On an Infectious Model for Default Crisis

Wai-Ki Ching [*] Tak-Kuen Siu [†] Li-min Li[‡] Tang Li[§] Wai-Keung Li [¶]

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Abstract

In this paper, we look at the problem of modelling the temporal dependence of defaults and introduce a novel approach for describing the chain reaction of infectious defaults. We extend a Markov chain model for crisis management in epidemiology, namely, Greenwood's model, to describe the chain reaction of infectious defaults of bonds across any pair of industrial sectors. The development of the extended version of Greenwood's model contributes to the literature in two major aspects. First, it contributes to the credit risk literature by providing a new framework for measuring risk of a credit portfolio over the duration of a default crisis, called a default cycle. This is different from the traditional approach for risk measurement in which the time horizon is fixed, say one day or one week, and leads to a new dimension for risk management. Second, it provides a new model for crisis management in epidemiology by extending an important model in the field. We employ recursive formulae for the joint probability distributions of the duration of a default crisis and the number of defaults over the crisis. Moreover, we employ two important risk measures, namely, Value-at-Risk (VaR) and Expected Shortfall (ES), as proxies of risk over a default cycle. Numerical experiments are given to illustrate the practical implementation of the model and identify some main features of the model. We also perform empirical studies of the model using real default data and analyze the empirical behaviors of the risk measures arising from the model.

[*]Advanced Modeling and Applied Computing Laboratory, Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong. E-mail: wching@hkusua.hku.hk. Research supported in part by RGC Grant 7017/07P, HKU CRCG Grants, HKU Strategic Research Theme Fund on Computational Sciences.

[†]Department of Actuarial Mathematics and Statistics, School of Mathematical and Computer Sciences and the Maxwell Institute for Mathematical Sciences, Heriot-Watt University, Edinburgh EH14 4AS, UK.

[‡]Advanced Modeling and Applied Computing Laboratory, Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong. E-mail: liminli@hkusua.hku.hk

[§]Advanced Modeling and Applied Computing Laboratory, Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong. E-mail: litang@hkusua.hku.hk

[¶]Department of Statistics and Actuarial Science, The University of Hong Kong, Pokfulam Road, Hong Kong. Email: hrntlwk@hkucc.hku.hk

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1 Introduction

Recently, there has been considerable interest in the use of copulas for modeling dependence of defaults and credit qualities of entities. Copulas can be used to model asymmetric dependency and non-linear association of variables, which cannot be described by correlations. Copulas is an important tool in statistics and has long been used in survival analysis. It has also been used in reliability analysis in engineering and in life contingency in actuarial science. Li [10] introduces the use of copulas for credit risk measurement and provides a comprehensive discussion on various copulas functions for modeling dependent credit risks. The applications of default correlations and copulas are traditional approaches to model dependent defaults. There are other models for dependence of defaults in the literature, including the mixture model approach, in particular the Poisson mixture model in CreditRisk+ [1] developed by Credit Suisse Financial Products in 1997, the infectious default models by Davis and Lo [4, 5] and the correlated default models in [6] and [7]. Much of the literature seem to focus on modelling the cross-sectional dependence of defaults. There is relatively little work on modelling the temporal dependence of defaults (see for instance, [11]).

Some recent literature show empirically that the default of a bond causes the widening of credit spreads of other firms and sudden jumps in credit spread indices, see for instance, Das et al. [3]. This reflects that the default of a bond has a significant impact on the likelihood of defaults of other bonds. The causality relationship goes from defaulted bonds to surviving bonds. Traditional approaches for modeling dependent defaults based on default correlations and copulas do not focus on modelling such a relationship. However, modeling the impact of defaulted bonds on the likelihood of defaults of other firms is certainly an important issue. It is especially the case in a default crisis during which this impact becomes more pronounced and defaulted bonds are more likely to cause the defaults of surviving bonds. A chain reaction of infectious defaults may be created during the crisis, and, this results in a default cycle. This situation is not unlike the situation of spreading of some critical infectious diseases, such as

Severe Acute Respiratory Syndrome (SARS). It is important to take the effect of spreading of defaults into account in risk management and crisis management of credit portfolios. From the practical perspective, understanding the duration of the default cycle can provide regulators with more information about how severe the default crisis is and some insights in evaluating the performance of some measures or policies that are imposed to shorten the duration or reduce the severity of the crisis. From the academic perspective, it provides some insights in understanding the underlying mechanism of the temporal dependent of defaults. In practice, it may be the case that defaults of bonds from one industrial sector have effect on the likelihood of defaults of bonds from another industrial sector, especially when the two sectors are related industrial sectors. It is of practical importance and relevance to develop a model, which allows the flexibility to model the chain reaction of infectious defaults of bonds across any two industrial sectors.

In this paper, we consider a new perspective for modelling the temporal dependence of defaults and introduce a novel approach for modelling the chain reaction of infections by exploiting the state of art of mathematical models in epidemiology. More specifically, we extend a Markov chain model, namely, the Greenwood's model, which is a classical model in epidemiology (see [2, p.108]), to describe the chain reaction of infectious defaults of bonds across any pair of industrial sectors. The extended version of Greenwood's model can be considered as a kind of Markovian regime-switching model. It contributes to the literature in two major aspects. First, it contributes to the credit risk literature by providing a new framework for measuring risk of a credit portfolio over the duration of a default crisis called a default cycle. This is different from the traditional approach for risk measurement in which the time horizon is fixed, say one day or one week, and opens up a new domain for credit risk management. Second, it provides a new model for crisis management in epidemiology by extending an important model in the field. We derive recursive formulae for the joint probability distributions of the duration of a default crisis and the severity of default measured by the number of defaults over the crisis. The recursive formulae provide market practitioners with a handful way to evaluate the joint probability distribution functions. We employ two well-known measures, namely, Value-at-Risk (VaR) and Expected Shortfall (ES), over a default cycle, as proxies of risk. These measures provide a summary on the joint probability distributions of the severity of defaults and the default cycle. They not only provide some insights in risk management, but also crisis management. In fact, crisis management can be viewed as a specific type of risk management. It takes care of those risks which are serious, destructive or of disastrous nature. In our current setting, the chain reaction of infectious defaults lead to a default disaster or crisis and it is not certain how long the crisis will last for. So, special attention needs to pay for evaluating how destructive this crisis is and how long it will last. The extended model also provides some insights in highlighting the interplay between the crisis management in credit markets and that in epidemiology. We believe that this paves a way for further development of the interaction between the two fields. The model is parsimonious and there are only a few parameters needed to be estimated irrespective of the size of the Markov chain. Also, the recursive formulae provide an efficient and convenient way to evaluate the joint probability distributions of the duration of a default crisis and the severity of default measured by the number of defaults over the crisis even when the size of the Markov chain is large. The model is applicable to real portfolios consisting of many bonds, say hundreds or thousands of bonds, with reasonable computational time and effort. The model parameters can be estimated easily by some standard statistical procedures, such as the maximum likelihood method. This makes the practical implementation of the model easier. We conduct numerical experiments to illustrate the practical implementation of our model and the capability of the model to describe the chain of reaction of infectious defaults of portfolios with many bonds. We perform empirical studies of the model using practical default data. In particular, a likelihood ratio test is performed to test our model (the two-sector model) against its special case (the one-sector model) when the chain of reaction of infectious defaults across two different sectors is not taken into account. The test results reveal that the two-sector model significantly outperforms the one-sector model. This justifies the necessity to consider the extended version of Greenwood's model. We also document empirical behaviors of the VaR and the ES evaluated from both the two-sector model and the one-sector model. We find that without incorporating the chain of reaction across two sectors leads to either underestimation or overestimation of the two crisis measures.

This paper is structured in the sequel. Section 2 presents the extended version of Greenwood's model for modelling the chain reaction of infectious defaults across two industrial sectors. We also derive a recursive formula for the joint probability distribution for the default cycle and the number of defaults during the crisis and outline the estimation procedure. In Section 3, we present and discuss the results of numerical experiments. The results of empirical analysis are then presented and discussed in Section 4. The final section concludes the paper.

2 The extended version of Greenwood's model

We consider a discrete time economy with different industrial sectors and develop an extended version of Greenwood's model (a two-sector model) to describe the chain reaction of infectious defaults across any two related industrial sectors, say sector A and sector B. Suppose that there are N entities, which are corporate bonds issued by firms from sector A. The key ideas of the extended version of the Greenwood's model are the followings: (i) to model the chain reaction of infectious defaults in a sector, say sector A; (ii) to describe the impact of the default state of another sector, say sector B, on the likelihood of the default of a bond from sector A. For illustration, we suppose that there are two default states in sector B. We say that the default state of sector B at time t is in state 0 if there is no default observed and that the default state of sector B is 1 if there is at least one default observed.

2.1 The model

We present the key ideas of the two-sector model. First, we fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability. Let $\mathcal{T}$ denote the time index set $\{0, 1, 2, \ldots,\}$ of our model. Suppose $\{H_t\}_{t \in \mathcal{T}}$ denotes a stochastic process on $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $\{0, 1\}$, where H_t represents the default state of sector B at time t. We further assume that $\{H_t\}_{t \in \mathcal{T}}$ follows a two-state, first-order observable Markov chain with the following transition probability matrix:

$$\begin{pmatrix} p^{(00)} & p^{(01)} \\ p^{(10)} & p^{(11)} \end{pmatrix}$$

where

$$p^{(00)} + p^{(01)} = 1 \quad \text{and} \quad p^{(10)} + p^{(11)} = 1 \ . \tag{1}$$

Suppose that

$$X := \{X_t\}_{t\in\mathcal{T}} \quad \text{and} \quad Y := \{Y_t\}_{t\in\mathcal{T}}$$

denote two stochastic processes on $(\Omega, \mathcal{F}, \mathcal{P})$, where X_t and Y_t represent the numbers of surviving bonds and the defaulted bonds at time $t \in \mathcal{T}$ from sector A, respectively. Here we assume the initial conditions $X_0 = x_0$ and $Y_0 = y_0$ are given, where of course $x_0 + y_0 = N$.

For each $s = 0, 1$, let α_s denote the probability that the default of a surviving bond is infected by the defaulted bonds in sector A if the default state of sector B is s, where $\alpha_s \in (0, 1)$, for $s = 1, 2$. Here, the chain reaction of infectious defaults of bonds from sector A depends on the default state of sector B. Thus both the chain reaction of infectious defaults in sector A and the impact of default state in sector B on the likelihood of defaults of sector A are modelled here.

For each $t \in \mathcal{T}$, the sum of the numbers of the defaulted bonds and the surviving bonds from sector A at time $t + 1$ equals to the number of surviving bonds from sector A at time t, i.e,

$$X_{t+1} + Y_{t+1} = X_t.$$

Then, under the two-sector model, the joint probability distribution of $(X_{t+1}, Y_{t+1}, H_{t+1})$ given (X_t, Y_t, H_t) is:

$$\begin{aligned}
& p_{x_t,y_t,h_t}(x_{t+1}, y_{t+1}, h_{t+1}) \\
= \; & P\left\{(X_{t+1}, Y_{t+1}, H_{t+1}) = (x_{t+1}, y_{t+1}, h_{t+1}) | (X_t, Y_t, H_t) = (x_t, y_t, h_t)\right\} \\
= \; & \begin{pmatrix} x_t \\ y_{t+1} \end{pmatrix} \alpha_{h_t}^{y_{t+1}} (1 - \alpha_{h_t})^{x_{t+1}} p^{(h_t, h_{t+1})} \qquad (2) \\
= \; & \begin{pmatrix} x_t \\ x_t - x_{t+1} \end{pmatrix} \alpha_{h_t}^{x_t - x_{t+1}} (1 - \alpha_{h_t})^{x_{t+1}} p^{(h_t, h_{t+1})} \; , \quad h_t, h_{t+1} = 0, 1 \; ,
\end{aligned}$$

where $p^{(h_t,h_{t+1})}$ is given by (1).

When there is only one default state in sector B, the two-sector model reduces to the following one-sector model, under which the joint probability distribution of X_{t+1} and Y_{t+1}

given $X_t = x_t$ and $Y_t = y_t$ is

$$
\begin{aligned}
p_{x_t,y_t}(x_{t+1}, y_{t+1}) &= P\{(X_{t+1}, Y_{t+1}) = (x_{t+1}, y_{t+1}) \mid (X_t, Y_t) = (x_t, y_t)\} \\
&= \begin{pmatrix} x_t \\ y_{t+1} \end{pmatrix} \alpha^{y_{t+1}}(1-\alpha)^{x_{t+1}} \\
&= \begin{pmatrix} x_t \\ x_t - x_{t+1} \end{pmatrix} \alpha^{x_t - x_{t+1}}(1-\alpha)^{x_{t+1}} ,
\end{aligned} \tag{3}
$$

where α is the probability that the default of a surviving bond is infected by the defaulted bonds in sector A. This is Greenwood's model in epidemiology.

Under the two-sector model, we focus on modelling the causal relationship of defaults in one direction, say the defaults of bonds from sector A caused by the defaulted bonds from sector B. It might be more realistic to model the causal relationship of defaults in both directions, from sector B and sector A, and vice versa. However, in this case, one might need to model the feedback effect of the regime-switching Markov chain by considering a kind of interactive Markov chain model. The estimation procedure of the interactive regime-switching Markov chain model is much more complicated than that of the regime-switching Markov chain in the current setting. Therefore, to make the estimation procedure more tractable and the model easier to implement in practice, we consider here the causality relationship in one direction. One possible remedy might be to fit the model two times, one from sector B to sector A and one from sector A to sector B, and get some insights into the causality relationship as we will demonstrate in our empirical experiment in Section 4. In some cases, one may also make some assumptions for the causality relationship based on some economic reasons. For example, based on some economic intuition, it is more reasonable to assume that the default of bonds from Transport sector is more likely to be caused by the defaulted bonds from the Energy Sector than vice versa (see Section 4 for instance).

The two-sector model can be considered as a Markovian regime-switching Markov chain modulated by another Markov chain with two regimes. From (2), the transition probability matrix of the Markov chain model is a $2(x_0 + 1) \times 2(x_0 + 1)$ matrix given as follows:

$$
\begin{pmatrix} p^{(00)}P(\alpha_0) & p^{(01)}P(\alpha_0) \\ p^{(10)}P(\alpha_1) & p^{(11)}P(\alpha_1) \end{pmatrix}
$$

where x_0 is the number of survival bonds in sector A at time $t = 0$, and, for $s = 0, 1$,

$$P(\alpha_s) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \alpha_s & 1-\alpha_s & 0 & \cdots & 0 \\ \alpha_s^2 & \begin{pmatrix} 2 \\ 1 \end{pmatrix} \alpha_s(1-\alpha_s) & (1-\alpha_s)^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \alpha_s^{x_0} & \begin{pmatrix} x_0 \\ 1 \end{pmatrix} \alpha_s^{x_0-1}(1-\alpha_s) & \begin{pmatrix} x_0 \\ 2 \end{pmatrix} \alpha_s^{x_0-2}(1-\alpha_s)^2 & \cdots & (1-\alpha_s)^{x_0} \end{pmatrix}.$$

Here the elements of the i^{th} row of $P(\alpha_s)$ are given by the i^{th} term in the expansion of $(\alpha_s + 1 - \alpha_s)^{i-1}$, for $i = 1, 2, \ldots, x_0 + 1$ and $s = 1, 2$.

2.2 The Joint pdf for the Duration and Severity of Default Crisis

We shall derive the joint probability distribution function (pdf) for the duration of the default crisis, namely, the default cycle, and the severity of defaults during the crisis period in sector A. First, we provide a precise definition of the default crisis in sector A. Let

$$T := \inf\{t \in \mathcal{T} | Y_t = 0\} = \inf\{t \in \mathcal{T} | \{X_t = X_{t-1}\} \cup \{X_t = 0\}\}.$$

Write $\mathcal{F}^X$ and $\mathcal{F}^Y$ for the complete, right-continuous filtration generated by the processes X or Y, respectively. Then, T is a stopping time with respect to $\mathcal{F}^X$ or $\mathcal{F}^Y$. Here, T represents the duration of the crisis period. In epidemiology, one is interested in measuring how long a spreading of some critical infectious diseases lasts for. This provides medical practitioners with additional information to assess the damage of the infectious diseases in a given population. In credit risk management, the duration T provides market practitioners with additional information about the severity and impact of a default cycle or crisis. It serves as an additional indicator for assessing credit risk and is not unlike the use of the duration of the trough period in a business cycle to provide indication on the economic situation. The severity of default over the crisis period in sector A is measured by the number of defaults W_T in sector A over the duration T. Note that conditional on $T = t$, W_t represents the number of defaults in sector A over the time duration $[0, t]$.

For $s = 0, 1$ and $0 < t \le T$, let

$$\begin{aligned} p_j^s(t) &= P\{X_t = j, H_t = s, Y_t > 0\} \\ &= \sum_{i=j+1}^{x_0-(t-1)} (p_i^0(t-1)p_{ij}^{(0s)} + p_i^1(t-1)p_{ij}^{(1s)}) \\ &= \sum_{r=0}^{1} \sum_{i=j+1}^{x_0-(t-1)} p_i^r(t-1)p_{ij}^{(rs)} , \end{aligned} \tag{4}$$

$$\begin{aligned} p_{ij}^{(rs)} &= P\{(X_{t+1}, H_{t+1}) = (j, s)|(X_t, H_t) = (i, r)\} \\ &= \begin{pmatrix} i \\ j \end{pmatrix} \alpha_r^{i-j}(1-\alpha_r)^j p^{(rs)} , \end{aligned} \tag{5}$$

and

$$p_i^s(0) = \begin{cases} 1 & \text{if } i = x_0, s = H_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, $p_j^s(t)$ can be obtained recursively using the above initial conditions and Equations (1) and (4) as follows:

$$\begin{aligned} & P\{(W_T, T) = (k, t)|X_0 = i, Y_0 > 0\} \\ = \;& p_{i-k}^1(t-1)p_{i-k,i-k}^{(10)} + p_{i-k}^0(t-1)p_{i-k,i-k}^{(00)} \\ & + p_{i-k}^1(t-1)p_{i-k,i-k}^{(11)} + p_{i-k}^0(t-1)p_{i-k,i-k}^{(01)} \\ = \;& p_{i-k}^1(t-1)(1-\alpha_1)^{i-k} + p_{i-k}^0(t-1)(1-\alpha_0)^{i-k}. \end{aligned} \tag{6}$$

We remark that even when the size of the Markov chain is large, the closed-form recursive formulae also provide an efficient and convenient way to evaluate the joint probability distribution W_T and T. This makes our model practically useful even when we have a large number of bonds in the portfolio.

2.3 Estimation

We outline the estimation method of the extended version of Greenwood's model. The model has four key unknown parameters $p_{(00)}$, $p_{(10)}$, α_0 and α_1 to be estimated irrespective to the number of bonds in both sector A and sector B since there are two default states in sector B and the Markov chain for sector A only depends on α_0 and α_1.

Given the default data of sector B, the transition probabilities $p^{(ij)}$ $(i, j = 0, 1)$ are estimated by the proportion of observed frequencies of transitions from state i to state i and state j. We then employ the maximum likelihood method to estimate the parameters α_0 and α_1 based on the observed numbers of defaults in both sector A and sector B. Given these observations $x_0, x_1, \ldots, x_N$ and $h_0, h_1, \ldots, h_N$, the likelihood function $L(\alpha|x_0, x_1, \ldots, x_N, h_0, h_1, \ldots, h_N)$ is:

$$\begin{aligned} & L(\alpha|x_0, x_1, \ldots, x_N, h_0, h_1, \ldots, h_N) \\ = & \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} (1-\alpha_{h_0})^{x_1} \alpha_{h_0}^{x_0-x_1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (1-\alpha_{h_1})^{x_2} \alpha_{h_1}^{x_1-x_2} \cdots \begin{pmatrix} x_{N-1} \\ x_N \end{pmatrix} (1-\alpha_{h_N})^{x_N} \alpha_{h_N}^{x_{N-1}-x_N} . \end{aligned}$$

Then, the maximum likelihood estimates of α_0 and α_1 are:

$$\hat{\alpha}_0 = \frac{\sum_{i=0}^{N}(1-h_i)y_i}{\sum_{i=0}^{N}(1-h_i)(x_i+y_i)} \quad \text{and} \quad \hat{\alpha}_1 = \frac{\sum_{i=0}^{N} h_i y_i}{\sum_{i=0}^{N} h_i(x_i+y_i)}.$$

We remark that for the one-sector model (3), the model parameter α can be estimated similarly by the maximum likelihood estimator.

3 Numerical experiments

We conduct numerical experiments to illustrate the practical implementation of our model and to demonstrate that our model is applicable for a large portfolio consisting of hundreds of bonds from sector A.

First, we describe two measures of risk to be considered here, namely, the VaR and the ES. Since we are going to use these measures of risk to describe the risk over the whole crisis period, we suppose that the loss variables of the portfolio are functions of the duration of the default crisis T and the severity of defaults measured by the number of defaulted bonds

W_T over the whole crisis period. Let

$$L(\cdot,\cdot)(\omega) : \mathcal{T} \times \mathcal{R} \times \Omega \to \mathcal{R}$$

denote a real-valued function $L(T, W_T)(\omega)$ of T and W_T. We then suppose that for a fixed $\omega \in \Omega$,

$$T(\omega) = t, \quad W_t(\omega) = w \quad \text{and} \quad L(t, w)(\omega) = l(t, w) \in \mathcal{R}.$$

That is, the loss from the portfolio over the default crisis is $l(t, w)$ when the duration of the default crisis $T = t$ and the number of defaulted bonds from sector A in the crisis $W_t = w$.

The VaR with probability level β is the β-quantile of the distribution of the loss variable $L(T, W_T)$ under the real-world probability $\mathcal{P}$. The ES with probability level β is the average of the loss from the portfolio over the default crisis when the loss exceeds the VaR of the default crisis with probability level β under $\mathcal{P}$.

Now, we consider a hypothetical portfolio consisting of five hundreds bonds from sector A. So, the initial number X_0 of surviving bonds in the portfolio is 500. We note that the loss variable is a function of the number of defaults W_T over the default cycle and the default cycle T. From an economic perspective, the loss from a portfolio over a default crisis increases if either the number of defaulted bonds W_T increases or the duration of the default crisis T becomes longer. Thus it is reasonable to assume the loss function $L(W_T, T)$ to be an increasing function of W_T (T) when T (W_T) is fixed.

Then, the duration of the default crisis must be less than or equal to $X_0 = 500$ since we must have at least one default in each period during a default crisis. Also, if $W_T = w$, the duration of the default crisis T should not be larger than w. In the numerical experiments, we assume some hypothetical values for the loss $L(W_T, T)$, for each $T = 0, 1, \ldots, 500$ and $W_T = 0, 1, \ldots, 500$, as below:

$$\begin{cases} L(i, 0) = 0, \text{ for each } i = 0, 1, \ldots, 500; \\ L(0, j) = j - 1 + 0.1, \text{ for each } j = 1, 2, \ldots, 500; \\ L(i, j) = L(0, j) + i - 1, \text{ for each } j = 1, 2, \ldots, 500 \text{ and } i = 1, 2, \ldots, j; \\ L(i, j) = 0, \text{ otherwise.} \end{cases}$$

The set of hypothetical values for the loss variable is used here for illustration. These values are consistent with the economic intuition that the loss from a portfolio increases if either the number of defaulted bonds W_T increases or the duration of the default crisis becomes longer. We then consider some specimen values for the model parameters and assume that

$$\alpha_0 = 0.0005,\ \alpha_1 = 0.005,\ p^{(00)} = 0.8 \text{ and } p^{(10)} = 0.3.$$

With these parameter values, one can compute the exact values of

$$\mathcal{P}\{(W_T, T) = (k, n) | X_0 = 500, Y_0 > 0\}$$

for different values of (k, n).

Table 1

Thus we see that the ES is substantially higher than the VaR at each of the probability levels.

4 Empirical results using practical default data

In this section, we present the empirical results of the two infectious models using real default data extracted from the figures in [8]. The estimation results for the two models presented here are obtained from the maximum likelihood method described in Section 2.3. We shall perform a likelihood ratio test for the one-sector model against the two-sector model and provide empirical evidence to support the use of the two-sector model based on the default data. We also illustrate the evaluation of the VaR and the ES from two models and compare the results obtained from the two models. In [8], Giampieri et al. apply the hidden Markov model to analyze the default data of bonds from four sectors (customer/service sector, energy sector, media sector and transportation) taken from the Standard & Poor's CreditPro 6.2 database in the period from 1 January 1981 to 31 December 2002. The following table gives the default data (we extract the data from [8]) in the consumer/service sector, the energy and natural resources sector, the leisure time/media sector and the transportation sector.

Table 2

The proportions of defaults for Consumer, Energy, Media and Transport are 24.11%, 16.9%, 20.46% and 20.00%, respectively. The default situation is the most serious in the consumer sector while it is the least in the energy sector. Also, the probability of defaults for all sectors are significantly different from zero. This reflects that the default risk of each of the four sectors is substantial.

For the evaluation the VaR and the ES in this section, we consider the hypothetical values of the loss $L(W_T, T)$ assumed in Section 3.

All computations were done on a Pentium 4HT PC with MATLAB. Table 3 presents the estimation results of the one-factor infectious model for the four sectors and the VaR and the ES for the four sectors with probability levels (β) 0.01 and 0.05.

Table 3

From the results in Table 3, we see that the values of the ES are always greater than those of the VaR in all cases. The values of the ES (VaR) with probability level 0.01 are always greater than those of the ES (VaR) with probability level 0.05 for all sectors. These results are consistent with intuition.

We then consider a two-sector infectious disease model. First, we compute the correlations of the default data from each pair of the sectors and present the results in Table 4. From Table 4, we see that all correlations are positive. This provides some preliminary evidence for supporting the use of the two-sector model from the perspective of descriptive statistical analysis. We shall provide more empirical evidence for supporting the use of the two-sector model by the results of likelihood later in this section. The arterisk "*" in the table indicates the pair of sectors which has the largest correlation. For example, from the first row of Table 4, the correlation between Consumer and Media is 0.6013 and is the largest among all the others. To build a two-sector model, for each row (Sector A), we may find the partner (Sector B) by searching the one with the largest correlation in magnitude (i.e. with the arterisk "*"). For example, Consumers sector (Sector A) may find its partner Media sector (Sector B).

Table 4

Then, we present the estimates of α_0, α_1 and the values of CRVaR and CRES for the four pairs of sectors with probability levels 0.01 and 0.05 in Table 5. From Table 5, we see that the values of the CRVaR and the CRES with probability levels 0.01 and 0.05 in the two-sector model are higher than the corresponding figures in the one-sector model for all sectors, except the Consumer sector in Table 3.

Table 5

Now, we conduct a Likelihood Ratio Test (LRT) for the one-sector model against the two-sector model. For detailed discussions on the LRT, we refer readers to the monograph by Hoel [9] (Section 9.1.4 therein). The test statistic of the LRT is the log-likelihood ratio, which follows a χ^2-distribution with one degree of freedom. The critical values are 3.843 and 6.637 at 95% and 99% significant levels, respectively. Those log-likelihood ratios greater than the critical value 6.637 are signified with a bracket $(\cdot)$ in Table 6. Table 6 presents the log-likelihood ratios of the two models for different pairs of sectors A and B.

Table 6

The arterisk "*" in Table 6 marks the pair of sectors which has the largest log-likelihood ratio. From the results in Table 6, we remark that for all of the sectors, except Energy, we can find a pair of sectors such that the two-sector model is statistically better than the one-sector model at both significant levels 99% and 95%. This provides empirical evidence for supporting the use of the two-sector model.

Table 7 presents the estimates of α_0 and α_1, the figures of the VaR and the ES for the four sectors with probability levels 0.01 and 0.05 from the two-sector model, where Sectors A and B are chosen by the pairs with arterisk "*" in Table 6, having the largest log-likelihood ratio. The figures in Table 7 are the same as those in Table 5, except in their third columns from which we can only see a small difference in the values of the probability α_0 and α_1. In the third column of Table 5, Energy (Sector A) is coupled with Media (Sector B) since they have the largest correlation as indicated by the arterisk "*" in Table 4. However, in the third column of Table 7, Energy (Sector A) is coupled with Transportation (Sector B) since they have the largest log-likelihood ratio as indicated by the arterisk "*" in Table 6. Overall, the values of the VaR and the ES in Table 7 are very close to and equal, up to one decimal place, the corresponding values in Table 5. We can also see that the other three sectors, namely, Consumer, Media and Transport, found the same partners in both cases of using correlation and log-likelihood ratio as criteria.

By focusing on the second column and the fourth column of Table 7, we further observe that the dependency of Media and Consumer is not symmetric. In particular, the values of the VaR and the ES in the fifth column are increased by incorporating one more factor by the two-sector model, but these values in the third column are reduced by adding the factor. The asymmetric dependence of Media and Consumer might be due to the situation that the causal relationship between Media and Consumer in one direction might be stronger than that in the other direction. For example, the causal relationship from Consumer to Media is stronger than that from Media to Consumer. This reveals that it might not be appropriate to use a symmetric measure of dependence for modeling the chain reaction of infections across two sectors since it cannot describe the asymmetric dependency of two sectors. Since the two-sector model describes explicitly the direction of the causal relationship, say from Sector A to Sector B, it can describe the asymmetric dependency of two sectors.

Now, we compare the figures for the two-sector model in Table 7 with those for the one-sector model in Table 3. We can see that the two-sector model gives marginally higher values for the VaR and the ES than the one-sector model in the last two columns. By comparing the second columns in Table 5 and Table 7 that the values of VaR and the ES can be substantially reduced with the incorporation of one more factor by the two-sector model. This reveals that adding one more factor might not necessarily increase the values of the VaR and the ES. This has important implications for an appropriate quantitative model to be used for evaluating both the VaR and the ES more accurately. From our empirical analysis based on the default data of the four sectors, the two-sector model is statistically better than the one-sector model. This provides empirical evidence that the chain of reaction of defaults across two different sectors exist in the dataset and the two-sector model seems to be a more appropriate model than the one-sector model. So, without incorporating the chain of reaction of defaults across two different sectors might lead to either underestimation or overestimation of the VaR and the ES. This has serious consequences and implications for risk and crisis management.

Table 7

5 Conclusion

We introduced a two-sector infectious disease model for the chain reaction of infectious defaults across any two related industrial sectors by extending Greenwood's model for crisis management in epidemiology. The model can describe (i) the chain reaction of infections in a sector; (ii) the impact of the default state of another sector on the likelihood of the default of a bond from the sector in (i). We considered both the duration of default crisis or the default cycle and the severity of defaults over the default cycle under the two-sector infectious model. A recursive formula for the joint probability distribution of the default cycle and the number of defaults over the cycle was derived for the two-sector model. This result is also novel and interesting from the perspective of crisis management in epidemiology. We employed the maximum likelihood method to estimate the model parameters and adopted the VaR and the ES as proxies for risk. We conducted numerical experiments to illustrate the practical implementation of the model and to demonstrate its applicability to a large portfolio consisting of hundreds of bonds.

We conducted empirical studies on the two models using real default data. The estimation results for the two models and the numerical results of the two crisis measures evaluated from the two models were presented and discussed. We also performed a likelihood ratio test for the one-sector model against the two-sector model and found that the two-sector model is statistically better than the one-sector model at significance levels 95% and 99%. In other words, the two-sector model outperforms the one-sector model in fitting the default data. We also provided analysis for the VaR and the ES evaluated from the two models and found that without incorporating the chain reaction of infections across two sectors leads to either underestimation or overestimation of the two measures of risk. This has serious consequences and important implications for risk and crisis management.

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Table 1: The VaR and ES.

	VaR	ES
β=0.01	19.1	28.4
β=0.05	5.1	13.9

Table 2: The default data (Taken from [8]).

Sector	Total	Default
Consumer	1041	251
Energy	420	71
Media	650	133
Transport	281	59

Table 3: The one-sector model

Sectors	Consumer	Energy	Media	Transport
Default probability α	0.0030	0.0021	0.0025	0.0026
CRVaR($\beta = 0.01$)	284.1	19.1	58.1	15.1
CRES ($\beta = 0.01$)	322.0	24.1	69.8	18.5
CRVaR($\beta = 0.05$)	208.1	12.1	38.1	9.1
CRES ($\beta = 0.05$)	254.4	16.6	50.2	12.6

Table 4: Correlations of the sectors

	Consumer	Energy	Media	Transport
Consumer	-	0.0224	0.6013*	0.3487
Energy	0.0224	-	0.1258*	0.1045
Media	0.6013*	0.1258	-	0.3708
Transport	0.3487	0.1045	0.3708*	-

Table 5: The two-sector model

Sector A	Consumer	Energy	Media	Transport
Sector B	Media	Media	Consumer	Media
α_0	0.0013	0.0018	0.0005	0.0013
α_1	0.0043	0.0023	0.0033	0.0036
CRVaR ($\beta = 0.01$)	166.1	20.1	52.1	16.1
CRES ($\beta = 0.01$)	195.6	24.5	63.3	20.2
CRVaR ($\beta = 0.05$)	114.1	12.1	34.1	10.1
CRES ($\beta = 0.05$)	146.1	17.1	45.7	14.1

Table 6: The log-likelihood ratio: one-sector model to two-sector model

Sector A	Consumer	Energy	Media	Transport
Sector B	Media	Media	Consumer	Media
Log-likelihood Ratio	(68.99^{*})	1.21	(43.44^{*})	(12.84^{*})

Table 7: Two-sector model

Sector A	Consumer	Energy	Media	Transport
Sector B	Media	Transport	Consumer	Media
α (in one-sector model)	0.0030	0.0021	0.0025	0.0026
α_0	0.0013	0.0018	0.0005	0.0013
α_1	0.0043	0.0025	0.0033	0.0036
CRVaR ($\beta = 0.01$)	166.1	20.1	52.1	16.1
CRES ($\beta = 0.01$)	195.6	24.5	63.3	20.2
CRVaR ($\beta = 0.05$)	114.1	12.1	34.1	10.1
CRES ($\beta = 0.05$)	146.1	17.1	45.7	14.1