$.** Consider the isomorphism

$$(4.5)\qquad \gamma:\ (G\times G)/B_\Delta \longrightarrow (G/B)\times G:\ \ (g_1,g_2).B_\Delta \longmapsto (g_1.B,\, g_1g_2^{-1}).$$

Composing γ with the embedding $\psi: G\times_B B\to (G\times G)/B_\Delta$ in (4.1) gives

$$\gamma\psi:\ G\times_B B \longrightarrow (G/B)\times G:\ \ [g,b]\longmapsto (g.B,\, gbg^{-1}),\ \ g\in G, b\in B$$

with

$$\gamma\psi(G\times_B B)=\{(g_1.B,g_2): g_1,g_2\in G, g_1^{-1}g_2g_1\in B\}.$$

Using the embedding $\gamma\psi$, we can regard $(G\times_B B,\Pi)$ as a Poisson submanifold of $((G/B)\times G,\, \gamma\phi(\pi_D^+))$, where recall from §4.1 that ϕ is the projection $G\times G\to (G\times G)/B_\Delta$.

In this subsection, we prove some properties of the Poisson structure $\gamma\phi(\pi_D^+)$ on $(G/B)\times G$ that are helpful in understanding the Poisson structure Π on $G\times_B B$. For notational simplicity, set

$$\Pi_1=\gamma\phi(\pi_D^+).$$

Let $\zeta: G\to G/B$ be the projection. Since B is a Poisson subgroup of (G,π_G), $\zeta(\pi_G)$ is a well-defined Poisson structure on G/B, which we denote by $\pi_{G/B}$, and $(G/B,\pi_{G/B})$ is a Poisson (G,π_G)-homogeneous space. In particular, $\pi_{G/B}$ is invariant under translation by elements in H. It is shown in [18] that the H-orbits of symplectic leaves of $\pi_{G/B}$ in G/B are precisely the intersections $(Bu.B)\cap(B_-v.B)$ for $u,v\in W, v\le u$. In particular, $\pi_{G/B}$ vanishes at $u.B$ for every $u\in W$.

Proposition 4.13. *Both of the following projections are Poisson:*

$$p_1:\ ((G/B)\times G,\ \Pi_1)\longrightarrow (G/B,\ \pi_{G/B}):\ \ (g_1.B,\ g_2)\longmapsto g_1.B$$
$$p_2:\ ((G/B)\times G,\ \Pi_1)\longrightarrow (G,\ \pi):\ \ (g_1.B,\ g_2)\longmapsto g_2.$$

Proof. The map $p_2\gamma\phi: G\times G\to G$ is η as in (2.6), so $p_2(\Pi_1)=\eta(\pi_D^+)=\pi$ by the definition of π. Let $p_1': G\times G\to G$ be the projection to the first factor. Then $p_1(\Pi_1)=(p_1\gamma\phi)(\pi_D^+)=(\zeta p_1')(\pi_D^+)$. Note that p_1' is a group homomorphism, and for $\Lambda\in\wedge^2(\mathfrak{g}\oplus\mathfrak{g})$ in (2.9), $p_1'(\Lambda)=\Lambda_0$, where $\Lambda_0\in\wedge^2\mathfrak{g}$ is given in Proposition 2.2. Thus $p_1'(\pi_D^+)=\Lambda_0^R+\Lambda_0^L$ (see §1.1), and

$$p_1(\Pi_1)=(\zeta p_1')(\pi_D^+)=\zeta(\Lambda_0^R+\Lambda_0^L)=\zeta(\Lambda_0^R)=\pi_{G/B}.$$

□

Corollary 4.14. *The projection $q:(G\times_B B,\Pi)\to(G/B,\ \pi_{G/B}):\ [g,b]\mapsto g.B$ is Poisson.*

5. Birational isomorphisms between $X_{t,w}, F_{t,w}$ and $G^{1,w^{-1}w_0}\times_H G^{1,w_0}/H$

5.1. **Relation between X_{t,w_0} and $X_{t,w}$.** Let w_0 be the longest element in W. Recall that $\sigma: G\times(G\times_B B)\to G\times_B B$ is the action of G on $G\times_B B$ given by (3.3) and that $G^{u,v}=BuB\cap B_-vB_-\subset G$ for $u,v\in W$.

Lemma 5.1. *For any $w\in W$,*

$$\sigma\left(G^{1,w^{-1}w_0}\ \times\ X_{t,w_0}\right)\subset X_{t,w}.$$

Proof. Let $[g, tn] \in X_{t,w_0}$, so $g \in G$, $n \in U$, and $gtng^{-1} \in Bw_0B_- = Bw_0$. If $g_1 \in G^{1,w^{-1}w_0}$, then $g_1^{-1} \in B_-w_0^{-1}wB_-$. Thus

$$\mu([g_1g, b]) = g_1gtng^{-1}g_1^{-1} \in Bw_0B_-w_0^{-1}wB_- = BwB_-.$$

□

By Lemma 2.6 and Theorem 4.8, $G^{1,w^{-1}w_0}$ and X_{t,w_0} are Poisson submanifolds of (G, π_G) and (X_t, Π) respectively. Proposition 4.5 implies the following

Proposition 5.2. *For $w \in W$, equip $G^{1,w^{-1}w_0}$ with the Poisson structure π_G and $X_{t,w}$ the Poisson structure Π. Then*

$$\sigma: \ (G^{1,w^{-1}w_0}, \pi_G) \times (X_{t,w_0}, \Pi) \longrightarrow (X_{t,w}, \Pi)$$

is a Poisson map.

5.2. **A decomposition of X_{t,w_0}.** We partition X_{t,w_0} into smooth subvarieties. Recall that $q: G \times_B B \to G/B : [g, b] \mapsto g.B$ is the projection. Fix $t \in H$. Set

$$X_{t,w_0}^u = X_{t,w_0} \cap q^{-1}(Bw_0u \cdot B) \quad \text{for } u \in W. \tag{5.1}$$

The Bruhat decomposition gives

$$X_{t,w_0} = \bigsqcup_{u \in W} X_{t,w_0}^u \quad \text{(disjoint union)}. \tag{5.2}$$

Lemma 5.3. *Let $t \in H$ and $u \in W$. If $X_{t,w_0}^u \neq \emptyset$, then $u \leq w_0u$, where $\leq$ is the Bruhat order on W, and*

$$q\left(X_{t,w_0}^u\right) \subset (B_-u.B) \cap (Bw_0u.B).$$

Proof. Assume that $[g, tn] \in X_{t,w_0}^u$, where $g \in G$ and $n \in U$. Since $g \in Bw_0uB$ and $gtng^{-1} \in Bw_0$, $g \in (Bw_0)^{-1}Bw_0uB = B_-uB$. Thus $g \in B_-uB \cap Bw_0uB$ and $u \leq w_0u$ by [7, Corollary 1.2]. □

For $u \in W$ such that $u \leq w_0u$, set

$$\mathcal{R}_{u,w_0u} = (B_-u.B) \cap (Bw_0u.B) \subset G/B.$$

We now establish an isomorphism between X_{t,w_0}^u and $\mathcal{R}_{u,w_0u} \times U \cap u^{-1}U_-u$. To this end, first parametrize $Bw_0u.B$ by the isomorphism

$$U_{w_0u} \longrightarrow Bw_0u.B: \quad m \longmapsto m\dot{w}_0\dot{u}.B,$$

where $U_{w_0u} = U \cap (w_0uU_-u^{-1}w_0^{-1})$. Note that for $m \in U_{w_0u}$, $m\dot{w}_0\dot{u}.B \in B_-u.B$ if and only if $m \in U_{w_0u} \cap B_-uBu^{-1}w_0^{-1}$. Thus

$$U_{w_0u} \cap B_-uBu^{-1}w_0^{-1} \longrightarrow \mathcal{R}_{u,w_0u}: \ m \longmapsto m\dot{w}_0\dot{u}.B \tag{5.3}$$

is an isomorphism. Let $m \in U_{w_0u} \cap B_-uBu^{-1}w_0^{-1}$. By the unique factorization

$$B_- \times (U \cap u^{-1}Uu) \longrightarrow B_-uB, \ (b_-, n) \longmapsto b_-\dot{u}n, \quad b_- \in B_-, n \in U \cap u^{-1}Uu,$$

$$m\dot{w}_0\dot{u} = b_-\dot{u}n_m, \quad \text{for unique } b_- \in B_- \text{ and } n_m \in U \cap u^{-1}Uu. \tag{5.4}$$

Define

$$\xi^u: \ U_{w_0u} \cap B_-uBu^{-1}w_0^{-1} \longrightarrow U \cap u^{-1}Uu: \ m \longmapsto n_m.$$

Notation 5.4. Let $G_0 = B_-B$. For $g \in B_-B$, write

$$g = (g)_-(g)_0(g)_+, \quad \text{where } (g)_- \in U_-, (g)_0 \in H, (g)_+ \in U.$$

For $m \in U_{w_0u} \cap B_-uBu^{-1}w_0^{-1}$ with the decomposition of $m\dot{w}_0\dot{u}$ in (5.4), since $m\dot{w}_0 = b_-\dot{u}n_m\dot{u}^{-1} \in G_0$, $n_m = \dot{u}^{-1}(m\dot{w}_0)_+\dot{u}$. Thus the map ξ^u is also given by

$$(5.5) \qquad \xi^u : \ U_{w_0u} \cap B_-uBu^{-1}w_0^{-1} \longrightarrow U \cap u^{-1}Uu : \ \ m \longmapsto \dot{u}^{-1}(m\dot{w}_0)_+\dot{u}.$$

Consider now the morphism

$$J_t^u : \ \mathcal{R}_{u,w_0u} \times (U \cap u^{-1}U_-u) \longrightarrow X_t : \ (m\dot{w}_0\dot{u}.B,\, m_1) \longmapsto [m\dot{w}_0\dot{u},\ tm_1\xi^u(m)]$$

where $m \in U_{w_0u} \cap B_-uBu^{-1}w_0^{-1}$ and $m_1 \in U \cap u^{-1}U_-u$.

Lemma 5.5. *The image of J_t^u is contained in X_{t,w_0}^u.*

Proof. Let $m \in U_{w_0u} \cap B_-uBu^{-1}w_0^{-1}$, and write $m\dot{w}_0\dot{u} = b_-\dot{u}\xi^u(m)$ for unique $b_- \in B_-$. Then for any $m_1 \in U \cap \dot{u}^{-1}U_-\dot{u}$,

$$(m\dot{w}_0\dot{u})(tm_1\xi^u(m))(m\dot{w}_0\dot{u})^{-1} = m\dot{w}_0\dot{u}tm_1\xi^u(m)\xi^u(m)^{-1}\dot{u}^{-1}b_-^{-1} \in Bw_0B_- = Bw_0.$$

Thus $[m\dot{w}_0\dot{u},\ tm_1\xi^u(m)] \in X_{t,w_0} \cap q^{-1}(Bw_0u.B) = X_{t,w_0}^u$. □

Proposition 5.6. *For any $u \in W$ such that $u \leq w_0u$,*

$$(5.6) \qquad J_t^u : \ \mathcal{R}_{u,w_0u} \times (U \cap u^{-1}U_-u) \longrightarrow X_{t,w_0}^u$$

is an isomorphism. In particular, X_{t,w_0}^u is smooth and irreducible and

$$\dim X_{t,w_0}^u = l(w_0) - l(u).$$

Proof. Since $U_{w_0u} \times U \to X_t : (m,\, n) \longmapsto [m\dot{w}_0\dot{u},\ tn]$ is an embedding, J_t^u is an embedding. Define $K_t^u : X_{t,w_0}^u \to \mathcal{R}_{u,w_0u} \times (U \cap u^{-1}U_-u)$ as follows: Let $[g, tn] \in X_{t,w_0}^u$. Assume without loss of generality that $g = m\dot{w}_0\dot{u}$ with $m \in U_{w_0u} \cap B_-uBu^{-1}w_0^{-1}$ and write $n = m_1n_1$ with $m_1 \in U \cap u^{-1}U_-u$ and $n_1 \in U \cap u^{-1}Uu$. Let

$$K_t^u([m\dot{w}_0\dot{u}, tm_1n_1]) = (m\dot{w}_0\dot{u}.B, m_1).$$

It is straightforward to check that K_t^u and J_t^u are inverse isomorphisms. By [24], $\mathcal{R}_{u,w_0u}$ is smooth and irreducible and has dimension $l(w_0) - 2l(u)$. It follows that X_{t,w_0}^u is smooth and irreducible, and that $\dim X_{t,w_0}^u = l(w_0) - l(u)$. □

5.3. **The open subvariety X_{t,w_0}^1 of X_{t,w_0}.** For $u = 1 \in W$, the open subset X_{t,w_0}^1 of X_{t,w_0} is especially simple. Indeed, $R_{1,w_0} = B_-.B \cap Bw_0.B$ is parametrized by

$$(5.7) \qquad U \cap B_-w_0B_- \longrightarrow \mathcal{R}_{1,w_0} : \quad m \longmapsto m\dot{w}_0.B,$$

and the isomorphism $J_t := J_t^1$ in (5.6) simplifies to

$$(5.8) \quad J_t : \ \ \mathcal{R}_{1,w_0} \longrightarrow X_{t,w_0}^1 : \ m\dot{w}_0.B \longmapsto [m\dot{w}_0,\ t(m\dot{w}_0)_+], \qquad m \in U \cap B_-w_0B_-$$

The inverse of J_t is the restriction to $X_{t,w}^1$ of the projection $q : G \times_B B \to G/B$. Identify $U \cap B_-w_0B_-$ with $G^{1,w_0}/H$ by $m \mapsto m.H$ for $m \in U \cap B_-w_0B_-$. Then the isomorphism in (5.7) can be replaced by

$$(5.9) \qquad \psi_{w_0} : \ \ G^{1,w_0}/H \longrightarrow \mathcal{R}_{1,w_0} : \quad g.H \longmapsto g\dot{w}_0.B.$$

Lemma 5.7. *The composition $J_t\psi_{w_0}$ is given by*

$$J_t\psi_{w_0} : \quad G^{1,w_0}/H \longrightarrow X_{t,w_0}^1 : \ \ g.H \mapsto [g\dot{w}_0,\ t(g\dot{w}_0)_+].$$

Proof. For $g \in G^{1,w_0}$, write $g = mh$, where $m \in U \cap (B_-w_0B_-)$ and $h \in H$. It is straightforward to check that $[m\dot{w}_0,\, t(m\dot{w}_0)_+] = [g\dot{w}_0,\, t(g\dot{w}_0)_+]$. □

Since X^1_{t,w_0} is open in X_{t,w_0}, X^1_{t,w_0} is a Poisson subvariety of $(G \times_B B, \Pi)$. Recall that $\pi_{G/B}$ is the projection to G/B of the Poisson structure π_G on G, and that $\mathcal{R}_{1,w_0}$ is a Poisson subvariety of $(G/B, \pi_{G/B})$ by [18]. Moreover, since the Poisson structure π_G on G is invariant under right multiplication by elements in H, the quotient G/H has a well-defined Poisson structure which we denote by $\pi_{G/H}$. Clearly $G^{u,v}/H$ is a Poisson subvariety of $(G/H, \pi_{G/H})$ for any $u, v \in W$.

Lemma 5.8. *Both*

$$\psi_{w_0} :\ (G^{1,w_0}/H,\ \pi_{G/H}) \longrightarrow (\mathcal{R}_{1,w_0},\ \pi_{G/B}) \quad \text{and} \quad J_t :\ (\mathcal{R}_{1,w_0},\ \pi_{G/B}) \longrightarrow (X^1_{t,w_0},\ \Pi)$$

are Poisson isomorphisms.

Proof. By §4.5, $\pi_{G/B}$ vanishes at $\dot{w}_0.B \in G/B$. Since the action of (G, π_G) on $(G/B,\ \pi_{G/B})$ by left translation is Poisson, the map

$$(G^{1,w_0},\ \pi_G) \longrightarrow (\mathcal{R}_{1,w_0},\ \pi_{G/B}) :\ \ g \longmapsto g\dot{w}_0.B$$

is Poisson, and so is ψ_{w_0}. For any $t \in H$, the projection $q : (G \times_B tU, , \Pi) \to (G/B, \pi_{G/B})$ is Poisson by Corollary 4.14. Thus $J_t^{-1} = q|_{X^1_{t,w_0}} : (X^1_{t,w_0}, \Pi) \to (\mathcal{R}_{1,w_0}, \pi_{G/B})$ is Poisson. □

We now state a fact that will be used in §5.4. Let

$$\xi := \xi^1 :\ \ U \cap B_-w_0B_- \longrightarrow U :\ \ m \longmapsto (m\dot{w}_0)_+. \tag{5.10}$$

The following Lemma 5.9 can be checked directly.

Lemma 5.9. *The image of* ξ *in* (5.10) *is again* $U \cap B_-w_0B_-$ *and*

$$\xi :\ \ U \cap B_-w_0B_- \longrightarrow U \cap B_-w_0B_- :\ \ m \longmapsto (m\dot{w}_0)_+ \tag{5.11}$$

is biregular with inverse given by

$$\xi^{-1} :\ \ U \cap B_-w_0B_- \longrightarrow U \cap B_-w_0B_- :\ \ n \longmapsto (n\dot{w}_0^{-1})_+. \tag{5.12}$$

5.4. **A birational isomorphism between $X_{t,w}$ and $G^{1,w^{-1}w_0} \times_H G^{1,w_0}/H$.** For $t \in H$, consider the Zariski open subset X^0_t of X_t given by

$$X^0_t = \{[n_1\dot{w}_0,\, tn_2] :\ n_1 \in U, n_2 \in U \cap (B_-w_0B_-)\}.$$

For $w \in W$, $X_{t,w} \cap X^0_t$ is then a Zariski open subset of $X_{t,w}$, and we show below that it is nonempty. In addition, consider the free right action of $H \times H$ on $G^{1,w^{-1}w_0} \times G^{1,w_0}$ given by

$$(g_1, g_2) \cdot (h_1, h_2) = (g_1h_1,\ h_1^{-1}g_2h_2), \qquad g_1, g_2 \in G, h_1, h_2 \in H. \tag{5.13}$$

The action preserves the Poisson structure $\pi_G \oplus \pi_G$. Denote the quotient Poisson variety by $(G^{1,w^{-1}w_0},\ \pi_G) \times_H (G^{1,w_0},\ \pi_G)/H$. We can now state and prove the main result of §5.

Theorem 5.10. *For any* $t \in H$ *and* $w \in W$,

$$\rho : (G^{1,w^{-1}w_0},\ \pi_G) \times_H (G^{1,w_0},\ \pi_G)/H \longrightarrow (X_{t,w},\ \Pi) :$$
$$[g_1,\ g_2.H] \longmapsto [g_1g_2\dot{w}_0,\ t(g_2\dot{w}_0)_+]$$

is a biregular Poisson isomorphism from $(G^{1,w^{-1}w_0}, \pi_G) \times_H (G^{1,w_0}, \pi_G)/H$ *to the Zariski open subset* $X_{t,w} \cap X_t^0$ *of* $X_{t,w}$*. Moreover,*

$$X_{t,w} \cap X_t^0 = \{[n(m\dot{w}_0^{-1})_+\dot{w}_0,\, tm] :\, n \in U \cap (B_-w^{-1}w_0B_-), m \in U \cap (B_-w_0B_-)\}.$$

Proof. Recall the Poisson map σ in Proposition 5.2. Replace X_{t,w_0} in σ by its open subvariety X_{t,w_0}^1. It follows from Lemma 5.8 that ρ is Poisson. Identify

$$(U \cap (B_-w^{-1}w_0B_-)) \times (U \cap (B_-w_0B_-)) \stackrel{\cong}{\longrightarrow} (G^{1,w^{-1}w_0} \times G^{1,w_0})/(H \times H)$$

by $(n,m) \mapsto (n,m).(H \times H)$ for $n \in U \cap (B_-w^{-1}w_0B_-)$ and $m \in U \cap (B_-w_0B_-)$, so that ρ becomes

$$\rho' :\ (U\cap(B_-w^{-1}w_0B_-))\times(U\cap(B_-w_0B_-)) \longrightarrow X_{t,w} :\ (n,m) \longmapsto [nm\dot{w}_0,\, t(m\dot{w}_0)_+].$$

Let $\xi : U \cap (B_-w_0B_-) \to U \cap (B_-w_0B_-) : m \mapsto (m\dot{w}_0)_+$ be the isomorphism from Lemma 5.9. The composition $\rho'' := \rho'(\mathrm{id} \times \xi^{-1})$ is then given by

$$\rho'' :\ (U\cap(B_-w^{-1}w_0B_-))\times(U\cap(B_-w_0B_-)) \longrightarrow X_{t,w} :\ (n,m) \longmapsto [n\xi^{-1}(m)\dot{w}_0,\, tm].$$

By Lemma 5.9, the image of ρ'' is given by

$$\mathrm{Im}\rho'' = \{[n(m\dot{w}_0^{-1})_+\dot{w}_0,\, tm] :\, n \in U \cap (B_-w^{-1}w_0B_-), m \in U \cap (B_-w_0B_-)\}.$$

To prove the theorem, it now suffices to show that ρ'' is injective with image $X_{t,w}\cap X_t^0$, using the fact that $X_{t,w}$ is smooth. Note that $\mathrm{Im}\rho''$ lies in the affine chart $X_t' := \{[n\dot{w}_0, tm] : n,m \in U\}$ of X_t. It is clear from the explicit formula of ρ'' that it is injective. It remains to show that $X_{t,w} \cap X_t^0 = \mathrm{Im}\rho''$. Clearly, $X_{t,w} \cap X_t^0 \supset \mathrm{Im}\rho''$. Suppose that $[n_1\dot{w}_0, tn_2] \in X_{t,w} \cap X_t^0$. Then $n_2 \in U \cap (B_-w_0B_-)$ and $n_1\dot{w}_0tn_2\dot{w}_0^{-1}n_1^{-1} \in BwB_-$. Thus

$$(n_2\dot{w}_0^{-1})_+n_1^{-1} \in B_-w_0BwB_- = B_-w_0wB_-.$$

Let $n = n_1(n_2\dot{w}_0^{-1})_+^{-1}$. Then $n \in U \cap (B_-w^{-1}w_0B_-)$ and $n_1 = n(n_2\dot{w}_0^{-1})_+$. Then $[n_1\dot{w}_0, tn_2] = \rho''(n, n_2) \in \mathrm{Im}\rho''$. □

Recall that $\mu : G \times_B B \to G$ is the Grothendieck map.

Corollary 5.11. *Let G be simply connected. For any $t \in H$ and $w \in W$, the map*

$$\mu\rho :\ (G^{1,w^{-1}w_0},\ \pi_G) \times_H (G^{1,w_0},\ \pi_G)/H \longrightarrow (F_{t,w},\ \pi) :$$
$$[g_1,\ g_2.H] \longmapsto g_1g_2\dot{w}_0t(g_2\dot{w}_0)_+(g_2\dot{w}_0)^{-1}g_1^{-1}$$

is a biregular Poisson isomorphism from $(G^{1,w^{-1}w_0}, \pi_G) \times_H (G^{1,w_0}, \pi_G)/H$ *to a Zariski open subset of* $(F_{t,w}, \pi)$*.*

Remark 5.12. In a future paper, we will use the rational isomorphisms ρ and $\mu\rho$ to study log-canonical coordinates for the Poisson varieties $(X_{t,w}, \Pi)$ and $(F_{t,w}, \pi)$ and study the associated cluster algebras.

6. Appendix

6.1. **Poisson Lie groups.** In this appendix, we recall some general facts on Poisson Lie groups that are used in the construction of the Poisson structure π on G in §2 and the Poisson structure Π on $G \times_B B$ in §4. Some of the omitted details in this section can be found in [1] and [26].

Recall that a Poisson bi-vector field π_G on a Lie group G is said to be *multiplicative* if the map $m : G \times G \to G : (g_1, g_2) \mapsto g_1 g_2$ is Poisson with respect to π_G. A *Poisson Lie group* is a Lie group G with a multiplicative Poisson bi-vector field π_G. An action $\sigma : G \times P \to P$ of a Poisson Lie group (G, π_G) on a Poisson manifold (P, π_P) is said to be Poisson if σ is a Poisson map.

If (G, π_G) is a Poisson Lie group, then $\pi_G(e) = 0$, where $e \in G$ is the identity element. The linearization of π_G at e is the linear map $\delta_{\mathfrak{g}} : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ given by $\delta_{\mathfrak{g}}(x) = [\tilde{x}, \pi_G](e)$, where for $x \in \mathfrak{g}$, $\tilde{x}$ is any vector field on G with $\tilde{x}(e) = x$, and $[\tilde{x}, \pi_G]$ is the Lie derivative of π_G by $\tilde{x}$.

Two Poisson Lie groups (G, π_G) and (G^*, π_{G^*}) are said to be *dual* to each other if their Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ are dual to each other and if the dual map of $\delta_{\mathfrak{g}} : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ is the Lie bracket on $\mathfrak{g}^*$ and the dual map of $\delta_{\mathfrak{g}^*} : \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*$ is the Lie bracket on $\mathfrak{g}$.

One important class of Poisson Lie groups is constructed from *Manin triples.* Recall that a Manin triple is a quadruple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*, \langle \, , \, \rangle)$, where $\mathfrak{d}$ is an even dimensional Lie algebra, $\langle \, , \, \rangle$ is a symmetric non-degenerate invariant bilinear form on $\mathfrak{d}$, $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras of $\mathfrak{d}$, both maximally isotropic with respect to $\langle \, , \, \rangle$, and $\mathfrak{g} \cap \mathfrak{g}^* = 0$. The bilinear form $\langle \, , \, \rangle$ gives rise to a non-degenerate pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, so $\mathfrak{g}^*$ can indeed be regarded as the dual space of $\mathfrak{g}$.

Assume that $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*, \langle \, , \, \rangle)$ is a Manin triple. Let $\{x_i\}$ be a basis of $\mathfrak{g}$ and let $\{\xi_i\}$ be the dual basis of $\mathfrak{g}^*$. Let

$$\Lambda = \frac{1}{2} \sum_i (\xi_i \wedge x_i) \in \wedge^2 \mathfrak{d}.$$

Then Λ is independent of the choices of the bases, and the Schouten bracket $[\Lambda, \Lambda] \in \wedge^3 \mathfrak{d}$ is ad-invariant. Let D be a connected Lie group with Lie algebra $\mathfrak{d}$. Define the bi-vector fields $\pi_D^{\pm}$ on D by

$$\pi_D^{\pm} = \Lambda^R \pm \Lambda^L,$$

where Λ^R and Λ^L are respectively the right and left invariant bi-vector fields on D with $\Lambda^R(e) = \Lambda^L(e) = \Lambda$. Then both π_D^- and π_D^+ are Poisson structures on D (see [26, Proposition 3.4.1] for a proof that π_D^- is Poisson, and use the fact that left and right-invariant vector fields commute to see $[\pi_D^+, \pi_D^+] = [\pi_D^-, \pi_D^-]$, so π_D^+ is Poisson). Let G and G^* be the connected subgroups of D with Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. One then checks that both G and G^* are Poisson submanifolds of (D, π_D^-). Set

$$\pi_G = \pi_D^-|_G \quad \text{and} \quad \pi_{G^*} = -\pi_D^-|_{G^*}.$$

Then (G, π_G) and (G^*, π_{G^*}) is a pair of dual Poisson Lie groups, and (G, π_G) and $(G^*, -\pi_{G^*})$ are Poisson subgroups of (D, π_D^-). In particular, every Poisson action of (D, π_D^-) on a Poisson manifold restricts to Poisson actions of (G, π_G) and $(G^*, -\pi_{G^*})$.

The following Lemma 6.1 is immediate from definitions.

Lemma 6.1. *The following two actions are Poisson:*

$$(D, \pi_D^-) \times (D, \pi_D^+) \longrightarrow (D, \pi_D^+): \quad (d_1, d_2) \longmapsto d_1 d_2$$
$$(D, \pi_D^+) \times (D, -\pi_D^-) \longrightarrow (D, \pi_D^+): \quad (d_1, d_2) \longmapsto d_1 d_2.$$

A proof of the following Proposition 6.2 can be found in [1]. See also [5] and [29]. We give an outline of the proof for completeness.

Proposition 6.2. *Assume that G is closed in D, and let $p : D \to D/G$ be the natural projection. Then*

$$p(\pi_D^{\pm}) = p(\Lambda^R)$$

is a well-defined Poisson structure on D/G, and the actions

$$(G, \pi_G) \times (D/G,\ p(\pi_D^{\pm})) \longrightarrow (D/G,\ p(\pi_D^{\pm})): \quad (g, d.G) \longmapsto gd.G,$$
$$(G^*, -\pi_{G^*}) \times (D/G,\ p(\pi_D^{\pm})) \longrightarrow (D/G,\ p(\pi_D^{\pm})): \quad (u, d.G) \longmapsto ud.G$$

are Poisson. Moreover, symplectic leaves of $p(\pi_D^{\pm})$ in D/G are precisely the connected components of the nonempty intersections of G and G^*-orbits in D/G.

Proof. The element $\Lambda \in \wedge^2 \mathfrak{d}$ is mapped to 0 under the projection $\mathfrak{d} \to \mathfrak{d}/\mathfrak{g}$. Thus $p(\Lambda^L) = 0$. Since $p(\Lambda^R)$ is clearly well-defined, $p(\pi_D^+) = p(\pi_D^-)$ are well-defined. Now the Poisson action of (D, π_D^-) on (D, π_D^+) by left multiplication restricts to give Poisson actions of (G, π_G) and $(G^*, -\pi_{G^*})$, which clearly descend to give Poisson actions on $(D/G,\ p(\pi_D^{\pm}))$.

Since $\mathfrak{g} + \mathfrak{g}^*_{\mathrm{st}} = \mathfrak{d}$, the G-orbit $G.\underline{d}$ and the G^*-orbit $G^*.\underline{d}$ intersect transversally at $\underline{d} = d.G$ for any $d \in D$. In particular,

$$T_{\underline{d}}((G.\underline{d}) \cap (G^*.\underline{d})) = T_{\underline{d}}(G.\underline{d}) \cap T_{\underline{d}}(G^*.\underline{d}).$$

To prove the statement about the symplectic leaves of $p(\pi_D^+)$, it is thus enough to check that $T_{\underline{d}}(G.\underline{d}) \cap T_{\underline{d}}(G^*.\underline{d})$ coincides with the tangent space to the symplectic leaf of $p(\pi_D^+)$ at $\underline{d}$. To this end, identify $T_{\underline{d}}(D/G) \cong \mathfrak{d}/\mathrm{Ad}_d\mathfrak{g}$. Then $p(\pi_D^+)(\underline{d})$ becomes the element $p_d(\Lambda) \in \wedge^2(\mathfrak{d}/\mathrm{Ad}_d\mathfrak{g})$, where $p_d : \mathfrak{d} \to \mathfrak{d}/\mathrm{Ad}_d\mathfrak{g}$ is the projection. Let $\tilde{\Lambda} : \mathfrak{d} \to \mathfrak{d}$ be the map

$$\tilde{\Lambda}(x + \xi) = \frac{1}{2}\sum_{i=1}^{n}(\langle x + \xi,\ \xi_i\rangle x_i - \langle x + \xi,\ x_i\rangle \xi_i) = \frac{1}{2}(x - \xi), \qquad x \in \mathfrak{g}, \xi \in \mathfrak{g}^*.$$

Let $S_{\underline{d}}$ be the symplectic leaf of $p(\pi_D^+)$ through $\underline{d}$. By definition,

$$T_{\underline{d}} S_{\underline{d}} = p_d\left(\tilde{\Lambda}(\mathrm{Ad}_d\mathfrak{g})\right) \subset \mathfrak{d}/\mathrm{Ad}_d\mathfrak{g}.$$

Since for any $x + \xi \in \mathrm{Ad}_d\mathfrak{g}$,

$$\tilde{\Lambda}(x + \xi) = \frac{1}{2}(x - \xi) = x - \frac{1}{2}(x + \xi) = -\xi + \frac{1}{2}(x + \xi), \tag{6.1}$$

one sees that $p_d(\tilde{\Lambda}(\mathrm{Ad}_d\mathfrak{g})) = p_d(\mathfrak{g}) \cap p_d(\mathfrak{g}^*) \cong T_{\underline{d}}(G.\underline{d}) \cap T_{\underline{d}}(G^*.\underline{d})$. □

Lemma 6.3. *The local diffeomorphism $p|_{G^*} : (G^*, -\pi_{G^*}) \to (D/G, p(\pi_D^+)) : u \mapsto u.G$ is Poisson.*

Proof. This is because $(G^*, -\pi_{G^*})$ is a Poisson subgroup of (D, π_D^-). □

The following Lemma 6.4 on the symplectic leaves of π_D^+ in D is proved in [1]. We give a slightly different proof here for completeness. Moreover, (6.3) in our proof of Lemma 6.4 is used in the proof of Proposition 4.2.

Lemma 6.4. *Symplectic leaves of π_D^+ in D are precisely the connected components of nonempty intersections of (G, G^*)-double cosets and (G^*, G)-double cosets in D.*

Proof. Let $d \in D$. Since

$$T_d(G^*dG) + T_d(GdG^*) = r_d\mathfrak{g}^* + l_d\mathfrak{g} + r_d\mathfrak{g} + l_d\mathfrak{g}^* = r_d\mathfrak{d} = T_dD,$$

the two cosets G^*dG and GdG^* intersect transversally at d. Let Σ be the symplectic leaf of π_D^+ through d. It is enough to show that

$$T_d\Sigma = T_d(G^*dG) \cap T_d(GdG^*).$$

Let $\tilde{\pi}_D^+ : T^*D \to TD$ be the bundle map defined by π_D^+ (see (1.1)). Identify $\mathfrak{d}$ with $\mathfrak{d}^*$ via $\langle\ ,\ \rangle$, and for $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, let $\alpha_{x+\xi}$ be the right invariant 1-form on D with value $x + \xi$ at the identity element of D. By (6.1),

$$\tilde{\pi}_D^+(\alpha_{x+\xi})(d) = r_d\tilde{\Lambda}(x + \xi) + l_d\tilde{\Lambda}(\mathrm{Ad}_d^{-1}(x + \xi)) \tag{6.2}$$

$$= -r_d(\xi) + l_d\left(p_{\mathfrak{g}}\mathrm{Ad}_d^{-1}(x + \xi)\right) \tag{6.3}$$

$$= r_d(x) - l_d\left(p_{\mathfrak{g}^*}\mathrm{Ad}_d^{-1}(x + \xi)\right), \tag{6.4}$$

where $p_{\mathfrak{g}} : \mathfrak{d} \to \mathfrak{g}$ and $p_{\mathfrak{g}^*} : \mathfrak{d} \to \mathfrak{g}^*$ are the projections with respect to the decomposition $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$. Thus $T_d\Sigma \subset T_d(G^*dG) \cap T_d(GdG^*)$. Conversely, if $v_d \in T_d(G^*dG) \cap T_d(GdG^*)$, then

$$v_d = -r_d(\xi) + l_d(x_1) = r_d(x) - l_d(\xi_1)$$

for some $x, x_1 \in \mathfrak{g}$ and $\xi, \xi_1 \in \mathfrak{g}^*$, so $x_1 + \xi_1 = \mathrm{Ad}_d^{-1}(x + \xi)$, so $v_d = \tilde{\pi}_D^+(\alpha_{x+\xi})(d) \in T_d\Sigma$. Hence $T_d\Sigma = T_d(G^*dG) \cap T_d(GdG^*)$.

□

6.2. **Coisotropic reduction.** In this section, we prove Proposition 6.7, which is used in the study of the Poisson structure π on $G \times_B B$ in §4. Proposition 6.7 can be proved by combining Corollary 2.3 in [31] and several examples therein, but we give a direct proof here for completeness.

A *Poisson vector space* is by definition a pair (V, π), where V is a vector space and $\pi \in \wedge^2 V$. Let (V, π) be a finite dimensional Poisson vector space. Let

$$\tilde{\pi} : \quad V^* \longrightarrow V : \quad (\tilde{\pi}(\xi),\ \eta) = \pi(\xi, \eta), \qquad \xi, \eta \in V^*$$

and set $V_\pi = \tilde{\pi}(V^*) \subset V$. A subspace V_1 of V is called a Poisson subspace of (V, π) if $V_1 \supset V_\pi$, or, equivalently, if $\pi \in \wedge^2 V_1$. In this case, (V_1, π) is a Poisson vector space. Recall [41] that a subspace U of V is said to be coisotropic with respect to π if $\tilde{\pi}(U^0) \subset U$, where $U^0 = \{\xi \in V^* : \ (\xi, x) = 0,\ \forall x \in U\}$.

Remark 6.5. It is easy to see that $U \subset V$ is coisotropic if and only if π is in the subspace $U \wedge V$ of $\wedge^2 V$.

Lemma 6.6. *Let U be a coisotropic subspace of (V, π), let $\phi : V \to V/\tilde{\pi}(U^0)$ be the projection, and set $\varpi = \phi(\pi)$. Then*

$$(V/\tilde{\pi}(U^0))_\varpi = \phi(U \cap V_\pi) \subset U/\tilde{\pi}(U^0) \subset V/\tilde{\pi}(U^0). \tag{6.5}$$

In particular, $(U/\tilde{\pi}(U^0),\ \varpi)$ is a Poisson vector subspace of $(V/\tilde{\pi}(U^0),\ \varpi)$.

Proof. By a linear algebra computation, $(\tilde{\pi}(U^0))^0 = \tilde{\pi}^{-1}(U) = \{\xi \in V^* : \tilde{\pi}(\xi) \in U\}$. By identifying $(V/\tilde{\pi}(U^0))^* = (\tilde{\pi}(U^0))^0$, one has

$$(V/\tilde{\pi}(U^0))_{\varpi} = \tilde{\varpi}((V/\tilde{\pi}(U^0))^*) = \phi(\tilde{\pi}(\tilde{\pi}^{-1}(U))) = \phi(U \cap V_\pi).$$

□

Let (P, π_P) be a Poisson manifold. A submanifold $Q \subset P$ is said to be coisotropic if $\tilde{\pi}_P((T_qQ)^0) \subset T_qQ$ for every $q \in Q$, where $(T_qQ)^0$ is the conormal bundle of Q in P and $\tilde{\pi}_P$ is the bundle map from T^*P to TP given in (1.1). In this case, $\tilde{\pi}_P((T_qQ)^0)$ is called the *characteristic distribution* of π_P on Q.

Proposition 6.7. *Let (P, π_P) and (R, π_R) be two Poisson manifolds with a surjective Poisson submersion $\phi : (P, \pi_P) \to (R, \pi_R)$. Assume that*

1) Q is a coisotropic submanifold of (P, π_P) such that the characteristic distribution of π_P on Q coincides with the distribution defined by the tangent spaces to the fibers of ϕ and that $\phi(Q)$ is smooth submanifold of R;

2) for every $q \in Q$, Q intersects with the symplectic leaf S_q of π_P at q cleanly, i.e., $Q \cap S_q$ is a submanifold and $T_q(Q \cap S_q) = T_qQ \cap T_qS_q$.

Then $\phi(Q)$ is a Poisson submanifold of (R, π_R), and for each $q \in Q$, the symplectic leaf of π_R in $\phi(Q)$ at $\phi(q)$ is the connected component of $\phi(Q \cap S_q)$ containing $\phi(q)$.

Proof. To show $\phi(Q)$ is a Poisson submanifold, it suffices to show $T_{\phi(q)}(\phi(Q))$ is a Poisson subspace of $T_{\phi(q)}(R)$ for each $q \in Q$. This last assertion is a consequence of the last statement of Lemma 6.6 and assumption 1), since Q is coisotropic. Furthermore, by 2), (6.5) in Lemma 6.6 gives the assertion on symplectic leaves. □

6.3. **Singularities of intersections of Bruhat cells and Steinberg fibers.** For an affine variety X with ring of regular functions $O(X)$ and $g_1, \ldots, g_k \in O(X)$, let $V(g_1, \ldots, g_k)$ denote the common vanishing set of $g_1, \ldots, g_k$, and let $(g_1, \ldots, g_k)$ denote the ideal in $O(X)$ generated by $g_1, \ldots, g_k$. If $Y \subset X$ is Zariski closed, let $I(Y)$ be the ideal of regular functions vanishing on Y.

Assume G is simply connected for the remainder of the section. Let $r = \dim H$, let $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ be the set of simple roots, and let $\omega_1, \omega_2, \ldots, \omega_r$ be the corresponding *fundamental weights*, i.e., $\omega_j \in \mathfrak{h}^*$ for each $1 \leq j \leq r$ and $\omega_j(h_{\alpha_k}) = \delta_{jk}$ for $1 \leq j, k \leq r$. Denote by χ_j the character of the irreducible representation with ω_j as the highest weight. Then the *Steinberg map* is the map [21]

$$\chi : \ G \longrightarrow \mathbb{C}^r : \ \ \chi(g) = (\chi_1(g), \chi_2(g), \ldots, \chi_r(g)). \tag{6.6}$$

For $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$, let $F_z = F_t$ for any $t \in H$ such that $\chi(t) = z$. For a Bruhat variety $\overline{BwB_-}$, let $f_i = \chi_i|_{\overline{BwB_-}} - z_i$.

Proposition 6.8. *1)* $F_z \cap \overline{BwB_-} = V(f_1, \ldots, f_r)$.
2) $F_z \cap \overline{BwB_-}$ *is nonempty and irreducible*
3) $\dim(F_z \cap \overline{BwB_-}) = d = \dim(G) - r - l(w)$.

Proof. Since $F_z = V(\chi_1 - z_1, \ldots, \chi_r - z_r)$, part 1) is clear. By Proposition 3.3, $F_z \cap \overline{BwB_-}$ is nonempty. Let V be an irreducible component of $V(f_1, \ldots, f_r)$ and note that $\dim(V) \geq \dim(G) - r - l(w)$. Let $F_z = \cup_{i=1}^n C_{z_i}$ be the decomposition of F_z into conjugacy classes with $C_{z_1} = R_z$ the unique regular conjugacy class in F_z.

Then $F_z \cap \overline{BwB_-} = \cup_{i=1,\dots,n,y\geq w} C_{z_i} \cap ByB_-$. If $i > 1$ or $y > w$, then by Proposition 2.15,

$$\dim(C_{z_i} \cap ByB_-) = \dim(C_{z_i}) - l(y) < \dim(R_z) - l(w) = d.$$

It follows from Lemma 3.2 that $V(f_1, \dots, f_r)$ is irreducible, and part 3) is an easy consequence. □

Lemma 6.9. *$\overline{BwB_-}$ is Cohen-Macaulay for all $w \in W$.*

Proof. By a Theorem of Ramanathan, $\overline{B_-wB_-}/B_-$ is Cohen-Macaulay in G/B_- [33]. The result now follows easily by using the fact that the smooth morphism $G \to G/B_-$ is a locally trivial bundle in the Zariski topology and using the isomorphism $\overline{B_-wB_-} \cong \overline{Bw_0wB_-}$ given by left multiplication by $\dot{w}_0$. □

Lemma 6.10. *(see* [22, Lemma 7.1]*) Let Y be an irreducible Cohen-Macaulay affine variety of dimension n, and let $f_1, \dots, f_r \in O(Y)$. Let $X = V(f_1, \dots, f_r)$. Suppose*
1) X is irreducible and
2) there is a smooth point $y \in X$ such that $(df)_1(y), \dots, (df)_r(y)$ are linearly independent.

Then $\dim(X) = n - r$ *and the ideal* $(f_1, \dots, f_r) = I(X)$.

Remark 6.11. Our statement is more general than the statement in [22]. The proof is identical, once we recall a basic fact about Cohen-Macaulay varieties. The ideal $(f_1, \dots, f_r) = Q_1 \cap \dots \cap Q_s$ has a minimal primary decomposition. The Cohen-Macaulay condition ensures that if $P_i = \sqrt{Q_i}$ for $i = 1, \dots, s$, the varieties $V(P_i)$ all have the same dimension (by [32, Theorem 17.6]).

Theorem 6.12. *The ideal $(f_1, \dots, f_r)$ is the ideal of functions vanishing on the irreducible variety $F_z \cap \overline{BwB_-}$ in $\overline{BwB_-}$. Moreover, $F_z \cap \overline{BwB_-}$ is Cohen-Macaulay.*

Proof. By Lemma 6.9, $\overline{BwB_-}$ is Cohen-Macaulay. By Proposition 6.8, $F_z \cap \overline{BwB_-}$ is irreducible. Recall that $(\chi_1 - z_1, \dots, \chi_r - z_r)$ is the ideal of F_z and R_z is the smooth locus of F_z ([21], Theorem 4.24). In particular,

$$R_z = \{y \in F_z : (d\chi)_1(y), \dots, (d\chi)_r(y) \text{ are linearly independent on } T_y(G)\}$$

and $T_y(R_z)$ is defined in $T_y(G)$ by the vanishing of $(d\chi)_1(y), \dots, (d\chi)_r(y)$.

Since R_z and BwB_- are smooth locally closed subvarieties of G and R_z meets BwB_- transversally, $R_z \cap BwB_-$ is smooth and locally closed in $F_z \cap \overline{BwB_-}$. Let $y \in R_z \cap BwB_-$ and let λ be a nonzero covector in the span of $(d\chi)_1(y), \dots, (d\chi)_r(y)$. Since $T_y(R_z) + T_y(BwB_-) = T_y(G)$, it follows that λ is nonzero on $T_y(BwB_-)$. Thus, the restrictions $(df)_1(y), \dots, (df)_r(y)$ of $(d\chi)_1(y), \dots, (d\chi)_r(y)$ to $T_y(BwB_-)$ are linearly independent. Since $T_y(BwB_-) = T_y(\overline{BwB_-})$, we can apply Lemma 6.10 to deduce the first assertion. Since $\overline{BwB_-}$ is Cohen-Macaulay, $F_z \cap \overline{BwB_-}$ is Cohen-Macaulay using [36, Corollary, page 65]. □

Proposition 6.13. *1) Let $\overline{BwB_-}_{\mathrm{ns}}$ be the smooth locus of $\overline{BwB_-}$. Then $R_z \cap \overline{BwB_-}_{\mathrm{ns}}$ is the smooth locus of $F_z \cap \overline{BwB_-}_{\mathrm{ns}}$.*
2) The singular locus of $F_z \cap \overline{BwB_-}$ has codimension at least 2.

Proof. By Theorem 6.12, the ideal sheaf of $F_z \cap \overline{BwB_-}_{\mathrm{ns}}$ is generated by $f_1, \dots, f_r$. By the Jacobian criterion, the smooth locus of $F_z \cap \overline{BwB_-}_{\mathrm{ns}}$ is the set of points $y \in F_z \cap \overline{BwB_-}_{\mathrm{ns}}$ where $(df)_1(y), \dots, (df)_r(y)$ are linearly independent on $T_y(\overline{BwB_-}_{\mathrm{ns}})$.

Let $y \in R_z \cap \overline{BwB_-}_{\rm ns}$ be in BvB_-. Using transversality as in the proof of Theorem 6.12, it follows that $(df)_1(y), \ldots, (df)_r(y)$ are linearly independent on $T_y(BvB_-)$, so they are linearly independent on $T_y(\overline{BwB_-}_{\rm ns}) \supset T_y(BvB_-)$. Conversely, if $y \in F_z - R_z$, $(d\chi)_1(y), \ldots, (d\chi)_r(y)$ are linearly dependent in $T_y(G)$. As a consequence, their restrictions $(df)_1(y), \ldots, (df)_r(y)$ are linearly dependent on $T_y(\overline{BwB_-}_{\rm ns})$, which gives 1). For 2), note that if y is a singular point of $F_z \cap \overline{BwB_-}$, then either y is a singular point of $\overline{BwB_-}$ or y is a singular point of $F_z \cap \overline{BwB_-}_{\rm ns}$. Since the singular set of $\overline{BwB_-}$ has codimension at least two [3], it suffices to show that the singular set of $F_z \cap \overline{BwB_-}_{\rm ns}$ has codimension at least two. Let $F_z = \cup_{i=1}^n C_{z_i}$ be the decomposition into conjugacy classes with $C_{z_1} = R_z$. By 1), the singular set of $F_z \cap \overline{BwB_-}_{\rm ns}$ is contained in

$$\cup_{v \geq z, i \geq 2} C_{z_i} \cap BvB_-.$$

By [21, 4.24], if $i \geq 2$, $\dim(C_{z_i}) \leq \dim(R_z) - 2$. Since $\dim(C_{z_i} \cap BvB_-) = \dim(C_{z_i}) - l(v)$ by Proposition 2.15, it follows that if $i \geq 2$,

$$\dim(C_{z_i} \cap BvB_-) \leq \dim(R_z \cap BvB_-) - 2 \leq \dim(F_z \cap \overline{BwB_-}) - 2.$$

Part 2) follows. □

Theorem 6.14. *$F_z \cap \overline{BwB_-}$ is normal.*

Proof. Since $F_z \cap \overline{BwB_-}$ is Cohen-Macaulay by Theorem 6.12, condition S_2 of Serre is satisfied (see [32, page 183]). Part 2) of Proposition 6.13 is equivalent to condition R_1 of Serre, so the theorem follows using Serre's normality criterion ([32, Theorem 23.8]). □

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