Characterization of standard embeddings between complex Grassmannians by means of varieties of minimal rational tangents

Ngaiming Mok*

Abstract. Tsai [Ts, 1993] proved that a proper holomorphic mapping $f : \Omega \to \Omega'$ from an irreducible bounded symmetric domain Ω of rank ≥ 2 into a bounded symmetric domain Ω' is necessarily totally geodesic provided that $r' := \text{rank}(\Omega') \leq \text{rank}(\Omega) := r$, proving a conjecture of the author's [Mk1, 1989] motivated by Hermitian metric rigidity. As a first step in the proof, Tsai showed that df preserves almost everywhere the set of tangent vectors of rank 1. Identifying bounded symmetric domains as open subsets of their compact duals by means of the Borel embedding, this means that the germ of f at a general point preserves the varieties of minimal rational tangents (VMRTs).

In another completely different direction Hwang-Mok (cf. [HM6, 2004]) established with very few exceptions the Cartan-Fubini Extension Priniciple for germs of local biholomorphisms between Fano manifolds of Picard number 1, showing that the germ of map extends to a global biholomorphism provided that it preserves VMRTs. We propose to isolate the problem of characterization of special holomorphic embeddings between Fano manifolds of Picard number 1, especially in the case of classical manifolds such as rational homogeneous spaces of Picard number 1, by a non-equidimensional analogue of the Cartan-Fubini Extension Principle. As an illustration we show along this line that standard embeddings between complex Grassmann manifolds of rank ≤ 2 can be characterized by the VMRT-preserving property and a non-degeneracy condition, giving a new proof of a result of Neretin's [Ne, 1999] which on the one hand paves the way for far-reaching generalizations to the context of rational homogeneous spaces and more generally Fano manifolds of Picard number 1, on the other hand should be applicable to the study of proper holomorphic mappings between bounded domains carrying some form of geometric structures.

Keywords: minimal rational curves, varieties of minimal rational tangents, analytic continuation, rigidity, bounded symmetric domains, proper holomorphic maps

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This article is motivated by two distinct topics of research: proper holomorphic maps between bounded symmetric domains in Several Complex Variables; and varieties of minimal rational tangents, *alias* VMRTs (cf. Hwang-Mok [HM3]), on uniruled projective manifolds in Algebraic Geometry. By embedding bounded symmetric domains into their compact duals, we relate rigidity results for proper holomorphic maps between bounded symmetric domains of rank ≥ 2 to the study of local properties of VMRTs in terms of differential projective geometry.

Motivated by Hermitian metric rigidity, the author conjectured in [Mk1, 1989] that a proper holomorphic mapping $f : \Omega \to \Omega'$ from an irreducible bounded symmetric domain Ω of rank ≥ 2 into a bounded symmetric domain Ω' is necessarily totally geodesic provided that $r' := \text{rank}(\Omega') \leq \text{rank}(\Omega) := r$. In 1993, Tsai [Ts] resolved the conjecture. Following a scheme for studying holomorphic maps on bounded symmetric domains of rank ≥ 2 in Mok-Tsai [MT], Tsai showed that *df* preserves almost everywhere the set of vectors tangent to minimal disks, i.e., tangent to lines (minimal rational curves) on the

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compact duals in terms of the Borel embedding. In other words, f is VMRT-preserving at a general point. The hard part of Tsai's proof then proceeded by exploiting the complex geometry of bounded symmetric domains with respect to the Bergman metric.

In another completely different direction, in the context of Algebraic Geometry Hwang-Mok studied the geometry of uniruled projective manifolds, including especially Fano manifolds, in terms of VMRTs. Especially we established with very few exceptions the Cartan-Fubini Extension Priniciple for germs of local biholomorphisms between Fano manifolds of Picard number 1, showing that the germ of a VMRT-preserving local biholomorphism extends to a global biholomorphism (Hwang-Mok [HM4, 1999] [HM5, 2001] [HM6, 2004]). We propose to isolate the problem of characterization of special holomorphic embeddings between Fano manifolds of Picard number 1, especially in the case of classical manifolds such as rational homogeneous spaces of Picard number 1, by a non-equidimensional analogue of the Cartan-Fubini Extension Principle. In the special case of irreducible Hermitian symmetric spaces of the compact type we establish such a non-equidimensional result on analytic continuation. Using this we show in the current article that standard embeddings between Grassmann manifolds of rank ≥ 2 can be characterized by the VMRT-preserving property and a non-degeneracy condition, giving a new proof of a result of Neretin's [Ne, 1999]. Our method serves to give a proof of Tsai's rigidity theorem on proper holomorphic maps in the case of classical symmetric domains of type I (which generalizes to many cases). We present the proof of the special case to illustrate a schematic approach in the study of proper holomorphic maps between domains carrying certain geometric structures in which rigidity properties of proper holomorphic maps are established in two steps: a first step which consists of the study of boundary values of such maps with an aim to force local differential-geometric properties on the map, and a second step which derives rigidity properties of the proper map from such local differential-geometric properties by means of a non-equidimensional analogue of the Cartan-Fubini Extension Principle. Domains carrying some form of geometric structures include first of all irreducible bounded symmetric domains of rank ≥ 2, but should extend more generally at least to certain classes of bounded homogeneous domains.

Recently Hong [Ho, 2007] established a very strong form of rigidity characterizing certain totally geodesic complex submanifolds of irreducible Hermitian symmetric spaces of Picard number 1, which in the special case of a Grassmann manifold X says that Grassmann submanifolds S of rank ≥ 2 are characterized by the fact that X induces in a canonical way a Grassmann structure of rank ≥ 2 on S. Although this is not how the proof goes, such a rigidity result can be understood as ascertaining first of all that the induced Grassmann structure on S is automatically flat, so that the germ of S at a point can be identified as the holomorphic image of a VMRT-preserving holomorphic map f from a germ of Grassmann manifold satisfying a certain non-degeneracy condition, and secondly that the image of such a germ of map must necessarily be the germ of a Grassmann submanifold. In a joint work with Hong [HoM], we extend the method of analytic continuation to yield a far-reaching generalization of the second half of the result to the case of rational homogeneous spaces of Picard number 1.

By way of explicit examples the author wishes to relate function-theoretic problems on certain bounded domains to the geometric theory on Fano manifolds basing on the study of minimal rational curves and their varieties of minimal rational tangent, the two being linked to each other via a non-equidimensional theory for local holomorphic maps respecting geometric structures. To make the article accessible the author has provided more background on the various sides of the subject than is absolutely necessary. Especially, we give a brief introduction to the geometric theory of Fano manifolds basing on VMRTs sufficient for our purpose, and sometimes give direct and elementary proofs of basic facts about Grassmann manifolds on top of resorting to general theory about geometric structures modeled on irreducible Hermitian symmetric spaces of the compact type.

§1 Background and motivation

(1.1) Motivated by Hermitian metric rigidity, the author formulated in [Mk1, 1989] a conjecture on the rigidity of proper holomorphic maps between irreducible bounded symmetric domains under certain rank conditions. Tsai [Ts, 1993] completely resolved the Conjecture in the affirmative, as follows.

Theorem A (Tsai [Ts, Main Theorem]). *Let $f : \Omega \to \Omega'$ be a proper holomorphic map between two bounded symmetric domains such that Ω is irreducible and of rank ≥ 2, and such that* $\mathrm{rank}(\Omega') \leq \mathrm{rank}(\Omega)$. *Then,* $\mathrm{rank}(\Omega') = \mathrm{rank}(\Omega)$, and $f : \Omega \to \Omega'$ is a totally geodesic embedding.

Tsai's proof broke down into two parts. In the first part, by studying boundary values of proper holomorphic maps as in Mok-Tsai [MT, 1992] and using the structure of boundary components of bounded symmetric domains (cf. Wolf [Wo]) he showed that the rank condition in the hypothesis of Theorem A forces $\mathrm{rank}(\Omega') = \mathrm{rank}(\Omega)$ and that at almost every point $x \in \Omega$, df_x must transform vectors of rank 1 to vectors of rank 1, where the rank is a general notion for tangent vectors on a bounded symmetric domain which agrees with the notion of ranks of matrices in the case of classical symmetric domains of type I, II or III defined in terms of matrices.

In what follows bounded symmetric domains are realized as Euclidean domains by means of the Harish-Chandra embedding. In the usual notations we write $\Omega = G/K$, $\Omega' = G'/K'$ as homogeneous manifolds, where G resp. G' is the group of biholomorphic automorphisms of Ω resp. Ω', and $K \subset G$ resp. $K' \subset G'$ is the isotropy subgroup at the origin 0 with respect to Harish-Chandra coordinates. At any $x \in \Omega$ denote by $K_x \subset G$ the isotropy subgroup. Then, we have also $\Omega \cong G/K_x$. The association of each $\gamma \in K_x$ to the differential $d\gamma_x \in \mathrm{End}(T_x(\Omega))$ defines a faithful linear representation of K_x. K_x is thus isomorphic to a real linear subgroup of the complex general linear group $\mathrm{GL}(T_x(\Omega))$, and its complexification $K_x^{\mathbb{C}}$ is isomorphic to a complex linear subgroup of the same group $\mathrm{GL}(T_x(\Omega))$. Given $x \in \Omega$ and $x' \in \Omega'$, and complex linear maps $\lambda_1, \lambda_2 : T_x(\Omega) \to T_{x'}(\Omega')$, in what follows we say that λ_1 is complex equivalent to λ_2 if and only if there exists $\varphi \in K_x^{\mathbb{C}}$, $\varphi' \in K_{x'}'^{\mathbb{C}}$ such that $\lambda_2 = d\varphi'_{x'} \circ \lambda_1 \circ d\varphi_x$.

Let $f : \Omega \to \Omega'$ be a proper holomorphic map between two irreducible bounded symmetric domains of the same rank $r \geq 2$. Knowing that df_x transforms rank-1 vectors to rank-1 vectors at almost every point $x \in \Omega$, in many cases one can deduce that at almost every point $x \in \Omega$ the differential $df_x : T_x(\Omega) \to T_{x'}(\Omega'), x' := f(x)$ is complex equivalent to the differential $df_0(x) : T_x(\Omega) \to T_{x'}(\Omega')$ for some equivariant holomorphic totally geodesic embedding $f_0 : \Omega \to \Omega'$ such that $f_0(x) = f(x) = x'$. This is the case for instance when Ω and Ω' are both Type-1 bounded symmetric domains of the same rank $r \geq 2$. One purpose of the current article is to isolate this part of the proof of Tsai's theorem, and to give a characterization of equivariant holomorphic totally geodesic embeddings between Ω and Ω' for such pairs (Ω, Ω') completely in terms of the local differential-geometric property as described. For this characterization we consider certain pairs (Ω, Ω') of irreducible bounded symmetric domains of rank ≥ 2 equipped with an equivariant holomorphic totally geodesic embedding $f_0 : \Omega \to \Omega'$. Assume without loss of generality that $f_0(0) = 0$. Our problem is to characterize the germ of map $f_0 : (\Omega, 0) \to (\Omega', 0)$ among all germs of holomorphic embeddings $f : (\Omega, 0) \to (\Omega', 0)$ in terms of conditions on the tangent subspaces $df_x\big(T_x(\Omega)\big) \subset T_{f(x)}(\Omega')$ at points where f is defined. For this type of characterization it is essential to assume that f_0 transforms rank-1 vectors to rank-1 vectors but it will no longer be necessary to assume that Ω and Ω' are of the same rank. Some supplementary non-degeneracy condition needs to be placed on the embedding f_0. In the examples we are discussing in the current article, we will not need to spell out the condition as it is always satisfied and will be used in the proof only implicitly.

To illustrate the methods for characterizing $f_0 : \Omega \to \Omega'$ we will only consider in this article the case of bounded symmetric domains of type I and of rank ≥ 2. For its definition the bounded symmetric domain $D(p,q)$ of type I is represented by the set of p-by-q matrices Z with complex coefficients such that $I - \overline{Z}^t Z$ is positive definite. The identity component $\mathrm{Aut}_0(D(p,q))$ is the quotient of the special unitary group $\mathrm{SU}(p,q)$ of the standard Hermitian form on $\mathbb{C}^{p+q}$ of signature (p,q) by a finite cyclic group. For positive integers r, s such that $p \leq r, q \leq s$, there is a standard embedding $\tau : D(p,q) \hookrightarrow D(r,s)$ given by $\tau(Z) = \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}$. Obviously $\tau : D(p,q) \to D(r,s)$ is holomorphic and Φ-equivariant with respect to a group homomorphism $\Phi : \mathrm{Aut}_0(D(p,q))$ and $\mathrm{Aut}_0(D(r,s))$ induced by an obvious group homomorphism of $\mathrm{SU}(p,q)$ into $\mathrm{SU}(r,s)$, and the image $\tau(D(p,q)) \subset D(r,s)$ is a totally geodesic complex submanifold.

(1.2) We are going to relate irreducible bounded symmetric domains Ω of rank ≥ 2 to the geometry of Fano manifolds. Viewing a bounded symmetric domain $\Omega = G/K$ as a homogeneous manifold, at every point $x \in \Omega$ there is the isotropy representation of the isotropy subgroup $K_x \subset G$ on the holomorphic tangent space $T_x(\Omega)$.

Let n be a positive integer. Fix an n-dimensional complex vector space V and let M be any n-dimensional complex manifold. In what follows all bundles are understood to be holomorphic. The frame bundle $\mathcal{F}(M)$ is a principal $GL(V)$-bundle with the fiber at x defined as $\mathcal{F}(M)_x = \mathrm{Isom}(V, T_x(M))$, the set of linear isomorphisms from V to the holomorphic tangent space at x.

Definition 1 (G-structures). *Let* $\mathrm{G} \subset GL(V)$ *be any complex Lie subgroup. A* *holomorphic* G*-structure is a* G*-principal subbundle* $\mathcal{G}(M)$ *of* $\mathcal{F}(M)$*. An element of* $\mathcal{G}_x(M)$ *will be called a* G*-frame at* x*. For* $\mathrm{G} \neq GL(V)$ *we say that* $\mathcal{G}(M)$ *defines a holomorphic reduction of the tangent bundle to* G*.*

On an m-dimensional smooth manifold M, a Riemannian metric g on M gives a reduction of the structure group of the (real) tangent bundle from the general linear group $\mathrm{GL}(m, \mathbb{R})$ to the orthogonal group $\mathrm{O}(m)$. Riemannian geometry may be regarded as the geometry of smooth $\mathrm{O}(m)$-structures. A Riemannian manifold (M, g) is locally isometric to the Euclidean space if and only if there exists on M an atlas of coordinate charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$, $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$, on which orthonormal frames can be chosen to consist of constant vector fields in terms of the chosen systems of coordinates. We say in this case that the smooth $\mathrm{O}(m)$-structure is flat. On complex manifolds we have the following analogous notion of flat holomorphic G-structures.

Definition 2 (flat G-structures). *Let* M *be a complex manifold and* $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ *be any atlas of holomorphic coordinate charts on* M*. In terms of Euclidean coordinates we identify* $\mathcal{F}(U_\alpha)$ *with the product* $\mathrm{GL}(V) \times U_\alpha$*. We say that a holomorphic* G*-structure* $\mathcal{G}(M)$ *on* M *is flat if and only if there exists an atlas of holomorphic coordinate charts* $\{\varphi_\alpha : U_\alpha \to V\}$ *such that the restriction* $\mathcal{G}(U_\alpha)$ *of* $\mathcal{G}(M)$ *to* U_α *is the product* $\mathrm{G} \times U_\alpha \subset \mathrm{GL}(V) \times U_\alpha$*.*

Let (S, g) be an irreducible Hermitian symmetric space of the compact type and of rank≥ 2. Let $G_c = \mathrm{Aut}_0(S, g_c)$ and write $K \subset G_c$ for the isotropy subgroup at an arbitrary base point $0 \in S$. For $x \in S$ write $K_x \subset G_c$ for the isotropy subgroup at $x \in S$, so that $K_0 = K$. Denote by $\mathcal{W}_x \subset \mathbb{P}T_x(S)$ the variety of highest weight tangents of K_x on $T_x(S)$. Let $L_x \subset \mathrm{GL}(T_x(S))$ be the identity component of the linear subgroup consisting of linear isomorphisms preserving $\mathcal{W}_x$. L_x is isomorphic to $K_x^{\mathbb{C}}$, where $\gamma \in K_x^{\mathbb{C}}$ corresponds to $d\gamma(x) \in L_x$. By an S-structure we mean a G-structure with $G = L_0 \subset \mathrm{GL}(T_0(S))$. As L_0 is identified with $K^{\mathbb{C}}$ we also called an S-structure a $K^{\mathbb{C}}$ structure. (Note that the notation G is a generic symbol for a group in 'G-structures' and has nothing to do with the notations G and G_c.) In a slight variation to the terminology of Mok [Mk 1] a highest weight vector in $\widetilde{W_x}$ will be called a minimal characteristic vector. (In [Mk 1] it is called a characteristic vector.)

The S-structure thus defined on S is flat in the sense of Definition 2. That this is so can be seen by using Harish-Chandra coordinates, by which S is realized as a compactification of a complex vector space $\mathbb{U}$ such that for each $x \in \mathbb{U}$, the Euclidean translation $T_x(z) = z + x$ extends holomorphically to a biholomorphic automorphism of S. In particular, the holomorphic reduction of the frame bundle $\mathcal{F}_S$ to $\mathcal{G}_S$ is realized over $\mathbb{U}$ by a constant subbundle in terms of Harish-Chandra coordinates.

Given an S-structure on a complex manifold X we have an associated bundle $\mathcal{W} \subset \mathbb{P}T_X$ of varieties of highest weight tangents of the isotropy representations of (the semisimple parts of) the reductive groups K_x on $T_x(X)$. The assumption that X admits a flat S-structure means that given any $x \in X$, some neighborhood U_x of x can be identified with an open set U on S in such a way that $\mathcal{W}|_{U_x}$ agrees with the bundle

$\mathcal{W}|_U$ over U of the varieties of highest weight tangents on S. In this direction we have the following basic result of Ochiai [Oc] in the theory of geometric structures.

Theorem B (Ochiai [Oc]). *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. Denote by $\pi : \mathcal{W} \to S$ the bundle of varieties of highest weight tangents. Let U, $V \subset S$ be two connected open sets and $f : U \to V$ be a biholomorphism such that $f_*\mathcal{W}|_U = \mathcal{W}|_V$. Then, f extends to a biholomorphic automorphism of S.*

An immediate corollary is the characterization of the model spaces S in terms of flat S-structures.

Corollary to Theorem B. *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. A simply connected compact complex manifold carrying a flat S-structure must be biholomorphically isomorphic to S.*

Proof of Corollary. Let X be a complex manifold carrying a flat S-structure. X is covered by open subset $\{U_\alpha\}_{\alpha\in A}$. Each U_α, $\alpha \in A$, is identified with some open subset $V_\alpha \subset S$ by coordinate charts $\varphi_\alpha : U_\alpha \to V_\alpha$ such that $d\varphi_\alpha$ transforms the given flat S-structure on $U_\alpha \subset X$ to the standard S-structure on V_α. Starting with one chart, one can correct overlapping charts as follows. Given $\varphi_\beta : U_\beta \to V_\beta$, on the overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$, the two charts $\varphi_\alpha|_{U_{\alpha\beta}} : U_{\alpha\beta} \to S$, $\varphi_\beta|_{U_{\alpha\beta}} : U_{\alpha\beta} \to S$ are related by $\varphi_\beta|_{U_{\alpha\beta}} = f_{\alpha\beta} \circ \varphi_\alpha|_{U_{\alpha\beta}}$ for some $f_{\alpha\beta} : \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta})$. By Theorem B, $f_{\alpha\beta}$ extends to a biholomorphism $F_{\alpha\beta} : S \to S$. If $U_{\alpha\beta}$ is connected then we have a holomorphic mapping $h : U_\alpha \cup U_\beta \to S$ given by $h|_{U_\alpha} = \varphi_\alpha; h|_{U_\beta} = F_{\alpha\beta}^{-1} \circ \varphi_\beta$. When X is simply connected, starting with a single φ_α, by the usual argument of developing the initial map $\varphi_\alpha : U_\alpha \to V_\alpha \subset S$ along paths, the contractibility of each closed path allows one to prove that extension to any point is independent of the choice of the path and open subsets covering the path, thereby defining a global local biholomorphism. $\Phi : X \to S$. When X is compact, Φ is a covering map, and must therefore be a biholomorphism since S is simply connected. □

We now specialize to the situation of Grassmann manifolds. Let p, q be positive integers and W be a $(p+q)$-dimensional complex vectors space. We denote by $\mathrm{Gr}(p, W)$ the Grassmann manifold of p-planes in W. When the reference to a specific background vector space W is unimportant we write $G(p, q)$ for $\mathrm{Gr}(p, W)$. Denote by $M(q, p)$ the complex vector space of q-by-p matrices, and by I_p the p-by-p identity matrix. Consider the subset $\mathbb{U} \subset G(p, q)$ consisting of all p-planes E_Z generated by the column vectors of $\begin{bmatrix} Z \\ I_p \end{bmatrix}$, $Z \in M(q, p)$, with respect to a fixed ordered basis $(e_{p+1}, \ldots, e_{p+q}; e_1, \ldots, e_p)$. Then the map $\varphi : M(q, p) \cong \mathbb{C}^{pq} \subset G(p, q)$ is such a chart. An open subset $\mathbb{U} \subset G(p, q)$ obtained with respect to some choice of ordered basis of W will be called a Euclidean cell.

Any linear automorphism of W induces a biholomorphic automorphism of $G(p, q)$. All biholomorphic automorphisms of $G(p, q)$ in the identity component $\mathrm{Aut}_0(G(p, q))$ are obtained this way, and we have $\mathrm{Aut}_0(G(p, q)) \cong \mathrm{GL}(p+q; \mathbb{C})/\mathbb{C}^*$. For $\Phi \in \mathrm{Aut}_0(G(p, q))$, the restriction of Φ to a Euclidean cell $\mathbb{U} \cong M(q,p)$ can be described as a fractional linear transformation, as follows. $\Phi \in \mathrm{Aut}_0(G(p,q))$ is defined by a linear transformation $\Phi_0 \in \mathrm{GL}(W)$, represented by $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with respect to the ordered basis $(e_{p+1}, \ldots, e_n; e_1, \ldots, e_p)$, where $A \in \mathrm{GL}(q, \mathbb{C})$, etc. Then Φ_0 transforms the p-plane E_Z, represented by $\begin{bmatrix} Z \\ I_p \end{bmatrix}$, to the p-plane spanned by the column vectors of $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Z \\ I \end{bmatrix} = \begin{bmatrix} AZ+B \\ CZ+D \end{bmatrix}$. Provided that $CZ+D$ is invertible, $\Phi_0(E_Z) = E_{\Phi(Z)}$, where $\Phi(Z) = (AZ+B)(CZ+D)^{-1}$.

$G(p,q)$ is covered by a finite number of charts consisting of Euclidean cells $\mathbb{U}$ obtained by permutations on the ordered basis. On the overlapping regions the transition maps, which are induced by automorphisms of W corresponding to a change of basis, are thus given by fractional linear transformations. The Jacobian matrices of the transition maps are of a particular type, as follows. Let $Z' = \Phi(Z) = (AZ+B)(CZ+D)^{-1}$, $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GL}(p+q, \mathbb{C})$, be a fractional linear transformation. The tangent space at each point of a Euclidean cell can be identified with the complex vector space $M(q,p)$. Whenever $\det(CZ+D) \neq 0$, Φ is holomorphically defined at Z and $d\Phi_Z$ is invertible. For the differential $d\Phi_Z$, identified as a Jacobian matrix, we have

$$\begin{aligned} d\Phi(z)(X) &= AX(CZ+D)^{-1} - (AZ+B)(CZ+D)^{-1}CX(CZ+D)^{-1} \\ &= [A - (AZ+B)(CZ+D)^{-1}C]X(CZ+D)^{-1} \; . \end{aligned}$$

Hence, $d\Phi(z)(X) = Q(Z)XP(Z)$, where $Q(Z) \in \mathrm{GL}(q,\mathbb{C})$, $P(Z) \in \mathrm{GL}(p,\mathbb{C})$. Thus the charts consisting of Euclidean cells endow $G(p,q)$ with a special structure as a complex manifold. It gives in particular a trivialization of the holomorphic tangent bundle over each Euclidean cell, so that the transition functions for the holomorphic tangent bundle takes values in a proper subgroup $\mathrm{G} \subsetneq \mathrm{GL}(pq, \mathbb{C})$ where G consists of linear transformations γ on $M(q,p) \cong \mathbb{C}^{pq}$ of the form $\gamma(X) = QXP$, $Q \in \mathrm{GL}(q,\mathbb{C})$, $P \in \mathrm{GL}(p,\mathbb{C})$. This gives an S-structure in the case where S is the Grassmannian $G(p,q)$, which will be refereed to as a Grassmann structure. Here we take $V = M(q,p)$, and $\mathrm{G} \subset \mathrm{GL}(V,\mathbb{C})$ to be the image under the homomorphism $\Theta : \mathrm{GL}(q,\mathbb{C}) \times \mathrm{GL}(p,\mathbb{C}) \to \mathrm{GL}(M(q,p))$ defined by $\Theta(P,Q)(X) = QXP$ for all X. G is isomorphic to the quotient of $\mathrm{GL}(q,\mathbb{C}) \times \mathrm{GL}(p,\mathbb{C})$ by a copy of $\mathbb{C}^*$, and is reductive, with the semisimple part being isomorphic to the quotient of $\mathrm{SL}(q,\mathbb{C}) \times \mathrm{SL}(p,\mathbb{C})$ by a finite group. That the standard Grassmann structure on $G(p,q)$ thus defined is flat follows from the fact that translations on the Euclidean cells $\mathbb{U}$ are special cases of fractional linear transformations, which extends to automorphisms of $G(p,q)$. Grassmann structures are thus G-structures with $\mathrm{G} = \Theta(\mathrm{GL}(q,\mathbb{C}) \times \mathrm{GL}(p,\mathbb{C}))$. On $G(p,q)$ we have a holomorphic decomposition $T_{G(p,q)} = \mathbb{A}^{(p)} \otimes \mathbb{B}^{(q)}$ of the holomorphic tangent bundle into the tensor product of universal vector bundles, where the superscript denotes the rank of the bundle. A complex manifold M admits a Grassmann structure if and only if the holomorphic tangent bundle T_M admits a non-trivial decomposition into the tensor product of two holomorphic vector bundles each of rank ≥ 2.

Given a $(p+q)$-dimensional complex vector space W the Grassmannian $\mathrm{Gr}(p,W)$ of p-planes π in W is canonically isomorphic to the Grassmannian $\mathrm{Gr}(q,W^*) = G(q,p)$ of q-planes in the dual vector space W^*, by the isomorphism which assigns the p-plane $\pi \subset W$ to its annihilator $\pi^\perp \subset W^*$, $\pi^\perp \cong \mathbb{C}^q$. In the sequel following standard practice we will interchange the roles of p and q (and thus the roles of P and Q) and make use of p-by-q matrices to parametrize Euclidean cells in $G(q,p)$, which is identified with $G(p,q)$ as described.

In the case of Grassmann structures we have the following non-equidimensional version of Ochiai's Theorem due to Neretin ([Ne, 1999]). For its formulation by a standard embedding $\tau : \mathrm{Gr}(p,W) \to \mathrm{Gr}(p',W')$, $p' \geq p$, between Grassmann manifolds we mean a holomorphic embedding induced by a linear embedding of $\lambda : W \to W'$ and given by $\tau(\pi) = \lambda(\pi) \oplus \nu$, where $\nu \subset W'$ is a $(p'-p)$-dimensional vector subspace transversal to $\lambda(W)$, or one of the form $\tau = \tau_0 \circ \iota$, where $\iota : \mathrm{Gr}(p,W) \cong \mathrm{Gr}(q,W^*)$ in which $q = \dim(W) - p$, and $\tau_0 : \mathrm{Gr}(q,W^*) \to \mathrm{Gr}(p',W')$ is obtained as in the former procedure. In terms of Harish-Chandra coordinates, an example of a standard embedding from $G(p,q)$ to $G(r,s)$ is obtained $p \leq r, q \leq s$ is given by $\tau_1(Z) = \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}$, or by $\tau_2(Z) = \begin{bmatrix} Z^t & 0 \\ 0 & 0 \end{bmatrix}$, where Z^t stands for the transpose of Z, in the event that $p \leq s, q \leq r$.

Theorem C (from [Ne, Thm. 2.3]). *Let p,q,r,s be integers such that $2 \leq p \leq r$ and $2 \leq q \leq s$. Let $U \subset M(p,q)$ be an open connected subset containing 0, and $\Psi : U \to M(r,s)$ be a holomorphic immersion such that at every $Z \in U$ and for every $X \in T_Z(U) \cong M(p,q)$, we have*

$$d\Psi(Z)(X) = P(Z) \cdot X \cdot Q(Z)$$

for some matrices $P(Z) \in M(r,p)$ of rank r and $Q(Z) \in M(q,s)$ of rank q. Then, there exists $K \in M(r,s)$, $L \in M(r,p)$, $N \in M(q,p)$, and $M \in M(q,s)$ such that L is of rank p and M is of rank q, and such that $\Psi(Z) = K + LZ(I - NZ)^{-1}M$. In particular, Ψ is of the form $\Psi = \mu \circ \Theta \circ \gamma$, where γ lies in the parabolic subgroup $P \subset \mathrm{Aut}_0(G(p,q))$ at $0 \in G(p,q)$, $\gamma \in \mathrm{Aut}_0(G(r,s))$ and $\Theta : G(p,q) \to G(r,s)$ is a standard embedding.

In the equidimensional case note that

$$\begin{aligned}(AZ+B)(CZ+D)^{-1} &= BD^{-1} + \big[(AZ+B) - BD^{-1}(CZ+D)\big]((CZ+D))^{-1} \\ &= BD^{-1} + (A - BD^{-1}C)Z(CZ+D)^{-1} \\ &= BD^{-1} + (A - BD^{-1}C)Z(I + D^{-1}CZ)^{-1}D^{-1}\end{aligned}$$

which is of the form as in Theorem C. In the conclusion of Theorem C, in view of the hypothesis on $d\Psi$ the standard embedding is given by $\tau_1(Z) = \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}$ in terms of Harish-Chandra coordinates.

Identify the Grassmannian $G(r,s)$ as a projective submanifold of some $\mathbb{P}^N$ by means of the Plücker embedding. Neretin's original result is more general, and is applicable under additional assumptions also to holomorphic maps $\Psi : U \to M(r,s)$ which are not necessarily immersions. He required $f : U \to M(r,s)$ to be non-degenerate in the sense that (a) df is of rank at least 2 at some point; and (b) that the image of f does not lie in any projective linear subspace Π of $\mathbb{P}^N$ such that $\Pi \subset G(r,s)$. In this general set-up the hypothesis on $d\Psi$ is the same, except that $A(z)$ resp. $B(z)$ is not necessarily of rank p resp. q. The conclusion is also the same, except that L resp. M in the conclusion is not necessarily of rank p resp. q. A mapping of the form $\Phi(Z) = K + LZ(I - NZ)^{-1}M$ without assuming that L and M are invertible is called in [Ne] a generalized linear-fractional map of Krein-Smul'yan. When we impose the condition that Ψ is an immersion and that $d\Psi(Z)(X) = P(Z) \cdot X \cdot Q(Z)$, then it follows easily that $P(Z)$ and $Q(Z)$ can be taken to be invertible for every $Z \in U$.

(1.3) We now turn to the other domain of research which concerns us, viz., the geometric theory of projective uniruled manifolds (e.g., Fano manifolds) based on the study of varieties of minimal rational tangents. We give a brief introduction here and refer the reader to Hwang-Mok [HM2, (2.3)] and [HM4] for an introduction to the basics of the theory sufficient for our purpose. A general reference for rational curves in Algebraic Geometry is Kollár [Ko].

By a parametrized rational curve on a projective manifold X we mean a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$ from the Riemann sphere $\mathbb{P}^1$ into X. We say that two parametrized rational curves f_1 and f_2 are equivalent if and only if they are the same up to a reparametrization of $\mathbb{P}^1$, i.e., if and only if there exists $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$ such that $f_2 = f_1 \circ \gamma$. A rational curve is an equivalence class $[f]$ of parametrized rational curves $f : \mathbb{P}^1 \to X$ under this equivalence relation. We will sometimes also refer to a nontrivial holomorphic image $f(\mathbb{P}^1) = C$ (as a cycle) of the Riemann sphere as a rational curve, to be noted as $[C]$. Let X be a uniruled projective manifold, i.e., a projective manifold covered by rational curves. Fix an ample line bundle L on X. Without explicitly mentioning it, the degree of an algebraic curve C on X will be measured with respect to L, i.e., it is the integral of a (positive) curvature form of L over C. The uniruled projective manifold X is covered by rational curves of degree bounded by some integer depending on X and L. Let $f : \mathbb{P}^1 \to X$ be a parametrized rational curve. Then, by the Grothendieck Splitting Theorem the pull-back f^*T_X of the holomorphic tangent bundle T_X splits over $\mathbb{P}^1$ into the direct sum of holomorphic line bundles. By definition the equivalence class $[f]$ of a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$ is a free rational curve if and only if all Grothendieck direct summands of f^*T_X are of nonnegative degree. By a minimal rational curve on X we will mean a free rational curve of minimal degree among all free rational curves on X. The set of minimal rational curves can be given naturally the structure of a complex manifold, a connected component of which will be called a minimal rational component $\mathcal{K}$. A rational curve belonging to $\mathcal{K}$ will sometimes be called a $\mathcal{K}$-rational curve. The degree of $\mathcal{K}$, to be denoted by $\deg(\mathcal{K})$, is the degree of one and hence any $\mathcal{K}$-rational curve.

Associated to $(X, \mathcal{K})$ there is the universal family $\rho : \mathcal{U} \to \mathcal{K}$ of rational curves belonging to $\mathcal{K}$, where $\mathcal{U}$ is smooth, $\rho : U \to \mathcal{K}$ is a holomorphic $\mathbb{P}^1$-bundle, and the fiber $\rho^{-1}(\kappa) \cong \mathbb{P}^1$ of a point $\kappa \in \mathcal{K}$ gives a copy of the Riemann sphere $\mathbb{P}^1$ corresponding to the rational curve represented by κ. From any choice of parametrization $f : \mathbb{P}^1 \to X$ of κ a point on $\rho^{-1}(\kappa)$ gives a point of the cycle $C = f(\mathbb{P}^1) \subset X$ defined independent of the choice of parametrization, and we have in fact a canonical holomorphic map $\mu : \mathcal{U} \to X$. Thus, the universal family comes equipped with a double fibration $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$. The canonical map $\mu : \mathcal{U} \to X$ is a holomorphic submersion, and its image contains in general a non-empty Zariski open subset of X. For a point $x \in X$, the μ-fiber $\mathcal{U}_x$ corresponds to equivalence classes of parametrized rational curves $f : \mathbb{P}^1 \to X$ such that $f(0) = x$, where two such parametrized rational curves f_1, f_2 are taken to be equivalent if and only if they are the same up to reparametrization via an automorphism of $\mathbb{P}^1$ fixing 0. For a general point $x \in X$, every rational curve passing through x of degree $\leq \deg(\mathcal{K})$ is free. For such a point x there is no possibility of a decomposition of minimal rational curves passing through x under deformations fixing x, and $\mathcal{U}_x \subset \mathcal{U}$ is smooth and compact, hence projective.

By Mori's Breaking-up Lemma, over a general point of X, a general member $[f] \in \mathcal{K}$ is standard in the sense that $f^*T_X \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$. Note that a standard rational curve is in particular immersed, since any nontrivial holomorphic bundle homomorphism $\lambda : \mathcal{O}(2) \cong T_{\mathbb{P}'} \to f^*T_X \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ is injective at every point. Over a general point $x \in X$ we have a rational map called the tangent map $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$ defined by assigning each rational curve $[f]$ marked at x to the complex line $\mathbb{C}df(T_0(\mathbb{P}^1)) \subset T_x(X)$. The total transform $\mathcal{C}_x := \overline{\tau_x(\mathcal{U}_x)} \subset \mathbb{P}T_x(X)$ is called the variety of minimal rational tangents, *alias* VMRT, of $(X, \mathcal{K})$ at x. The tangent map τ_x is holomorphic whenever every $\mathcal{K}$-rational curve marked at x is nonsingular. By Kebekus [Ke, 2002] this is the case at a general point. In [Ke], it is actually proven that the tangent map is a finite holomorphic map at a general point $x \in X$. By Hwang-Mok [HM5] the tangent map is birational onto its image under some non-degeneracy assumption on the Gauss map at a general smooth point of $\mathcal{C}_x$. In conjunction with [Ke] and Cho-Miyaoka-Shepherd-Barron, Hwang-Mok [HM6] showed that the tangent map $\tau_x : \mathcal{U}_x \to \mathcal{C}_x$ is a finite birational holomorphic map at a general point $x \in X$. In the special case where $X \subset \mathbb{P}_N$ is uniruled by projective lines by elementary arguments it is known that at a general point $x \in X$, $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is nonsingular and the tangent map τ_x is a biholomorphism.

For a projective subvariety $\mathcal{A} \subset \mathbb{P}^N$ we will sometimes consider its lifting $\widetilde{\mathcal{A}} := \pi^{-1}(\mathbb{P}^N)$ to $\mathbb{C}^{N+1}$ under the canonical map $\pi : \mathbb{C}^{N+1} - \{0\} \to \mathbb{P}^N$. $\widetilde{A} \cup \{0\} \subset \mathbb{C}^{N+1}$ is called the affine cone over $\mathcal{A}$. Thus at general point $x \in X$, $\widetilde{\mathcal{C}_x} \subset T_x(X)$ is the set of nonzero vectors tangent to $\mathcal{K}$-rational curves passing through x.

Now let X be a Fano manifold of Picard number 1, $\mathcal{K}$ be a minimal rational component on X. We say that Cartan-Fubini extension holds for the pair $(X, \mathcal{K})$, if for any Fano manifold X' of Picard number 1 equipped with a minimal rational component $(X', \mathcal{K}')$, any VMRT-preserving biholomorphism $\varphi : U \cong U' \subset X'$ on a domain $U \subset X$ extends to a global biholomorphism $\Phi : X \to X'$. In this direction our main result is is the following.

Theorem D (Hwang-Mok [HM6]). *Let X be a Fano manifold of Picard number 1. Suppose there exists a minimal rational component $\mathcal{K}$ such that for a general point $x \in X$ the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is not a finite union of projective linear subspaces. Then, Cartan-Fubini extension holds for $(X, \mathcal{K})$. Namely, for any choice of Fano manifold X' of Picard number 1, any minimal rational component $\mathcal{K}'$ with $\mathcal{C}' \subset \mathbb{P}T(X')$ and any connected open subsets $U \subset X, U' \subset X'$, if there exists a biholomorphic map $\varphi : U \to U'$ satisfying $\varphi_*(\mathcal{C}_x) = \mathcal{C}'_{\varphi(x)}$ for all generic $x \in U$, then there exists a biholomorphic map $\Phi : X \to X'$ such that φ is the restriction of Φ to U.*

Regarding the Cartan-Fubini Extension Principle most relevant to our discussion in the current article is the special case, first proven in Hwang-Mok [5], where the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is of dimension $p \geq 1$ satisfying the additional condition (†) that the Gauss map is generically finite. This is always the case by a result of Ein [Ei] whenever each irreducible component of $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is nonsingular and different from a projective linear subspace. (In fact the Gauss map is actually finite by Zak's Tangency Theorem [Za].) The condition (†) is equivalently the requirement that at a general smooth point of $[\alpha] \in \mathcal{C}_x$, the kernel of the projective second fundamental form $\sigma_{[\alpha]}$ is trivial.

The special case of Theorem D under the additional hypothesis (†) is already a far-reaching generalization of [(1.2), Theorem B] of Ochiai [Oc]. In fact, representing an irreducible bounded symmetric domain Ω of rank ≥ 2 as G/K in the usual notations as in (1.1), and embedding Ω as a domain of its compact dual M, the variety of highest weight tangents of the isotropy representation of K on $\mathbb{P}T_0(X)$ agrees with the variety of minimal rational tangents $\mathcal{C}_0$ (cf. Mok [Mk]). In particular $\mathcal{C}_0 \subset \mathbb{P}T_0(M)$ is a nonlinear homogeneous projective submanifold, and the kernel of the second fundamental form $\sigma_{[\alpha]}$ vanishes at every point $[\alpha] \in \mathcal{C}_0$.

§2 Rational saturation of a germ of complex submanifold of the Grassmann manifold

(2.1) We examine now the classification of complex linear maps between tangent spaces of bounded symmetric domains.

Definition 3. *Let* Ω resp. Ω' *be an irreducible bounded symmetric domain. Write* G resp. G' *for the identity component of the group of biholomorphic automorphisms of* Ω resp. Ω'. *Let* $x, y \in \Omega$ resp. $x', y' \in \Omega'$ *be arbitrary points, and write* $K_x \subset G$ resp. $K'_{x'} \subset G'$ *for the isotropy subgroup at* x resp. x'. *Let* $\lambda_1 : T_x(\Omega) \to T_{x'}(\Omega')$ *and* $\lambda_2 : T_y(\Omega) \to T_{y'}(\Omega')$ *be complex linear maps. We say that* λ_1 *is complex equivalent to* λ_2 *if and only if there exist* $\theta \in G$, $\theta' \in G'$, $\varphi \in K_x^{\mathbb{C}}$ *and* $\varphi' \in K'^{\mathbb{C}}_{x'}$ *such that* $\theta(x) = y$, $\theta'(y') = x'$ *and such that* $\lambda_1 = d\varphi'_{x'} \circ (d\theta'_{y'} \circ \lambda_2 \circ d\theta_x) \circ d\varphi_x$. *In particular, if* $x = x'$ *and* $y = y'$, *then* λ_1 *is complex equivalent to* λ_2 *if and only if there exists* $\varphi \in K_x^{\mathbb{C}}$, $\varphi' \in K'^{\mathbb{C}}_{x'}$ *such that* $\lambda_1 = d\varphi'_{x'} \circ \lambda_2 \circ d\varphi_x$.

The notion of complex equivalence of λ_1 and λ_2 is formulated in such a way that it is by definition invariant under $\mathrm{Aut}_0(\Omega)$ and $\mathrm{Aut}_0(\Omega')$. Embedding bounded symmetric domains canonically into their compact duals $\Omega \subset M$ resp. $\Omega' \subset M'$, the following lemma shows that in fact the notion of complex equivalence of linear maps between tangent spaces is properly a notion depending only on the compact duals M and M', and it is in fact invariant under automorphisms of M and those of M', noting that $\mathrm{Aut}_0(M)$ is the complexification of $\mathrm{Aut}_0(\Omega)$, etc.

Lemma 1. *Let* Ω resp. Ω' *be an irreducible bounded symmetric domain, and* M resp. M' *be its compact dual (which is an irreducible Hermitian symmetric manifold of the compact type). Denote by* $\Omega \subset M$ *resp.* $\Omega' \subset M'$ *the Borel embedding. Let* $x, x' \in \Omega \subset M$; $y, y' \in \Omega \subset M'$ *and let* $\lambda_1 : T_x(\Omega) \to T_{x'}(\Omega')$, $\lambda_2 : T_y(\Omega) \to T_{y'}(\Omega')$ *be complex linear maps. Then,* λ_1 *is complex equivalent to* λ_2 *if and only if there exists* $\psi \in \mathrm{Aut}_0(M)$, $\psi' \in \mathrm{Aut}_0(M')$ *such that* $\psi(x) = y$, $\psi'(y') = x'$ *and such that* $\lambda_1 = d\psi'_{y'} \circ \lambda_2 \circ d\psi_x$.

Proof. Suppose λ_1 is complex equivalent to λ_2 in the sense of Definition 3. Then, in the notations there we have

$$\lambda_1 = d\varphi'_{x'} \circ (d\theta'_{y'} \circ \lambda_2 \circ d\theta_x) \circ d\varphi_x = (d\varphi'_{x'} \circ d\theta'_{y'}) \circ \lambda_2 \circ (d\theta_x \circ d\varphi_x) = d\psi'_{y'} \circ \lambda_2 \circ d\psi_x,$$

where we define $\psi := \theta \circ \varphi$, $\psi' := \varphi' \circ \theta'$. Clearly, $\psi(x) = y$, $\psi'(y') = x'$ and $\psi \in \mathrm{Aut}_0(M)$, $\psi' \in \mathrm{Aut}_0(M')$. Conversely, given $\lambda_1 = d\psi'_{y'} \circ \lambda_2 \circ d\psi_x$ as in the statement of Lemma 1, we assert that λ_1 is complex equivalent to λ_2. By the homogeneity of Ω resp. Ω' under $\mathrm{Aut}_0(\Omega) = G$ resp. $\mathrm{Aut}_0(\Omega') = G'$, we deduce that $\lambda_1 = d\widetilde{\varphi}'_{x'} \circ (d\theta' \circ \lambda_2 \circ d\theta) \circ d\widetilde{\varphi}_x$ for some $\theta \in G, \theta(x) = y; \theta' \in G', \theta'(y') = x'$; and $\widetilde{\varphi} \in P_x$ resp. $\widetilde{\varphi}' \in P_{x'}$, where $P_x \subset G^{\mathbb{C}}$ is the (parabolic) isotropy subgroup at x, and $P_{x'} \subset G^{\mathbb{C}'}$ is the (parabolic) isotropy subgroup at x'. $K_x^{\mathbb{C}} \subset P_x$ is a Levi factor of the parabolic subgroup $P_x \subset \mathrm{Aut}_0(M) = G^{\mathbb{C}}$; an analogous statement applies to $x' \in \Omega' \subset M'$. We have a Levi decomposition $P_x = K_x^{\mathbb{C}} \cdot M^-$, where $M^- \subset P_x$ is the unipotent radical (which is abelian). Here $M^- = \exp(\mathfrak{m}^-)$, where $\mathfrak{m}^-$ is the vector space of holomorphic vector fields vanishing at x to the order 2. It follows that $d\gamma_x = \mathrm{id}_{T_x(M)}$. Thus, writing $\widetilde{\varphi} = \varphi \cdot \gamma$, where $\varphi \in K_x^{\mathbb{C}}$ and $\gamma \in M^-$, it follows that $d\widetilde{\varphi}_x = d\varphi_x$. The same argument applies to $d\widetilde{\varphi}' \in P_{x'}$, and we conclude that there exists $\varphi \in K_x^{\mathbb{C}}$ and $\varphi' \in K'^{\mathbb{C}}_{x'}$ such that $\lambda_1 = d\varphi'_{x'} \circ (d\theta'_{y'} \circ \lambda_2 \circ d\theta_y) \circ d\varphi_x$, i.e., $\lambda_i : T_x(M) \to T_{x'}(M')$; $i = 1, 2$; are complex equivalent to each other according to Definition 1, as asserted. □

Regarding linear maps between Grassmann manifolds the following two lemmas are well-known. To be self-contained we include elementary and direct proofs. Lemma 2 is purely a statement about linear maps between nontrivial tensor product spaces.

Lemma 2. *Let* $A^{(p)}$, $B^{(q)}$, $E^{(r)}$, $F^{(s)}$ *be finite-dimensional complex vector spaces, where the superscript indicates the complex dimension and will sometimes be omitted. Assume* $p, q, r, s \geq 2$. *Suppose* $\lambda : A \otimes B \to E \otimes F$ *is an injective complex linear map such that* λ *transforms decomposable vectors in* $A \otimes B$ *to decomposable vectors in* $E \otimes F$. *In other words, for any nonzero vectors* $\alpha \in A$, $\beta \in B$, $\lambda(\alpha \otimes \beta) = \gamma(\alpha, \beta) \otimes \delta(\alpha, \beta)$ *for some nonzero vectors* $\gamma(\alpha, \beta) \in E$, $\delta(\alpha, \beta) \in F$. *Suppose there exist* $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in A \times B$ *such that* $\gamma(\alpha_1, \beta_1)$ *and* $\gamma(\alpha_2, \beta_2)$ *are linearly independent, and likewise there exist* (α'_1, β'_1), $(\alpha'_2, \beta'_2) \in A \times B$ *such that* $\delta(\alpha'_1, \beta'_1)$ *and* $\delta(\alpha'_2, \beta'_2)$ *are linearly independent. Then, either there exist complex linear maps* $\mu : A^{(p)} \to E^{(r)}$, $\nu : B^{(q)} \to F^{(s)}$ *such* *that* $\lambda(\alpha \otimes \beta) = \mu(\alpha) \otimes \nu(\beta)$*; or there exist complex linear maps* $\varphi : A^{(p)} \to F^{(s)}$ *and* $\psi : B^{(q)} \to E^{(r)}$ *such that* $\lambda(\alpha \otimes \beta) = \psi(\beta) \otimes \varphi(\alpha)$.

In the statement of Lemma 2 note that $\gamma(\alpha, \beta)$ and $\delta(\alpha, \beta)$ are each well-defined only up to multiplication by a nonzero complex number.

Proof of Lemma 2. Pick $\alpha_1, \alpha_2 \in A^{(p)}$ which are linearly independent, and $\beta_0 \in B^{(q)}$ to be a nonzero vector. Then $\lambda((\alpha_1 + \alpha_2) \otimes \beta_0) = \lambda(\alpha_1, \beta_0) + \lambda(\alpha_2, \beta_0) = \gamma(\alpha_1, \beta_0) \otimes \delta(\alpha_1, \beta_0) + \gamma(\alpha_2, \beta_0) \otimes \delta(\alpha_2, \beta_0)$. By hypothesis the latter is decomposable. As a consequence, either (a) $\delta(\alpha_1, \beta_0)$ and $\delta(\alpha_2, \beta_0)$ are proportional to each other, or (b) $\gamma(\alpha_1, \beta_0)$ and $\gamma(\alpha_2, \beta_0)$ are proportional to each other. Interchanging $E^{(r)}$ and $F^{(s)}$ if necessary we may assume without loss of generality that (a) holds, and we may write $\lambda(\alpha_i, \beta_0) = \gamma(\alpha_i, \beta_0) \otimes \xi(\beta_0)$ for some $\xi(\beta_0) \in F$ and for both $i = 1, 2$. When $\beta_0 \in B$ is fixed, for each pair (α_1, α_2) of linearly independent elements of A either Alternative (a) or (b) but not both must hold. Let Γ_1 (resp. Γ_2) be the subset of the Grassmannian $\mathrm{Gr}(2, A)$ over which Alternative (a) holds (resp. over which Alternative (b) holds). It is clear that Γ_1 and Γ_2 are both closed. Since $\mathrm{Gr}(2, A)$ is the disjoint union of Γ_1 and Γ_2 it follows that one of them is all of $\mathrm{Gr}(2, A)$. Since Γ_1 is nonempty we must have $\Gamma_1 = \mathrm{Gr}(2, A)$. In other words, Alternative (a) holds for every $\alpha \in A$ when $\beta = \beta_0$ is fixed. and we may write $\lambda(\alpha, \beta_0) = \gamma(\alpha, \beta_0) \otimes \xi(\beta_0)$. If we allow β_0 to vary then a similar argument as in the above (in analogy to Alternatives (a) and (b)) shows that the same formula holds true with β_0 replaced by any nonzero $\beta \in B$, i.e., $\lambda(\alpha, \beta) = \gamma(\alpha, \beta) \otimes \xi(\beta)$ for any $\alpha \in A$.

Likewise fixing a nonzero $\alpha \in A$ by the same argument we must have (c) $\lambda(\alpha, \beta) = \eta(\alpha) \otimes \delta(\alpha, \beta)$ or (d) $\lambda(\alpha, \beta) = \gamma(\alpha, \beta) \otimes \chi(\alpha)$ for any $\beta \in B$, and Alternative (c) holds for every nonzero $\alpha \in A$ or Alternative (d) holds for every nonzero $\alpha \in A$. In the latter case since $\lambda(\alpha, \beta_0) = \gamma(\alpha, \beta_0) \otimes \xi(\beta_0)$ we may take $\chi(\alpha) = \xi(\beta_0)$, and we have $\lambda(\alpha, \beta) = \gamma(\alpha, \beta) \otimes \xi(\beta_0)$ for every $\alpha \in A$, $\beta \in B_0$, contradicting with the hypothesis that $\delta(\alpha_1', \beta_1')$ and $\delta(\alpha_2', \beta_2')$ are linearly independent for some (α_1', β_1') and (α_2', β_2'). Thus, Alternative (c) holds true for every nonzero α and we have $\lambda(\alpha, \beta) = \eta(\alpha) \otimes \delta(\alpha, \beta)$, so that for nonzero $\alpha \in A$, $\beta \in B$, $\lambda(\alpha, \beta)$ must both be of the form $\gamma(\alpha, \beta) \otimes \xi(\beta)$ and of the form $\eta(\alpha) \otimes \delta(\alpha, \beta)$. Pick any nonzero $\alpha_0 \in A$, $\beta_0 \in B$. In some open neighborhood $U = U_1 \times U_2$ of (α_0, β_0) in $A \times B$ we can choose $\eta(\alpha)$ varying holomorphically with α, $\xi(\beta)$ varying holomorphically with β and a holomorphic function $f : U \to \mathbb{C}^*$ such that $\lambda(\alpha, \beta) = f(\alpha, \beta)\eta(\alpha) \otimes \xi(\beta)$. Finally, as $\eta(\alpha)$ and $\xi(\beta)$ are only determined up to scalar multiples we fix them by writing $\lambda(\alpha, \beta_0) = \eta(\alpha) \otimes \xi(\beta_0)$ for $\alpha \in U_1$ (so that in particular $\eta(\alpha_0)$ is fixed, and then writing $\lambda(\alpha_0, \beta) = \eta(\alpha_0) \otimes \xi(\beta)$ for $\beta \in U_2$. ¿From the linearity of λ it follows that both $\eta : A^{(p)} \to E^{(r)}$ and $\xi : B^{(q)} \to F^{(s)}$ are linear. Defining $\lambda_0 : A^{(p)} \otimes B^{(q)} \to E^{(p)} \otimes F^{(s)}$ by $\lambda_0(\alpha \otimes \beta) = \eta(\alpha) \otimes \xi(\beta)$, we have two linear maps λ, λ_0 such that $\lambda(\alpha \otimes \beta) = f(\alpha, \beta)\lambda_0$, where $f(\alpha_0, \beta_0) = 1$. Since any two pointwise proportional linear maps on a vector space must be the same up to a scalar multiple, restricting to the linear subspaces of the form $\mathbb{C}\alpha \otimes B$ and then $A \otimes \mathbb{C}\beta$ we conclude that $f(\alpha, \beta)$ is independent of both α and β. Thus, $f \equiv 1$ as $f(\alpha_0, \beta_0) = 1$. The proof of Lemma 2 is complete. $\square$

Lemma 3 concerns germs of complex submanifolds of the Grassmann manifold whose tangent spaces consist only of decomposable vectors. The question is whether they lie in some projective linear subspaces. The latter indeed holds true as proved by Mirollo [Mi, 1985], and it is a special case of Choe-Hong [CH, 2004]. In [CH] complex submanifolds of the Grassmann manifold X with the property stated above are called integral submanifolds of the cone structure. Here the cone structure refers to the affine cones $\widetilde{\mathcal{W}_x} \cup \{0\} \subset T_x(X)$. In what follows we say that a submanifold $\Pi \subset G(r,s)$ is a projective linear subspace if and only if, identifying $G(r,s)$ as a projective submanifold of some projective space $\mathbb{P}^N$ by means of the Plücker embedding, $\Pi \subset G(r,s) \subset \mathbb{P}^N$ is a projective linear subspace of $\mathbb{P}^N$. We note that, identifying a Euclidean cell $\mathbb{U} \subset G(r,s)$ as the tensor product $E^{(r)} \otimes F^{(s)}$, where $E^{(r)}$ is identified with the fiber $E_0^{(r)}$ of the universal bundle $\mathbb{F}^{(r)}$ at 0, and $F^{(s)}$ is identified with the fiber $F_0^{(s)}$ of the universal bundle $\mathbb{F}^{(s)}$ at 0, then the projective linear subspaces of $G(r,s)$ passing through 0 are the topological closures in $G(r,s)$ of a linear subspace of either of the forms $H \otimes \mathbb{C}\varphi_0$, $\mathbb{C}\epsilon_0 \otimes J \subset E^{(r)} \otimes F^{(s)} = \mathbb{U}$, where $H \subset E^{(r)}$ and $J \subset F^{(s)}$ are linear subspaces.

Lemma 3. *Let $r,s \geq 2$ be integers, $U \subset G(r,s)$ be an open subset and $S \subset U$ be an m-dimensional complex submanifold. Let $\mathbb{E}^{(r)}$ and $\mathbb{F}^{(s)}$ be universal bundles over $G(r,s)$ so that $T_{G(r,s)} \equiv \mathbb{E}^{(r)} \otimes \mathbb{F}^{(s)}$ canonically. Suppose $m > 1$ and assume that for any $x \in S, T_x(S) \subset E_x^{(r)} \otimes \mathbb{C}\xi(x)$ for some nowhere zero holomorphic section $\xi \in \Gamma(S, \mathbb{F}^{(s)})$. Then, S is contained in some projective linear subspace of $G(r,s)$.*

Proof. Without loss of generality assume that $U \subset \mathbb{U} \subset G(r,s)$ is an open subset of a Euclidean cell $\mathbb{U}$, on which we make use of Harish-Chandra coordinates $Z = \big(z_{ij}\big)_{1\leq i\leq r, 1\leq j\leq s}$ in the usual identification of points on $\mathbb{U}$ as r-by-s matrices. Let $(e_i)_{1\leq i\leq r}$ resp. $(\epsilon_j)_{1\leq j\leq s}$ be the standard basis of $\mathbb{E}^{(r)}|_W$ resp. $\mathbb{F}^{(s)}|_W$, $\frac{\partial}{\partial z_{ij}} = e_i \otimes \epsilon_j$. By assumption $T_S = H \otimes \mathbb{C}\xi$ for some rank-m holomorphic vector subbundle H. Let $x \in S$. Assume without loss of generality that $H_x = \mathrm{Span}\{e_1, \cdots e_m\} \subset E_x^{(r)}$, $m \geq 2$, $\xi(x) = \epsilon_1$. Let η and φ be linearly independent holomorphic sections of H on a neighborhood of x on S. Using the Einstein convention write $\widetilde{\eta} = \eta \otimes \xi = \eta^i e_i \otimes \xi^j \epsilon_j = \eta^i \xi^j \frac{\partial}{\partial z_{ij}}$, $\widetilde{\varphi} = \varphi \otimes \xi = \varphi^k e_k \otimes \xi^\ell \epsilon_\ell = \varphi^k \xi^\ell \frac{\partial}{\partial z_{k\ell}}$. We claim that $d\xi^\ell(x) = 0$ for $\ell > 1$. Assuming this, since $x \in S$ is arbitrary it follows that $\mathbb{C}\xi$ is constant over S with respect to Harish-Chandra coordinates, and hence S must lie in the projective linear subspace $\overline{E^{(r)} \times \mathbb{C}\xi_0} \subset G(r,s)$, where $\xi_0 = \xi(y)$ for any point $y \in S$, proving Lemma 3. To prove our claim we compute Lie brackets of the vector fields $\widetilde{\eta}$ and $\widetilde{\varphi}$. We have

$$
\begin{aligned}
[\widetilde{\eta}, \widetilde{\varphi}] &= \left[\eta^i \xi^j \frac{\partial}{\partial z_{ij}}, \varphi^k \xi^\ell \frac{\partial}{\partial z_{k\ell}}\right] \\
&= \eta^i \xi^j \frac{\partial}{\partial z_{ij}}(\varphi^k \xi^\ell) \frac{\partial}{\partial z_{k\ell}} - \varphi^i \xi^j \frac{\partial}{\partial z_{ij}}(\eta^k \xi^\ell) \frac{\partial}{\partial z_{k\ell}} \\
&= \xi^j (\eta^i \varphi^k - \varphi^i \eta^k) \frac{\partial \xi^\ell}{\partial z_{ij}} \frac{\partial}{\partial z_{k\ell}} + \Big(\eta^i \xi^j \frac{\partial \varphi^k}{\partial z_{ij}} - \varphi^i \xi^j \frac{\partial \eta^k}{\partial z_{ij}}\Big)\Big(\xi^\ell \frac{\partial}{\partial z_{k\ell}}\Big) \\
&= \xi^j (\eta^i \varphi^k - \varphi^i \eta^k) \frac{\partial \xi^\ell}{\partial z_{ij}} \frac{\partial}{\partial z_{k\ell}} + \mu^k \xi^\ell \frac{\partial}{\partial z_{k\ell}} \qquad (1)
\end{aligned}
$$

for some holomorphic vector field μ on a neighborhood of x on S. Since by definition T_S is integrable, it follows that

$$[\widetilde{\eta}, \widetilde{\varphi}](x) \in E_x^{(r)} \otimes \mathbb{C}\xi(x) \ . \tag{2}$$

By (1) and specializing to the case where $\eta(x) = e_a$, $\varphi(x) = e_b$, $1 \le a, b \le m$, $a \ne b$, and noting that $\xi(x) = \epsilon_1$, we have

$$[\widetilde{\eta}, \widetilde{\varphi}](x) = \frac{\partial \xi^\ell}{\partial z_{a1}} \frac{\partial}{\partial z_{b\ell}} - \frac{\partial \xi^\ell}{\partial z_{b1}} \frac{\partial}{\partial z_{a\ell}} + \mu^k \xi^\ell \frac{\partial}{\partial z_{k\ell}} \ . \tag{3}$$

Since $\mu^k \xi^\ell \frac{\partial}{\partial z_{k\ell}} \in \mathbb{E}^{(r)} \otimes \mathbb{C}\xi$ it follows from (2) and (3) that

$$\frac{\partial \xi^\ell}{\partial z_{a1}} = \frac{\partial \xi^\ell}{\partial z_{b1}} = 0 \quad \text{for all} \quad \ell > 1 \ . \tag{4}$$

Now $T_x(S) = H_x \otimes \mathbb{C}\epsilon_1 = \text{Span}\{e_1, \dots, e_m\} \otimes \mathbb{C}\epsilon_1 = \text{Span}\{\frac{\partial}{\partial z_{c1}} : 1 \le c \le m\}$. It follows from (4) that

$$d\xi^\ell(x) = 0 \quad \text{for all} \quad \ell > 1 \ . \tag{5}$$

This proves our claim and completes the proof of Lemma 3. $\square$.

Combining Lemma 2 and Lemma 3 we have obtained

Proposition 1. *Let M and M' be Grassmann manifolds of rank ≥ 2, $0 \in M$ and $0 \in M'$ be arbitrary base points. Let $f : (M, 0) \to (M', 0)$ be a germ of holomorphic embedding such that df transforms decomposable tangent vectors into decomposable tangent vectors. Then, either df_x is complex equivalent at every point x on the domain of definition of f to a standard holomorphic embedding $\tau_{x,x'} : M \to M'$ between Grassmannians, $\tau_{x,x'}(x) = x'$, $x' := f(x)$, or the image of f lies on some projective linear subspace of M'.*

(2.2) To prove Theorem B (from Ochiai [Oc]) by means of varieties of minimal rational tangents we note first of all that on an irreducible Hermitian symmetric space S of rank ≥ 2, the variety of highest weight tangents $\mathcal{W}_x \subset \mathbb{P}T_x(S)$ at $x \in S$ agrees with the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(S)$ (Hwang-Mok [HM1], cf. Mok [Mk2, (1.4)]). Here S is embedded by the first canonical embedding into some projective space $\mathbb{P}^N$, and the minimal rational component $\mathcal{K}$ is the space of projective lines lying on S.

In Hwang-Mok [HM5] as a first step towards proving Cartan-Fubini type extension results we proved the uniqueness of tautological foliations on fibered spaces $\mathcal{C}$ of varieties of minimal rational tangents under a non-degeneracy assumption on Gauss maps. The proof relies on determining Cauchy characteristics of distributions defined on $\mathcal{C}$, and as such does not readily apply to the non-equidimensional case. To proceed along the argument of [HM5] one would either have to work on the submanifold and prove that the tautological foliation of the ambient manifold restricts to one of the submanifold, or else one works on the ambient manifold and finds a method of extending the tautological foliation on the submanifold to a neighborhood of the submanifold in such a way that the extended foliation (equivalently 1-dimensional distribution) belongs to the Cauchy characteristic of the relevant distribution defined on the cone of the ambient manifold. Neither of these appears to be an easy problem. In their place in the case of germs of holomorphic mappings between irreducible Hermitian symmetric spaces of the compact type and of rank ≥ 2 we derived in Mok [Mk2] a local differential-geometric method to compare tautological foliations under the effect of a local holomorphic maps which respects the varieties of minimal rational tangents. This proof leads especially to a new proof of Ochiai's Theorem and readily to a partial non-equidimensional generalization. We recall first of all the relevant result of Mok [Mk2].

Proposition 2 (Mok [Mk2]). *Let X and Z be two irreducible Hermitian symmetric spaces of the compact type and of rank ≥ 2. Let $\mathbb{U} \subset Z$ be a Euclidean cell in Harish-Chandra coordinates, $U \subset \mathbb{U}$ be a connected open neighborhood of $0 \in \mathbb{U}$, and $f : U \to X$ be a holomorphic map such that $f(0) = 0$ and $df_z\big(\widetilde{\mathcal{C}}_z(Z)\big) \subset \widetilde{\mathcal{C}}_{f(z)}(X)$ for every $z \in U$. For any $x \in X$, $\beta \in \widetilde{\mathcal{C}}_x(X)$, write*

$$\sigma_\beta : T_\beta\big(\widetilde{\mathcal{C}}_0(X)\big) \times T_\beta\big(\widetilde{\mathcal{C}}_0(X)\big) \to T_\beta\big(T_x(X)\big)/T_\beta\big(\widetilde{\mathcal{C}}_0(X)\big)$$

for the second fundamental form with respect to the Euclidean flat connection ∇ on $T_x(X)$. For any subspace $V \subset T_\beta\big(\widetilde{\mathcal{C}}_0(X)\big)$, define

$$\operatorname{Ker}\sigma_\beta(V,\cdot) := \big\{\delta \in T_\beta\big(\widetilde{\mathcal{C}}_0(\Omega_2)\big) : \sigma_\beta(\delta,\gamma) = 0\ ,\quad \forall\, \gamma \in V\big\}\ .$$

For any $2 \in U$, and $\alpha \in \widetilde{\mathcal{C}}_z(Z)$, denote by $\widetilde{\alpha}$ the constant vector field on Z which is α at x and identify $T_{df(\alpha)}\big(T_0(X)\big)$ with $T_0(X)$. Then, we have

$$\nabla_{df(\alpha)} df(\widetilde{\alpha}) \in \operatorname{Ker}\sigma_{df(\alpha)}\big(T_{df(\alpha)}\big(df(\widetilde{\mathcal{C}}_x(\Omega_1))\big),\cdot\big)\ .$$

Proposition 2 was used in Tu [Tu, 2002] to obtain a generalization in special cases of Tsai [Ts] to give rigidity results on proper holomorphic maps in which the domain and the target are both type I domains of rank ≥ 2, while the rank of the target exceeds that of the domain by 1.

We briefly recall how Ochiai's result [(1.2), Theorem B] follows from Proposition 2. In what follows we adopt the notations in Theorem B. On a Hermitian symmetric space S of the compact type and of rank ≥ 2 the varieties of highest weight vectors $\mathcal{W}$ agrees with the varieties of minimal rational tangents $\mathcal{C}$. Consider the 1-dimensional foliation $\mathcal{F}$ on $\pi : \mathcal{C} \to S$ whose integral curves are the tautological liftings of minimal rational curves (lines) on S. If $f : U \to V$ is VMRT-preserving, then Proposition 2 implies that the two foliations $\mathcal{F}|_{\pi^{-1}(V)}$ and $f_*\big(\mathcal{F}|_{\pi^{-1}(V)}\big)$ agree with each other. From this the arguments of Hwang-Mok [HM4,5] imply that f can be analytically continued along minimal rational curves emanating from U. There it was proven that the process of analytic continuation can be iterated to give a birational isomorphism F on S. It follows then from the VMRT-preserving property of F at general points of S and deformation theory of rational curves that F and its inverse are biregular outside subvarieties of codimension 2. Embed S by the anticanonical line bundle into a projective space. By pulling-back anticanonical sections and by Hartogs extension F induces a biholomorphic automorphism of S.

In the non-equidimensional case we will now apply Proposition 2 to study germs of holomorphic embeddings between bounded symmetric domains of type I satisfying local differential-geometric conditions. To start with in Hwang-Mok [HM5] (cf. (1.3) here) we introduced a non-degeneracy condition on the Gauss map of the variety of minimal rational tangents $\mathcal{C}_x$ at a general point x of a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component, viz., we require that (†) the Gauss map is generically finite on $\mathcal{C}_x$, equivalently (†) is satisfied if and only if at a general smooth point $[\alpha]$ of $\mathcal{C}_x$, the kernel $\mathrm{Ker}\overline{\sigma}_{[\alpha]} = 0$ for the projective second fundamental form $\overline{\sigma}_{[\alpha]}$ at $[\alpha] \in \mathrm{Reg}(\mathcal{C}_x)$. We extend this to the situation of a linear section of $\mathcal{C}_x$ and define a non-degeneracy condition (††) which reduces to (†) when the linear section is $\mathcal{C}_x$ itself. Recall that a complex-analytic variety is said to be of pure dimension n if and only if each irreducible component is of the same dimension n.

Definition 4. *Let $m \geq 2$, $\mathcal{A} \subset \mathbb{P}^m$ be a projective subvariety of pure dimension $a \geq 1$. Let $\Pi \subset \mathbb{P}^m$ be a projective linear subspace, and $\mathcal{B} := \Pi \cap \mathcal{A}$ be a non-empty projective subvariety of pure dimension $b \geq 1$. We say that the pair $(\mathcal{B}, \mathcal{A})$ satisfies the non-degeneracy condition (††) if and only if for every general smooth point $[\beta] \in \mathcal{B}$, $[\beta]$ is also a smooth point of $\mathcal{A}$ and $\mathrm{Ker}\overline{\sigma}_{[\beta]}\big(T_{[\beta]}(\mathcal{B}), \cdot\big) = 0$.*

Adopting the notations in Proposition 2 in the above, except that we pass from affine cones to their projectivizations and make use of the projective second fundamental form $\sigma_{[\beta]}$ of $\mathcal{A} \subset \mathbb{P}^m$ at $[\beta]$, the last sentence in Definition 4 means that $\overline{\sigma}_{[\beta]}(\xi, \eta) = 0$ for every $\xi \in T_{[\beta]}(\mathcal{B})$ implies that $\eta = 0$. Taking $\mathcal{B}$ to mean $\mathcal{C}_x$, when Π equals $\mathbb{P}^m$ the condition (††) reduces to (†).

Let $U \subset G(p, q)$ be a domain and $f : U \to G(r, s)$ be a holomorphic map such that $df(x) : T_x(G(p, q)) \to T_{f(x)}(G(r, s))$ is complex equivalent to $d\tau(x)$ for a standard embedding $\tau : G(p, q) \to G(r, s)$. To apply Proposition 2 in the non-equidimensional case the first problem is to check that the non-degeneracy condition (††) at a general point is satisfied for the pair $\big(f_*\mathcal{C}_x(G(p, q)), \mathcal{C}_{f(x)}(G(r, s))\big)$. For this purpose we note the following simple fact about the second fundamental form in the case of Grassmann manifolds, where the VMRT $\mathcal{C}_0$ at $0 \in G(r, s)$ is the image of the Segre embedding $\zeta : \mathbb{P}(E^r) \times \mathbb{P}(F^{(s)}) \hookrightarrow \mathbb{P}(E^{(r)} \otimes F^{(s)})$ defined by $\zeta([\alpha], [\beta]) = [\alpha \otimes \beta])$. Here $E^{(r)}$ resp. $F^{(s)}$ stands for $E^{(r)}_0$ resp. $F^{(s)}_0$, which is the fiber of the universal bundle $\mathbb{E}^{(r)}$ resp. $\mathbb{F}^{(s)}$ over 0.

Lemma 4. *Let $\gamma \in E^{(r)} \otimes F^{(s)}$ be a non-zero decomposable vector, $\gamma = a \otimes b$. Then, the tangent space $T_\alpha(\widetilde{\mathcal{C}_0})$ is naturally identified with $\mathbb{C}a \otimes F^{(s)} \oplus E^{(r)} \otimes \mathbb{C}b$, and the second fundamental form σ of $\widetilde{\mathcal{C}_0} \subset E^{(r)} \otimes F^{(s)} \cong M(r, s) \cong \mathbb{C}^{rs}$ is given by*

$$\begin{aligned} &\sigma_\gamma\big(a \otimes \varphi_1 + \epsilon_1 \otimes b, a \otimes \varphi_2 + \epsilon_2 \otimes b\big) \\ &= \varphi_1 \otimes \epsilon_2 + \epsilon_1 \otimes \varphi_2 \ \mathrm{mod}\big(a \otimes F^{(s)}\big) + \big(E^{(r)} \otimes b\big) \ . \end{aligned}$$

Here $\epsilon_1, \epsilon_2 \in E^{(r)}$, $\varphi_1, \varphi_2 \in F^{(s)}$ are arbitrary vectors.

Proof. We note that $\big(E^{(r)} \otimes \mathbb{C}b\big) \cap \big(\mathbb{C}a \otimes F^{(s)}\big) = \mathbb{C}(a \otimes b) = \mathbb{C}\gamma$, so that φ_i resp. ϵ_i; $i = 1, 2$; is uniquely determined only up to $\mathbb{C}a$, resp. $\mathbb{C}b$, It follows nonetheless that $\varphi_1 \otimes \epsilon_2 + \epsilon_1 \otimes \varphi_2$ is uniquely determined up to $\big(a \otimes F^{(s)}\big) + \big(E^{(r)} \otimes b\big)$, so that the expression for σ_γ in Lemma 4 is indeed well-defined. The Segre embedding ζ lifts to $\widetilde{\zeta} : \big(E^{(r)} - \{0\}\big) \times \big(F^{(s)} - \{0\}\big) \to E^{(r)} \otimes F^{(s)}$ given by $\widetilde{\zeta}(a, b) = a \otimes b$. By definition for $\epsilon \in E^{(r)}, \varphi \in F^{(s)}$ we have

$$\widetilde{\zeta}(a + \epsilon, b + \varphi) = (a + \epsilon) \otimes (b + \varphi) = a \otimes b + (a \otimes \varphi + \epsilon \otimes b) + \epsilon \otimes \varphi \ .$$

It follows from standard calculations that

$$\begin{aligned} &\sigma_\gamma\big(a \otimes \varphi + \epsilon \otimes b, a \otimes \varphi + \epsilon \otimes b\big) \\ &= \varphi \otimes \epsilon + \epsilon \otimes \varphi \ \mathrm{mod}\big(a \otimes F^{(s)}\big) + \big(E^{(r)} \otimes b\big) \ . \end{aligned}$$

Lemma 4 follows by polarization of the symmetric bilinear map σ_γ. □

Proposition 3. *Let p, q, r, s be integers such that $2 \le p \le r$ and $2 \le q \le s$. Let $f : \big(G(p,q); 0\big) \to \big(G(r,s); 0\big)$ be a germ of holomorphic embedding and denote by $S \subset G(r,s)$ the germ of pq-dimensional complex submanifold at $0 \in G(r,s)$ which is the germ of the image of f at 0. Assume that for any x in the domain of definition of f, $df_x : T_x(G(p,q)) \to T_y(G(r,s)), y := f(x)$, is complex equivalent to $d\tau_0 : T_0(G(p,q)) \to T_0(G(r,s))$ for a standard embedding $\tau : G(p,q) \to G(r,s), \tau(0) = 0$. Then, for any $y \in S$ and for any minimal characteristic vector β at y tangent to S, the germ of the line C_β at y actually lies on S.*

The assumption that $df_x : T_x(G(p,q)) \to T_y(G(r,s)), y := f(x)$, is complex equivalent to $d\tau_0 : T_0(G(p,q)) \hookrightarrow T_0(G(r,s))$ translates into an algebraic statement about df_x, as follows. Identify the holomorphic tangent bundle $T_{G(p,q)}$ as the tensor product $\mathbb{A}^{(p)} \otimes \mathbb{B}^{(q)}$ of its (nonnegative) universal bundles, so that $T_x\big(G(r,s)\big) \cong A^{(p)} \otimes B^{(q)}$ for the fibers $A^{(p)}$ resp. $B^{(q)}$ at x of $\mathbb{A}^{(p)}$ resp. $\mathbb{B}^{(q)}$. Similarly, identify the holomorphic tangent bundle $T_{G(r,s)}$ as the tensor product $\mathbb{E}^r \otimes \mathbb{F}^s$ of its (nonnegative) universal bundles, so that $T_y\big(G(r,s)\big) \cong E^{(r)} \otimes F^{(s)}$ for the fibers $E^{(r)}$ resp. $F^{(s)}$ at y of $\mathbb{E}^{(r)}$ resp. $\mathbb{F}^{(s)}$. Then df_x complex equivalent to $d\tau_0$ if and only if $df_x = \mu \otimes \nu$ for some injective linear maps $\mu : A^{(p)} \to E^{(r)}, \nu : B^{(q)} \to F^{(s)}$.

Proof of Proposition 3. The germ of holomorphic map $f : \big(G(p,q); 0\big) \to \big(G(r,s); 0\big)$ induces on S a subbundle $\mathcal{R} \subset \mathbb{P}T_S$ such that for x sufficiently close to 0 so that $y = f(x) \in S$, we have $\widetilde{\mathcal{R}}_y = df_x(\widetilde{\mathcal{C}}_x)$. $\mathcal{R}$ comes equipped with a tautological foliation $\mathcal{E}$ corresponding to the tautological foliation on $\mathcal{C}_{G(p,q)}$. Denote by $\mathcal{F}$ the tautological foliation on the target Grassmann manifold $G(r,s)$. A leaf of $\mathcal{F}$ is the tautological lifting of a line on $G(r,s)$, while a leaf of $\mathcal{E}$ on S is the tautological lifting of the image under f of a nonempty connected open subset of a line on $G(p,q)$. Let $x \in G(p,q)$ be sufficiently close to 0, and $\alpha \in T_x\big(G(p,q)\big)$ be a minimal characteristic vector. $\beta := df(\alpha) \in T\big(G(r,s)\big)$ is by hypothesis a minimal characteristic vector at $y = f(x)$.

Let $\beta^\sharp \in T_{[\beta]}(\mathcal{E})$ and $\beta^\flat \in T_{[\beta]}(\mathcal{F})$ be such that $d\pi(\beta^\sharp) = d\pi(\beta^\flat) = \beta$ for the canonical projection $\pi : \mathbb{P}T_{G(p,q)} \to G(p,q)$. Thus $\pi(\beta^\sharp - \beta^\flat) = 0$, $\beta^\sharp - \beta^\flat \in T_{[\beta]}\big(\mathbb{P}T_y\big(G(r,s)\big)\big)$, which is canonically identified with $T_y\big(G(r,s)\big)/\mathbb{C}\beta$. From the hypothesis we have $df\Big(\widetilde{\mathcal{C}}_y\big(G(p,q)\big)\Big) \subset \widetilde{\mathcal{C}}_x\big(G(r,s)\big)$. Proposition 2 applies to the germ of holomorphic embedding $f : \big(G(p,q);0\big) \to G\big((r,s);0\big)$. In the notations there, $\nabla_{df(\alpha)} df(\widetilde{\alpha})$, regarded as a vector in $T_y\big(G(r,s)\big)$, agrees mod $\mathbb{C}\beta$ with $\overline{\eta} := \beta^\sharp - \beta^\flat$, noting that $\beta = df(\alpha)$. Here $\overline{\eta} \in T_{[\beta]}\big(\mathcal{C}_0\big(G(r,s)\big)\big) = T_\beta\big(\widetilde{\mathcal{C}}_0\big(G(r,s)\big)\big)$ mod $\mathbb{C}\beta$. Pick any $\eta \in T_\beta\big(\widetilde{\mathcal{C}}_0(G(r,s)\big)$ such that $\overline{\eta} = \eta$ mod $\mathbb{C}\beta$. Then,

$$\eta \in \operatorname{Ker} \sigma_\beta\Big(T_\beta\big(df\big(\widetilde{\mathcal{C}}_0\big(G(r,s)\big)\big), \cdot\big)\Big) \ . \tag{1}$$

Note here that $\mathbb{C}\beta$ is in the kernel of σ_β at β, and the choice of the representative η modulo $\mathbb{C}\beta$ is immaterial. Now by hypothesis $df_x\big(\widetilde{\mathcal{C}_x}\big(G(p,q)\big)\big) = \big(V^{(p)} \otimes W^{(q)}\big) \cap \widetilde{\mathcal{C}}_y$ for some p-plane $V^{(p)} \subset E^{(r)}$ and some q-plane $W^{(q)} \subset F^{(s)}$. The second fundamental form on $\widetilde{\mathcal{C}}_y\big(G(r,s)\big)$ at $\gamma = a \otimes b$, $a \in E^{(r)}$, $b \in F^{(s)}$, is given as follows. We have

$$T_\gamma\big(\widetilde{\mathcal{C}}_y\big(G(r,s)\big)\big) = \big(a \otimes F^{(s)}\big) + \big(E^{(r)} \otimes b\big) \ . \tag{2}$$

Since $(a \otimes F^{(s)}) \cap (E^{(r)} \otimes b) = \mathbb{C}(a \otimes b) = \mathbb{C}\gamma$, so that $\dim T_\gamma\big(\widetilde{\mathcal{C}}_y\big(G(r,s)\big)\big) = r + s - 1$, which is $1 + \big((r-1) + (s-1)\big) = 1 + \dim\big(\mathcal{C}_y\big(G(r,s)\big)\big)$. For $\varphi_i \in F^{(s)}$, $\epsilon_i \in E$; $i = 1, 2$; we have by Lemma 4

$$\begin{aligned} &\sigma_\gamma\big(a \otimes \varphi_1 + \epsilon_1 \otimes b, a \otimes \varphi_2 + \epsilon_2 \otimes b\big) \\ &= \varphi_1 \otimes \epsilon_2 + \epsilon_1 \otimes \varphi_2 \ \mathrm{mod}\big(a \otimes F^{(s)}\big) + \big(E^{(r)} \otimes b\big) \ . \end{aligned} \tag{3}$$

Consider now $\gamma = \beta = df(\alpha)$. By (1), $\sigma_\beta(\xi, \eta) = 0$ for every

$$\xi \in T_\beta\big(df\big(\widetilde{\mathcal{C}}_x\big(G(p,q)\big)\big) = \big(a \otimes W^{(q)}\big) + \big(V^{(p)} \otimes b\big) \ . \tag{4}$$

In other words, for any $v \in V^{(p)}$, $w \in W^{(q)}$, writing $\eta = a \otimes \varphi + \epsilon \otimes b$, $\varphi \in F^{(s)}$, $\epsilon \in E^{(r)}$ we have

$$0 = \sigma_\beta(a \otimes w + v \otimes b, a \otimes \varphi + \epsilon \otimes b) = \epsilon \otimes w + v \otimes \varphi \ \mathrm{mod}\big(a \otimes F^{(s)}\big) + \big(E^{(r)} \otimes b\big) \ . \tag{5}$$

In particular, taking $v = 0$ we have

$$\epsilon \otimes w \in \big(a \otimes F^{(s)}\big) + \big(E^{(r)} \otimes b\big) \tag{6}$$

for every $w \in W^{(q)}$. By assumption, $\dim W^{(q)} \geq 2$. Taking w to be linearly independent of b, (6) is possible only if $\epsilon \in \mathbb{C}a$. Similarly, taking $w = 0$ in (5) we have

$$v \otimes \varphi \in \big(a \otimes F^{(s)}\big) + \big(E^{(r)} \otimes b\big) \tag{7}$$

for every $v \in V^{(p)}$, $\dim V^{(q)} \geq 2$, which is possible only if $\varphi \in \mathbb{C}b$. As a consequence, we must have

$$\eta = a \otimes \varphi + \epsilon \otimes b \in \big(a \otimes \mathbb{C}b\big) + \big(\mathbb{C}a \otimes b\big) = \mathbb{C}(a \otimes b) = \mathbb{C}\beta \ ;$$
$$\overline{\eta} = \eta \bmod \mathbb{C}\beta = 0 \ . \tag{8}$$

In other words, we have proven that $\beta^\sharp = \beta^\flat$, showing that the leaves of $\mathcal{E}$ and $\mathcal{F}$ at $[\beta]$ are tangent to each other at $[\beta]$. It follows by integrating holomorphic vector fields that a local leaf of $\mathcal{E}$ is necessarily a local leaf of $\mathcal{F}$, proving Proposition 3, as desired. □

(2.3) Let X be a Fano manifold of Picard number 1. There is a procedure of recovering X from a space of minimal rational curves by a process of adjunction, as follows (Hwang-Mok [HM2]). Let $\mathcal{K}$ be a minimal rational component on X. Let $x \in X$ be a general point and let $\mathcal{V}_1$ be the union of minimal rational curves emanating from x. At a general point x_1 of $\mathcal{V}_1$ let $\mathcal{V}_2(x_1)$ be the union of minimal rational curves emanating from x_1 and write $\mathcal{V}_2$ for the closure of $\bigcup\big\{\mathcal{V}_2(x_1) : x_1 \in \mathcal{V}_1\big\}$. This process can be iterated in an obvious way to define $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \dots, \mathcal{V}_k, \dots$. From the condition that X is of Picard number 1 it follows using deformation theory of rational curves that $\mathcal{V}_n = X$ ([HM2, (4.2), Proposition 13]).

With respect to the process of adjunction of minimal rational curves there is a class of projective subvarieties which deserve attention in the context of a geometric theory basing on varieties of minimal rational tangents. To formulate it we introduce a notion of saturation with respect to minimal rational curves which applies in general to the situation of Fano manifolds and more generally to uniruled projective manifolds, but we will specialize to the case of a rational homogeneous space of Picard number 1, in which case there is a unique choice of minimal rational component $\mathcal{K}$ whose members are precisely projective lines with respect to the first canonical embedding. In particular, all minimal rational curves are nonsingular. We will denote by $\mathcal{C}_X \subset \mathbb{P}T_X$ the total space of its varieties of minimal rational tangents, which is a holomorphic bundle of projective submanifolds. We have

Definition 5. *Let $(X, \mathcal{K})$ be a rational homogeneous space of Picard number 1 equipped with the unique minimal rational component $\mathcal{K}$ whose members are of degree 1 with respect to $\mathcal{O}(1)$. Denote by $\mathcal{C}_X \subset \mathbb{P}T_X$ the total space of its varieties of $\mathcal{K}$-tangents. Let $\Sigma \subset X$ be an irreducible projective subvariety and $\mathcal{E} \subset \mathcal{C}_X\big|_\Sigma$ be a subvariety. For $y \in \Sigma$ denote by $\mathcal{E}_y$ the fiber of $\mathcal{E}$ over y. We say that $(\Sigma, \mathcal{E}) \hookrightarrow (X, \mathcal{C}_X)$ is rationally saturated if*

(a) $\mathcal{E}_y = \mathbb{P}T_y(\Sigma) \cap \mathcal{C}_X \neq \emptyset$ *for a smooth point* $y \in \Sigma$, *and*

(b) *for every smooth point y on Σ, and for every minimal rational curve C (i.e. projective line) on X passing through y, C must necessarily lie on Σ whenever* $\big[T_y(C)\big] \in \mathcal{E}_y$.

More generally, if Σ is an irreducible complex-analytic subvariety of some domain $U \subset X$, we say that $(\Sigma, \mathcal{E}) \hookrightarrow (X, \mathcal{C}_X)$ is rationally saturated if and only if the same *conditions (a) and (b) are satisfied, except that for (b) we require only that the germ of the minimal rational curve C at y must necessarily lie on Σ whenever $[T_y(C)] \in \mathcal{E}_y$.*

If we take a minimal rational curve on X to play the role of a geodesic, the notion of rational saturation of $(\Sigma, \mathcal{E}) \hookrightarrow (X, \mathcal{C}_X)$ is the analogue of a totally geodesic subspace in Riemannian geometry, except that in the former notion the 'geodesics' (minimal rational curves) are only defined for a distinguished set of tangent directions corresponding to varieties of minimal rational tangents.

By an obvious adaptation of the method of proof of Hwang-Mok [HM5] on Cartan-Fubini type extension, [(2.2), Proposition 2], and the proof of [(2.2), Proposition 3], we have

Theorem 1. *Let X resp. Z be an irreducible Hermitian symmetric spaces of the compact type of rank ≥ 2 and write $\mathcal{C}_X$ resp. $\mathcal{C}_Z$ for the total space of its varieties of minimal rational tangents. Let $U \subset Z$ be a connected open subset and $f : U \to X$ be a holomorphic embedding onto a complex submanifold $S \subset V$ of some open subset $V \subset X$ such that $[df](\mathcal{C}_Z, z) \subset (\mathcal{C}_X, f(z))$. Suppose at a general point of S, the non-degeneracy condition (††) in [(2.2), Definition 4] on projective second fundamental forms is satisfied at a general point of $\mathcal{C}_X \cap \mathbb{P}T_S$. Then, $(S, \mathcal{C}_X \cap \mathbb{P}T_S)$ is rationally saturated with respect to $(X, \mathcal{C}_X)$. Furthermore, f extends to a rational map $F : Z \to X$.*

Since [(2.2), Proposition 2] holds true for holomorphic maps between irreducible Hermitian symmetric spaces of the compact type, the scheme of proof of [(2.2), Proposition 3] applies in the general set-up of Theorem 1 provided that (††) is satisfied. The bulk of the proof of Proposition 3 says that it is indeed satisfied in the special case of maps between Grassmannians of rank ≥ 2 which respect VMRTs and which satisfies an additional non-degeneracy assumption. Together with Theorem 1 this says that in the special case of maps between Grassmannians under consideration in Proposition 3, the germ of holomorphic map $f : (G(p,q), 0) \to (G(r,s), 0)$ extends to a rational map $F : G(p,q) \to G(r,s)$.

§3 Completion to a standard embedding by the adjunction of rational curves

(3.1) For the study of germs of holomorphic maps between irreducible Hermitian symmetric spaces of the compact type, in the last section we reach a point where one can in principle verify analytic continuation by checking the non-degeneracy condition (††) in terms of second fundamental forms. In the case of Grassmannians of rank ≥ 2 under consideration in [(2.2), Proposition 3], we are going to prove that the analytic continuation actually yields a standard embedding. We will accomplish this by introducing a notion of parallel transport of VMRTs along minimal rational curves. In general, this notion concerns the tangent spaces of VMRTs along a minimal rational curve, but in some cases such as the Grassmannians under consideration, the parallel transport allows us to identify VMRTs along a minimal rational curve on a given submanifold S with those of a model submanifold M, eventually leading to an identification of S as an open subset of M. We have

Proposition 4. *Let p, q, r, s be integers such that $2 \le p \le r$, $2 \le q \le s$. Let $S \subset G(r,s)$ be a germ of pq-dimensional complex submanifold at $0 \in G(r,s)$ such that $\mathcal{C} := \mathbb{P}T_S \cap \mathcal{C}_{G(r,s)} \subset \mathbb{P}T_S$ defines in a canonical way a Grassmann structure on S modeled on $G(p,q)$. Assume that $(S, \mathcal{C})$ is rationally saturated with respect to $\big(G(r,s), \mathcal{C}_{G(r,s)}\big)$. Then, S is an open subset of a complex submanifold $M \hookrightarrow G(r,s)$ such that M is biholomorphically isomorphic to the Grassmann manifold $G(p,q)$, and such that the embedding $M \hookrightarrow G(r,s)$ is a standard embedding.*

Proof. For convenience we make use of Harish-Chandra coordinates on a Zariski open neighborhood of $0 \in G(r,s)$ where the latter point is identified with the origin of $M(r,s) \cong \mathbb{C}^r \otimes \mathbb{C}^s$. By assumption $\mathcal{C}_0 := \mathbb{P}T_0(S) \cap \mathcal{C}_{G(r,s)} = \mathbb{P}(\mathbb{C}^p) \times \mathbb{P}(\mathbb{C}^q) \hookrightarrow \mathbb{P}(\mathbb{C}^r) \times \mathbb{P}(\mathbb{C}^s) = \mathcal{C}_0\big(G(r,s)\big) \subset \mathbb{P}T_0\big(G(r,s)\big)$. There exists a Grassmann submanifold $M \subset G(r,s)$, $M \cong G(p,q)$, such that $T_0(M) = \mathbb{C}^p \otimes \mathbb{C}^q \hookrightarrow \mathbb{C}^r \otimes \mathbb{C}^s \cong M(r,s) = T_0\big(G(r,s)\big)$. We proceed to prove that S is actually an open subset of M.

We introduce a process of partial adjunction of open subsets of minimal rational curves to recover some nonempty open subset of S, as follows. Denote by $\mathcal{K}$ the unique minimal rational component on $G(r,s)$. On $G(r,s)$ consider the double fibration $\mu : \mathcal{C} \to G(r,s)$ and $\rho : \mathcal{C} \to \mathcal{K}$, where $\mu = \pi\big|_{\mathcal{C}}$ for the canonical projection $\pi : \mathbb{P}T_{G(r,s)} \to G(r,s)$, $\mathcal{K}$ is the Chow component of lines on $G(r,s)$, and $\rho : \mathcal{C} \to \mathcal{K}$ defines the tautological $\mathbb{P}^1$-bundle over $\mathcal{K}$. We sometimes write $\rho : \mathcal{C}^\rho \to \mathcal{K}$ to emphasize the role of $\mathcal{C}$ as the total space of a holomorphic $\mathbb{P}^1$-bundle. For each $\alpha \in \widetilde{\mathcal{C}}_0$ recall that C_α is the unique line on $G(r,s)$ passing through 0 such that $T_0(C_\alpha) = \mathbb{C}\alpha$. Write $D_\alpha := S \cap C_\alpha$ and define $\mathcal{U}_1^\sharp = \bigcup\{D_\alpha : \alpha \in \widetilde{\mathcal{C}}_0\}$. Clearly there exists a non-empty subset $\mathcal{U}_1 \subset \mathcal{U}_1^\sharp$ such that $\mathcal{U}_1$ is a locally closed submanifold of $G(r,s)$, $\dim(\mathcal{U}_1) = p + q + 1$. Now for any $y \in S$ and $\beta \in \widetilde{\mathcal{C}}_y$ define analogously the line C_β passing through y, $T_y(C_\beta) = \mathbb{C}\beta$, and $D_\beta := S \cap C_\beta$.

Define $\mathcal{U}_2^\sharp := \bigcup\{D_\beta : \beta \in \widetilde{\mathcal{C}}_y, y \in \mathcal{U}_1\}$. Again choose a nonempty subset $\mathcal{U}_2 \subset \mathcal{U}_2^\sharp$ such that $\mathcal{U}_2$ is a locally closed submanifold of $G(r,s)$ of maximal possible dimension, in the following way. $\mathcal{U}_2^\sharp$ can be identified as the image under a canonical holomorphic map γ of some open subset $\mathcal{W}_0$ of the tautological $\mathbb{P}^1$-bundle over $\mathcal{C}\big|_{\mathcal{U}_1}$ i.e., $\mathcal{W}_0 \subset \rho^*\mathcal{C}^\rho\big|_{\mu^{-1}(\mathcal{U}_1)}$ is an open subset. Choose a point $w \in \mathcal{W}_0$ where γ is of maximal rank and define $\mathcal{U}_2 \subset \mathcal{U}_2^\sharp$ to be $\gamma(\mathcal{W})$ for a sufficiently small open neighborhood $\mathcal{W}$ of w in $\mathcal{W}_0$. We continue this process to obtain a sequence of locally closed complex submanifolds $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_i, \ldots$ with $\dim(\mathcal{U}_{k+1}) \ge \dim(\mathcal{U}_k)$ and stop whenever $\dim(\mathcal{U}_{k+1}) = \dim(\mathcal{U}_k)$. Since $\mathcal{C}_y \subset \mathbb{P}T_y(S)$ is linearly non-degenerate the process continues unless $\dim(\mathcal{U}_k) = \dim(S)$, i.e., $\mathcal{U}_k \subset S$ is an open subset. In what follows k will denote the first index such that $\dim(\mathcal{U}_k) = \dim(S)$.

The crux of our proof is an argument which allows us to show inductively that $\mathcal{U}_i \subset M$ and that for every $y \in \mathcal{U}_i$, $\mathcal{C}_y \subset \mathbb{P}T_y(M)$, so that $\mathcal{U}_{i+1}^\sharp \subset M$, and hence $\mathcal{U}_{i+1} \subset M$. For this it suffices to prove the following lemma, which may be regarded as a method of parallel transport of varieties of minimal tangents along a line. In what follows for clarity we write $\mathcal{C}_y(S)$ in place of $\mathcal{C}_y$ for $y \in S$.

Lemma 5. *In the notations of* Proposition 4 *let $M \subset G(r,s)$ be a Grassmann subman-*

ifold, $M \cong G(p,q)$. Let $y \in S$ be such that $\mathcal{C}_y(S) = \mathcal{C}_y(M)$, and C be a line on M passing through y. Then, for each $z \in C \cap S$, $\mathcal{C}_z(S) = \mathcal{C}_z(M)$.

Assuming the lemma, we proceed to prove that $\mathcal{U}_i \subset \mathcal{U}_i^\sharp \subset M$ for each $i = 1, 2, \dots, k$ by induction. Clearly $\mathcal{U}_1 \subset M$. For $i \geq 2$ assume inductively that $\mathcal{U}_i \subset M$ and that $\mathcal{C}_y(S) = \mathcal{C}_y(M)$ for every $y \in \mathcal{U}_{i-2}$ with the convention that $\mathcal{U}_0 = \{0\}$. By Lemma 5, $\mathcal{C}_y(S) = \mathcal{C}_z(M)$ for every $z \in \mathcal{U}_{i-1}$. It follows that lines emanating from $z \in \mathcal{U}_{i-1}$ and tangent to S at z must lie on M. Since $\mathcal{U}_i^\sharp$ is obtained from $\mathcal{U}_{i-1}$ by adjoining portions of lines emanating from $\mathcal{U}_{i-1}$, we have $\mathcal{U}_i \subset \mathcal{U}_i^\sharp \subset M$, as asserted. Finally, $\mathcal{U}_k \subset S$ is a non-empty open set, and it follows that $S \subset M$, as asserted in Proposition 4. It remains to establish Lemma 5.

Proof of Lemma 5. By assumption, S is the image of some germ of holomorphic embedding $f : \big(G(p,q), 0\big) \to \big(G(r,s), 0\big)$. Let $U \subset G(p,q)$ be the open neighborhood of 0 where f is defined such that f maps U biholomorphically onto S. Let now $y_0 \in U$ be close to 0, $y = f(y_0)$, such that $\mathcal{C}_y(S) = \mathcal{C}_y(M)$, and C_0 be a line on $G(p,q)$ passing through y_0, $C \subset G(r,s)$ a line passing through y such that $df(T_{y_0}(C_0)) = T_y(C) \subset T_y(S)$. Let $z \in C \cap S$ be distinct from y, $z_0 \in C_0$ be such that f is defined at z_0, $f(z_0) = z$. Write $T_{z_0}(C_0) = \mathbb{C}\gamma_0$. Let $\Sigma_0 \subset G(p,q)$ be the union of lines emanating from y_0, so that Σ_0 is nonsingular at z_0. Define $\Sigma = f(\Sigma_0 \cap U)$, so that Σ is nonsingular at z. $\Sigma \subset G(r,s)$ is an open subset of the union of lines C emanating from y such that $T_y(C) \subset T_y(S)$.

From the deformation theory of rational curves we have

$$T_{[\gamma_0]}\big(\mathcal{C}_{z_0}\big(G(p,q)\big)\big) = T_{z_0}(\Sigma_0)/T_{z_0}(C_0) = T_{z_0}(\Sigma_0)/\mathbb{C}\gamma_0 \ . \tag{1}$$

Then, writing $\gamma = df(\gamma_0)$ we have by (1)

$$T_{[\gamma]}\big(\mathcal{C}_z(S)\big) = T_z(\Sigma)/T_z(C_0) = T_z(\Sigma)/\mathbb{C}\gamma \ . \tag{2}$$

On the other hand since $\mathcal{C}_y(S) = \mathcal{C}_y(M)$ we conclude that $\Sigma \subset M$, so that $\Sigma \subset S \cap M$. Again by the deformation theory of rational curves we have

$$T_{[\gamma_0]}\big(\mathcal{C}_z(M)\big) = T_z(\Sigma)/\mathbb{C}\gamma = T_{[\gamma]}\big(\mathcal{C}_z(S)\big) \ . \tag{3}$$

Thus we have two projective submanifolds $\mathcal{C}_z(S), \mathcal{C}_z(M) \subset \mathcal{C}_z\big(G(r,s)\big)$, both of dimension $p+q-1$ and containing $[\gamma]$ such that (3) holds, i.e., $\mathcal{C}_z(M)$ and $\mathcal{C}_z(S)$ are tangent to each other at $[\gamma]$. We observe now that we must have from the hypothesis of Proposition 4 that $\mathcal{C}_z(S) = \mathbb{P}(A_S) \times \mathbb{P}(B_S) \subset \mathbb{P}(\mathbb{C}^r) \times \mathbb{P}(\mathbb{C}^s) \subset \mathbb{P}(\mathbb{C}^r \otimes \mathbb{C}^s) = \mathbb{P}T_z\big(G(r,s)\big)$ for some vector subspaces $A_S \subset \mathbb{C}^r$, $B_S \subset \mathbb{C}^s$, $\dim(A_S) = p$, $\dim(B_S) = q$. The analogue holds true for $\mathcal{C}_z(M)$ with $\mathcal{C}_z(M) = \mathbb{P}(A_M) \times \mathbb{P}(B_M) \subset \mathbb{P}(A_M \otimes B_M)$. The statement that $\mathcal{C}_z(S)$ and $\mathcal{C}_z(M)$ are tangent to each other at z translates into the fact that, writing $\gamma = a \otimes b$,

$$T_{[\gamma]}\big(\mathcal{C}_z(S)\big) = (a \otimes B_S) + (A_S \otimes b) = (a \otimes B_M) + (A_M \otimes b) = T_{[\gamma]}\big(\mathcal{C}_z(M)\big) \ . \tag{4}$$

which implies readily that $A_S = A_M$, $B_S = B_M$. In other words, tangency of $\mathcal{C}_z(S)$ and $\mathcal{C}_z(M)$ at $[\gamma]$ implies $\mathcal{C}_z(S) = \mathcal{C}_z(M)$, as asserted in Lemma 5. The proof of Lemma 5 is complete, from which Proposition 4 follows. □

In Proposition 4 the germ of complex submanifold $S \subset G(r,s)$ at $0 \in G(r,s)$ is by hypothesis the image of the germ of $G(p,q)$ at 0 under a holomorphic map f, and the Grassmann structure on S modeled on $G(r,s)$, defined by $\mathcal{C} := \mathbb{P}T_S \cap \mathcal{C}_{G(r,s)}$, was shown to agree with the Grassmann structure on S induced by f. Starting with $S \subset G(r,s)$ and forgetting about the germ of holomorphic mapping f, we have in the hypothesis of Proposition 4 equivalently a flat Grassmann structure on S. Combining Propositions 3 and 4 we have given a new proof of Theorem C which we formulate in terms of Grassmann structures in view of the nature of our proof.

Theorem 2. *Let X be a Grassmann manifold, $\mathcal{C}_X \subset \mathbb{P}T_X$ its total space of varieties of minimal rational tangents, and $S \subset X$ be a germ of complex submanifold such that $\mathcal{E} := T_S \cap \mathcal{C}_X$ defines canonically a flat Grassmann structure of rank ≥ 2. Then $(S, \mathcal{E}) \hookrightarrow (X, \mathcal{C}_X)$ is rationally saturated, and S is an open subset of a Grassmann submanifold $M \subset X$.*

Remarks As is apparent the proofs of Proposition 4 and Theorem 2 can be easily modified to yield Neretin's more general result referred to after the statement of [(1.2), Theorem C] in which the germ of map is not required to be an embedding. It suffices to consider foliations on the image of the germ of the holomorphic map.

§4 A perspective on the study of proper holomorphic maps in terms of geometric structure

(4.1) We are now in a position to give a proof of Tsai's Theorem ([1.1], Theorem A]) on proper holomorphic maps between bounded symmetric domains in the special case of irreducible bounded symmetric domains of type I, by a proof that exploits (a) boundary values of holomorphic functions on a bounded symmetric domain and (b) the study of geometric structures by means of rational curves.

Proof of Theorem A in the case of Type-I domains. To make the proof self-contained we recall the argument of taking boundary values of Mok-Tsai [MT] and Tsai [Ts] on product domains embedded in bounded symmetric domains. We have the Borel embedding $D(p,q) \subset G(p,q)$, $D(r,s) \subset G(r,s)$. Let p, q, r, s be positive integers such that $\min(p,q) = \min(r,s) \geq 2$ so that the Type-I domains $D(p,q)$ and $D(r,s)$ are of the same rank $\rho \geq 2$. Let $f : D(p,q) \to D(r,s)$ be a proper holomorphic map. For each minimal disk $\Delta \subset D(p,q)$ there exists a product domain $\Pi \subset D(p,q)$ which can be naturally identified with $\Delta \times D(p-1, q-1)$. Restricting to Π, we consider the holomorphic mappings $f_z(w) = f(z,w)$, $z \in \Delta$, $w \in D(p-1, q-1)$. For almost every boundary point $\zeta \in \Delta$, letting z converge to ζ non-tangentially we obtain a holomorphic mapping $f_\zeta : D(p-1, q-1) \to \partial D(r,s)$. We may regard f_ζ as being defined on a face Φ of $\partial D(p,q)$, $\Phi \cong D(p-1, q-1)$. From the boundary structure of bounded symmetric domains (cf. Wolf [Wo]) the image of f_ζ must lie in some face Ψ of $\partial D(r,s)$ such that $\Psi \cong D(r-1, s-1)$. It follows that every rank-1 vector α tangent to Φ at some $b \in \Phi$ is transformed under df_ζ to a vector of rank $\leq \rho - 1$. The condition on the rank of a vector, which is identified with a matrix, is given by the vanishing of a number of minors. We have a constant vector field $\widetilde{\alpha}$ on a neighborhood of the topological closure of the product domain Π which agrees with α at b. The vanishing of the minors for $df_\zeta(\alpha)$ for almost every $\zeta \in \partial\Delta$ forces the vanishing of $df(\widetilde{\alpha})$ on all of Π from the Cauchy integral formula in terms of non-tangential boundary values of bounded holomorphic functions on the unit disk, and we conclude that $df(\widetilde{\alpha})$ must also satisfy the same algebraic identities, i.e., the vanishing of the minors. The preceding argument can be applied to any product domain $\Pi \subset D(p,q)$ as defined, and we conclude that df transforms any rank-1 vector to a vector of rank $\leq \rho - 1$. An inductive argument allows us to conclude that in fact df transforms rank-1 vectors to either rank-1 vectors or 0. From the properness assumption on f it follows readily that df is injective outside some complex-analytic subvariety of $D(p,q)$. Thus, the map f defines a holomorphic embedding of some domain $U \subset D(p,q)$ which transforms rank-1 vectors to rank-1 vectors. By [(2.1), Lemma 3] either (a) $df(x)$ is complex equivalent to a standard embedding $\tau : D(p,q) \to D(r,s)$ for every $x \in U$, or else (b) the image of f must lie on some projective linear subspace $\mathbb{P}$ of $G(r,s)$. But $\mathbb{P} \cap D(r,s)$ is the complex ball $\mathbb{B}$, and all boundary components in $\partial\mathbb{B}$ must be 0-dimensional. By the preceding restriction argument on product domains $\Pi \subset D(p,q)$ we conclude that f is actually constant, a plain contradiction. Having proved that Alternative (a) holds true, by [(3.2), Theorem 2], $f : D(p,q) \to D(r,s)$ is in fact itself a standard embedding, proving Theorem A in the special case of Type-I domains. $\square$.

(4.2) We have given in the current article a proof of the special case of Theorem A (from Tsai [Ts]) pertaining to Type-I domains of E. Cartan (hence to Grassmannians) in the hope that this can serve as an example relating function theory on bounded symmetric domains to the geometric theory of Fano manifolds (including Hermitian symmetric spaces of the compact type). This involves on the one side harmonic analysis, more specifically integral representations of boundary values of bounded holomorphic functions on the one side, and on the other side the Cartan-Fubini Extension Principle in terms of minimal rational curves as a far-reaching generalization of Theorem B (from Ochiai [Oc]). The link between the two apparently distinct areas of research is through a proof of Ochiai's Theorem by means of holomorphic local differential geometry, where certain generalized conformal structures are studied by means of the geometry of varieties of minimal rational tangent as subvarieties of the projectivized tangent spaces, and where the primary differential projective-geometric invariant utilized in the current article is the projective second fundamental form.

The proof of Theorem A in the special case pertaining to Grassmannians suggests a scheme of research (a) to identify bounded domains Ω which admit natural embeddings into some (quasi-)projective manifolds X on which there exists a notion of minimal rational curves; (b) to study the holomorphic local differential geometry of geometric structures on X and its local complex submanifolds inherited from varieties of minimal rational tangents; (c) to develop harmonic analysis on the bounded domains Ω for the study of proper holomorphic maps by means of geometric structures. Possible examples of bounded domains susceptible to a study by means of geometric structures and minimal rational curves may include certain classes of bounded homogeneous spaces as defined by Pyatetskii-Shapiro [P-S] and the crown domains defined by Akhiezer and Gindikin [AG]. The scope of study of (b) includes rigidity problems on germs of holomorphic maps on rational homogeneous spaces independent of consideration of any bounded domains on them. In this direction Hong-Mok [HoM] is able to make use of geometric structures underlying varieties of minimal rational tangents to prove rigidity results for maps between certain pairs of rational homogeneous manifolds of Picard number 1. On the other hand for the special case of maps between irreducible Hermitian symmetric spaces of the compact type, Hong [Ho] had established stronger rigidity theorems characterizing certain complex submanifolds which do not necessarily come from the image of the map. In the context of [(3.2), Theorem 2] her result says that the complex submanifold $S \subset X$ is necessarily a Grassmann submanifold provided that it inherits a Grassmann structure of rank ≥ 2 without requiring that such a structure comes from the holomorphic image of a Grassmann manifold. This result suggests that for the case where ambient manifolds are irreducible Hermitian symmetric spaces of the compact type, rigidity results for submanifolds are amenable to a proof in two steps, viz., to prove first of all that G-structures inherited from ambient Hermitian symmetric spaces of the compact type are necessarily flat, so that the complex submanifold can be locally identified as the holomorphic image of some irreducible Hermitian symmetric manifold of the compact type, and then to make use of [(3.2), Theorem 2] and its generalizations to prove rigidity results. If this scheme is implementable, then extended to the class of rational homogeneous spaces (for which apparently the method of [Ho] is difficult to apply), one is led to the general question of (d) proving integrability of inherited geometric structures modeled on rational homogeneous spaces of Picard number 1.

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