A NOTE ON POISSON HOMOGENEOUS SPACES

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Abstract. We identify the cotangent bundle Lie algebroid of a Poisson homogeneous space G/H of a Poisson Lie group G as a quotient of a transformation Lie algebroid over G. As applications, we describe the modular vector fields of G/H, and we identify the Poisson cohomology of G/H with coefficients in powers of its canonical line bundle with relative Lie algebra cohomology of the Drinfeld Lie algebra associated to G/H. We also construct a Poisson groupoid over $(G/H, \pi)$ which is symplectic near the identity section. This note serves as preparation for forthcoming papers, in which we will compute explicitly the Poisson cohomology and study their symplectic groupoids for certain examples of Poisson homogeneous spaces related to semi-simple Lie groups.

1. Introduction

The cotangent bundle of a Poisson manifold (P, π) is naturally a Lie algebroid [21] called the cotangent bundle Lie algebroid of (P, π) and denoted by $T^*(P, \pi)$. Let $K_P = \wedge^{\text{top}} T^*P$ be the canonical line bundle over P. Then the Lie algebroid $T^*(P, \pi)$ has a natural representation on K_P. The Poisson cohomology of (P, π) as defined in [13], the Poisson homology of (P, π) as defined in [3], and the twisted Poisson cohomology of (P, π) as defined in [7], can be regarded as the Lie algebroid cohomology of $T^*(P, \pi)$ with coefficients in, respectively, the trivial line bundle, K_P and K_P^2 (see [7, 21, 26]). In general, one can consider the Lie algebroid cohomology of $T^*(P, \pi)$ with coefficients in K_P^N for any integer N, which we will denote by $H^\bullet(P, \pi; K_P^N)$ and refer to as *generalized Poisson cohomology* of (P, π). A symplectic groupoid of (P, π) is a Lie groupoid over P with Lie algebroid $T^*(P, \pi)$ and a compatible symplectic structure [24].

This note concerns the cotangent bundle Lie algebroids of Poisson homogeneous spaces of a Poisson Lie group (G, π_G). More precisely, by a theorem of Drinfeld [6], each Poisson homogeneous space $(G/H, \pi)$ of (G, π_G) corresponds to a Lie subalgebra $\mathfrak{l}$ of the double Lie algebra $\mathfrak{d}$ of (G, π_G). In this note, we identify the cotangent bundle Lie algebroid of $(G/H, \pi)$ with a *quotient* of the transformation Lie algebroid $G \rtimes_\lambda \mathfrak{l}$ over G associated to an infinitesimal action λ of $\mathfrak{l}$ on G. We also identify the representation of $T^*(G/H, \pi)$ on $K_{G/H}$ with a *quotient representation* of $G \rtimes_\lambda \mathfrak{l}$ (see §2.3 for the detail).

We give two applications. First, for any integer N, we identify the generalized Poisson cohomology $H^\bullet\left(G/H, \pi; K_{G/H}^N\right)$ with Lie algebra cohomology of $\mathfrak{l}$ relative to H with coefficients in $C^\infty(G)_N$, the space of smooth functions on G together with an $(\mathfrak{l}, H)$-module structure that depends on N (see Corollary 4.12 for detail). We also discuss the canonical pairing between $H^\bullet\left(G/H, \pi; K_{G/H}^N\right)$ and $H^\bullet\left(G/H, \pi; K_{G/H}^{2-N}\right)$ as a pairing on relative Lie algebra cohomology of $\mathfrak{l}$, and we compute the modular vector fields of $(G/H, \pi)$. The identifications of the Poisson cohomology and homology (i.e., when $N = 0$ and $N = 1$) with relative Lie algebra cohomology of $\mathfrak{l}$ have been established in [15] and [19] but by different methods.

As a second application, we construct a Poisson groupoid Γ over $(G/H, \pi)$ that is symplectic near the identity section, and we give conditions and examples when it is symplectic. The groupoid structure on Γ is a quotient of a transformation groupoid over G (see Mackenzie's book [18] for a general treatment of quotients of groupoids), while the Poisson structure on Γ is obtained by reduction of a quasi-Poisson manifold by an action of a quasi-Poisson Lie group, a theory developed by Alekseev and Kosmann-Schwarzbach in [1]. In the special case when (G, π_G) is complete and when H is a Poisson Lie subgroup of (G, π_G) with π being the projection of π_G to G/H, a symplectic groupoid of $(G/H, \pi)$ was constructed by P. Xu in [25].

There are many examples of Poisson homogeneous spaces associated to semi-simple Lie groups, and they are in general not of the type G/H with H being a Poisson Lie subgroup. See [8, 9, 16] for studies of certain varieties which can serve as moduli spaces of Poisson homogeneous spaces. In forthcoming papers, we will use results from this note to compute explicitly the Poisson cohomology and study their symplectic groupoids for certain examples of Poisson homogeneous spaces treated in [8, 9, 16]. Such examples included flag varieties of complex semi-simple groups [8] and semi-simple Riemannian symmetric spaces [10] (see Example 5.14).

1.1. **Notation.** For a smooth manifold P, the tangent and cotangent bundles of P are denoted by TP and T^*P respectively. For an integer $0 \leq k \leq \dim P$, $\mathcal{V}^k(P)$ and $\Omega^k(P)$ will denote respectively the spaces of smooth k-vector fields and smooth k-forms on P, and

$$\mathcal{V}(P) = \oplus_{k=0}^{\dim P} \mathcal{V}^k(P) \quad \text{and} \quad \Omega(P) = \oplus_{k=0}^{\dim P} \Omega^k(P).$$

If P and Q are smooth manifolds and $F : P \to Q$ is a smooth map, F_* will denote the induced map $TP \to TQ$.

For a vector bundle A over P, $\Gamma(A)$ will denote the space of smooth sections of A. If V is an n-dimensional vector space, $\wedge^{\mathrm{top}} V$ always denotes $\wedge^n V$. Let V^* be the dual space of V. For $x \in \wedge^k V$ and $\xi \in \wedge^j V^*$ with $k \leq j$, $\iota_x \xi \in \wedge^{j-k} V^*$ is defined by $(\iota_x \xi, y) = (\xi, x \wedge y)$ for all $y \in \wedge^{j-k} V$. Unless otherwise specified, all vector spaces are real.

For a Lie group G and $g \in G$, l_g and r_g denote respectively the left and right translation on G by g. The identity element of a group is always denoted by e.

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2. Some basic facts on Lie algebroids

We refer to [17, 18] for details on the facts reviewed in this section.

2.1. **Lie algebroids and Lie algebroid cohomology.** Recall that a Lie algebroid over a manifold P is a vector bundle A over P together with a vector bundle homomorphism $\rho_A : A \to TP$ and a Lie bracket $[\, ,\,]$ on $\Gamma(A)$ such that

1) $[fa_1, a_2] = f[a_1, a_2] - \rho_A(a_2)(f)a_1$ for all $f \in C^\infty(P)$ and $a_1, a_2 \in \Gamma(A)$;

2) $\rho_A[a_1, a_2] = [\rho_A(a_1), \rho_A(a_2)]$ for all $a_1, a_2 \in \Gamma(A)$.
Let A be a Lie algebroid over P. A *representation* of A on a vector bundle E over P is an $\mathbb{R}$-bilinear map $D : \Gamma(A) \times \Gamma(E) \to \Gamma(E) : (a, s) \mapsto D_a s$, such that for any $a, b \in \Gamma(A)$, $s \in \Gamma(E)$, and $f \in C^\infty(P)$,

$$\begin{array}{ll} 1) & D_{fa}s = fD_a s; \\ 2) & D_a(fs) = fD_a s + (\rho(a)f)s; \\ 3) & D_a(D_b s) - D_b(D_a s) = D_{[a,b]}s. \end{array}$$

The *trivial representation* of A is the one on the trivial line bundle $E = P \times \mathbb{R}$ given by $D_a f = \rho(a)(f)$ for $a \in \Gamma(A)$ and $f \in \Gamma(E) \cong C^\infty(P)$. One has the natural notion of tensor products and duals of representations of A. In particular, a representation D of A on a line bundle L gives rise to a representation of A on the N-th power L^N of L for any integer $N \geq 0$. For a negative integer N, we use the natural identification between L^N and$(L^{-N})^*$ and thus have a representation of A on L^N as well.

For a representation D of A on E, and for $k \geq 0$, define

$$\begin{aligned} d_{A,E} :\ \Gamma(\mathrm{Hom}(\wedge^k A, E)) &\longrightarrow \Gamma(\mathrm{Hom}(\wedge^{k+1} A, E)) \\ (d_{A,E}\phi)(a_1, a_2, \cdots, a_{k+1}) &= \sum_{j=1}^{k+1} (-1)^{j+1} D_{a_j} \phi(a_1, \cdots, \hat{a}_j, \cdots, a_{k+1}) \\ &\quad + \sum_{i<j} (-1)^{i+j} \phi([a_i, a_j], \cdots, \hat{a}_i, \cdots \hat{a}_j, \cdots, a_{k+1}) \end{aligned}$$

for $a_1, \ldots, a_{k+1} \in \Gamma(A)$. Then $d_{A,E}^2 = 0$. The cohomology of the cochain complex

$$(\Gamma(\mathrm{Hom}(\wedge A, E)),\ d_{A,E}),$$

which will be denoted by $H^\bullet_{\mathrm{Lie}}(A; E)$, is called the *Lie algebroid cohomology of A with coefficients in E*. When E is the trivial representation, we set $H^\bullet(A; E) = H^\bullet_{\mathrm{Lie}}(A)$.

2.2. **Relative Lie algebra cohomology.** Our reference for this section is [2]. A Lie algebra $\mathfrak{l}$ can be regarded as a Lie algebroid over a one point space, so for every $\mathfrak{l}$-module V, we have the coboundary operators

$$d_{\mathfrak{l},V} :\ \mathrm{Hom}(\wedge^k \mathfrak{l}, V) \longrightarrow \mathrm{Hom}(\wedge^{k+1} \mathfrak{l}, V), \qquad k \geq 0.$$

Let $\mathfrak{h} \subset \mathfrak{l}$ be a Lie subalgebra, H a Lie group with Lie algebra $\mathfrak{h}$, and $H \to \mathrm{Aut}(\mathfrak{l}) : h \mapsto \mathrm{Ad}_h$ a group homomorphism integrating the adjoint action of $\mathfrak{h}$ on $\mathfrak{l}$.

Definition 2.1. An $(\mathfrak{l}, H)$*-module* is a topological vector space V which is both an $\mathfrak{l}$-module and an H-module such that

1) for every $v \in V$, the map $H \to V : h \mapsto hv$ is smooth, and that the restriction to $\mathfrak{h}$ of the action of $\mathfrak{l}$ on V coincides with the one induced from the H-action;
2) for every $v \in V, x \in \mathfrak{l}$, and $h \in H$, $h(x(h^{-1}(v))) = (\mathrm{Ad}_h x)(v)$.

Let V be an $(\mathfrak{l}, H)$-module. For $k \geq 0$, let

$$C^k_{\mathfrak{l},H;V} = \left(\wedge^k (\mathfrak{l}/\mathfrak{h})^* \otimes V\right)^H,$$

where the superscript H denotes the subspace of H-invariants. Identify $(\mathfrak{l}/\mathfrak{h})^* \cong \{\xi \in \mathfrak{l}^* \mid \xi|_{\mathfrak{h}} = 0\} \subset \mathfrak{l}^*$ and regard $C^k_{\mathfrak{l},H;V}$ as in $\wedge^k \mathfrak{l}^* \otimes V \cong \mathrm{Hom}(\wedge^k \mathfrak{l}, V)$. Then

$$\bigoplus_{k\geq 0} C^k_{\mathfrak{l},H;V} \subset \bigoplus_{k\geq 0} \mathrm{Hom}(\wedge^k \mathfrak{l}, V)$$

is invariant under $d_{\mathfrak{l},V}$. The cohomology of the cochain complex $(C^\bullet_{\mathfrak{l},H;V}, d_{\mathfrak{l},V})$, which will be denoted by $H^\bullet_{\mathrm{Lie}}(\mathfrak{l}, H; V)$, is called the *Lie algebra cohomology of* $\mathfrak{l}$ *relative to* H *with coefficients in* V.

Suppose that U and V are two $(\mathfrak{l}, H)$-modules. Then $U \otimes V$ is naturally an $(\mathfrak{l}, H)$-module. For any $0 \leq j, k \leq n = \dim(\mathfrak{l}/\mathfrak{h})$, define

$$C^j_{\mathfrak{l},H;U} \times C^k_{\mathfrak{l},H;V} \longrightarrow C^{j+k}_{\mathfrak{l},H;U\otimes V}: \quad (c_1, c_2) \longmapsto c_1 \otimes c_2 := \phi \wedge \psi \otimes u \otimes v,$$

where $c_1 = \phi \otimes u, c_2 = \psi \otimes v$ with $\phi \in \wedge^j(\mathfrak{l}/\mathfrak{h})^*$, $\psi \in \wedge^k(\mathfrak{l}/\mathfrak{h})^*$, $u \in U$, and $v \in V$. It is easy to check that

$$d_{\mathfrak{l},U\otimes V}(c_1 \otimes c_2) = d_{\mathfrak{l},U}(c_1) \otimes c_2 + (-1)^j c_1 \otimes d_{\mathfrak{l},V}(c_2) \tag{2.1}$$

if $c_1 \in C^j_{\mathfrak{l},H;U}$. Assume that $\nu \in \left(C^n_{\mathfrak{l},H;U\otimes V}\right)^*$ is such that

$$\nu\left(d_{\mathfrak{l},U\otimes V}(C^{n-1}_{\mathfrak{l},H;U\otimes V})\right) = 0. \tag{2.2}$$

For $0 \leq k \leq n$, define the pairing $(\, , \,)_\nu$ between $C^k_{\mathfrak{l},H;U}$ and $C^{n-k}_{\mathfrak{l},H;V}$ by

$$(c_1, \ c_2)_\nu = \nu(c_1 \otimes c_2).$$

It follows from (2.1) that

$$(d_{\mathfrak{l},U}(c_1), \ c_2)_\nu + (-1)^{k-1}(c_1, \ d_{\mathfrak{l},V}(c_2))_\nu = 0$$

for all $c_1 \in C^{k-1}_{\mathfrak{l},H;U}$ and $c_2 \in C^{n-k}_{\mathfrak{l},H;V}$. Thus $(\, , \,)_\nu$ induces a well-defined pairing, still denoted by $(\, , \,)_\nu$, between $H^k_{\mathrm{Lie}}(\mathfrak{l}, H; U)$ and $H^{n-k}_{\mathrm{Lie}}(\mathfrak{l}, H; V)$ for every $0 \leq k \leq n$.

2.3. **Quotients of transformation Lie algebroids.** Let again $\mathfrak{l}$ be a Lie algebra, $\mathfrak{h} \subset \mathfrak{l}$ a Lie subalgebra, H a Lie group with Lie algebra $\mathfrak{h}$, and $H \to \mathrm{Aut}(\mathfrak{l}) : h \to \mathrm{Ad}_h$ a group homomorphism integrating the adjoint action of $\mathfrak{h}$ on $\mathfrak{l}$.

Definition 2.2. An $(\mathfrak{l}, H)$-*space* is a smooth manifold M together with a Lie algebra homomorphism $\lambda : \mathfrak{l} \to \mathcal{V}^1(M)$ and a right action of H on M such that

1) the restriction of λ on $\mathfrak{h}$ coincides with the infinitesimal action of $\mathfrak{h}$ on M induced by the right H-action, and

2) for all $m \in M, x \in \mathfrak{l}$ and $h \in H$, $\lambda_x(mh) = h_*\lambda_{\mathrm{Ad}_h x}(m)$, where h_* is the differential of the map $h : M \to M : m_1 \mapsto m_1 h$ for $m_1 \in M$.

We will sometimes denote an $(\mathfrak{l}, H)$-space by the pair (M, λ) without explicitly mentioning the action of H on M.

Let (M, λ) be an $(\mathfrak{l}, H)$-space. Using the action λ of $\mathfrak{l}$ on M, one can form the *transformation Lie algebroid* $M \rtimes_\lambda \mathfrak{l}$ over M, which is the trivial vector bundle $M \times \mathfrak{l}$ over M with the anchor map

$$M \times \mathfrak{l} \longrightarrow TM : \ (m, x) \longmapsto \lambda_x(m), \ m \in M, x \in \mathfrak{l},$$

and the Lie bracket $[\, , \,]_{M\rtimes_\lambda \mathfrak{l}}$ on $\Gamma(M \times \mathfrak{l}) \cong C^\infty(M, \mathfrak{l})$ determined by

$$[\bar{x}_1, \bar{x}_2]_{M\rtimes_\lambda \mathfrak{l}} = \overline{[x_1, x_2]},$$

where for $x \in \mathfrak{l}$, $\bar{x}$ is the constant function on M with value x.

Assume in addition that the H-action on M is free and proper so that the quotient M/H is a smooth manifold. Consider the associated vector bundle $A = M \times_H (\mathfrak{l}/\mathfrak{h})$ over M/H, where $h \in H$ acts on $\mathfrak{l}/\mathfrak{h}$ by Ad_h. Points in A will be denoted by $[m, x+\mathfrak{h}]$, where $m \in M$ and $x \in \mathfrak{l}$. Note that

$$\rho_A: \ A \longrightarrow T(M/H): \ \ [m, \ x+\mathfrak{h}] \longmapsto q_*\lambda_x(m)$$

is a well-defined bundle map, where $q: M \to M/H$ is the natural projection, and

$$\begin{aligned} \Gamma(A) &= C^\infty(M, \mathfrak{l}/\mathfrak{h})^H \\ &= \{a \in C^\infty(M, \mathfrak{l}/\mathfrak{h}) \mid a(mh) = \mathrm{Ad}_{h^{-1}} a(m), \ \forall m \in M, h \in H\}. \end{aligned}$$

Let

$$\Gamma(M \rtimes_\lambda \mathfrak{l})^H = \{a \in C^\infty(M, \mathfrak{l}) \mid a(mh) = \mathrm{Ad}_{h^{-1}} a(m), \ \forall m \in M, h \in H\}.$$

For $a_1, a_2 \in \Gamma(A)$, let $\tilde{a}_1, \tilde{a}_2 \in \Gamma(M \rtimes_\lambda \mathfrak{l})^H$ be such that $\mathfrak{p}(\tilde{a}_1) = a_1$ and $\mathfrak{p}(\tilde{a}_2) = a_2$, where $\mathfrak{p}: M \rtimes_\lambda \mathfrak{l} \to A: (m, x) \mapsto [m, x+\mathfrak{h}]$ is the natural vector bundle projection. Define $[a_1, a_2] \in \Gamma(A)$ by

$$[a_1, a_2] = \mathfrak{p}([\tilde{a}_1, \tilde{a}_2]_{M \rtimes_\lambda \mathfrak{l}}). \tag{2.3}$$

The proof of the following lemma is omitted since it is straightforward.

Lemma 2.3. *Formula* (2.3) *is a well-defined Lie bracket on* $\Gamma(A)$. *With the Lie bracket in* (2.3) *on* $\Gamma(A)$ *and* ρ_A *as the anchor map,* A *is a Lie algebroid over* M/H. *Moreover, the bundle map* $\mathfrak{p}: M \rtimes_\lambda \mathfrak{l} \to A$ *is a Lie algebroid morphism.*

Definition 2.4. The Lie algebroid A in Lemma 2.3 is called the *H-quotient* of the transformation Lie algebroid $M \rtimes_\lambda \mathfrak{l}$ and will be denoted by $M \rtimes_{\lambda,H} (\mathfrak{l}/\mathfrak{h})$.

Example 2.5. If G is a Lie group and $H \subset G$ a closed subgroup, the tangent bundle Lie algebroid $T(G/H)$ is a quotient by H of the tangent bundle Lie algebroid TG. A more general discussion on quotients of Lie algebroid can be found in [18, Chap. 4].

We now turn to a special class of representations of $M \rtimes_{\lambda,H} (\mathfrak{l}/\mathfrak{h})$ that arise from representations of $M \rtimes_\lambda \mathfrak{l}$.

Definition 2.6. An $(\mathfrak{l}, H)$*-vector bundle* is an H-equivariant vector bundle E over an $(\mathfrak{l}, H)$-space (M, λ) together with a representation of $\mathfrak{l}$ on $\Gamma(E)$ such that

1) $x \cdot (fs) = \lambda_x(f)s + f(x \cdot s)$ for all $x \in \mathfrak{l}, f \in C^\infty(M)$, and $s \in \Gamma(E)$;

2) the $\mathfrak{l}$-action and the H-action on $\Gamma(E)$ induced from the H-action on E make $\Gamma(E)$ into an $(\mathfrak{l}, H)$-module (see Definition 2.1).

Let E be an $(\mathfrak{l}, H)$-vector bundle over M such that the H-action on M is free and proper. One then has the representation $\widetilde{D}$ of $M \rtimes_\lambda \mathfrak{l}$ on E given by

$$(\widetilde{D}_b s)(m) = (b(m) \cdot s)(m), \quad b \in \Gamma(M \rtimes_\lambda \mathfrak{l}) = C^\infty(M, \mathfrak{l}), \ m \in M, s \in \Gamma(E).$$

Let E/H be the quotient bundle over M/H with $\Gamma(E/H) = \Gamma(E)^H$, the space of H-invariant smooth sections of E. For $a \in \Gamma(A)$, let $\tilde{a} \in \Gamma(M \rtimes_\lambda \mathfrak{l})^H$ be such that $\mathfrak{p}(\tilde{a}) = a$. It is easy to see that $\widetilde{D}_{\tilde{a}} s \in \Gamma(E)$ is H-invariant for any $s \in \Gamma(E/H) \cong \Gamma(E)^H$, so we can regard $\widetilde{D}_{\tilde{a}} s$ as in $\Gamma(E/H)$. Define

$$D_a s = \widetilde{D}_{\tilde{a}} s, \quad s \in \Gamma(E/H) \cong \Gamma(E)^H. \tag{2.4}$$

The proof of the following Lemma 2.7 is straightforward.

Lemma 2.7. *Formula* (2.4) *is a well-defined representation of the quotient Lie algebroid* $M \rtimes_{\lambda,H} (\mathfrak{l}/\mathfrak{h})$ *on* E/H*, and we call it the* H*-quotient of the representation* $\widetilde{D}$ *of* $M \rtimes_\lambda \mathfrak{l}$ *on* E*.*

Lemma 2.8. *The Lie algebroid cohomology of* $A = M \rtimes_{\lambda,H} (\mathfrak{l}/\mathfrak{h})$ *with coefficient in* E/H *is isomorphic to the Lie algebra cohomology of* $\mathfrak{l}$ *relative to* H *with coefficients in* $\Gamma(E)$*, i.e.,*

$$H^k_{\rm Lie}(A; E/H) \;\cong\; H^k_{\rm Lie}(\mathfrak{l}, H; \Gamma(E)), \qquad \forall k \geq 0.$$

Proof. Let $\mathcal{T}$ be the trivial vector bundle over M with fiber $\mathfrak{l}/\mathfrak{h}$. Then for every $k \geq 0$, the vector bundle ${\rm Hom}(\wedge^k A, E/H)$ over M/H is the quotient by H of the H-equivariant vector bundle ${\rm Hom}(\wedge^k \mathcal{T}, E)$, so

$$\Gamma({\rm Hom}(\wedge^k A, E/H)) \cong \Big(\wedge^k (\mathfrak{l}/\mathfrak{h})^* \otimes \Gamma(E)\Big)^H \cong C^k_{\mathfrak{l},H;\Gamma(E)}. \tag{2.5}$$

By following the definitions of the Lie algebroid structure on A and the representation of A on E/H, it is straightforward to check that the identifications in (2.5) give an isomorphism of cochains

$$\left(\bigoplus_{k\geq 0} \Gamma({\rm Hom}(\wedge^k A, E/H)),\; d_{A,E/H}\right) \longrightarrow \left(\bigoplus_{k \geq 0} C^k_{\mathfrak{l},H;\Gamma(E)},\; d_{\mathfrak{l},H}\right).$$

□

Remark 2.9. Suppose that F is an $(\mathfrak{l}, H)$-line bundle over an $(\mathfrak{l}, H)$-space (M, λ) and that E is an H-equivariant square root of F, i.e., $E^2 \cong F$. Then E is naturally an $(\mathfrak{l}, H)$-line bundle with the $\mathfrak{l}$-action on $\Gamma(E)$ uniquely defined as follows: if t is a nowhere vanishing local section of E, then $x \cdot t = \frac{1}{2}\frac{x \cdot t^2}{t}$ for any $x \in \mathfrak{l}$ (see [7]). Consequently, one has the quotient representation of the quotient Lie algebroid $A = M \rtimes_{\lambda,H} (\mathfrak{l}/\mathfrak{h})$ on E/H.

3. Poisson cohomology and modular vector fields

3.1. **The cotangent bundle Lie algebroid and Poisson cohomology.** The cotangent bundle Lie algebroid of a Poisson manifold (P, π), denoted by $T^*(P, \pi)$, is the cotangent bundle T^*P of P with the anchor map

$$\tilde{\pi}: \quad T^*P \longrightarrow TP: \quad \tilde{\pi}(\alpha)(\beta) = \pi(\alpha, \beta), \qquad \alpha, \beta \in \Omega^1(P),$$

and the Lie bracket $\{\, ,\, \}_\pi$ on $\Omega^1(P)$ given by

$$\{\alpha, \beta\}_\pi = d(\pi(\alpha, \beta)) + \iota_{\tilde{\pi}(\alpha)} d\beta - \iota_{\tilde{\pi}(\beta)} d\alpha, \qquad \alpha, \beta \in \Omega^1(P). \tag{3.1}$$

Let

$$K_P = \wedge^{\rm top} T^* P$$

be the canonical line bundle over P. It is shown in [7, 26] that there is a representation of the Lie algebroid $T^*(P, \pi)$ on K_P given by

$$D_\alpha \mu = L_{\tilde{\pi}(\alpha)} \mu + (\pi, d\alpha)\mu = \{\alpha, \mu\}_\pi - (\pi, d\alpha)\mu = \alpha \wedge d(i_\pi \mu), \;\; \mu \in \Omega^{\rm top}(P), \tag{3.2}$$

where $\{\, ,\, \}_\pi$ is the Schouten bracket on the space $\Omega(P)$ induced from the bracket in (3.1) on $\Omega^1(P)$.

Definition 3.1. The representation of $T^*(P, \pi)$ on K_P is called the *canonical representation* of $T^*(P, \pi)$ on K_P.

For any integer N, let K_P^N be the N-th power of K_P, equipped with the natural extension of the representation of $T^*(P,\pi)$. When N is negative, we will understand K_P^N as $(K_P^{-N})^*$.

Definition 3.2. For a Poisson manifold (P,π) and for any integer N, we define the *Poisson cohomology of* (P,π) *with coefficients* K_P^N to be the Lie algebroid cohomology of $T^*(P,\pi)$ with coefficients in K_P^N, and we denote it by $H^\bullet(P,\pi;K_P^N)$. When $N=0$, we simply write $H^\bullet(P,\pi;K_P^N)$ as $H^\bullet(P,\pi)$. The totality of $H^\bullet(P,\pi;K_P^N)$ for all integers N is called the *generalized Poisson cohomology* of (P,π).

Remark 3.3. The Poisson cohomology of (P,π) defined in [13] is $H^\bullet(P,\pi)$. It is shown in [7, 26] that the Poisson homology of (P,π) defined in [3] is isomorphic to $H^\bullet(P,\pi;K_P)$. In [7], the cohomology $H^\bullet(P,\pi;K_P^2)$ is called the twisted Poisson cohomology of (P,π).

3.2. **The canonical pairing on Poisson cohomology.** Suppose that P is compact and oriented. For $0\le k\le n=\dim P$ and an integer N, set

$$C^k_{P,N}=\Gamma(\mathrm{Hom}(\wedge^k T^*P,\ K_P^N))\ \cong\ \Gamma(\wedge^k TP\otimes K_P^N).$$

The natural identifications of bundles

$$\wedge^k TP\otimes\wedge^{n-k}TP\cong\wedge^n TP,\qquad K_P^N\otimes K_P^{2-N}\cong K_P^2,\qquad \wedge^n TP\otimes K_P^2\cong K_P$$

give rise to an identification

$$J:\ \left(\wedge^k TP\otimes K_P^N\right)\otimes\left(\wedge^{n-k}TP\otimes K_P^{2-N}\right)\longrightarrow K_P$$

and thus an $\mathbb{R}$-bilinear pairing

$$(c_1,\ c_2):=\int_P J(c_1,c_2),\qquad c_1\in C^k_{P,N},\ c_2\in C^{n-k}_{P,2-N}.$$

A proof similar to that of Theorem 5.1 of [7] shows that (,) induces a well-defined pairing between $H^k(P,\pi;K_P^N)$ and $H^{n-k}(P,\pi;K_P^{2-N})$. We will refer to (,) the *canonical pairing* on the generalized Poisson cohomology of (P,π).

3.3. **Modular vector fields.** Let (P,π) be an orientable Poisson manifold, and let μ be a volume form of P. The *modular vector field of* π *with respect to* μ (see [23]) is defined to be the vector field θ_μ on P such that

$$D_\alpha\mu=(\theta_\mu,\alpha)\mu,\quad\forall\alpha\in\Omega^1(P),$$

where $D_\alpha\mu\in\Omega^{\mathrm{top}}(P)$ is given in (3.2). For an integer N, set $d_N=d_{T^*P,K_P^N}\in\mathrm{End}(C^\bullet_{P,N})$.

Proposition 3.4. *Let N be any integer. For any volume form μ, the action of the modular vector field θ_μ on $C^\bullet_{P,N}=\oplus_{k\ge0}C^k_{P,N}$ by Lie derivative commutes with the operator d_N. When $N\neq1$, the induced action of θ_μ on $H^\bullet(P,\pi;K_P^N)$ is trivial.*

Proof. Consider the identification

$$\mathcal{I}:\quad\mathcal{V}^k(P)\longrightarrow C^k_{P,N}:\quad V\longmapsto V\otimes\mu^N.$$

Since $L_{\theta_\mu}\mu=0$, $L_{\theta_\mu}\circ\mathcal{I}=\mathcal{I}\circ L_{\theta_\mu}$. It is also easy to show (see [7, Lemma 4.4]) that the operator $\delta_N:=\mathcal{I}^{-1}\circ d_N\circ\mathcal{I}$ is given by

$$\delta_N:\ \mathcal{V}^k(P)\longrightarrow\mathcal{V}^{k+1}(P):\quad V\longmapsto[\pi,V]+N\theta_\mu\wedge V.$$

Since $L_{\theta_\mu}\pi = 0$, it is clear that L_{θ_μ} commutes with δ_N. Consider the operator

$$b_\mu : \ \mathcal{V}^k(P) \longrightarrow \mathcal{V}^{k-1}(P) : \ \iota_{b_\mu V}\mu = (-1)^k d(\iota_V \mu).$$

It is easy to see that $b_\mu^2 = 0$ and that $\theta_\mu = b_\mu \pi$. Moreover, for $V_1 \in \mathcal{V}^k(P)$ and $V_2 \in \mathcal{V}(P)$,

$$\begin{aligned} b_\mu(V_1 \wedge V_2) &= b_\mu(V_1) \wedge V_2 + (-1)^k V_1 \wedge b_\mu(V_2) + (-1)^k [V_1, V_2] \\ b_\mu[V_1, V_2] &= [b_\mu(V_1), V_2] + (-1)^{k-1}[V_1, b_\mu(V_2)] \end{aligned}$$

It follows that $b_\mu \delta_N + \delta_N b_\mu = (1-N)L_{\theta_\mu}$. Thus θ_μ acts trivially on $H^\bullet(P, \pi; K_P^N)$ when $N \neq 1$. □

4. The cotangent bundle Lie algebroids and generalized Poisson cohomology of Poisson homogeneous spaces

4.1. **Review on Poisson Lie groups.** Recall that [5, 11, 14, 20] a *Poisson Lie group* is a Lie group G with a Poisson structure π_G such that the group multiplication map $(G, \pi_G) \times (G, \pi_G) \to (G, \pi_G) : \ (g, h) \mapsto gh$ is Poisson. Let (G, π_G) be a Poisson Lie group. Then π_G necessarily vanishes at the identity element e of G. Let $\delta : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ be the linearization of π_G at e. Then the dual map

$$\delta^* : \ \wedge^2 \mathfrak{g}^* \longrightarrow \mathfrak{g}^* : \ \ \xi \wedge \eta \longmapsto [\xi, \eta]$$

of δ defines a Lie bracket on $\mathfrak{g}^*$, and the pair $(\mathfrak{g}, \delta)$ becomes a *Lie bialgebra* [5]. For $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, define $\mathrm{ad}_x^* \xi \in \mathfrak{g}^*$ and $\mathrm{ad}_\xi^* x \in \mathfrak{g}$ by

$$(ad_x^* \xi, \, y) = (\xi, \, [y, x]) \quad \text{ and } \quad (\mathrm{ad}_\xi^* x, \, \eta) = (x, \, [\eta, \xi]), \quad \text{ where } \ y \in \mathfrak{g}, \eta \in \mathfrak{g}^*.$$

Let $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$. Then the bracket on $\mathfrak{d}$ given by

$$[x + \xi, \, y + \eta] = [x, y] + \mathrm{ad}_\xi^* y - \mathrm{ad}_\eta^* x + [\xi, \eta] + \mathrm{ad}_x^* \eta - \mathrm{ad}_y^* \xi, \quad x, y \in \mathfrak{g}, \xi, \eta \in \mathfrak{g}^*,$$

is a Lie bracket, and the bilinear form $\langle \, , \, \rangle$ on $\mathfrak{d}$ given by

$$\langle x + \xi, \ y + \eta \rangle = (x, \eta) + (y, \xi), \quad x, y \in \mathfrak{g}, \ \xi, \eta \in \mathfrak{g}^*,$$

is ad-invariant with respect to $[\, , \,]$. The pair $(\mathfrak{d}, \langle \, , \, \rangle)$ is called the double of the Lie bialgebra $(\mathfrak{g}, \delta)$. The adjoint action of $\mathfrak{g}$ on $\mathfrak{d}$ integrates to an action of G on $\mathfrak{d}$, still denoted by $\mathrm{Ad}_g : \mathfrak{d} \to \mathfrak{d}$ for $g \in G$, which is given by [6]

$$\mathrm{Ad}_g(x + \xi) =: \mathrm{Ad}_g x + \iota_{\mathrm{Ad}^*_{g^{-1}}\xi}(r_{g^{-1}} \pi_G(g)) + \mathrm{Ad}^*_{g^{-1}} \xi, \tag{4.1}$$

where $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ and $\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^*$ are the adjoint and co-adjoint actions of $g \in G$ on $\mathfrak{g}$ and on $\mathfrak{g}^*$ respectively. A subspace $\mathfrak{l}$ of $\mathfrak{d}$ is said to be Lagrangian if $\langle x, y \rangle = 0$ for all $x, y \in \mathfrak{l}$ and if $\dim \mathfrak{l} = \dim \mathfrak{g}$.

4.2. **Drinfeld Lagrangian subalgebras.** Let H be a closed subgroup of G.

Definition 4.1. [6] A (G, π_G)-homogeneous Poisson structure on G/H is a bivector field π on G/H such that 1) π is Poisson, and 2) the map

$$\sigma : \ (G, \pi_G) \times (G/H, \pi) \longrightarrow (G/H, \pi) : \ \ (g_1, g_2 H) \longmapsto g_1 g_2 H \tag{4.2}$$

is Poisson.

By definition, the map σ in (4.2) is Poisson if and only if

$$\pi(gH) = (\sigma_g)_*\pi(eH) + q_*\pi_G(g), \quad \forall g \in G, \tag{4.3}$$

where $q : G \to G/H$ is the projection, and for $g \in G$, $\sigma_g : G/H \to G/H$ is defined by $g_1H \to gg_1H$ for $g_1 \in G$. Thus, π is uniquely determined by $\pi(eH) \in \wedge^2 T_{eH}(G/H)$, and Conditions 1) and 2) on π in Definition 4.1 become the following two conditions on $\pi(eH) \in \wedge^2 T_{eH}(G/H)$:

(i) $\pi(eH) = (\sigma_h)_*\pi(eH) + q_*\pi_G(h)$ for all $h \in H$ (so that π given by (4.3) is well-defined); and

(ii) the bi-vector field π on G/H determined by $\pi(eH)$ via (4.3) is Poisson.

Let $\mathfrak{h}$ be the Lie algebra of H. Simple linear algebra arguments show that there is a one to one correspondence between $\wedge^2(\mathfrak{g}/\mathfrak{h})$ and the set of Lagrangian subspaces $\mathfrak{l}$ of $\mathfrak{d}$ such that $\mathfrak{l} \cap \mathfrak{g} = \mathfrak{h}$. The explicit correspondence is given by

$$\wedge^2\left(\mathfrak{g}/\mathfrak{h}\right) \ni r \mapsto \mathfrak{l}_r := \{x + \xi \mid x \in \mathfrak{g}, \xi \in \mathfrak{g}^*, \xi|_{\mathfrak{h}} = 0, \iota_\xi r = x + \mathfrak{h}\}. \tag{4.4}$$

Identify $T_{eH}(G/H) \cong \mathfrak{g}/\mathfrak{h}$. Then an element $\pi(eH) \in \wedge^2 T_{eH}(G/H) \cong \wedge^2(\mathfrak{g}/\mathfrak{h})$ corresponds to the Lagrangian subspace $\mathfrak{l}_{\pi(eH)}$ of $\mathfrak{d}$. Drinfeld showed [6] that Conditions (i) and (ii) on $\pi(eH) \in \wedge^2 T_{eH}(G/H)$ are respectively equivalent to

(a) $\mathrm{Ad}_h \mathfrak{l}_{\pi(eH)} = \mathfrak{l}_{\pi(eH)}$ for all $h \in H$, where $\mathrm{Ad}_h : \mathfrak{d} \to \mathfrak{d}$ is given in (4.1), and

(b) $\mathfrak{l}_{\pi(eH)}$ is a Lie subalgebra of $\mathfrak{d}$.

Definition 4.2. When $(G/H, \pi)$ is a Poisson homogeneous space of (G, π_G), the Lie subalgebra $\mathfrak{l}_{\pi(eH)}$ of $\mathfrak{d}$ is called the Drinfeld Lagrangian subalgebra associated to $\pi(eH)$.

Let $(G/H, \pi)$ be a Poisson homogeneous space of (G, π_G). Let q also denote the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Let $\Lambda \in \wedge^2\mathfrak{g}$ be any element such that

$$q(\Lambda) = \pi(eH) \in \wedge^2 T_{eH}(G/H) \cong \wedge^2(\mathfrak{g}/\mathfrak{h}). \tag{4.5}$$

The following Lemma 4.3 is straightforward to prove [4].

Lemma 4.3. *Conditions (i) and (ii) on $\pi(eH)$ are equivalent to*

1) $\mathrm{Ad}_h\Lambda - \Lambda + (r_{h^{-1}})_*\pi_G(h) \in \mathfrak{h} \wedge \mathfrak{g}$ *for all $h \in H$;*

2) $[\Lambda, \Lambda] + 2\delta(\Lambda) \in \mathfrak{h} \wedge \mathfrak{g} \wedge \mathfrak{g}$,

where $[\,,\,]$ is the Schouten bracket on $\wedge\mathfrak{g}$ and $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ is the linearization of π_G at e as well as its extension $\delta : \wedge^2\mathfrak{g} \to \wedge^3\mathfrak{g}$ given by

$$\delta(x \wedge y \wedge z) = \delta(x) \wedge y \wedge z - x \wedge \delta(y) \wedge z + x \wedge y \wedge \delta(z), \quad x, y, z \in \mathfrak{g}.$$

For $\Lambda \in \wedge^2\mathfrak{g}$ as in (4.5), define the bi-vector field π_Λ on G by

$$\pi_\Lambda = \Lambda^l + \pi_G, \tag{4.6}$$

where Λ^l is the left invariant bi-vector field on G with value Λ at e. Condition 1) on Λ in Lemma 4.3 implies that $q_*\pi_\Lambda$ is a well-defined bi-vector field on G/H. In fact,

$$q_*\pi_\Lambda = \pi.$$

Let $\Lambda\xi = \iota_\xi\Lambda$ for $\xi \in \mathfrak{g}^*$. The Drinfeld Lagrangian subalgebra $\mathfrak{l}_{\pi(eH)}$ is also given by

$$\mathfrak{l}_{\pi(eH)} = \{x + \Lambda\xi + \xi \mid x \in \mathfrak{h}, \xi \in \mathfrak{g}^*, \xi|_{\mathfrak{h}} = 0\}. \tag{4.7}$$

Remark 4.4. Although the bi-vector field π_Λ on G is not necessarily Poisson, we can still define the skew-symmetric bracket $\{\, ,\, \}_{\pi_\Lambda}$ on $\Omega^1(G)$ by replacing π by π_Λ in (3.1). Moreover, the space of left invariant 1-forms on G is invariant under $\{\, ,\, \}_{\pi_\Lambda}$. In fact, it is easy to show that

$$\{\xi^l, \eta^l\}_{\pi_\Lambda} = ([\xi, \eta]_\Lambda)^l, \qquad \xi, \eta \in \mathfrak{l},$$

where for $\zeta \in \mathfrak{g}^*$, ζ^l is the left invariant 1-form on G with value ζ at e, and

$$[\xi, \eta]_\Lambda \stackrel{\text{def}}{=} [\xi, \eta] + \mathrm{ad}^*_{\Lambda\xi}\,\eta - \mathrm{ad}^*_{\Lambda\eta}\,\xi, \qquad \xi, \eta \in \mathfrak{g}^*. \tag{4.8}$$

Lemma 4.5. *Let $\mathfrak{h}^0 = \{\xi \in \mathfrak{g}^* \mid \xi|_{\mathfrak{h}} = 0\}$. Then $[\xi, \eta]_\Lambda \in \mathfrak{h}^0$ for all $\xi, \eta \in \mathfrak{h}^0$.*

Proof. The condition $\mathrm{Ad}_h \mathfrak{l}_{\pi(eH)} = \mathfrak{l}_{\pi(eH)}$ for all $h \in H$ implies that $[x, \mathfrak{l}_{\pi(eH)}] \subset \mathfrak{l}_{\pi(eH)}$ for all $x \in \mathfrak{h}$, so $[x, \Lambda\xi + \xi] \in \mathfrak{l}_{\pi(eH)}$ for all $x \in \mathfrak{h}$ and $\xi \in \mathfrak{h}^0$, from which it follows that $[\xi, \eta]_\Lambda \in \mathfrak{h}^0$ for all $\xi, \eta \in \mathfrak{h}^0$. See also [4]. □

Let $\chi_{\mathfrak{h}^0,\Lambda} \in (\mathfrak{h}^0)^*$ be defined by

$$\chi_{\mathfrak{h}^0,\Lambda}(\xi) = \mathrm{tr}(T_\xi), \qquad \xi \in \mathfrak{h}^0,$$

where $T_\xi \in \mathrm{End}(\mathfrak{h}^0) : \eta \mapsto [\xi, \eta]_\Lambda$ for $\xi, \eta \in \mathfrak{h}^0$. Let $\chi_{\mathfrak{l}} \in \mathfrak{l}^*, \chi_{\mathfrak{g}} \in \mathfrak{g}^*$, and $\chi_{\mathfrak{g}^*} \in \mathfrak{g}$ be the adjoint characters of $\mathfrak{l}, \mathfrak{g}$ and $\mathfrak{g}^*$ respectively. Let $b\Lambda = \sum_i [x_i, y_i] \in \mathfrak{g}$ if $\Lambda = \sum_i x_i \wedge y_i$. We now prove a fact that will be used in the proof of Theorem 4.7.

Lemma 4.6. *For every $\xi \in \mathfrak{h}^0$,*

$$\chi_{\mathfrak{h}^0,\Lambda}(\xi) + (b\Lambda,\ \xi) = \frac{1}{2}\left(\chi_{\mathfrak{l}}(\Lambda\xi + \xi) - \chi_{\mathfrak{g}}(\Lambda\xi) + \chi_{\mathfrak{g}^*}(\xi)\right). \tag{4.9}$$

Proof. For $\xi \in \mathfrak{g}^*$, consider the operator $T_\xi \in \mathrm{End}(\mathfrak{g}^*) : T_\xi(\eta) = [\xi,\ \eta]_\Lambda$, and define $\chi_{\mathfrak{g}^*,\Lambda}(\xi) = \mathrm{tr}(T_\xi \in \mathrm{End}(\mathfrak{g}^*))$. It is easy to see that

$$\chi_{\mathfrak{g}^*,\Lambda}(\xi) = \chi_{\mathfrak{g}^*}(\xi) - \chi_{\mathfrak{g}}(\Lambda\xi) - 2(b\Lambda, \xi), \qquad \xi \in \mathfrak{g}^*.$$

For $\xi \in \mathfrak{h}^0$, since $T_\xi(\mathfrak{h}^0) \subset \mathfrak{h}^0$, we have an induced map $T_\xi \in \mathrm{End}(\mathfrak{g}^*/\mathfrak{h}^0)$. Define $\chi(\xi) = \mathrm{tr}(T_\xi \in \mathrm{End}(\mathfrak{g}^*/\mathfrak{h}^0))$ for $\xi \in \mathfrak{h}^0$. Then

$$\chi_{\mathfrak{h}^0,\Lambda}(\xi) = \chi_{\mathfrak{g}^*,\Lambda}(\xi) - \chi(\xi) = \chi_{\mathfrak{g}^*}(\xi) - \chi_{\mathfrak{g}}(\Lambda\xi) - 2(b\Lambda, \xi) - \chi(\xi) \tag{4.10}$$

for all $\xi \in \mathfrak{h}^0$. On the other hand, consider the embedding $\kappa : \mathfrak{h}^0 \hookrightarrow \mathfrak{l}$ by $\xi \mapsto \Lambda\xi + \xi$, and let $\mathfrak{p}_{\mathfrak{h}} : \mathfrak{l} \to \mathfrak{h}$ be the projection with respect to the decomposition $\mathfrak{l} = \mathfrak{h} + \kappa(\mathfrak{h}^0)$. For $\xi \in \mathfrak{h}^0$, let $S_\xi \in \mathrm{End}(\mathfrak{h})$ be the operator $S_\xi(x) = \mathfrak{p}_{\mathfrak{h}}[\Lambda\xi + \xi,\ x]$ for $x \in \mathfrak{h}$. Then

$$\chi_{\mathfrak{h}^0,\Lambda}(\xi) = \chi_{\mathfrak{l}}(\Lambda\xi + \xi) - \mathrm{tr}(S_\xi \in \mathrm{End}(\mathfrak{h})), \qquad \forall \xi \in \mathfrak{h}^0.$$

By identifying $\mathfrak{g}^*/\mathfrak{h}^0 \cong \mathfrak{h}^*$, one can show that $-S^*_\xi = T_\xi \in \mathrm{End}(\mathfrak{g}^*/\mathfrak{h}^0)$, and so $\mathrm{tr}(S_\xi \in \mathrm{End}(\mathfrak{h})) = -\chi(\xi)$ for all $\xi \in \mathfrak{h}^0$. Thus

$$\chi_{\mathfrak{h}^0,\Lambda}(\xi) = \chi_{\mathfrak{l}}(\Lambda\xi + \xi) + \chi(\xi), \qquad \forall \xi \in \mathfrak{h}^0. \tag{4.11}$$

Adding (4.10) and (4.11), we get (4.9). □

4.3. **The cotangent bundle Lie algebroid of** $(G/H, \pi)$**.** Let (G, π_G) be a Poisson Lie group. For $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, let x^l (resp. ξ^l) be the left invariant vector field (resp. 1-form) on G with value x (resp. ξ) at e. Then [15] the map

$$\lambda : \ \mathfrak{d} \longrightarrow \mathcal{V}^1(G) : \ x + \xi \longmapsto \lambda_{x+\xi} := x^l + \tilde{\pi}_G(\xi^l) \tag{4.12}$$

is a Lie algebra homomorphism from $\mathfrak{d}$ to the space $\mathcal{V}^1(G)$ of vector fields on G. Let $\mathfrak{p}_{\mathfrak{g}} : \mathfrak{d} \to \mathfrak{g}$ be the projection along $\mathfrak{g}^*$. By (4.1), we also have

$$\lambda_{x+\xi}(g) = (r_g)_* p_{\mathfrak{g}} \mathrm{Ad}_g(x + \xi), \qquad g \in G,\, x \in \mathfrak{g},\, \xi \in \mathfrak{g}^*. \tag{4.13}$$

Let now $(G/H, \pi)$ be a (G, π_G)-homogeneous Poisson space, and let $\mathfrak{l} = \mathfrak{l}_{\pi(eH)}$ be the Drinfeld Lagrangian subalgebra of $\mathfrak{d}$ as in Definition 4.2. Then G, with the right action of H by right translations and the infinitesimal action of $\mathfrak{l}$ by λ, becomes an $(\mathfrak{l}, H)$-space in the sense of Definition 2.2. Let $G \rtimes_\lambda \mathfrak{l}$ be the corresponding transformation Lie algebroid over G.

Theorem 4.7. *The cotangent bundle Lie algebroid of* $(G/H, \pi)$ *is isomorphic to the* H*-quotient* $A = G \rtimes_{\lambda, H} (\mathfrak{l}/\mathfrak{h})$ *of the transformation Lie algebroid* $G \rtimes_\lambda \mathfrak{l}$*.*

Proof. Let $\Lambda \in \wedge^2 \mathfrak{g}$ be any element with $q(\Lambda) = \pi(eH) \in \wedge^2 T_{eH}(G/H) \cong \wedge^2(\mathfrak{g}/\mathfrak{h})$. Recall that $\mathfrak{h}^0 = \{\xi \in \mathfrak{g}^* \mid \xi|_{\mathfrak{h}} = 0\}$. The projection $\mathfrak{l} \to \mathfrak{g}^* : x + \xi \to \xi$ gives an H-equivariant isomorphism $\mathfrak{l}/\mathfrak{h} \to \mathfrak{h}^0$ whose inverse is $\mathfrak{h}^0 \to \mathfrak{l}/\mathfrak{h} : \xi \mapsto \xi + \Lambda\xi + \mathfrak{h}$.

Using left translations by elements in G and the identification $T^*_{eH}(G/H) \cong \mathfrak{h}^0$, we have the vector bundle isomorphism

$$I : \ T^*(G/H) \longrightarrow G \times_H \mathfrak{h}^0 \cong G \times_H (\mathfrak{l}/\mathfrak{h}). \tag{4.14}$$

It remains to show that I is a Lie algebroid isomorphism. Recall that $\pi = q_* \pi_\Lambda$, where π_Λ is the bi-vector field on G given by $\pi_\Lambda = \Lambda^l + \pi_G$.

Let $n = \dim \mathfrak{h}^0$, and let $\xi_1, \xi_2, \ldots, \xi_n$ be a basis of $\mathfrak{h}^0$. For $\alpha \in \Omega^1(G/H)$, write

$$q^*\alpha = \sum_{j=1}^n f_{\alpha,j} \xi_j^l \ \in \ \Omega^1(G), \qquad \text{where} \qquad f_{\alpha,j} \in C^\infty(G), \ j = 1, \ldots, n. \tag{4.15}$$

Then $I(\alpha) = \sum_{j=1}^n f_{\alpha,j} \xi_j \in C^\infty(G, \mathfrak{h}^0)^H \cong \Gamma(A)$, and $b_\alpha = \sum_{j=1}^n f_{\alpha,j}(\Lambda\xi_j + \xi_j) \in C^\infty(G, \mathfrak{l})^H = \Gamma(G \rtimes_\lambda \mathfrak{l})^H$ is an H-invariant lifting of $I(a)$. Using $q_*\pi_\Lambda = \pi$, one has

$$\begin{aligned} \tilde{\pi}(\alpha) &= q_* \tilde{\pi}_\Lambda(q^*\alpha) = q_* \left(\sum_{j=1}^n f_{\alpha,j} \tilde{\pi}_\Lambda(\xi_j^l) \right) = q_* \left(\sum_{j=1}^n f_{\alpha,j} ((\Lambda\xi_j)^l + \tilde{\pi}_G(\xi_j^l)) \right) \\ &= q_* \left(\sum_{j=1}^n f_{\alpha,j} \lambda_{\Lambda\xi_j + \xi_j} \right) = \rho_A(I(\alpha)). \end{aligned}$$

Thus I maps the anchor map of $T^*(G/H)$ to the anchor map ρ_A of A.

It remains to show that $I\{\alpha, \beta\}_\pi = [I(\alpha), I(\beta)]$ for any $\alpha, \beta \in \Omega^1(G/H)$. Let $\{\, ,\, \}_{\pi_\Lambda}$ be the skew-symmetric bracket on $\Omega^1(G)$ defined by replacing π by π_Λ in

(3.1). Using again the fact that $\pi = q_*\pi_\Lambda$, we have

$$\begin{aligned} q^*\{\alpha,\beta\}_\pi &= \{q^*\alpha, q^*\beta\}_{\pi_\Lambda} = \sum_{j,k}\{f_{\alpha,j}\xi_j^l,\, f_{\beta,k}\xi_k^l\}_{\pi_\Lambda} \\ &= \sum_{j,k}\left(f_{\alpha,j}\tilde{\pi}_\Lambda(\xi_j^l)(f_{\beta,k})\xi_k^l\right) - \sum_{j,k}\left(f_{\beta,k}\tilde{\pi}_\Lambda(\xi_k^l)(f_{\alpha,j})\xi_j^l\right) \\ &\quad + \sum_{j,k}\left(f_{\alpha,j}f_{\beta,k}\{\xi_j^l,\xi_k^l\}_{\pi_\Lambda}\right). \end{aligned}$$

Thus, by Remark 4.4,

$$\begin{aligned} I\{\alpha,\beta\}_\pi &= \sum_{j,k}\left(f_{\alpha,j}\lambda_{\Lambda\xi_j+\xi_j}(f_{\beta,k})\xi_k\right) - \sum_{j,k}\left(f_{\beta,k}\lambda_{\Lambda\xi_k+\xi_k}(f_{\alpha,j})\xi_j\right) \\ &\quad + \sum_{j,k}\left(f_{\alpha,j}f_{\beta,k}[\xi_j,\xi_k]_\Lambda\right), \end{aligned}$$

where the bracket $[\,,\,]_\Lambda$ on $\mathfrak{g}^*$ is defined in (4.8). On the other hand, using

$$b_\alpha = \sum_{j=1}^n f_{\alpha,j}(\Lambda\xi_j + \xi_j) \quad \text{and} \quad b_\beta = \sum_{k=1}^n f_{\beta,k}(\Lambda\xi_k+\xi_k)$$

as H-invariant liftings of $I(\alpha)$ and $I(\beta)$ to smooth sections of $G\rtimes_\lambda \mathfrak{l}$, one can compute $[I(\alpha), I(\beta)] \in \Gamma(A)$ and see that $I\{\alpha,\beta\} = [I(\alpha), I(\beta)]$. This completes the proof that I is a Lie algebroid isomorphism. □

4.4. **The canonical representation of $T^*(G/H,\pi)$ on $K_{G/H}$.** Let $(G/H,\pi)$ be a Poisson homogeneous space of (G,π_G). Let

$$E = G \times \wedge^{\rm top}\mathfrak{h}^0$$

be the trivial H-equivariant line bundle over G, where

$$(g,\,Y)\cdot h = (gh,\, \mathrm{Ad}_h^* Y), \qquad g\in G,\, Y \in \wedge^{\rm top}\mathfrak{h}^0.$$

Then the identification $I: T^*(G/H) \to G\times_H \mathfrak{h}^0$ by left translation induces an identification $I: K_{G/H} \to E/H$. In this section, we show that E is naturally an $(\mathfrak{l},H)$-line bundle and that the canonical representation of $T^*(G/H)$ on $K_{G/H} \cong E/H$ can be identified with the H-quotient of the representation of $G\rtimes_\lambda \mathfrak{l}$ on E, where $\mathfrak{l}$ is the Drinfeld Lagrangian subalgebra of $\mathfrak{d}$ associated to $\pi(eH)$, and λ is the infinitesimal action of $\mathfrak{l}$ on G given in (4.12) (see Definition 3.1 and Lemma 2.7).

Let $\wedge^{\rm top}\mathfrak{l}$ be the 1-dimensional $(\mathfrak{l},H)$-module, on which $\mathfrak{l}$ acts by the adjoint character $\chi_\mathfrak{l}$ and $h\in H$ acts by $\mathrm{Ad}_h \in \mathrm{Aut}(\mathfrak{l})$. The trivial line bundle over G with fiber $\wedge^{\rm top}\mathfrak{l}$, still denoted by $\wedge^{\rm top}\mathfrak{l}$, is then an $(\mathfrak{l},H)$-line bundle. Regard $\wedge^{\rm top}T^*G$ as an (l,H)-line bundle, on which H acts by right translation and $\mathfrak{l}$ acts by Lie derivatives via λ. Set

$$F = \wedge^{\rm top}\mathfrak{l}\otimes\wedge^{\rm top}T^*G.$$

Then F is an $(\mathfrak{l},H)$-line bundle. Clearly, left translation in G gives rise to an H-equivariant trivialization

$$F \xrightarrow{\cong} G\times(\wedge^{\rm top}\mathfrak{l}\otimes\wedge^{\rm top}\mathfrak{g}^*),$$

where $h \in H$ acts on $G \times (\wedge^{\mathrm{top}} \mathfrak{l} \otimes \wedge^{\mathrm{top}} \mathfrak{g}^*)$ by

$$(g,\, X \otimes \mu) \cdot h = (gh,\, (\mathrm{Ad}_{h^{-1}} X) \otimes (\mathrm{Ad}_h^* \mu)), \qquad X \in \wedge^{\mathrm{top}} \mathfrak{l},\ \mu \in \wedge^{\mathrm{top}} \mathfrak{g}^*.$$

Lemma 4.8. $\wedge^{\mathrm{top}} \mathfrak{l} \otimes \wedge^{\mathrm{top}} \mathfrak{g}^* \cong (\wedge^{\mathrm{top}} \mathfrak{h}^0)^2$ *as H-modules, so $E^2 \cong F$ as H-equivariant line bundles over G.*

Proof. For $V \in \{\mathfrak{h}, \mathfrak{l}, \mathfrak{h}^0, \mathfrak{g}^*\}$, let $\chi_{H,V}$ be the character of the H-action on $\wedge^{\mathrm{top}} V$ induced from the adjoint and co-adjoint actions. It is easy to see that

$$\chi_{H,\mathfrak{l}} = \chi_{H,\mathfrak{h}} \chi_{H,\mathfrak{h}^0} \quad \text{ and } \quad \chi_{H,\mathfrak{g}^*} = \chi_{H,\mathfrak{h}}^{-1} \chi_{H,\mathfrak{h}^0}.$$

Thus $\chi_{H,\mathfrak{l}} \chi_{H,\mathfrak{g}^*} = \chi_{H,\mathfrak{h}^0}^2$. ☐

Since F is an (l, H)-line bundle, so is E as a square root of F by Remark 2.9. In the next Lemma 4.9, we determine the l-module structure on $\Gamma(E)$. Recall that $\chi_{\mathfrak{l}} \in \mathfrak{l}^*, \chi_{\mathfrak{g}} \in \mathfrak{g}^*$ and $\chi_{\mathfrak{g}^*}$ are the adjoint characters of $\mathfrak{l}, \mathfrak{g}$, and $\mathfrak{g}^*$ respectively.

Lemma 4.9. *Fix $Y_0 \in \wedge^{\mathrm{top}} \mathfrak{h}^0$, $Y_0 \neq 0$, and write elements in $\Gamma(E) = C^\infty(G, \wedge^{\mathrm{top}} \mathfrak{h}^0)$ as fY_0 for $f \in C^\infty(G)$. Then the l-module structure on $\Gamma(E)$ is given by*

$$(x + \xi) \cdot (fY_0) = \left(\lambda_{x+\xi}(f) + \frac{1}{2} \left(\chi_l(x + \xi) - \chi_{\mathfrak{g}}(x) + \chi_{\mathfrak{g}^*}(\xi) - 2(\pi_G, d\xi^l \right) f \right) Y_0$$

for any $x + \xi \in \mathfrak{l}$ and $f \in C^\infty(G)$.

Proof. Fix non-zero elements $X_0 \in \wedge^{\mathrm{top}} l$ and $\mu_0 \in \wedge^{\mathrm{top}} \mathfrak{g}^*$, and let μ_0^l be the left invariant volume form on G with $\mu_0^l(e) = \mu_0$. Then $X_0 \otimes \mu_0^l$ is a nowhere vanishing section of F. For $x + \xi \in \mathfrak{l}$, one has

$$\begin{aligned} (x + \xi) \cdot (X_0 \otimes \mu_0^l) &= \chi_{\mathfrak{l}}(x + \xi) X_0 \otimes \mu_0^l + X_0 \otimes L_{\lambda_{x+\xi}} \mu_0^l \\ &= (\chi_l(x + \xi) - \chi_{\mathfrak{g}}(x)) X_0 \otimes \mu_0^l + X_0 \otimes L_{\tilde{\pi}_G(\xi^l)} \mu_0^l. \end{aligned}$$

By (3.2),

$$L_{\tilde{\pi}_G(\xi^l)} \mu_0^l = \{\xi^l,\, \mu_0^l\}_{\pi_G} - 2(\pi_G, d\xi^l) \mu_0^l = (\chi_{\mathfrak{g}^*}(\xi) - 2(\pi_G, d\xi^l)) \mu_0^l,$$

from which the formula in Lemma 4.9 follows. ☐

By §2.3, the (l, H)-line bundle structure on E gives rise to a representation of the transformation Lie algebroid $G \rtimes_\lambda \mathfrak{l}$ on E and a representation of the H-quotient Lie algebroid $A = G \rtimes_{\lambda,H} (\mathfrak{l}/\mathfrak{h})$ on E/H.

Theorem 4.10. *Under the identification $I : T^*(G/H, \pi) \cong A = G \rtimes_{\lambda,H} (\mathfrak{l}/\mathfrak{h})$ of Lie algebroids and the identification $I : K_{G/H} \cong E/H$ of line bundles, the canonical representation of $T^*(G/H, \pi)$ on $K_{G/H}$ becomes the H-quotient representation of A on E/H.*

Proof. Denote by D both the canonical representation of $T^*(G/H, \pi)$ on $K_{G/H}$ and the quotient representation of A on E/H. We need to show that

$$D_{I(\alpha)} I(\mu) = I(D_\alpha \mu), \quad \forall\, \alpha \in \Omega^1(G/H),\ \mu \in \Omega^{\mathrm{top}}(G/H). \tag{4.16}$$

Let $Y_0 = \xi_1 \wedge \cdots \wedge \xi_n \in \wedge^{\mathrm{top}} \mathfrak{h}^0$, where $\xi_1, \ldots, \xi_n$ is a basis for $\mathfrak{h}^0$, and write

$$q^* \alpha = \sum_{j=1}^n f_{\alpha,j} \xi_j^l \ \in\ \Omega^1(G) \quad \text{ and } \quad q^* \mu = \phi \xi_1^l \wedge \cdots \xi_n^l \in \Omega^n(G),$$

where $f_{\alpha,j} \in C^\infty(G)$ for $j = 1, \ldots, n$, and $\phi \in C^\infty(G)$. Then

$$I(\alpha) = \sum_{j+1}^{n} f_{\alpha,j}\xi_j \in C^\infty(G, \mathfrak{h}^0)^H \quad \text{and} \quad I(\mu) = \phi Y_0 \in \Gamma(E)^H.$$

Moreover $b_\alpha := \sum_{j=1}^{n} f_{\alpha,j}(\Lambda\xi_j + \xi_j) \in \Gamma(G \bowtie_\lambda \mathfrak{l})^H$ is an H-invariant lifting of $I(\alpha)$ to a section of $G \bowtie_\lambda \mathfrak{l}$. Let $\widetilde{D}$ be the representation of $G \bowtie_\lambda \mathfrak{l}$ on E. By Lemma 4.9,

$$\begin{aligned}(4.17)\qquad \widetilde{D}_{b_\alpha} f_\mu \;=\; & \sum_{j=1}^{n} f_{\alpha,j}\left(\lambda_{\Lambda\xi_j+\xi_j}(\phi) - (\pi_G, d\xi_j^l)\phi\right) Y_0 \\ & + \frac{1}{2}\sum_{j=1}^{n} f_{\alpha,j}\left(\chi_{\mathfrak{l}}(\Lambda\xi_j + \xi_j) - \chi_{\mathfrak{g}}(\Lambda\xi_j) + \chi_{\mathfrak{g}^*}(\xi_j)\right)\phi Y_0.\end{aligned}$$

On the other hand, let Y_0^l be the left invariant n-form on G with $Y_0^l(e) = Y_0$. Then

$$\begin{aligned} q^* D_\alpha \mu &= q^*\left(\{\alpha,\, \mu\}_\pi - (\pi,\, d\alpha)\mu\right) = \{q^*\alpha,\, q^*\mu\}_{\pi_\Lambda} - (\pi_\Lambda,\, dq^*\alpha)q^*\mu \\ &= \sum_{j=1}^{n}\left(\{f_{\alpha,j}\xi_j^l,\, \phi Y_0^l\}_{\pi_\Lambda} - (\pi_\Lambda,\, d(f_{\alpha,j}\xi_j^l))\phi Y_0^l\right) \\ &= \sum_{j=1}^{n}\left(f_{\alpha,j}\tilde{\pi}_\Lambda(\xi_j^l)(\phi)Y_0^l + \{f_{\alpha,j}\xi_j^l,\, Y_0^l\}_{\pi_\Lambda}\phi\right) \\ &\quad - \sum_{j=1}^{n}(\pi_\Lambda,\, df_{\alpha,j}\wedge\xi_j^l + f_{\alpha,j}d\xi_j^l)\phi Y_0^l \\ &= \sum_{j=1}^{n} f_{\alpha,j}\left(\lambda_{\Lambda\xi_j+\xi_j}(\phi) - (\pi_G,\, d\xi_j^l)\phi\right)Y_0^l + \{f_{\alpha,j}\xi_j^l,\, Y_0^l\}_{\pi_\Lambda}\phi \\ &\quad + \sum_{j=1}^{n}\left(\tilde{\pi}_\Lambda(\xi_j^l)(f_{\alpha,j}) - f_{\alpha,j}(\Lambda^l,\, d\xi_j^l)\right)\phi Y_0^l. \end{aligned}$$

Using the properties of the Schouten bracket $\{\,,\,\}_{\pi_\Lambda}$ on $\Omega(G)$, one has

$$\sum_{j=1}^{n}\{f_{\alpha,j}\xi_j^l,\, Y_0^l\}_{\pi_\Lambda} = \sum_{j=1}^{n}\left(f_{\alpha,j}\{\xi_j^l,\, Y_0^l\}_{\pi_\Lambda} - \tilde{\pi}_\Lambda(\xi_j^l)(f_{\alpha,j})Y_0^l\right).$$

Thus

$$\begin{aligned}(4.18)\qquad q^* D_\alpha\mu \;=\; & \sum_{j=1}^{n} f_{\alpha,j}\left(\lambda_{\Lambda\xi_j+\xi_j}(\phi) - (\pi_G,\, d\xi_j^l)\phi\right)Y_0^l \\ & + \sum_{j=1}^{n} f_{\alpha,j}\left(\{\xi_j^l,\, Y_0^l\}_{\pi_\Lambda}\phi - (\Lambda^l,\, d\xi_j^l)\phi Y_0^l\right).\end{aligned}$$

By Lemma 4.6,

$$\{\xi,\, Y_0^l\}_{\pi_\Lambda} - (\Lambda^l,\, d\xi^l)Y_0^l = \frac{1}{2}\left(\chi_{\mathfrak{l}}(\Lambda\xi + \xi) - \chi_{\mathfrak{g}}(\Lambda\xi) + \chi_{\mathfrak{g}^*}(\xi)\right)Y_0^l, \quad \forall \xi \in \mathfrak{h}^0.$$

Comparing with (4.17) and (4.18), we see that (4.16) holds. □

4.5. **Poisson cohomology of** $(G/H,\pi)$**.** Let the notation be as in §4.3. For any integer N, since E is a trivial line bundle over G, $\Gamma(E^N)\cong C^\infty(G)$ as vector spaces. The induced (l,H)-module structure on $C^\infty(G)$ is specified as follows.

Notation 4.11. For an integer N, denote $C^\infty(G)_N$ the vector space $C^\infty(G)$ with the following $(\mathfrak{l},H)$-module structure: for $x+\xi\in\mathfrak{l}, h\in H$ and $f\in C^\infty(G)$,

$$\begin{aligned}(x+\xi)\cdot_N f &= \lambda_{x+\xi}(f)+\frac{N}{2}\left(\chi_{\mathfrak{l}}(x+\xi)-\chi_{\mathfrak{g}}(x)+\chi_{\mathfrak{g}^*}(\xi)-2(\pi_G,d\xi^l)\right)f,\\ h\cdot_N f &= \left(\chi_{H,\mathfrak{h}^0}(h)\right)^N(f\circ r_h),\end{aligned}$$

where $\chi_{H,\mathfrak{h}^0}(h)=\det(\mathrm{Ad}^*_{h^{-1}}:\mathfrak{h}^0\to\mathfrak{h}^0)$ and r_h is the right translation by h.

We can now identify the Poisson cohomology of G/H with relative Lie algebra cohomology. Corollary 4.12 follows directly from Lemma 2.8, Theorem 4.7, and Theorem 4.10.

Corollary 4.12. *For any integer N,*

$$H^\bullet\left(G/H,\pi;K^N_{G/H}\right)\cong H^\bullet_{\mathrm{Lie}}(\mathfrak{l},H;C^\infty(G)_N).$$

where the left hand side is the generalized Poisson cohomology of $(G/H,\pi)$ and the right hand side is the Lie algebra cohomology of $\mathfrak{l}$ relative to H with coefficients in $C^\infty(G)_N$.

The special case of Corollary 4.12 when $N=0$ was proved in [15].

4.6. **The pairing on the Poisson cohomology.** Assume that G/H is compact and orientable with a fixed orientation, so one has the map

$$\Omega^{\mathrm{top}}(G/H)\longrightarrow\mathbb{R}:\ \omega\longmapsto\int_{G/H}\omega. \tag{4.19}$$

By §3.2, for any integer N and any $0\le k\le n=\dim(G/H)$, there is a well-defined pairing $(\ ,\ )$ between $H^k\left(G/H,\pi;K^N_{G/H}\right)$ and $H^{n-k}\left(G/H,\pi;K^{2-N}_{G/H}\right)$. In view of Corollary 4.12, we now identify this pairing with a pairing on the corresponding relative Lie algebra cohomology spaces. Let the notation be as in §4.4. Then we have the identifications of H-modules:

$$\begin{aligned}\wedge^{\mathrm{top}}(\mathfrak{l}/\mathfrak{h})^*\otimes\Gamma(E^N)\otimes\Gamma(E^{2-N}) &\cong \wedge^{\mathrm{top}}(\mathfrak{l}/\mathfrak{h})^*\otimes\Gamma(E^2)\\ &\cong \wedge^{\mathrm{top}}(\mathfrak{l}/\mathfrak{h})^*\otimes\Gamma(F)\\ &\cong \wedge^{\mathrm{top}}(\mathfrak{l}/\mathfrak{h})^*\otimes\wedge^{\mathrm{top}}\mathfrak{l}\otimes\Omega^{\mathrm{top}}(G)\\ &\cong \wedge^{\mathrm{top}}(\mathfrak{l}/\mathfrak{h})^*\otimes\wedge^{\mathrm{top}}\mathfrak{l}\otimes\wedge^{\mathrm{top}}\mathfrak{g}^*\otimes C^\infty(G)\\ &\cong \wedge^{\mathrm{top}}(\mathfrak{l}/\mathfrak{h})^*\otimes(\wedge^{\mathrm{top}}\mathfrak{h}^0)^2\otimes C^\infty(G)\\ &\cong \wedge^{\mathrm{top}}\mathfrak{h}^0\otimes C^\infty(G),\end{aligned}$$

where we used Lemma 4.8 to identify $\wedge^{\mathrm{top}}\mathfrak{l}\otimes\wedge^{\mathrm{top}}\mathfrak{g}^*\cong(\wedge^{\mathrm{top}}\mathfrak{h}^0)^2$ and left translation in G to identify $\Omega^{\mathrm{top}}(G)\cong\wedge^{\mathrm{top}}\mathfrak{g}^*\otimes C^\infty(G)$. Thus we have an identification

$$(\wedge^{\mathrm{top}}(\mathfrak{l}/\mathfrak{h})^*\otimes\Gamma(E^N)\otimes\Gamma(E^{2-N}))^H\cong(\wedge^{\mathrm{top}}(\mathfrak{l}/\mathfrak{h})\otimes C^\infty(G))^H\cong\Omega^{\mathrm{top}}(G/H). \tag{4.20}$$

Let $\nu:(\wedge^{\mathrm{top}}(\mathfrak{l}/\mathfrak{h})^*\otimes\Gamma(E^N)\otimes\Gamma(E^{2-N}))^H\to\mathbb{R}$ be the composition of the identification in (4.20) with the integration map in (4.19). One checks directly that (2.2) holds and that, under the identifications in Corollary 4.12, the canonical pairing between

$H^k\left(G/H,\pi;K^N_{G/H}\right)$ and $H^{n-k}\left(G/H,\pi;K^{2-N}_{G/H}\right)$ coincides with the pairing between $H^k_{\rm Lie}(\mathfrak{l},H;C^\infty(G)_N)$ and $H^{n-k}_{\rm Lie}(\mathfrak{l},H;C^\infty(G)_{2-N})$ induced by ν (see §2.2).

4.7. **Modular vector fields of** $(G/H,\pi)$**.** Assume again that G/H is orientable and let μ be a fixed volume form on G/H. Fix a non-zero $Y_0\in\wedge^{\rm top}\mathfrak{h}^0$, and let Y_0^l be the corresponding left invariant form on G. Write $q^*\mu=\phi Y_0^l$, with $\phi\in C^\infty(G)$ everywhere non-zero. Let $\Lambda\in\wedge^2\mathfrak{g}$ be any element such that $q(\Lambda)=\pi(eH)\in\wedge^2T_{\underline{e}}G/H\cong\wedge^2\mathfrak{g}/\mathfrak{h}$, and let $\pi_\Lambda=\Lambda^l+\pi_G$ so that $q_*\pi_\Lambda=\pi$. Recall that $\chi_\mathfrak{l}\in\mathfrak{l}^*,\chi_\mathfrak{g}\in\mathfrak{g}^*$ and $\chi_{\mathfrak{g}^*}\in\mathfrak{g}$ are the adjoint characters of $\mathfrak{l},\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. Write $x_0=\chi_{\mathfrak{g}^*}\in\mathfrak{g}$, $\xi_0=\chi_\mathfrak{g}\in\mathfrak{g}^*$, and let $x_\mathfrak{l}$ be any element in $\mathfrak{g}^*$ such that $x_\mathfrak{l}(\xi)=\chi_\mathfrak{l}(\Lambda\xi+\xi)$ for $\xi\in\mathfrak{h}^0$. Recall that for $x\in\mathfrak{g}$ and $\xi\in\mathfrak{g}$, x^l (resp. x^r and ξ^l) is the left (resp. right) invariant vector field and one form on G with values x and ξ at $e\in G$.

Lemma 4.13. *Let the notation be as above. Let X be the vector field on G given by*

$$X=-\tilde{\pi}_\Lambda(d\log|\phi|)+\frac{1}{2}\left(x_\mathfrak{l}^l+x_0^r+\tilde{\pi}_\Lambda(\xi_0^l)\right).$$

Then q_*X is a well-defined vector field on G/H, and it is the modular vector field of π with respect to μ.

Proof. Let $\xi_1,\ldots,\xi_n$ be a basis of $\mathfrak{h}^0$ such that $\xi_1\wedge\cdots\wedge\xi_n=Y_0$. Let $\alpha\in\Omega^1(G/H)$, and write $q^*\alpha=\sum_{j=1}^n f_{\alpha,j}\xi_j^l\in\Omega^1(G)$. As in the proof of Theorem 4.10,

$$\begin{aligned} q^*D_\alpha\mu &= -\sum_{j=1}^n f_{\alpha,j}((\xi_j^l,\,\tilde{\pi}_\Lambda(d\log|\phi|)+(\pi_G,\,d\xi_j^l))Y_0^\mathfrak{l}\\ &\quad+\frac{1}{2}\sum_{j=1}^n f_{\alpha,j}(\chi_\mathfrak{l}(\Lambda\xi_j+\xi_j)-\chi_\mathfrak{g}(\Lambda\xi_j)+\chi_{\mathfrak{g}^*}(\xi_j))Y_0^l\\ &= \left(q^*\alpha,\ -\tilde{\pi}_\Lambda(d\log|\phi|)-F_0+\frac{1}{2}(x_\mathfrak{l}^l+(\Lambda\xi_0)^l+x_0^l)\right),\end{aligned}$$

where F_0 is the vector field on G such that $(F_0,\xi^l)=(\pi_G,d\xi^l)$ for all $\xi\in\mathfrak{g}^*$. It is shown in Proposition 4.7 of [7] that $F_0=\frac{1}{2}(x_0^l-x_0^r-\tilde{\pi}_G(\xi_0^l))$. Thus we have

$$q^*D_\alpha\mu=(q^*\alpha,\ X).$$

It follows that q_*X is a well-defined vector field on G/H and it is the modular vector field of π with respect to μ. □

Remark 4.14. Note that if μ is a G-invariant volume form on G/H, the modular vector field of π with respect to μ is

$$q_*X=\frac{1}{2}\left(x_\mathfrak{l}^l+x_0^r+\tilde{\pi}_\Lambda(\xi_0^l)\right).$$

This formula for the special case when $\mathfrak{h}^0$ is an ideal of $\mathfrak{g}^*$ has been obtained in [7].

5. A Poisson groupoid over $(G/H,\pi)$

When H is a Poisson Lie subgroup of (G,π_G) and $\pi=q_*\pi_G$, where $q:G\to G/H$ is the projection, a symplectic groupoid of $(G/H,\pi)$ was constructed in [25] (under the additional assumption that (G,π_G) is complete). In this section, let $(G/H,\pi)$ be an arbitrary Poisson homogeneous space of (G,π_G) with the Drinfeld Lagrangian subalgebra $\mathfrak{l} = \mathfrak{l}_{\pi(eH)}$. We assume that G is a closed subgroup of a connected Lie group D with Lie algebra $\mathfrak{d}$, $H = G \cap L$, where L is the connected subgroup of D with Lie algebra $\mathfrak{l}$, and that the infinitesimal action λ of $\mathfrak{l}$ on G in (4.12) integrates to an action of L on G. We will show that the associated space $\Gamma = G \times_H (L/H)$ is a Poisson groupoid over $(G/H, \pi)$. We also give conditions for Γ to be symplectic. The Poisson structure on Γ is obtained from reduction of a quasi-Poisson manifold by an action of a quasi-Poisson Lie group [1].

5.1. **The quasi-Poisson Lie group** $(G, \pi_{G,\Lambda}, \varphi)$**.** Let (G, π_G) be a Poisson Lie group corresponding to Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$. Then any $\Lambda \in \wedge^2\mathfrak{g}$ (not necessarily related to any Poisson homogeneous space of (G, π_G) as in §4.2) can be used to twist the Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ to a Manin quasi-triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}')$ [1], where

$$\mathfrak{g}' = \{\Lambda\xi + \xi \,|\, \xi \in \mathfrak{g}^*\}, \tag{5.1}$$

and thus defines a quasi-Poisson Lie group structure on G. More precisely, let $p_1 : \mathfrak{d} \to \mathfrak{g} : x + \xi \mapsto x - \Lambda\xi$, where $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, be the projection from $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}'$ to $\mathfrak{g}$ along $\mathfrak{g}'$, and define $\varphi \in \wedge^3\mathfrak{g}$ by

$$\varphi(\xi \wedge \eta \wedge \zeta) = \langle p_1[\Lambda\xi + \xi,\, \Lambda\eta + \eta],\, \Lambda\zeta + \zeta\rangle, \qquad \xi, \eta, \zeta \in \mathfrak{g}^*.$$

It is straightforward to check that, for any $\xi, \eta, \zeta \in \mathfrak{g}^*$,

$$\begin{aligned}\varphi(\xi \wedge \eta \wedge \zeta) = {} & \langle [\Lambda\xi, \Lambda\eta],\, \zeta\rangle + \langle [\Lambda\eta, \Lambda\zeta],\, \xi\rangle + \langle [\Lambda\zeta, \Lambda\xi],\, \eta\rangle \\ & + \langle \Lambda\xi,\, [\eta, \zeta]\rangle + \langle \Lambda\eta,\, [\zeta, \xi]\rangle + \langle \Lambda\zeta,\, [\xi, \eta]\rangle.\end{aligned}$$

In fact $\varphi = \frac{1}{2}[\Lambda, \Lambda] + \delta(\Lambda)$. Let Λ^l and Λ^r be respectively the left and right invariant bi-vector fields on G with value Λ at e, and define

$$\pi_{G,\Lambda} = \Lambda^l - \Lambda^r + \pi_G, \tag{5.2}$$

Lemma 5.1. [1] $(G, \pi_{G,\Lambda}, \varphi)$ *is a quasi-Poisson Lie group corresponding to the Manin quasi-triple* $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}')$ *in the sense that* $\pi_{G,\Lambda}$ *is multiplicative,*

$$\frac{1}{2}[\pi_{G,\Lambda},\, \pi_{G,\Lambda}] = \varphi^l - \varphi^r, \qquad \textit{and} \qquad [\pi_{G,\Lambda},\, \varphi^l] = 0,$$

where φ^r *(resp.* φ^l*) is the right (resp. left) invariant tri-vector field on* G *with value* φ *at* e*.*

Recall from [1] that a (right) quasi-Poisson action of $(G, \pi_{G,\Lambda}, \varphi)$ on a manifold P with a bi-vector field π_P is a right action $\rho : P \times G \to P$ of G on P such that

1) $[\pi_P, \pi_P] = 2\rho_\varphi$ and
2) $\rho : (P, \pi_P) \times (G, \pi_{G,\Lambda}) \to (P, \pi_P)$ is a bi-vector map,

where $\rho : x \mapsto \rho_x$ also denotes the Lie algebra homomorphism $\mathfrak{g} \to \mathcal{V}^1(P)$ given by

$$\rho_x(p) = \frac{d}{dt}|_{t=0} p \exp(tx), \qquad x \in \mathfrak{g},\, p \in P \tag{5.3}$$

as well as its multi-linear extension $\wedge^k\mathfrak{g} \to \mathcal{V}^k(P) : X \mapsto \rho_X$ for any integer $k \geq 1$. Left quasi-Poisson actions of $(G, \pi_{G,\Lambda}, \varphi)$ are similarly defined.

Example 5.2. Let $\pi_\Lambda = \Lambda^l + \pi_G$. It is easy to see that the action of $(G, \pi_{G,\Lambda}, \varphi)$ on (G, π_Λ) by right multiplication is a right quasi-Poisson action. For another example, assume that D is a connected Lie group with Lie algebra $\mathfrak{d}$ and that G is a closed subgroup of D. For $d \in D$, let $\underline{d} = dG \in D/G$, and for $x + \xi \in \mathfrak{d}$ with $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, let $\sigma_{x+\xi}$ be the vector field on D/G given by

$$\sigma_{x+\xi}(\underline{d}) = \frac{d}{dt}|_{t=0} \exp(t(x+\xi))\underline{d} \in T_{\underline{d}}(D/G), \qquad d \in D. \tag{5.4}$$

Let $\sigma : \wedge^k \mathfrak{d} \to \mathcal{V}^k(D/G) : X \mapsto \sigma_X$ also denote the multi-linear extension of σ. Let $\{x_i\}_{n=1}^n$ be a basis of $\mathfrak{g}$ and let $\{\xi_i\}_{i=1}^n$ be its dual basis of $\mathfrak{g}^*$. Define the bi-vector fields $\pi_{D/G}$ and $\pi_{D/G,\Lambda}$ on D/G respectively by

$$\pi_{D/G} = \frac{1}{2}\sum_i \sigma_{\xi_i} \wedge \sigma_{x_i} \quad \text{and} \quad \pi_{D/G,\Lambda} = \frac{1}{2}\sum_i \sigma_{\Lambda\xi_i+\xi_i} \wedge \sigma_{x_i} = \pi_{D/G} - \sigma_\Lambda. \tag{5.5}$$

Then [1] $\pi_{D/G}$ is Poisson and the action

$$(G, \pi_{G,\Lambda}, \varphi) \times (D/G,\, \pi_{D/G,\Lambda}) \longrightarrow (D/G,\, \pi_{D/G,\Lambda}) :\ (g, \underline{d}) \longmapsto g\underline{d}, \quad g \in G, d \in D,$$

is a left quasi-Poisson action of $(G, \pi_{G,\Lambda}, \varphi)$. In particular,

$$[\pi_{D/G,\Lambda},\ \pi_{D/G,\Lambda}] = -2\sigma_\varphi. \tag{5.6}$$

Moreover, let $\delta_{\mathfrak{g}'} : \mathfrak{g}' \to \wedge^2 \mathfrak{g}'$ be defined by

$$\langle \delta_{\mathfrak{g}'}(\Lambda\xi + \xi),\ x \wedge y\rangle = \langle \Lambda\xi + \xi,\ [x,y]\rangle = \langle \xi,\ [x,y]\rangle, \qquad \xi \in \mathfrak{g}^*,\ x, y \in \mathfrak{g}.$$

Then one can check that

$$[\sigma_{\Lambda\xi+\xi},\ \pi_{D/G,\Lambda}] = -\sigma_{\delta_{\mathfrak{g}'}(\Lambda\xi+\xi)} + \sigma_{\iota_\xi \varphi}, \qquad \forall \xi \in \mathfrak{g}^*. \tag{5.7}$$

5.2. **The bivector field π_P on $P = G \times (D/G)$.** Let the assumptions be as in §5.1. In particular, assume that D is a connected Lie group with Lie algebra $\mathfrak{d}$ and that G is a closed subgroup of D. Let $P = G \times (D/G)$. For any integer $k \geq 1$ and for a k-vector field V on G, let $(V, 0)$ be the corresponding k-vector field on P. Similarly a k-vector field U on D/G gives rise to the k-vector field $(0, U)$ on P. For $x \in \mathfrak{g}$, recall that x^l is the left invariant vector field on G with $x^l(e) = x$. Define the bi-vector field π_P on P by

$$\pi_P = (\pi_\Lambda, 0) - (0, \pi_{D/G,\Lambda}) + \sum_{i=1}^n (0, \sigma_{\Lambda\xi_i+\xi_i}) \wedge (x_i^l, 0). \tag{5.8}$$

Lemma 5.3. *The right action*

$$\rho :\ (P, \pi_P) \times (G, \pi_{G,\Lambda}, \varphi) \longrightarrow (P, \pi_P) :\ (g, \underline{d}) \cdot g_1 = (gg_1,\ g_1^{-1}\underline{d}), \qquad g, g_1 \in G, d \in D,$$

is a quasi-Poisson action of $(G, \pi_{G,\Lambda}, \varphi)$.

Proof. To show that $[\pi_P, \pi_P] = 2\rho_\varphi$, let $\varphi = \sum_k a_k \wedge b_k \wedge c_k$, where $a_k, b_k, c_k \in \mathfrak{g}$, and let

$$\rho'_\varphi = \sum_k \Big((0, \sigma_{a_k}) \wedge (b_k^l \wedge c_k^l, 0) + (0, \sigma_{b_k}) \wedge (c_k^l \wedge a_k^l, 0) + (0, \sigma_{c_k}) \wedge (a_k^l \wedge b_k^l, 0) \Big)$$

$$\rho''_\varphi = \sum_k \Big((a_k^l, 0) \wedge (0, \sigma_{b_k \wedge c_k}) + (b_k^l, 0) \wedge (0, \sigma_{c_k \wedge a_k}) + (c_k^l, 0) \wedge (0, \sigma_{a_k \wedge b_k}) \Big).$$

It is easy to see that $\rho_\varphi = (\varphi^l, 0) - (0, \sigma_\varphi) - \rho'_\varphi + \rho''_\varphi$. On the other hand, let $\pi_0 = \sum_{i=1}^n (0, \sigma_{\Lambda\xi_i+\xi_i}) \wedge (x_i^l, 0)$, so that $\pi_P = (\pi_\Lambda, 0) - (0, \pi_{D/G,\Lambda}) + \pi_0$, and

$$\begin{aligned}[]
[\pi_P,\ \pi_P] &= ([\pi_\Lambda, \pi_\Lambda], 0) + (0, [\pi_{D/G,\Lambda}, \pi_{D/G,\Lambda}]) + 2[\pi_0,\ (\pi_\Lambda, 0) - (0, \pi_{D/G,\Lambda})] + [\pi_0,\ \pi_0] \\
&= 2(\varphi^l, 0) - 2(0, \sigma_\varphi) + 2[\pi_0,\ (\pi_\Lambda, 0) - (0, \pi_{D/G,\Lambda})] + [\pi_0,\ \pi_0].
\end{aligned}$$

It is easy to see that $[\pi_0, \pi_0] = \pi_1 + \pi_2$, where

$$\pi_1 = -\sum_{i,j=1}^n (0, \sigma_{[\Lambda\xi_i+\xi_i, \Lambda\xi_j+\xi_j]}) \wedge (x_i^l \wedge x_j^l, 0),$$

$$\pi_2 = \sum_{i,j=1}^n ([x_i, x_j]^l, 0) \wedge (0, \sigma_{(\Lambda\xi_i+\xi_i)\wedge\Lambda\xi_j+\xi_j)}).$$

Thus $[\pi_P, \pi_P] = 2(\varphi^l, 0) - 2(0, \sigma_\varphi) + 2[\pi_0, (\pi_\Lambda, 0)] + \pi_1 - 2[\pi_0, (0, \pi_{D/G,\Lambda})] + \pi_2$. Now

$$2[\pi_0, (\pi_\Lambda, 0)] = 2\sum_{i=1}^n (0, \sigma_{\Lambda\xi_i+\xi_i}) \wedge (([x_i, \Lambda] + \delta(x_i))^l, 0).$$

Recall that $p_1 : \mathfrak{d} \to \mathfrak{g}$ is the projection along $\mathfrak{g}'$. Let $p' : \mathfrak{d} \to \mathfrak{g}'$ be the projection along $\mathfrak{g}$. It is easy to check that

$$\sum_{i,j=1}^n p'[\Lambda\xi_i + \xi_i,\ \Lambda\xi_j + \xi_j] \otimes x_i \wedge x_j = 2\sum_{i=1}^n (\Lambda\xi_i + \xi_i) \otimes ([x_i, \Lambda] + \delta(x_i))$$

$$\sum_{i,j=1}^n p_1[\Lambda\xi_i + \xi_i,\ \Lambda\xi_j + \xi_j] \otimes x_i \wedge x_j = 2\tilde{\varphi},$$

where $\tilde{\varphi} = \sum_k (a_k \otimes b_k \wedge c_k + b_k \otimes c_k \wedge a_k + c_k \otimes a_k \wedge b_k)$. Thus $2[\pi_0, (\pi_\Lambda, 0)] + \pi_1 = -2\rho'_\varphi$. Similarly, by (5.7),

$$[\pi_0, \pi_{D/G,\Lambda}] = \sum_{i=1}^n (x_i^l, 0) \wedge (0, -[\sigma_{\Lambda\xi_i+\xi_i},\ \pi_{D/G,\Lambda}]) = \sum_{i=1}^n (x_i^l, 0) \wedge (0, \sigma_{\delta_{\mathfrak{g}'}(\Lambda\xi_i+\xi_i)} - \sigma_{\iota_{\xi_i}\varphi}).$$

It is easy to check that $\sum_{i=1}^n x_i \otimes \iota_{\xi_i}\varphi = \tilde{\varphi}$ and that

$$2\sum_{i=1}^n x_i \otimes \delta_{\mathfrak{g}'}(\Lambda\xi_i + \xi_i) = \sum_{i,j=1}^n [x_i, x_j] \otimes (\Lambda\xi_i + \xi_i) \wedge (\Lambda\xi_j + \xi_j).$$

Thus $-2[\pi_0, \pi_{D/G,\Lambda}] + \pi_2 = 2\rho''_\varphi$. Hence $[\pi_P, \pi_P] = 2\rho_\varphi$. The proof that ρ is a bi-vector map is straightforward and we omit the details. □

We now study when π_P on $P = G \times (D/G)$ is nondegenerate. For $d \in D$, the linear map $\mathfrak{d} \to T_{\underline{d}}(D/G) : x + \xi \mapsto \sigma_{x+\xi}(\underline{d})$, where $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, induces an isomorphism $\mathfrak{d}/\mathrm{Ad}_d\mathfrak{g} \to T_{\underline{d}}(D/G)$. For $y \in \mathfrak{g}$ and $\eta \in \mathfrak{g}^*$, let $\alpha_{y+\eta}(\underline{d}) \in T^*_{\underline{d}}(D/G)$ be such that

$$(\alpha_{y+\eta}(\underline{d}),\ \sigma_{x+\xi}(\underline{d})) = \langle y + \eta,\ x + \xi\rangle, \qquad x \in \mathfrak{g}, \xi \in \mathfrak{g}^*. \tag{5.9}$$

Then we have the isomorphism

$$\mathrm{Ad}_d\mathfrak{g} \longrightarrow T^*_{\underline{d}}(D/G) :\ \ y + \eta \longmapsto \alpha_{y+\eta}(\underline{d}), \qquad y \in \mathfrak{g},\ \eta \in \mathfrak{g}^*,\ y + \eta \in \mathrm{Ad}_d\mathfrak{g}. \tag{5.10}$$

Note that when $y+\eta \in \mathrm{Ad}_d\mathfrak{g}$, $\sigma_{y+\eta}(\underline{d}) = 0$, so $\sigma_y(\underline{d}) = -\sigma_\eta(\underline{d})$. The proof of the first identity in the following Lemma 5.4 is straightforward and is omitted. The second identity follows from (4.13).

Lemma 5.4. *For $g \in G, d \in D, \xi \in \mathfrak{g}^*$ and $y + \eta \in \mathrm{Ad}_d\mathfrak{g}$ with $y \in \mathfrak{g}$ and $\eta \in \mathfrak{g}^*$,*

$$\begin{aligned}\tilde{\pi}_P(l^*_{g^{-1}}\xi,\, \alpha_{y+\eta}(\underline{d})) &= \left(\tilde{\pi}_G(l^*_{g^{-1}}\xi) + (l_g)_*(y + \Lambda\xi - \Lambda\eta),\ \ \sigma_{\Lambda\eta+\eta-\Lambda\xi-\xi}(\underline{d})\right) \\ &= (\lambda_{y-\Lambda\eta+\Lambda\xi+\xi}(g),\ \ -\sigma_{y-\Lambda\eta+\Lambda\xi+\xi}(\underline{d}))\end{aligned}$$

Lemma 5.5. *The bi-vector field π_P on $P = G \times (D/G)$ is nondegenerate at $(g, \underline{d})$, where $g \in G$ and $d \in D$, if*

$$\mathfrak{g}' \cap \mathrm{Ad}_d\mathfrak{g} = 0 \quad \text{and} \quad \mathfrak{g}^* \cap \mathrm{Ad}_{gd}\mathfrak{g} = 0. \tag{5.11}$$

In particular, π_P is nondegenerate at $(g, \underline{e})$ for any $g \in G$, where $e \in D$ is the identity.

Proof. Assume that (5.11) holds at $(g, \underline{d}) \in P$. Suppose that $\xi \in \mathfrak{g}^*$ and $y+\eta \in \mathrm{Ad}_d\mathfrak{g}$ are such that $\tilde{\pi}_P(l^*_{g^{-1}}\xi,\, \alpha_{y+\eta}(\underline{d})) = 0$. Then $\sigma_{\Lambda\eta+\eta-\Lambda\xi-\xi}(\underline{d}) = 0$ by Lemma 5.4, so $\Lambda\eta+\eta-\Lambda\xi-\xi \in \mathfrak{g}'\cap\mathrm{Ad}_d\mathfrak{g} = 0$. Thus $\xi = \eta$. By (4.1), $\tilde{\pi}_G(l^*_{g^{-1}}\xi) = (r_g)_* p_{\mathfrak{g}} \mathrm{Ad}_g \xi$, where $p_{\mathfrak{g}} : \mathfrak{d} \to \mathfrak{g}$ is the projection along $\mathfrak{g}^*$. Thus Lemma 5.4 implies that $p_{\mathfrak{g}}\mathrm{Ad}_g(y+\xi) = 0$, so $\mathrm{Ad}_g(y + \xi) \in \mathfrak{g}^* \cap \mathrm{Ad}_{gd}\mathfrak{g} = 0$. Thus $y = 0$ and $\xi = \eta = 0$. □

Remark 5.6. Let $N(\mathfrak{g}^*)$ be the normalizer subgroup of $\mathfrak{g}^*$ in D. Suppose that $D = N(\mathfrak{g}^*)G$ and that $\Lambda = 0$ (so $\pi(eH) = 0$). Then (5.11) holds for all $(g, d) \in G\times D$, and π_P is nondegenerate everywhere on P. See Example 5.14 for an example.

5.3. **The Poisson structure Π on $G\times_H(L/H)$.** Let the notation be as in §5.1 and §5.2, but assume now that $(G/H, \pi)$ is a Poisson homogeneous space of (G, π) and that $\Lambda \in \wedge^2\mathfrak{g}$ is such that $q(\Lambda) = \pi(eH) \in \wedge^2 T_{eH}(G/H) \cong \wedge^2(\mathfrak{g}/\mathfrak{h})$, where q denotes both projections $G \to G/H$ and $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. Let $(G, \pi_{G,\Lambda}, \varphi)$ be the quasi-Poisson Lie group defined using Λ as in §5.1.

Lemma 5.7. *Let P be any manifold with a bi-vector field π_P. Suppose that $\rho : (P, \pi_P) \times (G, \pi_{G,\Lambda}, \varphi) \to (P, \pi_P)$ is a right quasi-Poisson action of $(G, \pi_{G,\Lambda}, \varphi)$ and that ρ restricts to a free and proper action of H. Let $j : P \to P/H$ be the projection. Then $j_*\pi_P$ is a well-defined Poisson structure on P/H.*

Proof. By 1) in Lemma 4.3, $q_*\pi_{G,\Lambda}(h) = 0$ for all $h \in H$. It follows from the fact that ρ is a bi-vector map that $j_*\pi_P$ is well-defined. Since $\varphi \in \mathfrak{h} \wedge \mathfrak{g} \wedge \mathfrak{g}$ by 2) of Lemma 4.3, $[j_*\pi_P,\, j_*\pi_P] = j_*[\pi_P,\, \pi_P] = 2j_*\rho_\varphi = 0$, so $j_*\pi_P$ is Poisson. □

We now state a lemma from linear algebra.

Lemma 5.8. *Let (V, π) be a Poisson vector space. Suppose that U and W are subspaces of V such that $\tilde{\pi}(U^0) \subset W \subset U$, where $U^0 = \{\xi \in V^* \,|\, \xi|_U = 0\}$. Let $\phi : V \to V/W$ be the projection. Then U/W is a Poisson subspace of $(V/W, \phi(\pi))$.*

The following Lemma 5.9 follows immediately from Lemma 5.8.

Lemma 5.9. *Let the notation be as in Lemma 5.7. Suppose that Q is an H-invariant submanifold of P such that $\tilde{\pi}_P(T^0_qQ) \subset T_q(qH)$ for every $q \in Q$, where $T^0_qQ = \{\alpha \in T^*_qP |\, \alpha|_{T_qQ} = 0\}$ and qH is the H-orbit through q. Then Q/H is a Poisson submanifold of $(P/H,\, j_*\pi_P)$.*

We now apply Lemma 5.7 to $P = G \times (D/G)$ as in §5.2, π_P as in (5.8), and the action ρ as in Lemma 5.3. Denote by $G \times_H (D/G)$ the quotient of P by H with the projection $j : P \to G \times_H (D/G)$. By Lemma 5.7, $j_*\pi_P$ is a well-defined Poisson structure on $G \times_H (D/G)$. Set $[g, \underline{d}] = j(g, \underline{d})$ for $g \in G$ and $d \in D$.

Notation 5.10. The Poisson structure $j_*\pi_P$ on $G \times_H (D/G)$ will be denoted by Π.

Recall that $\mathfrak{l} = \mathfrak{l}_{\pi(eH)}$ is the Drinfeld Lie subalgebra of $\mathfrak{d}$ associated to $\pi(eH)$. Let L be the connected Lie subgroup of D with Lie algebra $\mathfrak{l}$ and assume that $H = G \cap L$. Let $\mathcal{O}$ be the L-orbit in D/G through $\underline{e} \in D/G$, where e is the identity element of D. Identify L/H with $\mathcal{O}$ and regard $G \times_H (L/H)$ as a submanifold of $G \times_H (D/G)$.

Lemma 5.11. *$G \times_H (L/H)$ is a Poisson submanifold of $(G \times_H (D/G),\ \Pi)$, and Π is nondegenerate at $[g, \underline{d}]$ for all $g \in G$ and $d \in L$ such that* (5.11) *holds.*

Proof. Let $Q = G \times \mathcal{O} \subset P$. Then Q is H-invariant. To see that Q/H is a Poisson submanifold of $(P/H, \Pi)$, it suffices, by Lemma 5.9, to show that $\tilde{\pi}_P(T_q^0 Q) \subset T_q(qH)$ for every $q = (g, \underline{d}) \in Q$, where $g \in G$ and $d \in L$. Using the isomorphism in (5.10), $T_q^0 Q = \{(0, \alpha_{y+\eta}(\underline{d})) \mid y + \eta \in \mathfrak{l} \cap \mathrm{Ad}_d \mathfrak{g}\}$, and by Lemma 5.4,

$$\tilde{\pi}_P(T_q^0 Q) = \{((l_g)_*(y - \Lambda\eta),\ \sigma_{\Lambda\eta - y}(\underline{d})) \mid y + \eta \in \mathfrak{l} \cap \mathrm{Ad}_d \mathfrak{g}\}.$$

By (4.7), $y + \eta \in \mathfrak{l}$ implies that $y - \Lambda\eta \in \mathfrak{h}$. Thus $\tilde{\pi}_P(T_q^0 Q) \subset T_q(qH)$.

By Lemma 5.5, π_P is nondegenerate at $(g, \underline{d})$ for all $g \in G$ and $d \in D$ such that (5.11) holds. At such a point $(g, \underline{d})$ where $d \in L$, $\mathfrak{l} \cap \mathrm{Ad}_d \mathfrak{g} = \mathrm{Ad}_d(\mathfrak{l} \cap \mathfrak{g}) = \mathrm{Ad}_d \mathfrak{h}$, and the map $\mathfrak{l} \cap \mathrm{Ad}_d \mathfrak{g} \to \mathfrak{h} : y + \eta \mapsto y - \Lambda\eta$ is an isomorphism, so $\tilde{\pi}_P(T_q^0 Q) = T_q(qH)$. It follows from a linear algebra argument that Π is nondegenerate at $[g, \underline{d}]$. □

5.4. **The Poisson groupoid** $(G \times_H (L/H), \Pi)$**.** Let the notation be as in §5.3. Recall that $\lambda : \mathfrak{d} \to \mathcal{V}^1(G)$ is the infinitesimal action of $\mathfrak{d}$ on G given in (4.12). Assume in addition that the restriction of λ to $\mathfrak{l}$ integrates to a right action of L on G, denoted by $(g, l) \mapsto g^l$ for $g \in G$ and $l \in L$, such that $g^h = gh$ for $g \in G$ and $h \in H$. Then G is a $(\mathfrak{d}, L)$-space (see Definition 2.2).

Let $\Gamma = G \times_H (L/H)$. It is straightforward to show (we omit the proof) that the following is a groupoid structure on Γ over G/H: for $g, g_1, g_2 \in G$ and $l, l_1, l_2 \in L$,

1) source map $s : \Gamma \to G/H : [g,\ lH] \mapsto gH$;

2) target map $t : \Gamma \to G/H : [g,\ lH] \mapsto g^l H$;

3) multiplication $\cdot_\Gamma$: $[g_1,\ l_1 H] \cdot_\Gamma [g_2,\ l_2 H] = [g_1,\ l_1 h l_2 H]$ when $g_1^{l_1} H = g_2 H$, where $h = (g_1^{l_1})^{-1} g_2$;

4) inverse $\tau : \Gamma \to \Gamma : [g,\ lH] \mapsto [g^l,\ l^{-1} H]$;

5) identity section $\epsilon : G/H \to \Gamma : gH \mapsto [g,\ eH]$.

Theorem 5.12. *With the groupoid structure described above, $(G \times_H (L/H), \Pi)$ is a Poisson groupoid over $(G/H, \pi)$.*

The proof of Theorem 5.12 will be given in §5.5.

Remark 5.13. Assume that $\pi(eH) = 0$, so that we can take $\Lambda = 0$. Recall that G^* is the connected subgroup of D with Lie algebra $\mathfrak{g}^*$. Assume further that the map $G^* \times G \to D : (u, g) \mapsto ug$ is a diffeomorphism. Identify G with $G^* \backslash D$. Then the restriction to L of the right action of D on $G \cong G^* \backslash D$ integrates the infinitesimal action λ of $\mathfrak{l}$ on G. By Remark 5.6 and Lemma 5.11, Π is nondegenerate everywhere on $G \times_H (L/H)$. Thus $(G \times_H (L/H), \Pi)$ is a symplectic groupoid over $(G/H, \pi)$. Note that in this case, the bi-vector field π_P on $P \cong D$ is Poisson by Lemma 5.3 and everywhere nondegenerate by Lemma 5.5. Moreover, by Lemma 4.5, $\mathfrak{h}^0$ is a subalgebra of $\mathfrak{g}^*$. Let H^0 be the connected subgroup of G^* with Lie algebra $\mathfrak{h}^0$. Then $L = HH^0$ is a coisotropic submanifold (D, π_P) and $L/H \cong H^0$. Our construction of the symplectic structure Π on $G \times_H (L/H) \cong G \times_H H^0$ is a special case of coisotropic reduction for symplectic manifolds. In the special case when H is a Poisson subgroup of (G, π_G) and when $\pi = q_* \pi_G$, this construction was carried out in [26]

Example 5.14. Let G be a connected and simply connected Lie group and let X be the variety of Borel subgroups of G. Let G_0 be a real form of G and K a compact real form of G such that $K_0 := G \cap K$ is a maximal compact subgroup of G_0. Choose an Iwasawa decomposition $G = KAN$ of G such that the Borel subgroup B of G containing AN lies in the unique closed G_0-orbit in X. This choice of B gives rise to a Poisson Lie group (K, π_K) with AN as a dual Poisson Lie group. Although K_0 is not a Poisson Lie subgroup of (K, π_K), it is shown in [10] that the projection π of π_K is a well-defined Poisson structure on K/K_0, making $(K/K_0, \pi)$ a Poisson homogeneous space of (K, π_K), and the Drinfeld Lagrangian subalgebra associated to $\pi(eK_0)$ is $\mathfrak{g}_0$, the Lie algebra of G_0. Let $T = K \cap B$, a maximal torus of K. The set of T-orbits of symplectic leaves of π in K/K_0 is shown in [10] to be in one to one correspondence with the set of G_0-orbits in X. Due to the importance in representation theory of G_0-orbits in X, the Poisson geometrical properties of $(K/K_0, \pi)$ are worth further study. Since $\pi(eK_0) = 0$ and since $G = KAN = ANK$, the conditions in Remark 5.6 are satisfied. By Remark 5.13 and Theorem 5.12, $K \times_{K_0} (G_0/K_0)$ has the structure of a symplectic groupoid over K/K_0. More details of this example, in particular, the generalized Poisson cohomology of $(K/K_0, \pi)$, will be studied in a future paper.

5.5. **Proof of Theorem 5.12.** Let the assumptions be as in §5.4. We need two lemmas. Recall that G^* is the connected subgroup of D with Lie algebra $\mathfrak{g}^*$.

Lemma 5.15. *For any $g \in G$ and $l \in L$, $g^l l^{-1} g^{-1} \in G^*$.*

Proof. Fix $g \in G$ and $l \in L$. To avoid confusion with the notation set up in §1.1 for left and right translations on G, if $v \in T_g G$, we let $g^{-1}v \in \mathfrak{g}$ and $vg^{-1} \in \mathfrak{g}$ be the left and right translation of v by g^{-1}.

Let $l(t)$ be a smooth curve in L such that $l(0) = e$ and $l(1) = l$, and let $u(t) = g^{l(t)} l(t)^{-1} g^{-1} \in D$. Let $u'(t) \in T_{u(t)} D$ and $l'(t) \in T_{l(t)} L$ be respectively the derivatives of $u(t)$ and $l(t)$ at t. Let $x(t) = l(t)^{-1} l'(t) \in l$. Then, for every t,

$$u'(t) = \lambda_{x(t)}(g^{l(t)}) l(t)^{-1} g^{-1} - g^{l(t)} x(t) l(t)^{-1} g^{-1}$$

so by (4.13), $u'(t)u(t)^{-1} = -\mathrm{Ad}_{g^{l(t)}} x(t) + p_{\mathfrak{g}} \mathrm{Ad}_{g^{l(t)}} x(t) = -p_{\mathfrak{g}^*} \mathrm{Ad}_{g^{l(t)}} x(t) \in \mathfrak{g}^*$, where $p_{\mathfrak{g}}$ and $\mathfrak{p}_{\mathfrak{g}^*}$ are projections from $\mathfrak{d}$ to $\mathfrak{g}$ and $\mathfrak{g}^*$ with respect to the decomposition $\mathfrak{d} = \mathfrak{g} + \mathfrak{g}^*$. It follows from $u(0) = e$ that $u(t) \in G^*$ for all t. In particular, $g^l l^{-1} g^{-1} = u(1) \in G^*$. □

The following Lemma 5.16 is equivalent to 1) in Lemma 4.3, and we omit its proof.

Lemma 5.16. *One has $\mathrm{Ad}_h \Lambda \xi + p_{\mathfrak{g}} \mathrm{Ad}_h \xi - \Lambda \mathrm{Ad}^*_{h^{-1}} \xi \in \mathfrak{h}$ for all $\xi \in \mathfrak{h}^0$ and $h \in H$.*

We can now start the proof of Theorem 5.12.

Let $\mathcal{G}_\Gamma = \{(\gamma_1, \gamma_2, \gamma_3) \in \Gamma \times \Gamma \times \Gamma \,|\, t(\gamma_1) = s(\gamma_2),\ \gamma_3 = \gamma_1 \cdot_\Gamma \gamma_2\}$. By the definition of Poisson groupoids [24], we need to show that $\mathcal{G}_\Gamma$ is coisotropic in $\Gamma \times \Gamma \times \Gamma$ with the Poisson structure $\Pi \oplus \Pi \oplus (-\Pi)$. Let (P, π_P) be as in §5.2 and recall that $j : P \to P/H$ is the projection. Since (Γ, Π) is a Poisson submanifold of $(P/H, \Pi)$, and since $j : (P, \pi_P) \to (P/H, \Pi)$ is a bi-vector map, it is enough [22, Corollary 2.2.5] to show that $\mathcal{G}_P := (j \times j \times j)^{-1}(\mathcal{G}_\Gamma)$ is coisotropic in $(P \times P \times P,\ \pi_P \oplus \pi_P \oplus (-\pi_P))$. Recall that $\mathcal{O}$ is the L-orbit in D/G through $\underline{e} \in D/G$ and that $Q = G \times \mathcal{O} \subset P$. Identify$L/H$ with $\mathcal{O}$ by identifying $lH \in L/H$ with $\underline{l} = lG \in D/G$ for $l \in L$. Then

$$\begin{aligned}\mathcal{G}_P = \{ &\big((g_1,\, l_1H),\, (g_2,\, l_2H),\, (g_1h_3,\, h_3^{-1}l_1hl_2H)\big) \mid \\ &g_1, g_2, g_2 \in G,\, l_1, l_2 \in L,\, h_3 \in H,\, g_1^{l_1}H = g_2H,\, h = (g_1^{l_1})^{-1}g_2\} \subset Q \times Q \times Q.\end{aligned}$$

We will first describe the tangent bundle of $\mathcal{G}_P$ and then the co-normal bundle of $\mathcal{G}_P$ in $P \times P \times P$.

Let $\mathcal{G}_2 = \{((g_1,\, l_1H),\, (g_2,\, l_2H)) \,|\, g_1, g_2 \in G,\, l_1, l_2 \in L,\, g_1^{l_1}H = g_2H\} \subset Q \times Q$. We now compute $T_{(q_1,q_2)}\mathcal{G}_2$ for $(q_1, q_2) = ((g_1,\, l_1H),\, (g_2,\, l_2H)) \in \mathcal{G}_2$. Define $\tilde{t}, \tilde{s} : Q \to G/H$ by $\tilde{t}(g, lH) = g^lH$ and $\tilde{s}(g, lH) = gH$ for $g \in G$ and $l \in L$. Then

$$T_{(q_1,q_2)}\mathcal{G}_2 = \{(v_1, v_2) \,|\, v_1 \in T_{q_1}Q,\, v_2 \in T_{q_2}Q,\, \tilde{t}_*(v_1) = \tilde{s}_*(v_2)\}.$$

Recall that $\sigma : \mathfrak{d} \to \mathcal{V}^1(D/G)$ is given in (5.4). Let $\kappa : \mathfrak{g} \to \mathcal{V}^1(G/H) : x \to \kappa_x$ be the Lie algebra anti-homomorphism given by

$$\kappa_x(gH) = \frac{d}{dt}|_{t=0} \exp(tx)gH, \qquad x \in \mathfrak{g},\, g \in G. \tag{5.12}$$

For $x, z \in \mathfrak{g}$, $\zeta \in \mathfrak{g}^*$ with $z + \zeta \in \mathfrak{l}$, and $q = (g, lH) \in Q$, let

$$v_{x,\, z+\zeta}(q) = ((l_g)_*x,\ \sigma_{z+\zeta}(lH)) \in T_qQ.$$

(Recall from §1.1 that the "l" in l_g denotes the left translation by g. This is not to be confused with an element in L.) Recall that $p_\mathfrak{g} : \mathfrak{d} \to \mathfrak{g}$ is the projection along $\mathfrak{g}^*$. Using the fact that G is a $(\mathfrak{d}, L)$-space via the infinitesimal action λ of $\mathfrak{d}$ and the action of L, one sees that

$$\tilde{t}_*v_{x_1,\, z_1+\zeta_1}(q_1) = \kappa_{p_\mathfrak{g}\mathrm{Ad}_{g_1^{l_1}l_1^{-1}}(x_1+z_1+\zeta_1)}(g_1^{l_1}H), \qquad \text{for } x_1 \in \mathfrak{g},\, z_1 + \zeta_1 \in \mathfrak{l}.$$

Since $\tilde{s}_*v_{x_2,\, z_2+\zeta_2}(q_2) = \kappa_{\mathrm{Ad}_{g_2}x_2}(g_2H)$ for $x_2 \in \mathfrak{g}$, $z_2 + \zeta_2 \in \mathfrak{l}$, we get

$$\begin{aligned}T_{(q_1,q_2)}\mathcal{G}_2 = \{&(v_{x_1,\, z_1+\zeta_1}(q_1),\ v_{x_2,\, z_2+\zeta_2}(q_2)) \mid x_1, x_2 \in \mathfrak{g},\ z_1 + \zeta_1, z_2 + \zeta_2 \in \mathfrak{l}, \\ &x_2 = \bar{x}_2 + \mathrm{Ad}_{g_2^{-1}}p_\mathfrak{g}\mathrm{Ad}_{g_1^{l_1}l_1^{-1}}(x_1 + z_1 + \zeta_1) \text{ for some } \bar{x}_2 \in \mathfrak{h}\}.\end{aligned} \tag{5.13}$$

Define

$$f :\ \mathcal{G}_2 \longrightarrow Q :\ ((g_1,\, l_1H),\, (g_2,\, l_2H)) \longmapsto (g_1,\, l_1(g_1^{l_1})^{-1}g_2l_2H).$$

Fix $g_i \in G, l_i \in L$, for $i = 1, 2$, such that $(q_1, q_2) = ((g_1,\, l_1H),\, (g_2,\, l_2H)) \in \mathcal{G}_2$. Let $x_i \in \mathfrak{g}$ and $z_i + \zeta_i \in \mathfrak{l}$ be such that $(v_{x_1,\, z_1+\zeta_1}(q_1),\ v_{x_2,\, z_2+\zeta_2}(q_2)) \in T_{(q_1,q_2)}Q$ as in (5.13). Let $g_1(t), l_1(t)$, and $l_2(t)$ be smooth curves in G and L respectively such that $g_1(0) = g_1$, $g_1'(0) = (l_{g_1})_*x_1$, and $l_i(0) = l_i$, $l_i'(0) = (r_{l_i})_*(z_i + \zeta_i)$ for $i = 1, 2$, where the superscript $'$ denotes derivative at 0. Let $g_2(t) = g_1(t)^{l_1(t)}h \exp t\bar{x}_2$, where $h = (g_1^{l_1})^{-1}g_2 \in H$. It is easy to see that $g_2'(0) = (l_{g_2})_*x_2$. Let

$$c(t) = ((g_1(t),\ l_1(t)H),\ (g_2(t),\ l_2(t)H)) \in Q \times Q.$$

Then $c(t) \in \mathcal{G}_2$ for all t, $c(0) = (q_1, q_2)$, and $c'(0) = (v_{x_1,\, z_1+\zeta_1}(q_1),\ \ v_{x_2,\, z_2+\zeta_2}(q_2))$. Since $f(c(t)) = (g_1(t),\ l_1(t) h \exp t\bar{x}_2 l_2(t) H)$, we have

$$f_*(v_{x_1,\, z_1+\zeta_1}(q_1),\ \ v_{x_2,\, z_2+\zeta_2}(q_2)) = \frac{d}{dt}|_{t=0} f(c(t)) = v_{x_1,\, z_3+\zeta_3}(g_1,\, l_1 h l_2 H)),$$

where $z_3 \in \mathfrak{g}$ and $\zeta_3 \in \mathfrak{g}^*$ are such that

$$z_3 + \zeta_3 = z_1 + \zeta_1 + \mathrm{Ad}_{l_1 h}(\bar{x}_2 + \mathrm{Ad}_{l_2}\bar{x}_3 + z_2 + \zeta_2) \quad \text{for some } \bar{x}_3 \in \mathfrak{h}. \tag{5.14}$$

Thus for $(q_1, q_2, q_3) = ((g_1, l_1 H),\ (g_2, l_2 H),\ (g_1 h_3,\ h_3^{-1} l_1 h l_2 H)) \in \mathcal{G}_P$, where $h_3 \in H$,

$$\begin{aligned} T_{(q_1,q_2,q_3)}\mathcal{G}_P = \Big\{ \Big(& v_{x_1,\, z_1+\zeta_1}(q_1),\ \ v_{x_2,\, z_2+\zeta_2}(q_2),\ \ v_{x_3 + \mathrm{Ad}_{h_3}^{-1} x_1,\, -x_3 + \mathrm{Ad}_{h_3}^{-1}(z_3+\zeta_3)}(q_3) \Big) \ | \\ & x_1, x_2 \in \mathfrak{g},\ x_3 \in \mathfrak{h},\ z_i \in \mathfrak{g}, \zeta_i \in \mathfrak{g}^*,\ z_i + \zeta_i \in \mathfrak{l},\ i = 1,2,3, \\ & x_2 = \bar{x}_2 + \mathrm{Ad}_{g_2^{-1}} p_{\mathfrak{g}} \mathrm{Ad}_{g_1^{l_1} l_1^{-1}}(x_1 + z_1 + \zeta_1) \text{ for some } \bar{x}_2 \in \mathfrak{h}, \\ & z_3 + \zeta_3 = z_1 + \zeta_1 + \mathrm{Ad}_{l_1 h}(\bar{x}_2 + \mathrm{Ad}_{l_2}\bar{x}_3 + z_2 + \zeta_2) \text{ for some } \bar{x}_3 \in \mathfrak{h} \Big\}. \end{aligned}$$

Let $T^0_{(q_1,q_2,q_3)}\mathcal{G}_P$ be the co-normal subspace of $T_{(q_1,q_2,q_3)}\mathcal{G}_P$ in $T^*_{(q_1,q_2,q_3)}(P \times P \times P)$. Recall that for $\underline{d} \in D/G$, $\alpha_{y+\eta}(\underline{d}) \in T^*_{\underline{d}}(D/G)$ is given in (5.9). For $y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$, and $q = (g, lH) \in Q$, let $\alpha_{\xi,\, y+\eta}(q) = (l^*_{g^{-1}}\xi,\ \alpha_{y+\eta}(lH)) \in T^*_q P$. Then for $x, z \in \mathfrak{g}$ and $\zeta \in \mathfrak{g}^*$ with $z + \zeta \in \mathfrak{l}$,

$$(\alpha_{\xi,\, y+\eta}(q),\ v_{x,\, z+\zeta}(q)) = (x, \xi) + \langle y + \eta,\ z + \zeta\rangle = (x, \xi) + (y, \zeta) + (z, \eta), \tag{5.15}$$

and the map $\mathfrak{g}^* \times \mathrm{Ad}_l \mathfrak{g} \to T^*_q P : (\xi,\ y + \eta) \mapsto \alpha_{\xi,\, y+\eta}(q)$ is an isomorphism. Let $l_3 = h_3 l_1 h l_2 \in L$, where $h = (g_1^{l_1})^{-1} g_2$. It follows from (5.15) that $T^0_{(q_1,q_2,q_3)}\mathcal{G}_P$ consists of all triples

$$(\alpha_{\xi_1,\, y_1+\eta_1}(q_1),\ \alpha_{\xi_2,\, y_2+\eta_2}(q_2),\ \alpha_{\xi_3,\, y_3+\eta_3}(q_3)) \in T^*_{(q_1,q_2,q_3)}(P \times P \times P), \tag{5.16}$$

where $\xi_i, \eta_i \in \mathfrak{g}^*$, $y_i \in \mathfrak{g}$ and $y_i + \eta_i \in \mathrm{Ad}_{l_i}\mathfrak{g}$ for $i = 1, 2, 3$, such that

$$\begin{aligned} 0 = & (\xi_1, x_1) + (\xi_2,\ \mathrm{Ad}_{g_2^{-1}} p_{\mathfrak{g}} \mathrm{Ad}_{g_1^{l_1} l_1^{-1}} x_1) + (\xi_3,\ \mathrm{Ad}_{h_3^{-1}} x_1) \\ & + \langle y_1 + \eta_1,\ z_1 + \zeta_1\rangle + (\xi_2,\ \mathrm{Ad}_{g_2^{-1}} p_{\mathfrak{g}} \mathrm{Ad}_{g_1^{l_1} l_1^{-1}}(z_1 + \zeta_1)) + \langle y_3 + \eta_3,\ \mathrm{Ad}_{h_3^{-1}}(z_1 + \zeta_1)\rangle \\ & + (\xi_2, \bar{x}_2) + \langle y_3 + \eta_3,\ \mathrm{Ad}_{h_3^{-1} l_1 h}\bar{x}_2\rangle \\ & + \langle y_2 + \eta_2,\ z_2 + \zeta_2\rangle + \langle y_3 + \eta_3,\ \mathrm{Ad}_{h_3^{-1} l_1 h}(z_2 + \zeta_2)\rangle \\ & + (\xi_3, x_3) - \langle y_3 + \eta_3,\ x_3\rangle \end{aligned}$$

for all $x_1 \in \mathfrak{g}$, $\bar{x}_2, x_3 \in \mathfrak{h}$, $\mathfrak{z}_1 + \zeta_1 \in \mathfrak{l}$ and $z_2 + \zeta_2 \in \mathfrak{l}$, which is equivalent to

$$\xi_1 + \mathrm{Ad}_{l_1(g_1^{l_1})^{-1}} \mathrm{Ad}^*_{g_2^{-1}} \xi_2 + \mathrm{Ad}_{h_3}\xi_3 \in \mathfrak{g}; \tag{5.17}$$

$$y_1 + \eta_1 + \mathrm{Ad}_{l_1(g_1^{l_1})^{-1}} \mathrm{Ad}^*_{g_2^{-1}} \xi_2 + \mathrm{Ad}_{h_3}(y_3 + \eta_3) \in \mathfrak{l}; \tag{5.18}$$

$$\xi_2 + \mathrm{Ad}_{h^{-1} l_1^{-1} h_3}(y_3 + \eta_3) \in \mathfrak{g} + \mathfrak{l}; \tag{5.19}$$

$$y_2 + \eta_2 + \mathrm{Ad}_{h^{-1} l_1^{-1} h_3}(y_3 + \eta_3) \in \mathfrak{l}; \tag{5.20}$$

$$\xi_3 - \eta_3 \in \mathfrak{h}^0, \tag{5.21}$$

where recall that $\mathfrak{h}^0 = \{\xi \in \mathfrak{g}^* | (\xi, \mathfrak{h}) = 0\}$. Since $\mathfrak{g} + \mathfrak{l} = \mathfrak{g} + \mathfrak{h}^0$ by (4.7), (5.19) and (5.20) imply that $\eta_2 - \xi_2 \in \mathfrak{h}^0$. Similarly, (5.17), (5.18), and (5.21) imply that

$\eta_1 - \xi_1 \in \mathfrak{h}^0$. Thus

$$\Lambda(\eta_i - \xi_i) + \eta_i - \xi_i \in \mathfrak{l} \quad \text{for } i = 1, 2, 3. \tag{5.22}$$

It remains to show that for any triple in (5.16) satisfying (5.17) - (5.21),

$$(\tilde{\pi}_P(\alpha_{\xi_1, y_1+\eta_1}(q_1)),\ \tilde{\pi}_P(\alpha_{\xi_2, y_2+\eta_1}(q_2)),\ -\tilde{\pi}_P(\alpha_{\xi_3, y_3+\eta_3}(q_3))) \in T_{(q_1,q_2,q_3)}\mathcal{G}_P. \tag{5.23}$$

Let $g_3 = g_1 h_3$. By Lemma 5.4 and (5.22),

$$\tilde{\pi}_P(\alpha_{\xi_i, y_i+\eta_i}(q_i)) = v_{x_i, z_i+\zeta_i}(q_i) \in T_{q_i}Q$$

for $i = 1, 2, 3$, and

$$x_i = y_i + \Lambda\xi_i - \Lambda\eta_i + \mathrm{Ad}_{g_i^{-1}} p_{\mathfrak{g}} \mathrm{Ad}_{g_i}\xi_i, \quad z_i + \zeta_i = \Lambda(\eta_i - \xi_i) + \eta_i - \xi_i. \tag{5.24}$$

Thus by our description of $T_{(q_1,q_2,q_3)}\mathcal{G}_P$, to show (5.23), it suffices to show

$$\bar{x}_2 := x_2 - \mathrm{Ad}_{g_2^{-1}} p_{\mathfrak{g}} \mathrm{Ad}_{g_1^{l_1} l_1^{-1}}(x_1 + z_1 + \zeta_1) \in \mathfrak{h}, \tag{5.25}$$

$$x_1 + \mathrm{Ad}_{h_3} x_3 \in \mathfrak{h} \tag{5.26}$$

$$\mathrm{Ad}_{h_3}(x_3 + z_3 + \zeta_3) + x_1 + z_1 + \zeta_1 + \mathrm{Ad}_{l_1 h}(\bar{x}_2 + z_2 + \zeta_2) \in \mathrm{Ad}_{l_1 h l_2}\mathfrak{h}. \tag{5.27}$$

By (5.24),

$$x_i + z_i + \zeta_i = y_i + \eta_i - \mathrm{Ad}_{g_i^{-1}} \mathrm{Ad}^*_{g_i^{-1}}\xi_i, \qquad i = 1, 2, 3. \tag{5.28}$$

We first prove (5.25). Since $\mathrm{Ad}_{l_1^{-1}}(y_1 + \eta_1) \in \mathfrak{g}$ and $g_1^{l_1} l_1^{-1} g_1^{-1} \in G^*$ by Lemma 5.15,

$$\bar{x}_2 = y_2 - \Lambda\eta_2 + \Lambda\xi_2 + \mathrm{Ad}_{g_2^{-1}} p_{\mathfrak{g}} \mathrm{Ad}_{g_2}\xi_2 - \mathrm{Ad}_{h^{-1} l_1^{-1}}(y_1 + \eta_1).$$

Note from (5.18) that

$$\mathrm{Ad}_{h^{-1} l_1^{-1}}(y_1 + \eta_1) + \xi_2 - \mathrm{Ad}_{g_2^{-1}} p_{\mathfrak{g}} \mathrm{Ad}_{g_2}\xi_2 + \mathrm{Ad}_{h^{-1} l_1^{-1} h_3}(y_3 + \eta_3) \in \mathfrak{l},$$

so by (5.20), $\mathrm{Ad}_{g_2^{-1}} p_{\mathfrak{g}} \mathrm{Ad}_{g_2}\xi_2 - \mathrm{Ad}_{h^{-1} l_1^{-1}}(y_1 + \eta_1) - \xi_2 + y_2 + \eta_2 \in \mathfrak{l}$. It follows from (4.7) that $\bar{x}_2 \in \mathfrak{h}$, so (5.25) holds. To prove (5.26), note that (5.17) implies that $p_{\mathfrak{g}^*} \mathrm{Ad}_{l_1 (g_1^{l_1})^{-1}} \mathrm{Ad}^*_{g_2^{-1}}\xi_2 = -\xi_1 - \mathrm{Ad}^*_{h_3^{-1}}\xi_3$, so by (5.18) and (4.7),

$$y_1 + p_{\mathfrak{g}} \mathrm{Ad}_{l_1 (g_1^{l_1})^{-1}} \mathrm{Ad}^*_{g_2^{-1}}\xi_2 + \mathrm{Ad}_{h_3} y_3 + p_{\mathfrak{g}} \mathrm{Ad}_{h_3}\eta_3 + \Lambda(\xi_1 - \eta_1) + \Lambda \mathrm{Ad}^*_{h_3^{-1}}(\xi_3 - \eta_3) \in \mathfrak{h}.$$

Thus by (5.17), (5.26) is equivalent to

$$\mathrm{Ad}_{h_3}\Lambda(\xi_3 - \eta_3) + p_{\mathfrak{g}} \mathrm{Ad}_{h_3}(\xi_3 - \eta_3) - \Lambda \mathrm{Ad}^*_{h_3^1}(\xi_3 - \eta_3) \in \mathfrak{h}$$

which holds because of Lemma 5.16. This proves (5.26). It remains to prove (5.27). Using Lemma 5.15 and (5.28), one sees that the left hand side of (5.27) is equal to

$$\mathrm{Ad}_{h_3}(y_3 + \eta_3) + \mathrm{Ad}_{l_1 h}(y_2 + \eta_2) - \mathrm{Ad}_{g_1^{-1}} p_{\mathfrak{g}^*} \mathrm{Ad}_{g_1}(\xi_1 + \mathrm{Ad}_{h_3}\xi_3) - \mathrm{Ad}_{l_1 (g_1^{l_1})^{-1}} \mathrm{Ad}^*_{g_2^{-1}}\xi_2.$$

By (5.17), $\mathrm{Ad}_{g_1^{-1}} p_{\mathfrak{g}^*} \mathrm{Ad}_{g_1}(\xi_1 + \mathrm{Ad}_{h_3}\xi_3) + \mathrm{Ad}_{l_1 (g_1^{l_1})^{-1}} \mathrm{Ad}^*_{g_2^{-1}}\xi_2 = 0$. Moreover, since $y_2 + \eta_2 \in \mathrm{Ad}_{l_2}\mathfrak{g}$ and $y_3 + \eta_3 \in \mathrm{Ad}_{h_3^1 l_1 h l_2}\mathfrak{g}$,

$$y_2 + \eta_2 + \mathrm{Ad}_{h^{-1} l_1^{-1} h_3}(y_3 + \eta_3) \in \mathrm{Ad}_{l_2}\mathfrak{g} \cap l = \mathrm{Ad}_{l_2}\mathfrak{h}$$

by (5.20). Thus the left hand side of (5.27) is in $\mathrm{Ad}_{l_1 h l_2}\mathfrak{h}$, so (5.27) holds.

This finishes the proof of Theorem 5.12.

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