A Higher-Order Markov-Switching Model for Risk Measurement

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Abstract

In this paper, we introduce a discrete-time higher-order Markov-switching (HMS) model for measuring the risk of a portfolio. We suppose that the logarithmic returns from a risky portfolio is governed by a HMS model with the drift and the volatility switch over time according to the states of a discrete-time higher-order hidden Markov model (HHMM). We interpret the states of the HHMM as unobservable states of an economy. The HHMM can incorporate the persistence or the long-range dependence of the economic states, which may due to business cycles. We adopt Value-at-Risk (VaR) as proxies of risk and investigate the impact of long-range dependence on risk measurement by comparing the VaR obtained from the HMS model and those evaluated from the first-order Markov-switching model through back-testing.

Keywords: Value at Risk; Long-range Dependence; Higher-order Markov Chain; Regime Switching.

1 Introduction

Value at Risk (VaR) has emerged as one of the most prominent tools in finance and insurance industries. Many regulatory bodies, financial institutions and insurance companies adopt VaR as a benchmark in their practices of risk measurement and management. VaR is a statistical estimation of a portfolio's loss in such a way that the owner of the portfolio prepares to incur that loss or more with a given (small) probability level over a given (short) time horizon for risk measurement. Jorion [19], Duffie and Pan [8], Best

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[5] and Embrechts [17] and J.P. Morgan's RiskMetrics - Technical Document for excellent and comprehensive accounts on the concept of VaR and its practical implementation. Basically, there are two common approaches to the VaR implementation, namely (i) the historical simulation and (ii) the model-based approach. The historical simulation is to calculate VaR based on the empirical distribution of historical data by bootstrapping it which is rather non-parametric in nature while the model-based method assumes a particular parametric form for the distributions of financial returns and estimate the corresponding unknown parameters from historical data. Although VaR remains as a popular and prominent tools for risk measurement and management in both the finance and insurance industries, some of the literature, including Artzner et al. [3, 4], Acerbi et al. [1, 2] and Yamai and Yoshiba [22], point out the theoretical shortcomings of VaR. Despite some theoretical shortcomings, VaR still remains its prominent role as a popular measure of risk in practice due to its simplistic interpretation which makes it a easily communicated risk measure in financial reporting and its computational tractability under some specific parametric assumptions, such as multivariate normality assumption.

Recently, there has been considerable interest in the applications of hidden Markov model (HMM) in finance. Elliott et al. [10] and Elliott and Kopp [15] provide an excellent overview of hidden Markov chain and its applications in finance. Some works on exploring the financial applications of hidden Markov chain include Elliott and van der Hoek [11] for portfolio optimization, Pliska [20] and Elliott et al. [13] for modelling short rate dynamics, Guo [18] and Elliott et al. [16] for option pricing under market incompleteness, Buffington and Elliott [6, 7] for pricing European and American options, Elliott et al. [14] for volatility estimation. There is a relatively little amount of work on exploring the use of the hidden Markov chain for risk measurement.

In this paper, we introduce a discrete-time higher-order Markov-switching (HMS) model for measuring the risk of a portfolio. We suppose that the logarithmic returns from a risky portfolio is governed by a HMS model with the drift and the volatility switch over time according to the states of a discrete-time higher-order hidden Markov chain model (HHMM). We interpret the states of the HHMM as unobservable states of an economy. Recently a double higher-order hidden Markov model has been proposed for extracting information from spot interest rates and credit ratings [21]. The HHMM can incorporate the persistence or the long-range dependence of the economic states, which may be due to business cycles. We adopt Value-at-Risk (VaR) as proxies of risk. We investigate the impact of long-range dependence on risk measurement by comparing the VaR evaluated from the HMS model and that obtained from the first-order hidden Markov chain model. In particular, we perform back-testing for the VaR obtained from the two models using simulated data.

The rest of the paper is organized as follows. In Section 2, we present the higher-order Markov-switching model for modelling the market values of a market portfolio and the corresponding risk measurement framework. In Section 3, we perform back-testing for the VaR evaluated from the the first-order and the second-order Hidden Markov chain models using simulated data. Finally, concluding remarks are given in Section 4.

2 A Higher-Order Markov-Switching Model for Portfolio Returns

In this section, we present a discrete-time Higher-order Markov-Switching (HMS) model for portfolio returns. First, we write $\mathcal{T}$ for the time index set

$$\{0, 1, \dots\}$$

of the economy. Fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability. We note that for risk measurement, we are working with the real-world probability $\mathcal{P}$ instead of the risk-neutral one. We suppose that the probability space is rich enough to incorporate the uncertainties due to the evolution of the hidden economic states and the fluctuations of market values of the risky portfolio.

In the sequel, we shall define a higher-order hidden Markov chain model for describing the hidden states of an economy. Let $\{V_t\}_{t\in\mathcal{T}}$ be a process representing the hidden states of an economy. We then assume that $\{V_t\}_{t\in\mathcal{T}}$ is an l^{th}-order discrete-time homogeneous Markov chain process taking values in the state space

$$\mathcal{V} := \{v_1, v_2, \dots, v_M\}.$$

We may interpret v_1 as the "best" economic condition, v_2 as the second "best" economic condition and v_M as the "worst" economic condition, etc. Now we let

$$\mathbf{i}(t, l) := (i_t, i_{t-1}, \dots, i_{t-l}),$$

where $t \geq l$, $l = 1, 2, \dots$ and $i_t, i_{t-1}, \dots, i_{t-l} \in \{1, 2, \dots, M\}$. Then, the state transition probabilities of the l^{th}-order hidden Markov chain process V are given by:

$$P(i_{t+1}|\mathbf{i}(t, l)) := P[V_{t+1} = V_{i_{t+1}}|V_t = v_{i_t}, \dots, V_{t-l} = v_{i_{t-l}}] \ , \quad i_{t+1} = 1, \dots, M. \tag{2.1}$$

The order l represents the degree of the long-range dependence of the hidden states of the economy. When $l = 1$, the l^{th}-order hidden Markov chain process V reduces to a first-order hidden Markov chain process.

In order to determine the HHMM completely, we need to specify the following initial distributions:

$$P(i_{l+1}|\mathbf{i}(l, l)) := \pi_{i_{l+1}|\mathbf{i}(l,l)}, \quad i_{t+1} = 1, 2, \dots, M. \tag{2.2}$$

We shall then describe the higher-order Markov-switching (HMS) model for portfolio returns. The main idea of the HMS model is that the drift and the volatility of the portfolio returns switch over time according to the states of the economy described by the HHMM.

Let $\{Y_t\}_{t\in\mathcal{T}}$ denote a stochastic process on $(\Omega, \mathcal{F}, \mathcal{P})$ such that Y_t represents the logarithmic return of a risky portfolio in the t^{th} period. We write $\mathbf{V}_{t,l}$ for $(V_t, V_{t-1}, \ldots, V_{t-l})$, for each $t \geq l$, $l = 1, 2, \ldots$. Let

$$\mu_t := \mu(\mathbf{V}_{t,l})$$

and

$$\sigma_t := \sigma(\mathbf{V}_{t,l})$$

be the drift and the volatility of the portfolio return in the t^{th} period respectively. We note that both the drift and the volatility depend on the current value and the past values of the HHMM up to lag l. In particular,

$$\mu(v_{i_t}, v_{i_{t-1}}, \ldots, v_{i_{t-l+1}}) = \mu_{\mathbf{i}(t,l)}, \tag{2.3}$$

and

$$\sigma(v_{i_t}, v_{i_{t-1}}, \ldots, v_{i_{t-l+1}}) = \sigma_{\mathbf{i}(t,l)}, \tag{2.4}$$

where $\mu_{\mathbf{i}(t,l)} \in \mathcal{R}$ and $\sigma_{\mathbf{i}(t,l)} > 0$.

Let $\{\xi_t\}_{t=1,2,\ldots}$ denote a sequence of i.i.d. random variables with common distribution $N(0,1)$, a standard normal distribution with zero mean and unit variance. We suppose that ξ and V are independent. Then, we suppose that the dynamics of the logarithmic returns of the portfolio are evoluted according to the following HMS model:

$$Y_t = \mu(\mathbf{V}_{t,l}) + \sigma(\mathbf{V}_{t,l})\xi_t, \ t = 1, 2, \ldots \tag{2.5}$$

By convention, $Y_0 = 0$, $\mathcal{P}$-a.s.

The structure of the HMS model resembles the continuous-state observation process in Elliott et al. [10]and Elliott et al. [12], in which first-order HMMs were considered and the drift and the volatility depend on the state V_{t-1}.

In the sequel, for simplicity of discussion, we shall consider the case when $l = 2$ (i.e., a second-order HMM). In this case, the dynamics of the logarithmic returns of the portfolio are given by:

$$Y_t = \mu(V_t, V_{t-1}) + \sigma(V_t, V_{t-1})\xi_t, \ t = 1, 2, \ldots \tag{2.6}$$

Now, we define a two-dimensional first-order HMM X embedding the second-order HMM V as follows:

$$X_t := (V_t, V_{t-1}). \tag{2.7}$$

Write $\mathcal{X}$ for a $(M \times M)$-matrix with the (i, j)-element

$$x_{ij} := (v_i, v_j),$$

for each $i, j = 1, 2, \ldots, M$. We note that $\mathcal{X}$ represents the state space of the two-dimensional first-order HMM X. Let

$$\tilde{\mathcal{X}} := vec(\mathcal{X}),$$

where $vec(\cdot)$ denotes the column-by-column vectorization function. We note that $\tilde{\mathcal{X}}$ denotes a M^2-dimensional vector. In particular, the $((j-1)M+i)^{th}$-element $\tilde{x}_{(j-1)M+i}$ of $\tilde{\mathcal{X}}$ is given by $x_{ij} := (v_i, v_j)$. Then, we define a one-dimensional first-order HMM $\tilde{X}$ induced by the two-dimensional first-order HMM X such that

$$\tilde{X}_t = \tilde{x}_{(j-1)M+i}$$

if $X_t = x_{ij}$. The state space of $\tilde{X}$ is given by $\tilde{\mathcal{X}}$. Following Elliott [9], we consider a function $\psi_j(\cdot)$ $(j = 1, 2, \ldots, M^2)$ defined by

$$\psi_j(\tilde{x}_k) := \delta_{jk}, \quad \text{where} \quad k = 1, 2, \ldots, M^2.$$

We define a vector function

$$\psi(\tilde{x}) := (\psi_1(\tilde{x}), \psi_2(\tilde{x}), \ldots, \psi_{M^2}(\tilde{x})),$$

for each $\tilde{x} \in \tilde{\mathcal{X}}$. Then, $\psi(\cdot)$ is a bijection of $\tilde{\mathcal{X}}$ and the set of unit basis vectors

$$\mathcal{E} := \{e_1, e_2, \ldots, e_{M^2}\}$$

with

$$e_j := (0, 0, \ldots, \underbrace{1}_{jth\ entry}, \ldots, 0, 0) \in \mathcal{R}^{M^2},$$

where $j = 1, 2, \ldots, M^2$ and the "1" is in the j^{th} position of e_j. Without loss of generality, the state space of $\tilde{X}$ can be taken to be the set $\mathcal{E}$. This is called the canonical representation of the state space of a Markov chain process. Let A be a $(M^2 \times M^2)$-matrix, which represents the time-independent transition probability matrix of the first-order Markov chain $\tilde{X}$. The (j, k)-element a_{jk} of A $(j, k = 1, 2, \ldots, M^2)$ is given by:

$$a_{jk} := P(\tilde{X}_t = e_j | \tilde{X}_{t-1} = e_k). \tag{2.8}$$

Following Elliott et al. [10], the semi-martingale representation for $\tilde{X}$ is given by:

$$\tilde{X}_t := A\tilde{X}_{t-1} + L_t, \tag{2.9}$$

where $\{L_t\}$ is a $\mathcal{R}^{M^2}$ martingale increment process with respect to the complete filtration $\mathcal{F}^{\tilde{X}}$ generated by the process $\tilde{X}$ and the measure $\mathcal{P}$.

Then, we specify the information structure of the model. Let $\mathcal{F}_t^Y$ and $\mathcal{F}_t^V$ denote the information set generated by the logarithmic return process Y and the hidden Markov chain process V up to and including time t, respectively, for each $t \in \mathcal{T}$. For the purpose of adaptive risk measurement, one is interested in computing the predictive distribution $F_{Y_{t+1}}(\cdot|\mathcal{F}_t^Y)$ of Y_{t+1} given the observable information $\mathcal{F}_t^Y$.

For each $i, j = 1, 2, \ldots, M$, let

$$\phi_{ij}(x) := \frac{1}{\sqrt{2\pi\sigma_{ij}^2}} \exp(-\frac{1}{2\sigma_{ij}^2} x^2),$$

which is the probability density function of a normal distribution $N(0, \sigma_{ij}^2)$ with mean zero and variance σ_{ij}^2. Then,

$$F_{Y_{t+1}}(y|\mathcal{F}_t^Y) = \sum_{i=1}^{M} \sum_{j=1}^{M} P[(V_t, V_{t-1}) = (v_i, v_j)|\mathcal{F}_t^Y] \int_{-\infty}^{y-\mu_{ij}} \phi_{ij}(x)dx. \quad (2.10)$$

The predictive density $f_{Y_{t+1}}(y|\mathcal{F}_t^Y)$ of Y_{t+1} given $\mathcal{F}_t^Y$ is given by:

$$f_{Y_{t+1}}(y|\mathcal{F}_t^Y) = \sum_{i=1}^{M} \sum_{j=1}^{M} P[(V_t, V_{t-1}) = (v_i, v_j)|\mathcal{F}_t^Y] \phi_{ij}(y - \mu_{ij}). \quad (2.11)$$

In order to compute either $F_{Y_{t+1}}(y|\mathcal{F}_t^Y)$ or $f_{Y_{t+1}}(y|\mathcal{F}_t^Y)$, one needs to determine

$$P[(V_t, V_{t-1}) = (v_i, v_j)|\mathcal{F}_t^Y].$$

We note that

$$P[(V_t, V_{t-1}) = (v_i, v_j)|\mathcal{F}_t^Y] = P(X_t = x_{ij}|\mathcal{F}_t^Y) = P(\tilde{X}_t = \tilde{x}_{(j-1)M+i}|\mathcal{F}_t^Y). \quad (2.12)$$

Hence,

$$F_{Y_{t+1}}(y|\mathcal{F}_t^Y) = \sum_{i=1}^{M} \sum_{j=1}^{M} P(\tilde{X}_t = \tilde{x}_{(j-1)M+i}|\mathcal{F}_t^Y) \int_{-\infty}^{y-\mu_{ij}} \phi_{ij}(x)dx,$$

and

$$f_{Y_{t+1}}(y|\mathcal{F}_t^Y) = \sum_{i=1}^{M} \sum_{j=1}^{M} P(\tilde{X}_t = \tilde{x}_{(j-1)M+i}|\mathcal{F}_t^Y) \phi_{ij}(y - \mu_{ij}).$$

Both $F_{Y_{t+1}}(y|\mathcal{F}_t^Y)$ and $f_{Y_{t+1}}(y|\mathcal{F}_t^Y)$ can be determined by the conditional probability distribution of $\tilde{X}_t$ given $\mathcal{F}_t^Y$.

Let $\tilde{X}_t^Y := E_{\mathcal{P}}(\tilde{X}_t|\mathcal{F}_t^Y)$. Then,

$$F_{Y_{t+1}}(y|\mathcal{F}_t^Y) = \sum_{i=1}^{M}\sum_{j=1}^{M}\left\langle \tilde{X}_t^Y, e_{(j-1)M+i}\right\rangle \int_{-\infty}^{y-\mu_{ij}} \phi_{ij}(x)dx,$$

and

$$f_{Y_{t+1}}(y|\mathcal{F}_t^Y) = \sum_{i=1}^{M}\sum_{j=1}^{M}\left\langle \tilde{X}_t^Y, e_{(j-1)M+i}\right\rangle \phi_{ij}(y-\mu_{ij}),$$

where $\langle \cdot, \cdot \rangle$ denotes an inner product in $\mathcal{R}^{M^2}$.

Then, following Elliott et al. (1994) (Chapter 3, Theorem 3.3) and the Bayes rule, we can get a recursive filter for $\tilde{X}_t^Y$ as follows:

$$\begin{aligned} \tilde{X}_{t+1}^Y &:= E_{\mathcal{P}}(\tilde{X}_{t+1}|\mathcal{F}_{t+1}^Y) \\ &= \frac{\sum_{i=1}^{M}\sum_{j=1}^{M}\left\langle \tilde{X}_t^Y, e_{(j-1)M+i}\right\rangle \phi_{ij}(y_{t+1}-\mu_{ij})Ae_{(j-1)M+i}}{\sum_{i=1}^{M}\sum_{j=1}^{M}\left\langle \tilde{X}_t^Y, e_{(j-1)M+i}\right\rangle \phi_{ij}(y_{t+1}-\mu_{ij})} . \end{aligned} \tag{2.13}$$

Let $q_{t+1|t}(\alpha)$ denote the α-quantile of the predictive distribution of Y_{t+1} given $\mathcal{F}_t^Y$. Then,

$$F_{Y_{t+1}}(q_{t+1|t}(\alpha)|\mathcal{F}_t^Y) = \alpha. \tag{2.14}$$

Let P_t denote the market value of the portfolio at time t. We suppose that the constant risk-free interest rate over a period is 5%. Then, the Value-at-Risk $VaR_{t+1|t}(\alpha)$ for the long position of the portfolio with probability level α is given by:

$$VaR_{t+1|t}(\alpha) = P_t[1 - \exp(q_{t+1|t}(\alpha) - r)]. \tag{2.15}$$

Now, we present the case of the first-order hidden Markov chain $Z := \{Z_t\}_{t\in\mathcal{T}}$ with state space $\mathcal{Z} := \{e_1, e_2, \ldots, e_M\} \subset \mathcal{R}^M$. Write $P := [p_{ij}]_{i,j=1,2,\ldots,M}$, which represents the transition probability matrix for Z. That is,

$$p_{ij} := P(Z_t = e_i|Z_{t-1} = e_j). \tag{2.16}$$

We suppose that the drift $\mu(Z_t)$ and the volatility $\sigma(Z_t)$ of the logarithmic return from the portfolio are given by:

$$\mu(Z_t) = \langle \mu, Z_t \rangle , \tag{2.17}$$

and

$$\sigma(Z_t) = \langle \mu, Z_t \rangle , \tag{2.18}$$

where $\mu := (\mu_1, \mu_2, \ldots, \mu_M)$ and $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_M)$. with $\mu_i \in \mathcal{R}$ and $\sigma_i > 0$ for each $i = 1, 2, \ldots, M$.

In this case, we assume that the logarithm return is given by:

$$Y_t = \mu(Z_t) + \sigma(Z_t)\xi_t, \tag{2.19}$$

Following Elliott et al. (1994), it can be shown that the probability distribution function of Y_{t+1} given $\mathcal{F}_t^Y$ is:

$$G_{Y_{t+1}}(y|\mathcal{F}_t^Y) = \sum_{i=1}^{M} \left\langle \tilde{Z}_t^Y, e_i \right\rangle \int_{-\infty}^{y-\mu_i} \phi_i(x)dx, \tag{2.20}$$

where $\phi_i(x)$ is defined as above and $\tilde{V}_t^Y$ is evaluated from the following recursive formula:

$$\tilde{Z}_{t+1}^Y = \frac{\sum_{i=1}^M \left\langle \tilde{Z}_t^Y, e_i \right\rangle \phi_i(y_{t+1} - \mu_i) P e_i}{\sum_{i=1}^M \left\langle \tilde{Z}_t^Y, e_i \right\rangle \phi_i(y_{t+1} - \mu_i)}. \tag{2.21}$$

Then, the α-quantile $\bar{q}_{t+1|t}(\alpha)$ of Y_{t+1} given $\mathcal{F}_t^Y$ is determined by:

$$G_{Y_{t+1}}(\bar{q}_{t+1|t}(\alpha)|\mathcal{F}_t^Y) = \alpha. \tag{2.22}$$

Then, the VaR in this case is given by:

$$VaR_{t+1|t}(\alpha) = P_t[1 - \exp(\bar{q}_{t+1|t}(\alpha) - r)]. \tag{2.23}$$

3 Back-testing

In this section, we shall investigate consequences for risk measurement of the presence of the long-range dependence in the price dynamics of the portfolio described by the higher-order Hidden Markov chain. In particular, we study the impact of the misspecification of the order of the hidden Markov chain (i.e. different levels of long-range dependence) on the performance of VaR forecasts via back-testing. We suppose that the order of HHMM under the "true" model is two and the length of the sequence T is chosen as 1000. We simulate the logarithmic returns from the "true" model and consider them as if they were the observed logarithmic returns from the portfolio. For simplicity, we assume the interest rate r is equal to 0. Then, we evaluate forecasting performances of the VaR obtained from the second-order HHMM and that obtained from its first-order counterpart. Here, we assume that the economy has two states, namely, "Good" state and "Bad" state. State "1" and State "2" represent "Good" state and "Bad" state. The numerical results were computed using CPU=2.66Ghz, Ram=1Ghz with Matlab environment.

First, we assume the specimen values of the "true" model. Here, we define

$$P(i|j,k) = P(V_{t+1} = i | V_t = j, V_t = k).$$

In particular, we suppose that the transition probabilities of the second-order HHMM are given:

$$\mathbf{P} = \{[P]_{(j-1)*l+k,i}\} = \begin{pmatrix} P(1|,1,1) & P(2|1,1) \\ P(1|,1,2) & P(2|1,2) \\ P(2|2,1) & P(2|2,1) \\ P(2|2,2) & P(2|2,2) \end{pmatrix} = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \\ 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}.$$

Then, we assume that

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} = \begin{pmatrix} 0.3968 & 0.1984 \\ 0.3175 & 0.1190 \end{pmatrix} \times 10^{-3}$$

and

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \begin{pmatrix} 0.63 & 1.57 \\ 0.94 & 2.52 \end{pmatrix} \times 10^{-2}.$$

Based on these parameter values, one can obtain the transition probability matrix A for the first order HMS as follows:

$$A = P$$

and the transition probability matrix A for the second-order HMS as follows:

$$A = \begin{pmatrix} 0.7 & 0.0 & 0.3 & 0.0 \\ 0.3 & 0.0 & 0.7 & 0.0 \\ 0.0 & 0.6 & 0.0 & 0.3 \\ 0.0 & 0.4 & 0.0 & 0.7 \end{pmatrix}.$$

We then assume that the initial values of the second-order HHMM are $V_0 = 2$ and $V_1 = 2$. We further assume that $Y(0) = 0$ and $\tilde{X}_0^Y = (0,0,0,1)^T$. Based on these parameters and the initial conditions, we shall simulate the second-order HHMM $\{V_t\}_{t\in\mathcal{T}}$ and the logarithmic returns of the portfolio $\{Y_t\}_{t\in\mathcal{T}}$. These simulated data will be used to perform back-testing for the second-order HMS and the first-order HMM.

Based on these parameter values, we can obtain the transition probability matrix A for the first order HMM as follows:

$$A = P$$

and the transition probability matrix A for the second order as follows:

$$A = \begin{pmatrix} 0.7 & 0.0 & 0.3 & 0.0 \\ 0.3 & 0.0 & 0.7 & 0.0 \\ 0.0 & 0.6 & 0.0 & 0.3 \\ 0.0 & 0.4 & 0.0 & 0.7 \end{pmatrix}$$

Now, for the first-order HMM, we suppose that

$$\mu_1 = \mu_{11}, \quad \mu_2 = \mu_{12}, \quad \sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{12},$$

$$p_{11} = P(1|1,1), \quad p_{21} = P(1|2,1) \quad \text{and} \quad \tilde{X}_0^Y = (0,1)^T.$$

In the sequel, we shall present the numerical results on the second-order and the first-order cases, In practice, the probability level α for VaR computation is usually set to be either 5% or 1% depending on various practical purposes. Here, we consider two scenarios, namely, Case I: $\alpha = 1\%$ and Case II: $\alpha = 5\%$.

3.1 Case I: $\alpha = 1\%$

Figure 1 displays the plots of the daily change in the market values of the portfolio and the 1% daily VaR obtained from the first-order HMM against the back-testing periods. Figure 2 presents the plots of the daily change in the market values of the portfolio and the 1% daily VaR obtained from the second-order HMM against the back-testing periods.

From these figures, we can see that the proportions of violations of the VaR obtained from the first-order and the second-order models are 4.1% and 1.3%, respectively. Hence this reveals that the first-order model seriously underestimates the VaR if the "true" model is the second-order one.

3.2 Case II: $\alpha = 5\%$

Figure 3 presents the plots of the daily change in the market values of the portfolio and the 5% daily VaR obtained from the first-order HMM against the back-testing periods. Figure 4 displays the plots of the daily change in the market values of the portfolio and the 5% daily VaR obtained from the second-order HMM against the back-testing periods.

From these figures, we observe that the proportions of violations of the VaR obtained from the first-order and the second-order models are 8.8% and 5.5%, respectively. This shows that the first-order model also seriously underestimates the VaR if the "true" model is the second-order one.

Based on these numerical results, we observe that the mis-specification of the order of the hidden Markov chain has significant impact on the VaR estimation. In particular, if the level of the long-range dependence is underestimated, the VaR will also be underestimated substantially. We also note that the computational time for both the first-order and the second-order models are less than one minute in an standard PC. Both models are computationally efficient.

4 Conclusion

In this paper, we proposed a discrete-time higher-order Markov-switching (HMS) model for measuring the risk of a portfolio. We adopt Value-at-Risk (VaR) as proxies of risk and investigate the impact of the long-range dependence on risk measurement by comparing the risk measures obtained from the HMS model and those evaluated from the first-order Markov-switching model through back-testing. Numerical results are given to illustrate that the mis-specification of the order of the hidden Markov chain has significant impact on VaR estimation. We have found that the underestimation of the level of the long-range dependence will lead to underestimation of the VaR.

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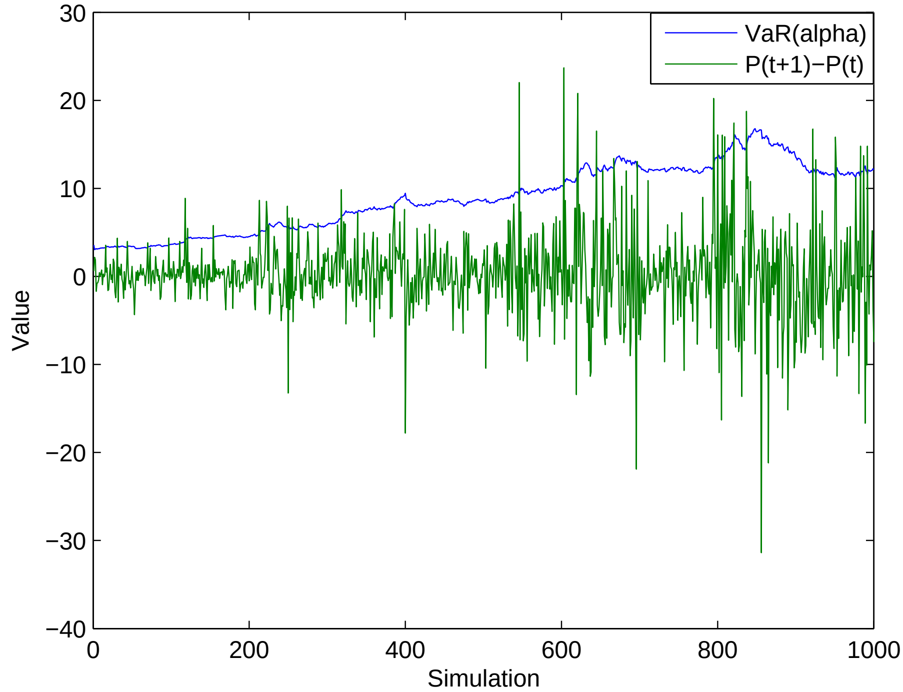

Figure 1: Back-testing for the first-order HHMM with $\alpha = 1\%$

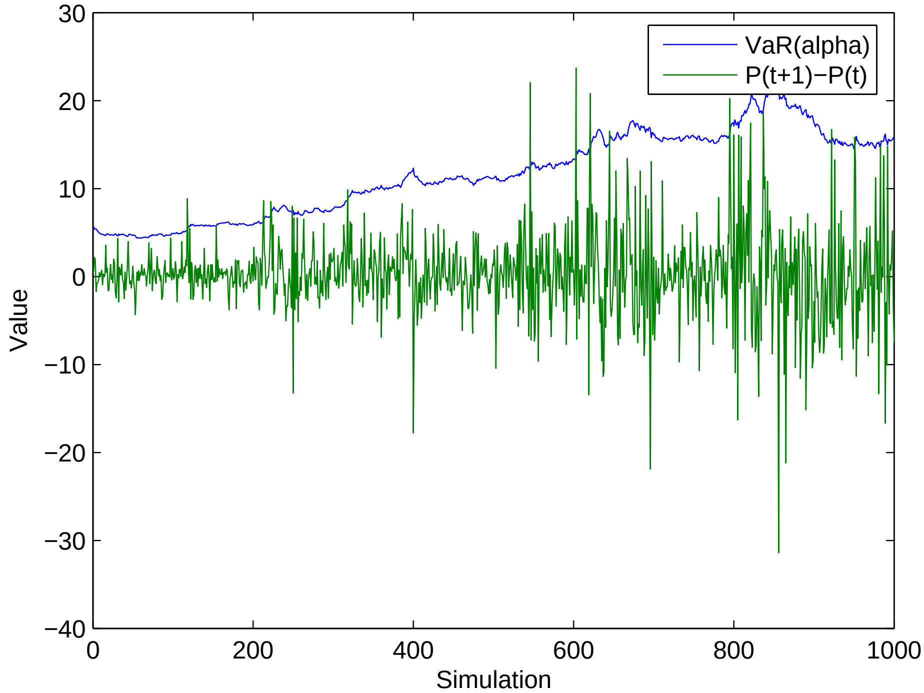

Figure 2: Back-testing for the second-order HHMM with $\alpha = 1\%$

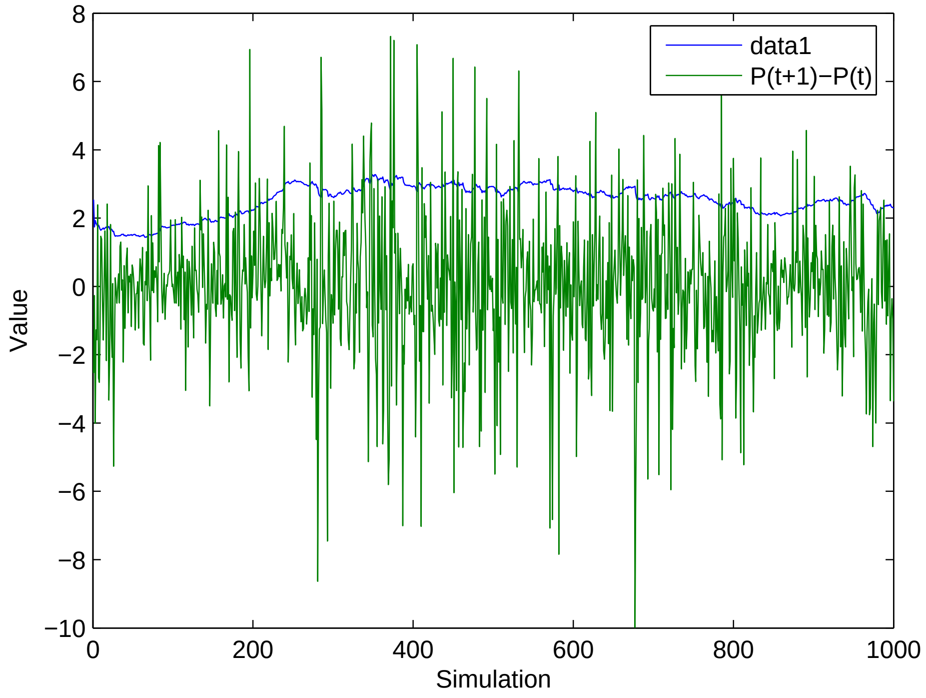

Figure 3: Back-testing for the first-order HHMM with $\alpha = 5\%$

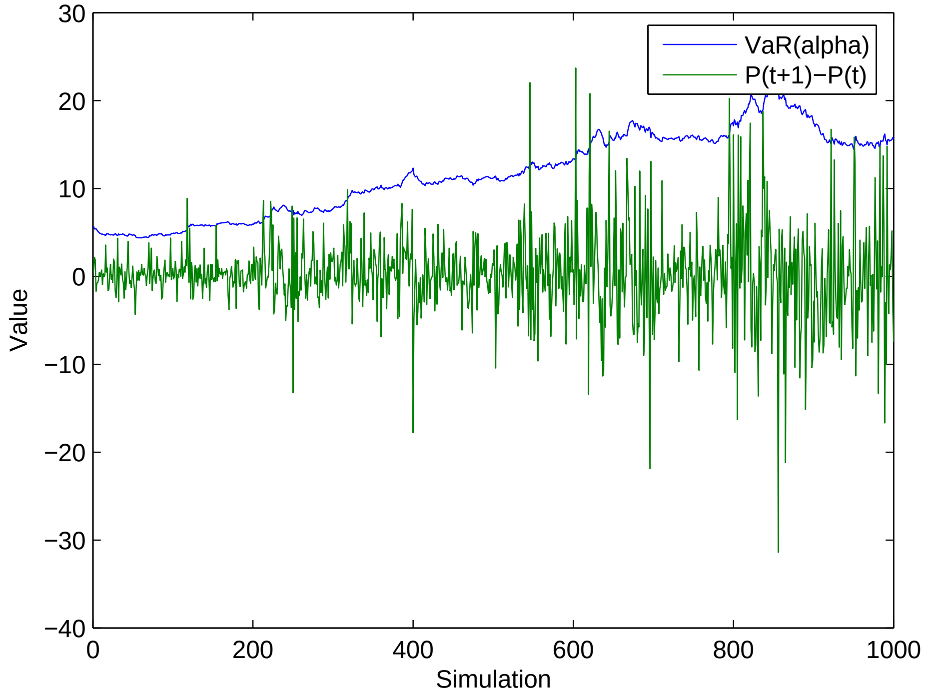

Figure 4: Back-testing for the second-order HHMM with $\alpha = 5\%$