A Min-Max Theorem on Tournaments

Xujin Chen[a*] Xiaodong Hu[a†] Wenan Zang[b‡§]

[a] Institute of Applied Mathematics, Chinese Academy of Sciences
Beijing 100080, China

[b] Department of Mathematics, The University of Hong Kong
Hong Kong, China

Abstract

We present a structural characterization of all tournaments $T = (V, A)$ such that, for any non-negative integral weight function defined on V, the maximum size of a feedback vertex set packing is equal to the minimum weight of a triangle in T. We also answer a question of Frank by showing that it is NP-complete to decide whether the vertex set of a given tournament can be partitioned into two feedback vertex sets. In addition, we give exact and approximation algorithms for the feedback vertex set packing problem on tournaments.

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§Corresponding author. E-mail: wzang@maths.hku.hk.

1 Introduction

A rich variety of combinatorial optimization problems falls within the general framework of packing and covering in hypergraphs. A *hypergraph* is a pair $\mathcal{H} = (V, \mathcal{E})$, where V is a finite set and $\mathcal{E}$ is a family of subsets of V. Elements of V and $\mathcal{E}$ are called the *vertices* and *edges* of $\mathcal{H}$, respectively. A *vertex cover* of $\mathcal{H}$ is a vertex subset that intersects all edges of $\mathcal{H}$. Let w be a nonnegative integral weight function defined on V. A family $\mathcal{S}$ of edges (repetition is allowed) of $\mathcal{H}$ is called a *w-packing* of $\mathcal{H}$ if each $v \in V$ belongs to at most $w(v)$ members of $\mathcal{S}$. Let $\nu_w(\mathcal{H})$ denote the maximum size of a w-packing of $\mathcal{H}$, and let $\tau_w(\mathcal{H})$ denote the minimum total weight of a vertex cover. Clearly $\nu_w(\mathcal{H}) \leq \tau_w(\mathcal{H})$; this inequality, however, need not hold equality in general. We say that $\mathcal{H}$ is *Mengerian* if the min-max relation $\nu_w(\mathcal{H}) = \tau_w(\mathcal{H})$ is satisfied for any nonnegative integral function w defined on V. Many celebrated results and conjectures in combinatorial optimization can be rephrased by saying that certain hypergraphs are Mengerian (See Section 79.1 of [19]), so Mengerian hypergraphs have been subjects of extensive research. As conjectured by Edmonds and Giles [9, 18] and proved recently by Ding, Feng, and Zang [4], the problem of recognizing Mengerian hypergraphs is NP-hard in general, and hence it cannot be solved in polynomial time unless $NP = P$. In this paper we study a special class of Mengerian hypergraphs; our work is a continuation of those done in [1, 2, 3, 5, 6].

Let $G = (V, E)$ be a graph (directed or undirected) and let $\mathcal{C}_G = (V, \mathcal{E})$, where $\mathcal{E}$ consists of $V(C)$, for all induced cycles C in G. Throughout this paper, by a cycle in a digraph we always mean a directed one. In [6], Ding and Zang obtained a structural description of all undirected graphs G for which $\mathcal{C}_G$ is Mengerian. Due to the long list of forbidden structures, to find a good characterization of all digraphs G with Mengerian $\mathcal{C}_G$ seems to be extremely difficult. While this characterization problem remains open in general, it was completely solved on tournaments by Cai, Deng, and Zang [1], where a *tournament* is an orientation of an undirected complete graph.

Theorem 1.1 [1] *Let T be a tournament. Then hypergraph $\mathcal{C}_T$ is Mengerian if and only if T has no subtournament isomorphic to F_1 nor F_2.*

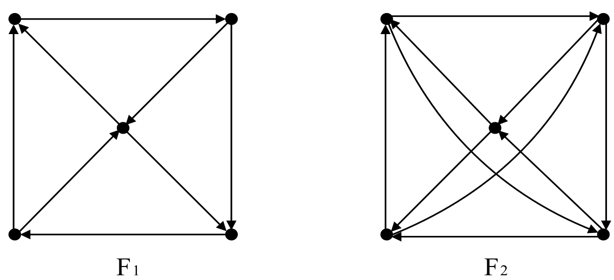

Figure 1. *Forbidden subtournaments F_1 and F_2, where the two arcs not shown in F_1 may take any directions.*

(Note that F_2 is the tournament in which every vertex is incident with precisely two incoming arcs and two outgoing arcs.) One objective of this paper is to establish a closely related min-max relation which is motivated as follows.

Every hypergraph $\mathcal{H} = (V, \mathcal{E})$ is naturally associated with another hypergraph $b(\mathcal{H}) = (V, \mathcal{E}')$, where $\mathcal{E}'$ consists of all minimal (with respect to set inclusion) vertex covers of $\mathcal{H}$. Usually $b(\mathcal{H})$ is called the *blocker* of $\mathcal{H}$. Although in general the blocker of a Mengerian hypergraph does not have to be Mengerian (see Section 79.2 of [19]), the famous max-flow-min-cut theorem and a Fulkerson theorem [11] (see Page 115 of [18]) assert that both the hypergraph of r-s paths in a graph and its blocker are Mengerian; so are the hypergraph of r-arborescences and its blocker by Edmond's disjoint arborescence theorem [7] and Fullerson's optimum arborescence theorem [12]. Recently, Chen *et al.* [3] managed to characterize all undirected graphs G for which $b(\mathcal{C}_G)$ is Mengerian; it turns out that $b(\mathcal{C}_G)$ is Mengerian if and only if $\mathcal{C}_G$ is. So a natural question is to ask: what is the blocker version of Theorem 1.1?

Theorem 1.2 *Let T be a tournament. Then hypergraph $b(\mathcal{C}_T)$ is Mengerian if and only if T has no subtournament isomorphic to F_1 nor F_2.*

An immediate corollary of Theorem 1.1 and Theorem 1.2 is the following.

Corollary 1.3 *Let T be a tournament. Then $b(\mathcal{C}_T)$ is Mengerian if and only if $\mathcal{C}_T$ is.*

Let us define a few terms before presenting an equivalent of the above statements. Let $G = (V, E)$ be a digraph with a nonnegative integral weight $w(v)$ on each vertex v. A *feedback vertex set* (FVS) of G is a vertex subset that intersects each cycle in G, and a *w-FVS packing* of G is a collection $\mathcal{F}$ of minimal FVS's (repetition is allowed) such that each vertex v is contained in at most $w(v)$ members of $\mathcal{F}$. Similarly, a *w-cycle packing* of G is a collection $\mathcal{C}$ of induced cycles (repetition is allowed) such that each vertex v is contained in at most $w(v)$ members of $\mathcal{C}$. The *weight* of a cycle (resp. an FVS) is the sum of weights of all vertices in this cycle (resp. FVS). Observe that every minimal FVS of G uniquely corresponds to an edge of $b(\mathcal{C}_G)$, and vice versa. So there is $1-1$ correspondence between a *w-FVS packing* of G and a w-packing of $b(\mathcal{C}_G)$, and $1-1$ correspondence between a *w-cycle packing* of G and a w-packing of $\mathcal{C}_G$. Moreover, if G is a tournament, then every cycle in a cycle packing is a triangle (a cycle of length three), and hence a cycle packing is actually a triangle packing.

Let $\boldsymbol{Z}_+$ denote the set of nonnegative integers. Then Theorem 1.1 and Theorem 1.2 can be restated as follows.

Theorem 1.4 *The following three statements are equivalent for a tournament $T = (V, A)$:*

(i) For any weight function $w \in \mathbf{Z}_+^V$, the minimum weight of an FVS in T is equal to the maximum size of a w-triangle packing of T;

(ii) For any weight function $w \in \mathbf{Z}_+^V$, the minimum weight of a triangle in T is equal to the maximum size of a w-FVS packing of T;

(iii) T has no subtournament isomorphic to F_1 nor F_2.

It is worthwhile pointing out that the above statement (i) is closely related to the famous Lucchesi-Younger theorem [15] which, when restricted to a planar digraph $G = (V, E)$, is equivalent to saying that for any $w \in \mathbf{Z}_+^E$, the minimum weight of a feedback arc set in G is equal to the maximum size of a cycle packing of G, where a *feedback arc set* of G is a set of arcs that intersects each cycle in G; and statement (ii) is closely related to the well-known Woodall conjecture [20] on packing feedback arc sets and Edmonds-Giles conjecture [8, 17] on packing directed cut covers.

Given a digraph $G = (V, E)$ with a nonnegative integral weight $w(v)$ on each vertex v, the *FVS packing problem* is to find a w-FVS packing of maximum size in G. In connection with this problem, Frank suggested the following question.

Question 1.5 [10] *Given a digraph G, can we decide in polynomial time whether each vertex of G can be colored by red or blue so that every cycle contains at least one red vertex and at least one blue vertex? Or is this an NP-complete problem?*

Our next theorem states that Frank's problem is NP-complete even when G is restricted to a tournament.

Theorem 1.6 *It is NP-complete to decide whether the vertex set of a given tournament can be partitioned into two feedback vertex sets.*

We shall also present algorithms for the FVS packing problem.

Theorem 1.7 *The FVS packing problem on a tournament $T = (V, A)$ with no F_1 nor F_2 can be solved exactly in $O(|V|^4)$ time.*

For the problem on a general tournament, we shall give an approximation algorithm.

Theorem 1.8 *The FVS packing problem on a general tournament can be approximated within a factor of* $2/5$.

The remainder of this paper is organized as follows: In Section 2, we give a proof of Theorem 1.2, which relies heavily on the structural description of tournaments with no F_1 nor F_2 obtained in [1]. In Section 3, we prove Theorem 1.6 by using the so-called Not-All-Equal 3-Satisfiability problem as the source problem. In section 4, we present an exact algorithm for the FVS packing problem on tournaments with no F_1 nor F_2, and describe a 2/5-approximation algorithm for the problem on general tournaments. In Section 5, we conclude this paper with some open problems.

2 Min-max relation

The purpose of this section is to prove Theorem 1.2. We break the proof into a series of lemmas, and shall implicitly and frequently use the fact that a vertex subset of a tournament is an FVS if and only if it intersects every triangle. As usual, a digraph G is called *strongly connected* if for any two vertices x and y, there exist a (directed) path from x to y and a (directed) path from y to x in G. Our proof relies heavily on the following structural description obtained in [1].

Lemma 2.1 [1] *Let $T = (V, A)$ be a strongly connected tournament. Then T has no subtournament isomorphic to F_1 nor F_2 if and only if V can be partitioned into $V_1, V_2, \ldots, V_k$ for some $3 \le k \le |V|$, which have the following properties:*

(i) For any i, j with $1 \le i \le j - 2 \le k - 2$, each arc between V_i and V_j is directed from V_j to V_i.

(ii) For any triangle $xyzx$ in T, there exists an i with $1 \le i \le k - 2$ such that $x \in V_i$, $y \in V_{i+1}$, and $z \in V_{i+2}$ (renaming x, y, and z if necessary).

We make two remarks on the above lemma: First, for notational convenience, the order of the indices used in the above partition $V_1, V_2, \ldots, V_k$ is precisely the reverse of the one used in [1]. Second, as depicted in Figure 2, the vertices of both F_1 and F_2 can be labeled as $u_1, u_2, \ldots, u_5$ such that $\{u_1, u_2, u_3\}$, $\{u_2, u_3, u_4\}$, and $\{u_1, u_4, u_5\}$ are vertex sets of three triangles. Using these triangles, we can immediately see the sufficiency.

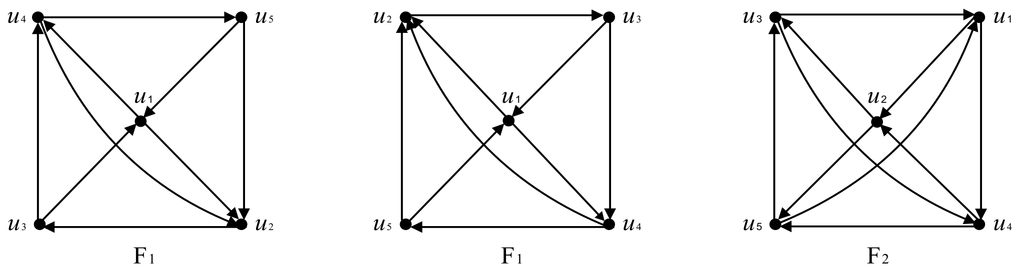

Figure 2. *Three triangles $u_1u_2u_3u_1$, $u_2u_3u_4u_2$, $u_1u_4u_5u_1$ in F_1 and F_2.*

Let $T = (V, A)$ be a tournament and $u, v \in V$. The arc in T with tail u and head v is written as (u, v) and called the arc from u to v. For any subtournament K of T, let $V(K)$ and $A(K)$ denote the vertex set and arc set of K, respectively. For any vertex u of T, let $T\backslash u$ denote the tournament obtained from T by deleting u, and let $T\langle u\rangle$ denote the tournament obtained from T by introducing a new vertex u' and then adding arcs in such a way that

$$\text{for each } v \in V - \{u\},\ (u', v) \text{ is an arc in } T\langle u\rangle \text{ iff } (u, v) \text{ is an arc in } T. \tag{1*}$$

(There is no direction constraint on the arc between u and u'.) We propose to call u' the *image* of u and call $T\langle u\rangle$ an *augmentation of* T (*with respect to* u). It can be seen from (1*) that

$$\text{No triangle in } T\langle u\rangle \text{ contains } \{u, u'\}. \tag{2*}$$

Lemma 2.2 *Let $T\langle u\rangle$ be an augmentation of a tournament $T = (V, A)$. If T contains no F_1 nor F_2, then neither does $T\langle u\rangle$.*

Proof. Assume the contrary: $T\langle u\rangle$ contains a subtournament F isomorphic to F_1 or F_2. Let u' be the image of u. Then F contains both u and u', for otherwise, by (1*), $V(F\backslash u') \cup \{u\}$ would induce a subtournament in T isomorphic to F, a contradiction.

(1) We may assume that T is strongly connected.

Suppose not, let K be the strongly connected component of $T\langle u\rangle$ that contains F (such K is available since F is strongly connected). Then $K\backslash u'$ is strongly connected, for otherwise the vertex set of $K\backslash u'$ can be partitioned into X and Y such that all arcs between X and Y are directed to Y. Without loss of generality, we assume that $u \in X$. Since u' is the image of u, all arcs in K between $X \cup \{u'\}$ and Y are directed to Y, contradicting the strong connectivity of K. Since K is an augmentation of $K\backslash u'$ (with respect to u), we get (1), otherwise replace T by $K\backslash u'$ and $T\langle u\rangle$ by K.

It follows from (1) that the vertex set V of T admits a partition $V_1, V_2, \ldots, V_k$ with properties (i) and (ii) as described in Lemma 2.1. Suppose $u \in V_h$. Let us partition the vertex set $V \cup \{u'\}$ of $T\langle u\rangle$ into k sets V_i' such that $V_h' = V_h \cup \{u'\}$ and $V_j' = V_j$ for all other j with $1 \le j \le k$. From (1*), we see that the partition $V_1', V_2', \ldots, V_k'$ satisfies (i) in Lemma 2.1 with respect to $T\langle u\rangle$. Since F is contained in $T\langle u\rangle$, Lemma 2.1 guarantees the existence of a triangle $xyzx$ in $T\langle u\rangle$ that violates (ii) in the lemma with respect to the partition $V_1', V_2', \ldots, V_k'$. Note that $\{x, y, z\}$ contains at most one of u and u' by (2*). Set $Q = \{x, y, z\}$ if $u' \notin \{x, y, z\}$ and $Q = (\{x, y, z\} - \{u'\}) \cup \{u\}$ otherwise. Then Q would induce a triangle in T that violates Lemma 2.1(ii) with respect to the partition $V_1, V_2, \ldots, V_k$, a contradiction. $\square$

Let $T = (V, A)$ be a tournament and let $S \subseteq V$. We shall use the following notations in our proof:

$$\mathscr{D}_S := \{C : C \text{ is a triangle in } T \text{ and } |V(C) \cap S| = 2\} \tag{3*}$$

$$\mathscr{F}_S := \{C : C \text{ is a triangle in } T,\ V(C) \subseteq S \text{ and } |V(C) \cap V(C')| \le 1 \text{ for every } C' \in \mathscr{D}_S\} \tag{4*}$$

$$\mathscr{F}_S^+ := \{C : C \text{ is a triangle in } T,\ V(C) \subseteq S \text{ and } |V(C) \cap V(C')| = 2 \text{ for some } C' \in \mathscr{D}_S\} \tag{5*}$$

Let $\mathscr{C}$ be a collection of some triangles in T. Write $V(\mathscr{C}) = \cup_{C \in \mathscr{C}} V(C)$. It follows from the definition that $V(\mathscr{D}_S) - S \neq \emptyset$ if $\mathscr{D}_S \neq \emptyset$ and that $V(\mathscr{F}_S) \cup V(\mathscr{F}_S^+) \subseteq S$.

Lemma 2.3 *Let $T = (V, A)$ be a tournament with no subtournament isomorphic to F_1 nor F_2. Suppose S is a subset of V such that $\mathscr{D}_S \neq \emptyset$ and that $|S \cap V(C)| \ge 2$ for every triangle C of T. Then there exists $R \subseteq S$ such that $|R \cap V(C)| = 1$ for every triangle $C \in \mathscr{D}_S$. Moreover, given S, such an R can be found in $O(|V|^3)$ time.*

Proof. Let us first construct an undirected graph G with vertex set S as follows: uv is an edge of G if and only if there is a triangle C in T such that $\{u, v\} = S \cap V(C)$. If G is a bipartite graph, let R be one color class of G, then R is as desired. So we assume that G is nonbipartite and aim to reach a contradiction. To this end, let $x_1 x_2 \ldots x_{2l+1} x_1$ be a shortest odd cycle of G. From the construction of G, we see that for every i with $1 \le i \le 2l+1$, there exists a vertex y_i in $V - S$ such that $\{x_i, x_{i+1}, y_i\}$ induces a triangle, denoted by $\triangle_i$, in $\mathscr{D}_S$. Note that y_i's may not be distinct.

Let T_0 denote the subtournament of T induced by vertex subset $\{x_i, y_i : 1 \le i \le 2l+1\}$. Then

(1) $\triangle_i$, for $i = 1, 2, \ldots, 2l+1$, are $2l+1$ triangles in T_0, where $x_{2l+2} = x_1$.

Let us perform a sequence of $2l+1$ augmentations in the following iterative way: $T_i := T_{i-1}\langle y_i\rangle$; that is, T_i is an augmentation of T_{i-1} with respect to y_i, for $i = 1, 2, \ldots, 2l+1$. Let y_i' be the image of y_i involved in the construction of T_i, and let C_i denote the triangle $y_i'x_ix_{i+1}y_i'$ if $\triangle_i = y_ix_ix_{i+1}y_i$ and $y_i'x_{i+1}x_iy_i'$ otherwise. Since $\{x_1, x_2, \ldots, x_{2l+1}\} \cap \{y_1, y_2, \ldots, y_{2l+1}\} = \emptyset = \{y_1', y_2', \ldots, y_{2l+1}'\} \cap \{y_1, y_2, \ldots, y_{2l+1}\}$, and since $x_1, x_2, \ldots, x_{2l+1}, y_1', y_2', \ldots, y_{2l+1}'$ are distinct vertices, by (1) we have

(2) C_i, for $i = 1, 2, \ldots, 2l+1$, are $2l+1$ triangles in T_{2l+1}, with the property that no vertex of T_{2l+1} is contained in more than two of them.

Since T_0 is a subtournament of T, it contains no F_1 nor F_2. Repeated applications of Lemma 2.2 yield the following.

(3) For any $0 \le i \le 2l+1$, tournament T_i contains no F_1 nor F_2.

Let us make one more simple observation.

(4) For any $0 \le i \le 2l+1$, every triangle in T_i contains at least two vertices from $\{x_1, x_2, \ldots, x_{2l+1}\}$.

To justify (4), we apply induction on i. For $i = 0$, since $S \cap V(T_0) = \{x_1, x_2, \ldots, x_{2l+1}\}$ and $|S \cap V(C)| \ge 2$ for every triangle C of T_0 (by hypothesis), the desired statement follows. Suppose we have established the assertion for T_{i-1}. Let us proceed to the induction step for T_i. Let $xyzx$ be an arbitrary triangle in T_i. Set $Q = \{x, y, z\}$ if $y_i' \notin \{x, y, z\}$ and $Q = (\{x, y, z\} - \{y_i'\}) \cup \{y_i\}$ otherwise. It follows from (1*) and (2*) that Q induces a triangle in T_{i-1}. So it contains at least two vertices from $\{x_1, x_2, \ldots, x_{2l+1}\}$ by induction hypothesis. We can thus deduce that the triangle $xyzx$ also contains at least two vertices from $\{x_1, x_2, \ldots, x_{2l+1}\}$ as y_i and y_i' are both outside $\{x_1, x_2, \ldots, x_{2l+1}\}$. So (4) is proved.

It can be seen from (2) that the minimum size of an FVS of T_{2l+1} is at least $l+1$. In view of (3), Theorem 1.1 (which is equivalent to Theorem 1.4(i)) guarantees the existence of at least $l+1$ vertex-disjoint triangles in T_{2l+1}. By (4), each of these $l+1$ triangles contains at least two vertices from $\{x_1, x_2, \ldots, x_{2l+1}\}$. Hence the size of $\{x_1, x_2, \ldots, x_{2l+1}\}$ is at least $2(l+1)$, a contradiction.

Since there are $O(|V|^3)$ triangles altogether in T, it takes $O(|V|^3)$ time to find the edge set of G . From the proof we see that G is a bipartite graph. Since the two color classes of G can be obtained in linear time by using depth first search, R can be found in $O(|V|^3)$ time. $\Box$

Lemma 2.4 *Let $T = (V, A)$ be a tournament with no subtournament isomorphic to F_1 nor F_2. Suppose S is a subset of V that contains at least two vertices from every triangle in T. Then $V(C) \nsubseteq V(\mathscr{D}_S)$ for every triangle $C \in \mathscr{F}_S$.*

Proof. We may assume that T is strongly connected, otherwise we turn to consider the strongly connected components of T separately. Thus V admits a partition $V_1, V_2, \ldots, V_k$ as described in Lemma 2.1. For every $v \in V$, we use $l(v)$ to denote the index i such that $v \in V_i$. Let $D = (V, B)$ be the digraph obtained from T by deleting all arcs from V_j to V_i with $i \le j-2$; in other words, $(u, v) \in B$ if and only if $(u, v) \in A$ and $|l(u) - l(v)| \le 1$. So each arc (u, v) in D falls into precisely one of the following three categories: we call (u, v) an *upward* arc if $l(u) = l(v) - 1$, a *downward* arc if $l(u) = l(v) + 1$, and a *level* arc if $l(u) = l(v)$. By Lemma 2.1(ii), we have

(1) D contains no triangle.

A (directed) path in D is called *upward* if it consists of three vertices and two upward arcs. It follows from Lemma 2.1 that an upward path P in D corresponds to a triangle in T (induced by $V(P)$), and vice versa. By the hypothesis on S, we get

(2) $|V(P) \cap S| \geq 2$ for any upward path P in D.

We prove the lemma by contradiction. Assume the contrary: $\{a, b, c\} \subseteq V(\mathscr{D}_S)$ for some triangle $abca \in \mathscr{F}_S$. Suppose $i = l(a) = l(b) - 1 = l(c) - 2$ for some $1 \leq i \leq k - 2$. Then (4*) guarantees the existence of three triangles $xx'x''x$, $yy'y''y$, and $zz'z''z$ in $\mathscr{D}_S$ such that $a \in \{x, x', x''\}$, $b \in \{y, y', y''\}$, and $c \in \{z, z', z''\}$ and that

(3) $xx'x''$, $yy'y''$, and $zz'z''$ are upward paths; that is, $l(x) = l(x') - 1 = l(x'') - 2$, $l(y) = l(y') - 1 = l(y'') - 2$, and $l(z) = l(z') - 1 = l(z'') - 2$.

Since each upward path in D corresponds to a triangle in T, it follows from (4*) that

(4) No upward path in D can go through two vertices in $\{a, b, c\}$ and a vertex in $V - S$. In particular, for any $u \in V - S$ and $v \in \{a, b, c\}$ with $|l(u) - l(v)| = 1$, the arc between u and v is downward unless $v \in \{a, c\}$ and $l(u) = i + 1$.

Using (1), (3), (4), and the fact $|\{x, x', x''\} \cap S| = |\{y, y', y''\} \cap S| = |\{z, z', z''\} \cap S| = 2$ (by (3*)), we can enumerate all possible configurations of the three triangles $xx'x''x$, $yy'y''y$, and $zz'z''z$, which are described in (5), (6), and (7), respectively; see Figure 3 for an illustration, where vertices in S are indicated by black points and those outside S by small circles.

(5) For triangle $xx'x''x$, exactly one of the following holds:
(5.1) $x \in V_{i-2} - S$, $x' \in V_{i-1} \cap S$, $x'' = a \in V_i \cap S$;
(5.2) $x \in V_{i-1} \cap S$, $x' = a \in V_i \cap S$, $x'' \in V_{i+1} - S$, and downward $(c, x'') \in B$, level $(b, x'') \in B$;
(5.3) $x = a \in V_i \cap S$, $x' \in V_{i+1} - S$, $x'' \in V_{i+2} \cap S$, and downward $(c, x') \in B$, level $(b, x') \in B$;
(5.4) $x = a \in V_i \cap S$, $x' \in V_{i+1} \cap S$, $x'' \in V_{i+2} - S$, and downward $(x'', b) \in B$, level $(x'', c) \in B$.

(6) For triangle $yy'y''y$, exactly one of the following holds:
(6.1) $y \in V_{i-1} - S$, $y' \in V_i \cap S$, $y'' = b \in V_{i+1} \cap S$, and downward $(a, y) \in B$;
(6.2) $y = b \in V_{i+1} \cap S$, $y' = V_{i+2} \cap S$, $y'' \in V_{i+3} - S$, and downward $(y'', c) \in B$.

(7) For triangle $zz'z''z$, exactly one of the following holds:
(7.1) $z \in V_i - S$, $z' \in V_{i+1} \cap S$, $z'' = c \in V_{i+2} \cap S$, and downward $(b, z) \in B$, level $(a, z) \in B$;
(7.2) $z \in V_i \cap S$, $z' \in V_{i+1} - S$, $z'' = c \in V_{i+2} \cap S$, and downward $(z', a) \in B$, level $(z', b) \in B$;
(7.3) $z \in V_{i+1} - S$, $z' = c \in V_{i+2} \cap S$, $z'' \in V_{i+3} \cap S$, and downward $(z, a) \in B$, level $(z, b) \in B$;
(7.4) $z = c \in V_{i+2} \cap S$, $z' \in V_{i+3} \cap S$, $z'' \in V_{i+4} - S$.

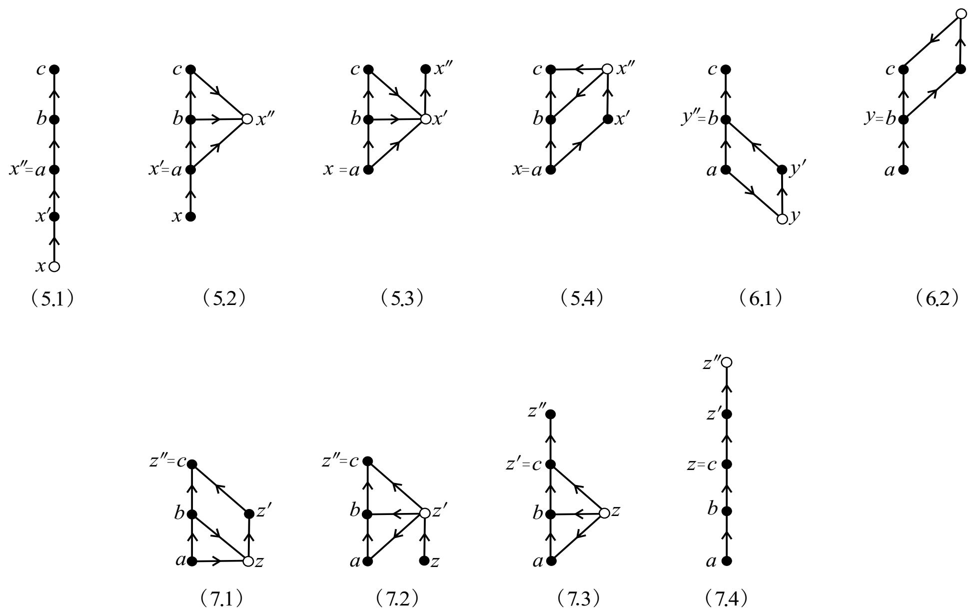

Figure 3. *Possible configurations of triangles $xx'x''x$, $yy'y''y$, and $zz'z''z$.*

(8) The following statements hold:

(8.1) Either (5.1) or (6.1) fails;

(8.2) Either (6.2) or (7.4) fails;

(8.3) Either (5.4) or (6.2) fails;

(8.4) Either (6.1) or (7.1) fails.

To justify (8.1), suppose to the contrary that both (5.1) and (6.1) hold. Using (1) and path $x'x''y = x'ay$, we get level $(x', y) \in B$, which in turn gives upward $(x, y) \in B$ (as path $xx'y$ does not correspond to a triangle in D). Thus xyy' is an upward path with x, y outside S, contradicting (2). Hence we have (8.1). Similarly, the violation of (8.2) (resp. (8.3), (8.4)) would give $\{(y'', z'), (y'', z'')\} \subseteq B$ (resp. $\{(x'', y'), (x'', y'')\} \subseteq B$, $\{(y', z), (y, z)\} \subseteq B$) and upward path $y'y''z''$ (resp. $x'x''y''$, yzz'), contradicting (2) again.

(9) The following statements hold:

(9.1) Either (5.1) or (7.1) fails;

(9.2) Either (5.4) or (7.4) fails.

Indeed, if both (5.1) and (7.1) hold then, using (1) and path $x'az$, we have upward $(x', z) \in B$, and hence have the upward path $xx'z$ with $\{x, z\} \subseteq V - S$, contradicting (2). Similarly, if both (5.4) and (7.4) hold then $(x'', z') \in B$, and so the upward path $x''z'z''$ contradicts (2).

(10) The following statements hold:

(10.1) Either (5.4) or (7.2) fails;

(10.2) Either (5.3) or (7.1) fails.

Indeed, if both (5.4) and (7.2) hold then, using (1) and path $z'ax'$, we have level $(z', x') \in B$. In view of path $z'x'x''$, we further have upward $(z', x'') \in B$. Thus $zz'x''$ contradicts (2). Similarly, if both (5.3) and (7.1) hold then we have level $(z', x') \in B$ and upward $(z, x') \in B$. It follows that the upward path $zx'x''$ contradicts (2).

(11) The following statements hold:

(11.1) Either (5.4) or (7.3) fails;

(11.2) Either (5.2) or (7.1) fails.

Indeed, if both (5.4) and (7.3) hold then, using (1) and paths $x''cz''$ and zax', we have upward $(x'', z'') \in B$ and level $(z, x') \in B$, respectively. Using path $zx'x''$, we obtain upward $(z, x'') \in B$. Thus the upward path $zx''z''$ contradicts (2). Similarly, if both (5.2) and (7.1) hold then we have $\{(x, z), (z', x''), (z, x'')\} \subseteq B$. Thus the upward path xzx'' contradicts (2).

(12) Either (5.4) or (6.2) holds.

Suppose otherwise, then from (6) and (5), we see that (6.1) and one of (5.1)-(5.3) hold. In view of (8.1), we further conclude that (5.2) or (5.3) holds. From (5.2) and (5.3), it follows that $\{u\} = \{x', x''\} \cap V_{i+1} \subseteq V - S$ and $(b, u) \in B$ is level. Using (1) and path $y'y''u$, we have upward $(y', u) \in B$ and hence the upward path $yy'u$ with $\{y, u\} \subseteq V - S$, contradicting (2).

(13) Either (6.1) or (7.1) holds.

Suppose otherwise, then (6) and (7) imply that (6.2) and one of (7.2)-(7.4) hold. Using (8.2), we further conclude that (7.2) or (7.3) holds. By (7.2) and (7.3), we have $\{u\} = \{z, z'\} \cap V_{i+1} - S$ and level $(u, b) \in B$. Using (1) and path $uby' = uyy'$, we get upward $(u, y') \in B$ and hence upward path $uy'y''$, which contradicts (2).

(14) (6.2) holds (so (6.1) fails).

Suppose otherwise, (6.2) fails (so (6.1) holds by (6)). It follows from (12) and (8.4) that (5.4) holds and (7.1) fails. Hence, by (7), one of (7.2), (7.3), and (7.4) holds, which leads to a contradiction to one of (10.1), (11.1), and (9.2).

It follows from (14) and (13) that (7.1) holds, which, together with (9.1) and (10.2), implies that neither (5.1) nor (5.3) holds. Moreover, the combination of (14) and (8.3) yields the failure of (5.4). Thus from (5) we see that (5.2) holds, contradicting (11.2). □

Lemma 2.5 *Let $T = (V, A)$ be a tournament with no subtournament isomorphic to F_1 nor F_2. Suppose S is a subset of V such that $\mathscr{D}_S \cup \mathscr{F}_S \neq \emptyset$ and that $|S \cap V(C)| \geq 2$ for every triangle C of T. Then there exists $R \subseteq S$ such that $|R \cap V(C)| = 1$ for every triangle C in $\mathscr{D}_S \cup \mathscr{F}_S$. Moreover, given S, such an R can be found in $O(|V|^3)$ time.*

Proof. We prove by contradiction. Assume (T, S) is a counterexample with minimum $|S|$. It follows instantly from Lemma 2.3 that $\mathscr{F}_S \neq \emptyset$. Let C_0 be a triangle in $\mathscr{F}_S$. Then Lemma 2.4 guarantees the existence of some $v \in V(C_0)$ with $v \notin V(\mathscr{D}_S)$. By considering $(T, S - \{v\})$, we deduce from the minimality of S that there exists $R \subseteq S - \{v\}$ which contains exactly one vertex from each triangle in $\mathscr{D}_{S-\{v\}} \cup \mathscr{F}_{S-\{v\}}$. Note that $C_0 \in \mathscr{D}_{S-\{v\}}$. So $|R \cap V(C)| = 1$ for every triangle C in $\mathscr{D}_S \cup \mathscr{F}_S$, a contradiction.

Let $s_1, s_2, \ldots, s_k$ be all the vertices in S. We apply the following algorithm to S. While $i \leq k$ do: set $S = S - \{s_i\}$ if s_i is contained in no triangle C such that $|V(C) \cap S| = 2$.

Since there are $O(|V|^2)$ triangles altogether in T containing s_i, each iteration takes $O(|V|^2)$ time and hence the whole algorithm runs in $O(|V|^3)$ time. Let S' denote the resulting S. From the above proof, we see that $\mathscr{F}_{S'} = \emptyset$ and that $\mathscr{D}_{S'} = \mathscr{D}_S \cup \mathscr{F}_S$, where S is the initial one. By Lemma 2.3 (with S' in place of S over there), we can find a subset R of S' in $O(|V|^3)$ time, such that $|R \cap V(C)| = 1$ for every triangle $C \in \mathscr{D}_{S'}$. This R is clearly as desired. $\square$

Now we are ready to establish the min-max relation.

Proof of Theorem 1.2. We shall actually show that statements (ii) and (iii) in Theorem 1.4 are equivalent. For convenience, we use the following notations in our proof. Given a tournament $T = (V, A)$ and a weight function $w \in \mathbf{Z}_+^V$, let τ_w denote the minimum weight of a triangle in T and let ν_w denote the maximum size of a w-FVS packing of T. Recall that we always have

(1) $\nu_w \leq \tau_w$.

(ii)$\Rightarrow$(iii) Suppose the contrary: $T = (V, A)$ contains a subtournament F isomorphic to F_1 or F_2. Define $w \in \boldsymbol{Z}_+^V$ as $w(v) = 1$ for each $v \in V$. Then $\tau_w = 3$. It is easy to see that each FVS of T contains at least two vertices in F. Since $|V(F)| = 5$, we have $\nu_w \leq 2$. Hence $\tau_w \neq \nu_w$, contradicting (ii).

(iii)$\Rightarrow$(ii) Let $T = (V, A)$ be a tournament with no F_1 nor F_2. To prove that $\nu_w = \tau_w$ for any $w \in \boldsymbol{Z}_+^V$, we apply induction on $|V|$.

The min-max relation holds trivially when $|V| \leq 3$. So we proceed to the induction step and assume that we have already proved the assertion for any tournament with no F_1 nor F_2 and with fewer vertices than T.

To establish the induction step, we apply induction on τ_w. Clearly, $\tau_w = \nu_w$ if $\tau_w = 0$. So we assume $\tau_w > 0$ and distinguish between two cases.

Case 1. $w(z) \geq \tau_w$ for some vertex $z \in V$.

Set $w' = w|_{V-\{z\}}$. By the induction hypothesis on $T \backslash z$ (with respect to the weight function w'), we get $\nu_{w'} = \tau_{w'}$. So it can be seen that

- either $T \backslash z$ is acyclic
- or there exists a w'-FVS packing $\mathcal{S}'$ of $T \backslash z$ with size τ_w (for $\tau_{w'} \geq \tau_w$).

In the former case, define $\mathcal{S}$ to be the multiset consisting of τ_w copies of $\{z\}$; in the latter case, define $\mathcal{S} := \{S' \cup \{z\} : S' \in \mathcal{S}'\}$. Then $\mathcal{S}$ is a collection of FVS's of T with size τ_w, which clearly yields a w-FVS packing of T with size τ_w (by the assumption of case 1). So by (1) we have $\nu_w = \tau_w$.

Case 2. $w(z) < \tau_w$ for any vertex $z \in V$.

Set $S := \{v \in V : w(v) \geq 1\}$. It follows from the assumption of the present case that

(2) $|S \cap V(C)| \geq 2$ for every triangle C in T.

In view of (2), the set of triangles of T is the disjoint union of three sets $\mathscr{D}_S$, $\mathscr{F}_S$, and $\mathscr{F}_S^+$ (recall (3*)-(5*)). It follows from the definition of $\mathscr{F}_S^+$ that

(3) All triangles C with $\sum_{v \in V(C)} w(v) = \tau_w$ are contained in $\mathscr{D}_S \cup \mathscr{F}_S$.

By Lemma 2.5, there exists $R \subseteq S$ such that $|R \cap V(C)| = 1$ for every triangle C in $\mathscr{D}_S \cup \mathscr{F}_S$. From the definition of $\mathscr{F}_S^+$, we see that R is an FVS of T. Set $\delta = \min\{w(v) : v \in R\}$. Then $\delta \geq 1$. Define $w' \in \boldsymbol{Z}_+^V$ as $w'(v) = w(v) - \delta|R \cap \{v\}|$ for all $v \in V$.

(4) $\tau_{w'} = \tau_w - \delta$.

To justify (4), it suffices to show that $|R \cap V(C)| \leq 2$ for every $C \in \mathscr{F}_S^+$ (by (3) and the selection of R). It is the case since any such C shares with some triangle in $\mathscr{D}_S$ two vertices, one of which is in $S - R$. So (4) follows.

By the induction hypothesis on $\tau_{w'}$ and by (4), T has a w'-FVS packing $\mathcal{S}$ of size $\tau_w - \delta$. Clearly, $\{R, R, \ldots, R\} \cup \mathcal{S}$ is a collection of FVS's of T with size τ_w, where the multiplicity of R is δ. This collection clearly yields a w-FVS packing of T with size τ_w. So by (1) we have $\nu_w = \tau_w$.

Combining the above two cases, we complete the proof of the induction step and hence our min-max theorem. □

3 NP-completeness

For convenience, let us call the problem addressed in Theorem 1.6 the *partition problem*. We show its NP-completeness in this section.

Proof of Theorem 1.6. Clearly, the partition problem is in NP. To prove the assertion, we appeal to the following Not-All-Equal 3-Satisfiability problem (Not-All-Equal-3SAT): Given n Boolean variables $\lambda_1, \lambda_2, \ldots, \lambda_n$ and m clauses $c_1, c_2, \ldots, c_m$ in CNF, each of which contains exactly three literals (variables or their negation), determine whether there exists an assignment of Boolean values to the variables such that for each clause at least one literal is true and at least one literal is false. It was shown by Schaefer [16] that Not-All-Equal-3SAT is NP-complete. Our objective is to reduce Not-All-Equal-3SAT to the partition problem.

For this purpose, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the set of variables and let $c_1, c_2, \ldots, c_m$ be the set of clauses in an arbitrary instance of Not-All-Equal-3SAT. We propose to construct a tournament T with $5n + 3m + 3$ vertices such that the vertex set of T can be partitioned into two FVS's if and only if $c_1 \wedge c_2 \wedge \cdots \wedge c_m$ is satisfiable (with respect to Not-All-Equal-3SAT). The construction goes as follows (see Figure 4 and Figure 5 for an illustration):

(i) To every variable λ_i, $1 \leq i \leq n$, we associate a tournament X_i with vertex set

$$V(X_i) = \{x_i^h : h = 1, 2, 3, 4, 5\}$$

and arc set

$$A(X_i) = \{(x_i^g, x_i^h) :\ 1 \le g < h \le 5 \text{ and } (g,h) \notin \{(1,5),(2,4)\}\} \cup \{(x_i^5, x_i^1), (x_i^4, x_i^2)\};$$

(ii) To every clause $c_j = c_j^1 \vee c_j^2 \vee c_j^3$, $1 \le j \le m$, we associate a triangle $Z_j = z_j^1 z_j^2 z_j^3 z_j^1$;

(iii) Let $V := (\cup_{i=1}^n V(X_i)) \cup V(Y) \cup (\cup_{j=1}^m V(Z_j))$, where $Y = y_1 y_2 y_3 y_1$ is a triangle, and all X_i's, Y, and Z_j's are pairwise disjoint;

(iv) To every $z = z_l^k \in \cup_{j=1}^m V(Z_j)$, $1 \le k \le 3$, $1 \le l \le m$, we associate an arc α_z from z_l^k to x_i^1 if $c_l^k = \lambda_i$ and from z_l^k to x_i^5 if $c_l^k = \bar{\lambda}_i$;

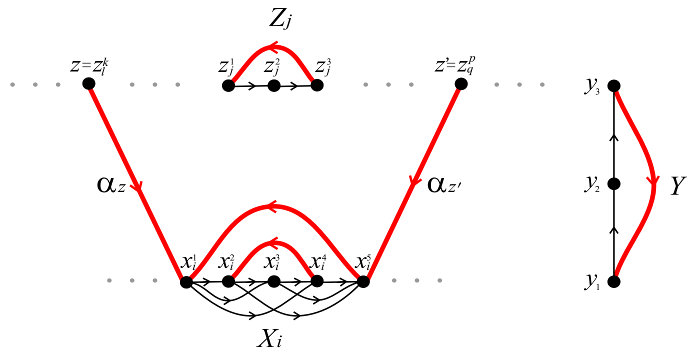

Figure 4. *An illustration of constructions (i)-(iv), where $c_l^k = \lambda_i$ and $c_q^p = \bar{\lambda}_i$.*

(v) Let A be the disjoint union of $\cup_{i=1}^n A(X_i)$, $A(Y)$, $\cup_{j=1}^m (A(Z_j) \cup \{\alpha_z : z \in V(Z_j)\})$, and $\{(u,v) :\ u \text{ and } v \text{ satisfy one of (a)-(e)}\}$:

(a) $u \in V(X_i)$, $v \in V(X_{i'})$, and $i < i'$;

(b) $u \in \cup_{i=1}^n V(X_i)$ and $v \in V(Y)$;

(c) $u \in \cup_{i=1}^n V(X_i)$, $v \in \cup_{j=1}^m V(Z_j)$, and α_v is not directed to u;

(d) $u \in V(Y)$ and $v \in \cup_{j=1}^m V(Z_j)$;

(e) $u \in V(Z_j)$, $v \in V(Z_{j'})$, and $j < j'$.

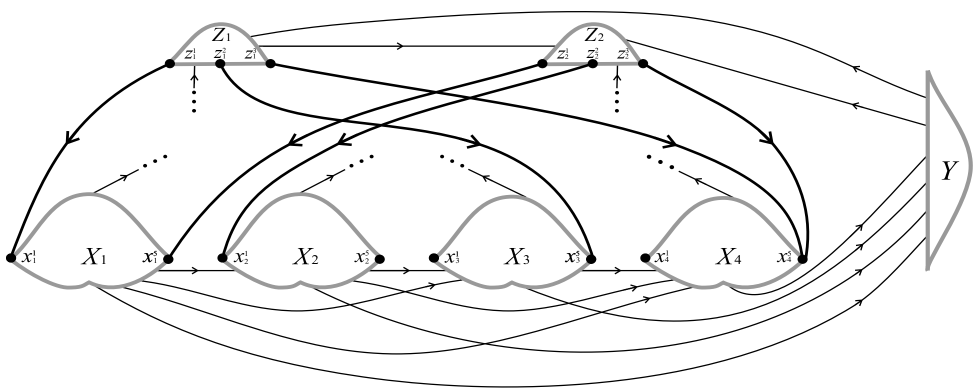

Figure 5. *Tournament T resulting from the instance* $(\lambda_1 \vee \bar{\lambda}_3 \vee \bar{\lambda}_4) \wedge (\bar{\lambda}_1 \vee \lambda_2 \vee \bar{\lambda}_4)$.

The construction is completed. It is easy to see that the construction can be accomplished in polynomial time and the resulting digraph $T = (V, A)$ is a tournament. The tournament T resulting from the NOT-ALL-EQUAL-3SAT instance with $n = 4$, $m = 2$, $c_1 = \lambda_1 \vee \bar{\lambda}_3 \vee \bar{\lambda}_4$, and $c_2 = \bar{\lambda}_1 \vee \lambda_2 \vee \bar{\lambda}_4$ is illustrated in Figure 5.

Let us define a linear order $\prec$ on the vertex set of T as follows: $x_1^1 \prec x_1^2 \prec x_1^3 \prec x_1^4 \prec x_1^5 \prec x_2^1 \prec x_2^2 \prec x_2^3 \prec x_2^4 \prec x_2^5 \prec \cdots \prec x_n^1 \prec x_n^2 \prec x_n^3 \prec x_n^4 \prec x_n^5 \prec y_1 \prec y_2 \prec y_3 \prec z_1^1 \prec z_1^2 \prec z_1^3 \prec z_2^1 \prec z_2^2 \prec z_2^3 \prec \cdots \prec z_m^1 \prec z_m^2 \prec z_m^3$. Observe that

(1) Set

$$B := \{(x_i^5, x_i^1), (x_i^4, x_i^2) : 1 \le i \le n\} \cup \{(y_3, y_1)\} \cup \{(z_j^3, z_j^1) : 1 \le j \le m\} \cup \{\alpha_z : z \in \cup_{j=1}^m V(Z_j)\}.$$

(In Figure 4, the arcs in B are bold lined.) Then for any $u, v \in V$ with $u \prec v$, arc $(v, u) \in A$ if and only if $(v, u) \in B$;

(2) For every $1 \le i \le n$, there are four triangles

$$X_i^1 = x_i^2 x_i^3 x_i^4 x_i^2 \text{ and } X_i^h = x_i^1 x_i^h x_i^5 x_i^1,\ h = 2, 3, 4,$$

altogether in tournament X_i; and

(3) For every $z \in \cup_{j=1}^m V(Z_j)$, there are three triangles Y_z^i, $i = 1, 2, 3$, altogether in T through α_z and y_i.

It follows from (1) that every triangle in T contains one or two arcs in B. Furthermore, since no two arcs in $B - \{\alpha_z : z \in \cup_{j=1}^m V(Z_j)\}$ have a common end, from the construction of T and (2) we see that

(4) Every triangle of T is either in $\{X_i^h : 1 \le h \le 4, 1 \le i \le n\} \cup \{Y\} \cup \{Z_j : 1 \le j \le m\}$ or contains α_z for some $z \in \cup_{j=1}^m V(Z_j)$.

Now we are ready to show that the vertex set of T can be partitioned into two FVS's if and only if the NOT-ALL-EQUAL-3SAT instance $c_1 \wedge c_2 \wedge \cdots \wedge c_m$ is satisfiable.

Sufficiency. Suppose there is a truth assignment for $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ such that each clause c_j, $1 \le j \le m$, contains at least one true literal and at least one false literal. Set

- $X := \{x_i^1 : \lambda_i \text{ is true}, 1 \le i \le n\} \cup \{x_i^5 : \lambda_i \text{ is false}, 1 \le i \le n\}$;
- $\bar{X} := \{x_i^1 : \lambda_i \text{ is false}, 1 \le i \le n\} \cup \{x_i^5 : \lambda_i \text{ is true}, 1 \le i \le n\}$;
- $Z := \{z \in \cup_{j=1}^m V(Z_j) : \alpha_z = (z, x), x \in X\}$; and
- $\bar{Z} := \cup_{j=1}^m V(Z_j) - Z = \{z \in \cup_{j=1}^m V(Z_j) : \alpha_z = (z, x), x \in \bar{X}\}$.

It is easy to see that

(5) $|X \cap V(X_i^h)| = |\bar{X} \cap V(X_i^h)| = 1$ for every $h = 2, 3, 4$ and $1 \le i \le n$;

(6) For every $z \in \cup_{j=1}^m V(Z_j)$, if the head of α_z is in X (resp. $\bar{X}$) then its tail is in Z (resp. $\bar{Z}$); and

(7) V is the disjoint union of two sets
- $S_1 := X \cup \bar{Z} \cup \{x_i^2 : 1 \le i \le n\} \cup \{y_1\}$ and
- $S_2 := \bar{X} \cup Z \cup \{x_i^3, x_i^4 : 1 \le i \le n\} \cup \{y_2, y_3\}$.

We claim that both S_1 and S_2 are FVS's of T. To justify this, let C be an arbitrary triangle C of T. Let us show that C meets both S_1 and S_2. By (5) and (6), the statement holds if $C \in \{X_i^h : h = 2,3,4;\ 1 \le i \le n\}$ or if C contains some α_z. If $C = X_i^1$ for some $1 \le i \le n$, then we derive from (2) and (7) that $x_i^2 \in V(C) \cap S_1$ and $x_i^3 \in V(C) \cap S_2$. If $C = Y$, then $y_1 \in V(C) \cap S_1$ and $y_2 \in V(C) \cap S_2$. So by (4) it remains to consider the case when $C = Z_j$ for some $1 \le j \le m$. Recall that c_j^h is true and c_j^i is false for some $1 \le h \ne i \le 3$. From (iv) and the definitions of X and $\bar{X}$, we deduce that $x \in X$ and $x' \in \bar{X}$, where (z_j^h, x) and (z_j^i, x') are arcs associated to z_j^h and z_j^i, respectively, as described in (iv). It follows from (6) and (7) that $z_j^h \in Z \cap V(C) \subseteq S_2 \cap V(C)$ and $z_j^i \in \bar{Z} \cap V(C) \subseteq S_1 \cap V(C)$. Therefore both S_1 and S_2 are FVS's of T, as claimed. By (7), we are done.

Necessity. Suppose the vertex set of T can be partitioned into two FVS's S_1 and S_2. For $1 \le i \le n$, set λ_i to be true if $x_i^1 \in S_1$ and false otherwise. Let us show that this assignment enables every c_j, $1 \le j \le m$, to contain at least one true literal and at least one false literal. To this end, we first show that

(8) $|\{x_i^1, x_i^5\} \cap S_1| = |\{x_i^1, x_i^5\} \cap S_2| = 1$ for all $1 \le i \le n$.

Indeed, by (2), we have $x_i^g \in S_1 \cap V(X_i^1)$ and $x_i^h \in S_2 \cap V(X_i^1)$ for some $2 \le g \ne h \le 4$. This, in turn, implies that $\{x_i^1, x_i^5\} \cap S_2 \ne \emptyset$ and $\{x_i^1, x_i^5\} \cap S_1 \ne \emptyset$ by considering triangles $X_i^g = x_i^1 x_i^g x_i^5 x_i^1$ and $X_i^h = x_i^1 x_i^h x_i^5 x_i^1$. So (8) is established.

Next we observe that

(9) $|\{x, z\} \cap S_1| = |\{x, z\} \cap S_2| = 1$ for all $\alpha_z = (z, x)$ with $z \in \cup_{j=1}^m V(Z_j)$ and $x \in \{x_i^1, x_i^5 : 1 \le i \le n\}$.

Indeed, by (iii) and the definition of S_1 and S_2, triangle $y_1 y_2 y_3 y_1$ contains $y_g \in S_1$ and $y_h \in S_2$ for some $1 \le g \ne h \le 3$. Recall (3), T contains triangles $Y_z^i = x y_i z x$, $i = 1, 2, 3$. Now using triangles Y_z^g and Y_z^h, we obtain $\{x, z\} \cap S_2 \ne \emptyset$ and $\{x, z\} \cap S_1 \ne \emptyset$. Hence (9) holds.

For $1 \le j \le m$, by (ii) triangle Z_j contains some $z_j^g \in S_1$ and $z_j^h \in S_2$. Suppose (z_j^g, x) and (z_j^h, x') are arcs associated to z_j^g and z_j^h, respectively, as described in (iv), where $\{x, x'\} \subseteq \{x_i^1, x_i^5 : 1 \le i \le n\}$. It follows from (9) that $x \in S_2$ and $x' \in S_1$. Suppose $x \in \{x_i^1, x_i^5\}$ for some $1 \le i \le n$.

- If $x = x_i^1$ then, by (iv), $c_j^g = \lambda_i$ is false as $x_i^1 \notin S_1$;
- If $x = x_i^5$ then, by (8), $x_i^1 \in S_1$. So λ_i is true and hence, by (iv), $c_j^g = \bar{\lambda}_i$ is false.

Therefore c_j^g is false (in either case). Similarly, it can be deduced from $x' \in S_1$ that c_j^h is true. Hence c_j contains both false literal c_j^g and true literal c_j^h; equivalently, $c_1 \wedge c_2 \wedge \cdots \wedge c_m$ is satisfiable, completing the proof. □

4 Algorithms

For simplicity, we use the same notations as introduced before. In particular, given a tournament $T = (V, A)$ and a weight function $w \in \mathbf{Z}_+^V$, let τ_w denote the minimum weight of a triangle in T and let ν_w denote the maximum size of a w-FVS packing of T.

For the case when T contains no F_1 nor F_2, we present the following algorithm for finding an optimal w-FVS packing of size ν_w.

Algorithm Opt_Pack OPTIMAL_FVS_PACKING

Input A tournament $T = (V, A)$ with no F_1 nor F_2 and a weight $w \in \boldsymbol{Z}_+^V$
Output A maximum w-FVS packing $\mathcal{S}$ of T with $|\mathcal{S}| = \nu_w$

1. $\tau_w \leftarrow$ the minimum weight of a triangle in T
2. **if** $\tau_w = 0$ or T is acyclic **then** return $\mathcal{S} \leftarrow \emptyset$
3. **if** $\exists\ z \in V$ with $w(z) \geq \tau_w$ **then**
4. **if** $T\backslash z$ is acyclic **then** return $\mathcal{S} \leftarrow \{S_i : S_i = \{z\},\ i = 1, 2, \ldots, \tau_w\}$
5. **else** $\{S_i : 1 \leq i \leq \tau_{w|_{V-\{z\}}}\} \leftarrow$ Opt_Pack$(T\backslash z, w|_{V-\{z\}})$
   return $\mathcal{S} \leftarrow \{\{z\} \cup S_i : 1 \leq i \leq \tau_w\}$
6. $S \leftarrow \{v \in V : w(v) \geq 1\}$, $R \leftarrow$ a subset of S with $|R \cap V(C)| = 1$ for all $C \in \mathscr{D}_S \cup \mathscr{F}_S$
7. $\delta \leftarrow \min\{w(v) : v \in R\}$, $w'(v) \leftarrow w(v) - \delta|R \cap \{v\}|$ for all $v \in V$
8. return $\mathcal{S} \leftarrow \{S_i : S_i = R,\ i = 1, 2, \ldots, \delta\} \cup$ Opt_Pack(T, w')

Remark. Note that $\mathcal{S}$ is a collection of FVS's of T with size ν_w, which obviously yields a w-FVS packing of T with the same size.

Theorem 4.1 *Let $T = (V, A)$ be a tournament with no F_1 nor F_2. Then Algorithm* OPTIMAL_FVS_PACKING *solves the FVS packing problem on T exactly in $O(|V|^4)$ time.*

Proof. The correctness of the algorithm follows instantly from the proof of Theorem 1.2. Let us now estimate the time complexity of the algorithm.

Note that either in Steps 3-5 one vertex z is eliminated from our consideration or in Step 7 the weight of at least one vertex becomes zero (from nonzero one). So the whole algorithm takes $O(|V|)$ iterations. From Lemma 2.5, we can conclude that each iteration takes $O(|V|^3)$ time. Hence the total running time of the algorithm is $O(|V|^4)$. $\square$

Let us proceed to the FVS packing problem on a general tournament T. For this general case, we can easily obtain a 1/3-approximation algorithm: Set $R = \emptyset$. While T contains a triangle C do: let v be a vertex in $V(C)$ of maximum weight. Set $R = R \cup \{v\}$ and $T = T\backslash v$. Obviously, $\{R, R, \ldots, R\}$, where the multiplicity of R is $\min\{w(v) : v \in R\}$, is an FVS packing the original T with size at least $\frac{1}{3}\nu_w$. By exploiting the structural characterization given in our min-max theorem and using the above exact algorithm as a subroutine, we can obtain a better approximation algorithm based on the subgraph removal technique.

Algorithm Apx_Pack APPROXIMATE_FVS_PACKING

Input A tournament $T = (V, A)$ and a weight $w \in \boldsymbol{Z}_+^V$

Output A w-FVS packing $\mathcal{S}$ of T with $|\mathcal{S}| \geq \frac{2}{5}\nu_w$

1. $\tau_w \leftarrow$ the minimum weight of a triangle in T, $R \leftarrow \emptyset$
2. **if** $\tau_w = 0$ or T is acyclic **then** return $\mathcal{S} \leftarrow \emptyset$
3. **while** T contains a subtournament F isomorphic to F_1 or F_2 **do**
4. $v \leftarrow$ a vertex in $V(F)$ of maximum weight, $R \leftarrow R \cup \{v\}$, $T \leftarrow T \backslash v$
5. **end-while**
6. $\mathcal{S}' = \{S_i : 1 \leq i \leq \tau_{w|_{V(T)}}\} \leftarrow$ Opt_Pack$(T, w|_{V(T)})$, $\delta \leftarrow \min\{w(v) : v \in R\}$
7. **if** $\mathcal{S}' = \emptyset$ **then** return $\mathcal{S} \leftarrow \{S_i : S_i = R, i = 1, 2, \ldots, \delta\}$
8. **else** return $\mathcal{S} \leftarrow \{S_i \cup R : i = 1, 2, \ldots, \min\{\tau_{w|_{V(T)}}, \delta\}\}$

Remark. Again $\mathcal{S}$ is a collection of FVS's of T, which obviously yields a w-FVS packing of T with the same size.

Theorem 4.2 *Let $T = (V, A)$ be an arbitrary tournament. Then Algorithm* APPROXIMATE_FVS_PACKING *approximates the FVS packing problem on T within a factor of $2/5$ in $O(|V|^4)$ time.*

Proof. Clearly, $\mathcal{S}$ is a collection of FVS's of T. To get the approximation ratio, it suffices to prove that

(1) $|\mathcal{S}| \geq \frac{2}{5}\nu_w$.

For this purpose, we turn to show that

(2) $\delta \geq \frac{2}{5}\nu_w$ if $\delta > 0$.

To justify (2), let u be a vertex in R with $w(u) = \delta$. Suppose u is added to R because of subtournament F (recall the while-loop of the algorithm), and suppose $\mathcal{S}^*$ is a w-FVS packing of T with size ν_w. Since we need to delete at least two vertices in F in order to destroy all triangles in F, each FVS in $\mathcal{S}^*$ contains at least two vertices in F. From the definition of a w-FVS packing, we deduce that $2|\mathcal{S}^*| \leq \sum_{v \in V(F)} w(v)$. Since u is a vertex with maximum weight in F and $|V(F)| = 5$, we have $2\nu_w = 2|\mathcal{S}^*| \leq 5w(u) = 5\delta$, yielding (2).

To establish (1), we may assume $\tau_w > 0$, for otherwise the statement holds trivially. So we have $\delta > 0$ when $R \neq \emptyset$. If $\mathcal{S}' = \emptyset$, then it follows from (2) and the first line of Step 7 of the algorithm that (1) holds. Otherwise, $\tau_{w|_{V(T)}}$ in Step 6 of the algorithm is at least $\tau_w (\geq \nu_w)$. Thus from the second line of Step 7 of the algorithm we can also conclude (1).

It was shown in [1] that F in Step 3 can be obtained in $O(|V|^2)$ time if it exists. Thus we deduce from Theorem 4.1 that APPROXIMATE_FVS_PACKING runs in $O(|V|^4)$ time. □

It is easy to see that Theorems 1.7 and 1.8 follow from the above two theorems, respectively.

5 Concluding remarks

In this paper we have characterized all tournaments T with Mengerian hypergraph $b(\mathcal{C}_T)$. Coincidently, $b(\mathcal{C}_T)$ is Mengerian if and only if $\mathcal{C}_T$ is. Major open problems in this research direction are to characterize all digraphs G with Mengerian $\mathcal{C}_G$ and those with Mengerian $b(\mathcal{C}_G)$. The arc versions of these problems are equally interesting. While these problems are extremely hard in general, Guenin and Thomas [14] successfully characterized all digraphs that pack, where a digraph G *packs* if for any subdigraph H of G, the maximum number of disjoint cycles is equal to the minimum number of vertices in a feedback vertex set in H. Guenin strongly believes that the blocker version of their theorem holds on exactly the same digraphs.

Conjecture 5.1 [13] *A digraph G packs if and only if for any subdigraph H of G, the maximum number of disjoint feedback vertex sets is equal to the length of a shortest cycle in H.*

We close this paper by the aforementioned Woodall's conjecture on packing feedback arc sets.

Conjecture 5.2 [20] *In any planar digraph the maximum number of disjoint feedback arc sets is equal to the length of a shortest cycle.*

Certainly, these two beautiful conjectures deserve arduous research efforts.

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