Pricing Exotic Options Under a Higher-order Hidden Markov Model

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Abstract

In this paper, we consider the pricing of exotic options when the price dynamic of the underlying risky asset is governed by a discrete-time Markov-modulated process driven by a Higher-order Hidden Markov Model (HHMM). We assume that the market interest rate, the drift and the volatility of the underlying risky asset's return switch over time according to the states of the HHMM. Here the states of the HHMM are interpreted as the states of an economy. The advantage of the HHMM is that it can capture the long-range dependence of the states of the economy. We will then employ the well-known tool in actuarial science, namely, the Esscher transform to determine an equivalent martingale measure for option valuation. Moreover, we will also investigate the impact of the long-range dependence of the states of the economy on the prices of some path-dependent exotic options, such as Asian options, lookback options and barrier options.

Keywords: Asian Options; Barrier Options; Lookback Options; HHMM; Long-range Dependence; Esscher Transform.

1 Introduction

Recently, there is a considerable interest in the applications of regime switching models driven by a hidden Markov chain to various financial problems. For an overview of hidden Markov chain and their financial applications, see Elliott et al [8], Elliott and Kopp [10] and Aggoun and Elliott [1]. Some works on the use of hidden Markov chain in finance include Buffington and Elliott [3, 4] for pricing European and American options, and Elliott et al [11] for option valuation in an incomplete market. Most of the literature concern the pricing of options under a continuous-time Markov-modulated process. There

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is not much work on the valuation of options under a discrete-time Markov-modulated framework. The presence of long-range dependence in the states of an economy is well-known in the economic literature. It is importance to investigate the impact of such well-known economic phenomenon on option valuation. There is a relatively little amount of literature on studying the impact of the long-range dependence in the states of an economy on option valuation. This paper helps to fill in the gap by investigating consequences for pricing some popular exotic (path-dependent) options of the presence of the long-range dependence in the states of an economy.

In this paper, we consider the pricing of exotic options when the price dynamic of the underlying risky asset is governed by a discrete-time Markov-modulated process driven by a Higher-order Hidden Markov Model (HHMM). The discrete-time framework provides a natural and intuitive way to incorporate the higher-order effect in the underlying hidden Markov chain. We assume that the market interest rates of a bank account, the drift and the volatility of the underlying risky asset's return switch over time according to one of the states of the HHMM. We interpret the states of the HHMM as the states of an economy. The use of HHMM to model the economic states can incorporate the long-range dependence of these states. We shall employ the well-known tool in actuarial science, namely, the Esscher transform to determine an equivalent martingale measure for option valuation in the incomplete market described by the HHMM. We shall investigate the impact of the long-range dependence of the economic state on the prices of some path-dependent exotic options, such as Asian options, lookback options and barrier options.

The rest of the paper is organized as follows. In Section 2, we present the Markov-modulated process with HHMM for modeling the price dynamic of the underlying risky asset. We shall illustrate the use of the Esscher transform to determine an equivalent martingale measure for option valuation in the HHMM setting in Section 3. Section 4 conducts some simulation experiments and investigates the impact of the long-range dependence of the economic state on the option prices. Finally, concluding remarks are given in Section 5.

2 Asset Price Dynamic by HHMM

In this section, we present a Markov-modulated process driven by a higher-order hidden Markov chain (HHMM) for modeling the asset price dynamic of an underlying risky asset. First, we consider a discrete-time economy with two primary traded assets, namely, a bank account and a share. Let $\mathcal{T}$ be the time index set $\{0, 1, \dots\}$ of the economy. Fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability. We suppose that the uncertainties due to the fluctuations of market prices and the hidden economic states are described by the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. In the sequel, we shall define a HHMM for describing the hidden states of an economy.

Let $X := \{X_t\}_{t \in \mathcal{T}}$ be an l^{th}-order discrete-time homogeneous HHMM, which takes values in the state space

$$\mathcal{X} := \{x_1, x_2, \ldots, x_M\}.$$

Write

$$\mathbf{i}(t, l) := (i_t, i_{t-1}, \ldots, i_{t-l}),$$

where $t \geq l$, $l = 1, 2, \ldots$ and $i_t, i_{t-1}, \ldots, i_{t-l} \in \{1, 2, \ldots, M\}$. The state transition probabilities of X are specified as follows:

$$P(i_{t+1}|\mathbf{i}(t,l)) := P[X_{t+1} = X_{i_{t+1}}|X_t = x_{i_t}, \ldots, X_{t-l} = x_{i_{t-l}}], \quad i_{t+1} = 1, 2, \ldots, M. \tag{2.1}$$

The order l represents the degree of long-range dependence of the hidden states of the economy. When $l = 1$, X becomes a short-memory or just the first-order hidden Markov model.

To determine the HHMM completely, we need to define the following initial distributions:

$$P(i_{l+1}|\mathbf{i}(l,l)) := \pi_{i_{l+1}|\mathbf{i}(l,l)}\ , \quad i_{t+1} = 1, 2, \ldots, M. \tag{2.2}$$

We shall then describe the Markov-modulated process for the price dynamic of the underlying risky asset. We assume that the market interest rate of the bank account, the drift and the volatility of the risky asset switch over time according to the states of the economy modeled by X.

Let $r_{t,j}$ be the market interest rate of the bank account in the t^{th} period. For each $j = 0, 1, \ldots, l$, we write $\mathbf{X}_{t,j}$ for $(X_t, X_{t-1}, \ldots, X_{t-j})$, for each $t \geq l$, $j = 0, 1, \ldots, l$. We suppose that r_t depends on the current value and the past values of the HHMM up to lag j, i.e.,

$$r_{t,j} := r(\mathbf{X}_{t,j})\ . \tag{2.3}$$

Then, the price dynamic $B := \{B_t\}_{t \in \mathcal{T}}$ of the bank account is given by:

$$B_t = B_{t-1}e^{r_{t,j}}\ , \quad B_0 = 1\ , \quad \mathcal{P}\text{-a.s.} \tag{2.4}$$

Let $S := \{S_t\}_{t \in \mathcal{T}}$ be the price process of the risky stock. For each $t \in \mathcal{T}$, let $Y_t := \ln(S_t/S_{t-1})$ be the logarithmic return in the t^{th}-period. We denote

$$\mu_{t,j} := \mu(\mathbf{X}_{t,j})$$

and

$$\sigma_{t,j} := \sigma(\mathbf{X}_{t,j})$$

the drift and the volatility of the risky stock in the t^{th}-period respectively. In other words, the drift and the volatility depend on the current value and the past values of the HHMM up to lag j. In particular,

$$\mu(x_{i_t}, x_{i_{t-1}}, \dots, x_{i_{t-j}}) = \mu_{\mathbf{i}(t,j)}, \tag{2.5}$$

and

$$\sigma(x_{i_t}, x_{i_{t-1}}, \dots, x_{i_{t-j}}) = \sigma_{\mathbf{i}(t,j)} \ , \tag{2.6}$$

where $\mu_{\mathbf{i}(t,j)} \in \mathcal{R}$ and $\sigma_{\mathbf{i}(t,j)} > 0$.

Let $\{\xi_t\}_{t=1,2,\dots}$ be a sequence of i.i.d. random variables with common distribution $N(0,1)$, a standard normal distribution with zero mean and unit variance. We assume that ξ and X are independent. Then, we suppose that the dynamic of Y is governed by the following Markov-modulated model:

$$Y_t = \mu(\mathbf{X}_{t,j}) - \frac{1}{2}\sigma^2(\mathbf{X}_{t,j}) + \sigma(\mathbf{X}_{t,j})\xi_t, \ t = 1, 2, \dots . \tag{2.7}$$

By convention, $Y_0 = 0$, $\mathcal{P}$-a.s.

When $j = 0$, the Markov-modulated model for Y becomes:

$$Y_t = \mu(X_t) - \frac{1}{2}\sigma^2(X_t) + \sigma(X_t)\xi_t \ , \ t = 1, 2, \dots \ , \tag{2.8}$$

where the drift and the volatility are driven by the current state of the Markov chain X only.

If we further assume that $l = 1$, the Markov-modulated model for Y resembles the first-order HMM for logarithmic returns in Elliott et al. [9].

3 Regime-switching Esscher Transform

The Esscher transform is a well-known tool in actuarial science. The seminal work of Gerber and Shiu [12] pioneers the use of the Esscher transform for option valuation. Their approach provides a convenient and flexible way for the valuation of options under a general asset price model. The use of the Esscher transform for option valuation can be justified by the maximization of the expected power utility. It also highlights the interplay between actuarial and financial pricing, which is an important topic for contemporary actuarial research as pointed out by Bühlmann et al. [5]. Elliott et al. [11] adopt the regime-switching version of the Esscher transform to determine an equivalent martingale measure for the valuation of options in an incomplete market described by a Markov-modulated geometric Brownian motion. Here, we consider a discrete-time version of the regime-switching Esscher transform and apply it to determine an equivalent martingale measure for pricing options in an incomplete market described by our model. In the sequel, we shall introduce the discrete-time regime-switching Esscher transform.

First, for each $t \in \mathcal{T}$, let $\mathcal{F}_t^X$ and $\mathcal{F}_t^Y$ denote the σ-algebras generated by the values of the Markov chain X and the observable logarithmic returns Y up to and including time t respectively. We write $\mathcal{G}_t$ for $\mathcal{F}_t^Y \vee \mathcal{F}_T^X$, for each $t \in \mathcal{T}$.

Let Θ_t be a $\mathcal{F}_T^X$-measurable random variable, for each $t = 1, 2, \ldots$. We interpret Θ_t as the regime-switching Esscher parameter at time t conditional on $\mathcal{F}_T^X$. Let $M_Y(t, \Theta_t)$ denote the moment generating function of Y_t given $\mathcal{F}_T^X$ evaluated at Θ_t under $\mathcal{P}$, i.e.,

$$M_Y(t, \Theta_t) := E(e^{\Theta_t Y_t} | \mathcal{F}_T^X) \ , \tag{3.1}$$

where $E(\cdot)$ is the expectation under $\mathcal{P}$.

Here we assume that there exists a Θ_t such that $M_Y(t, \Theta_t) < \infty$. Then, we define a process

$$\Lambda := \{\Lambda_t\}_{t \in \mathcal{T}}$$

with $\Lambda_0 = 1$, $\mathcal{P}$-a.s., as follows:

$$\Lambda_t := \prod_{k=1}^{t} \frac{e^{\Theta_k Y_k}}{M_Y(k, \Theta_k)}. \tag{3.2}$$

Lemma 3.1: Assume that Y_{t+1} is conditionally independent of $\mathcal{F}_t^Y$ given $\mathcal{F}_T^X$. Then, Λ is a $(\mathcal{G}, \mathcal{P})$-martingale.

Proof: We note that Λ_t is $\mathcal{G}_t$-measurable, for each $t \in \mathcal{T}$. Given that Y_{t+1} is conditionally independent of $\mathcal{F}_t^Y$ given $\mathcal{F}_T^X$,

$$\begin{aligned} E\left(\frac{\Lambda_{t+1}}{\Lambda_t} \middle| \mathcal{G}_t \right) &= E\left[\frac{e^{\Theta_{t+1} Y_{t+1}}}{M_Y(t+1, \Theta_{t+1})} \middle| \mathcal{F}_T^X \right] \\ &= 1 \ , \ \mathcal{P} - \text{a.s.} \end{aligned} \tag{3.3}$$

Hence, the result follows.

Now, we define a discrete-time version of the regime-switching Esscher transform in Elliott et al. [11] $\mathcal{P}^\Theta \sim \mathcal{P}$ on $\mathcal{G}_T$ associated with

$$(\Theta_1, \Theta_2, \ldots, \Theta_T)$$

as follows:

$$\mathcal{P}^\Theta(A) = E(\Lambda_T \cdot I_A) \ , \quad A \in \mathcal{G}_T \ . \tag{3.4}$$

Let $M_Y(t, z|\Theta)$ be the moment generating function of Y_t given $\mathcal{F}_T^X$ under $\mathcal{P}^\Theta$ evaluated at z, i.e.,

$$M_Y(t, z|\Theta) = E^\Theta(e^{zY_t} | \mathcal{F}_T^X) \ , \tag{3.5}$$

where $E^{\Theta}(\cdot)$ is an expectation under $\mathcal{P}^{\Theta}$.

Lemma 3.2: We have

$$M_Y(t, z|\Theta) = \frac{M_Y(t, \Theta_t + z)}{M_Y(t, \Theta_t)}. \tag{3.6}$$

Proof: By the Bayes' rule, Lemma 3.1 and the fact that Y_t is independent of $\mathcal{F}^Y_{t-1}$ given $\mathcal{F}^X_T$,

$$\begin{aligned} M_Y(t, z|\Theta) &= E^{\Theta}(e^{zY_t}|\mathcal{F}^Y_{t-1} \vee \mathcal{F}^X_T) \\ &= E\left(\frac{\Lambda_t}{\Lambda_{t-1}} e^{zY_t} \middle| \mathcal{G}_{t-1}\right) \\ &= \frac{E(e^{(z+\Theta_t)Y_t}|\mathcal{F}^Y_{t-1} \vee \mathcal{F}^X_T)}{M_Y(t, \Theta_t)} \\ &= \frac{M_Y(t, \Theta_t + z)}{M_Y(t, \Theta_t)} . \end{aligned} \tag{3.7}$$

The seminal works of Harrison and Pliska [13, 14] establish an important link between the absence of arbitrage and the existence of an equivalent martingale measure under which discounted price processes are martingales. This is known as the fundamental theorem of asset pricing and is then extended by several authors, including Dybvig and Ross [7], Back and Pliska [2] and Delbaen and Schachermayer [6]. In our case, we specify an equivalent martingale measure by the risk-neutral regime-switching Esscher transform and provide a necessary and sufficient condition on the regime-switching Esscher parameters $(\Theta_1, \Theta_2, \ldots, \Theta_T)$ for $\mathcal{P}^{\Theta}$ to be a risk-neutral regime-switching Esscher transform.

Proposition 3.3: The discounted price process $\{\frac{S_t}{B_t}\}_{t \in \mathcal{T}}$ is a $(\mathcal{G}, \mathcal{P}_{\Theta})$-martingale if and only if

$$\Theta_{t+1} := \Theta(\mathbf{X}_{t+1,j}) = \frac{r_{t+1,j} - \mu_{t+1,j}}{\sigma^2_{t+1,j}} , \quad t = 0, 1, \ldots, T-1. \tag{3.8}$$

Proof: By Lemma 3.2,

$$\begin{aligned} E^{\Theta}\left(\frac{S_{t+1}}{B_{t+1}} \middle| \mathcal{G}_t\right) &= \frac{S_t}{B_t} e^{-r_{t+1}} E^{\Theta}(e^{Y_{t+1}}|\mathcal{G}_t) \\ &= \frac{S_t}{B_t} e^{-r_{t+1}} M_Y(t+1, 1|\Theta) \\ &= \frac{S_t}{B_t} e^{-r_{t+1}} \frac{M_Y(t+1, \Theta_{t+1} + 1)}{M_Y(t+1, \Theta_{t+1})} \\ &= \frac{S_t}{B_t} , \ \mathcal{P} - \text{a.s.} , \end{aligned} \tag{3.9}$$

if and only if

$$\frac{M_Y(t+1,\Theta_{t+1}+1)}{M_Y(t+1,\Theta_{t+1})} = e^{r_{t+1}} \ . \tag{3.10}$$

Since $Y_{t+1}|\mathcal{F}_T^X \sim N(\mu_{t+1,j} - \frac{1}{2}\sigma^2_{t+1,j}, \sigma^2_{t+1,j})$,

$$M_Y(t+1,\Theta_{t+1}) = \exp\left[\Theta_{t+1}\left(\mu_{t+1,j} - \frac{1}{2}\sigma^2_{t+1,j}\right) + \frac{1}{2}\Theta^2_{t+1}\sigma^2_{t+1,j}\right]. \tag{3.11}$$

Then we have

$$\frac{M_Y(t+1,\Theta_{t+1}+1)}{M_Y(t+1,\Theta_{t+1})} = \exp(\mu_{t+1,j} + \Theta_{t+1}\sigma^2_{t+1,j}). \tag{3.12}$$

Hence, we have the result that

$$E^{\Theta}\left(\frac{S_{t+1}}{B_{t+1}}\bigg|\mathcal{G}_t\right) = \frac{S_t}{B_t} \ , \ \mathcal{P} - \text{a.s.} \ , \tag{3.13}$$

iff

$$\Theta_{t+1} = \frac{r_{t+1,j} - \mu_{t+1,j}}{\sigma^2_{t+1,j}}. \tag{3.14}$$

The risk-neutral dynamic of Y under $\mathcal{P}^{\Theta}$ is presented in the following corollary.

Corollary 3.4: Suppose $\nu := \{\nu_t\}_{t=1,2,\ldots,T}$ is a sequence of i.i.d. random variables such that $\nu_t \sim N(0,1)$ under $\mathcal{P}^{\Theta}$. Then, under $\mathcal{P}^{\Theta}$,

$$Y_{t+1} = r(\mathbf{X}_{t+1,j}) - \frac{1}{2}\sigma^2(\mathbf{X}_{t+1,j}) + \sigma(\mathbf{X}_{t+1,j})\nu_{t+1}, \quad t = 0, 1, \ldots, T-1 \ , \tag{3.15}$$

and the dynamic of X remains unchanged under the change of measures.

Proof: By Lemma 3.2,

$$M_Y(t+1, z|\Theta) = \exp\left[z\left(\mu_{t+1,j} - \frac{1}{2}\sigma^2_{t+1,j}\right) + \frac{1}{2}z(2\Theta_{t+1} + z)\sigma^2_{t+1,j}\right] . \tag{3.16}$$

By Proposition 3.3,

$$\Theta_{t+1} = \frac{r_{t+1,j} - \mu_{t+1,j}}{\sigma^2_{t+1,j}} \ . \tag{3.17}$$

This implies that

$$M_Y(t+1, z|\Theta) = \exp\left[z\left(r_{t+1,j} - \frac{1}{2}\sigma^2_{t+1,j}\right) + \frac{1}{2}z^2\sigma^2_{t+1,j}\right] . \tag{3.18}$$

Hence,

$$Y_{t+1} = r(\mathbf{X}_{t+1,j}) - \frac{1}{2}\sigma^2(\mathbf{X}_{t+1,j}) + \sigma(\mathbf{X}_{t+1,j})\nu_{t+1} \ , \quad t = 0, 1, \dots, T-1 \ . \tag{3.19}$$

Since the processes X and ξ are independent, the dynamic of X remains the same under the change of measures from $\mathcal{P}$ to $\mathcal{P}^{\Theta}$.

We shall consider the pricing of three different types of exotic options, namely, Asian options, lookback options and barrier options. First, we deal with an arithmetic average floating-strike Asian call option with maturity T. The payoff of the Asian option at the maturity T is given by:

$$P_{AA}(T) = \max(S_T - J_T, 0) \ , \tag{3.20}$$

where the arithmetic average J_T of the underlying stock price is:

$$J_T = \frac{1}{T}\sum_{t=0}^{T} S_t \ . \tag{3.21}$$

Then, we consider the pricing of a down-and-out European call option with barrier level L, strike price K and maturity at time T. The payoff of the barrier option at time T is

$$P_B(T) = \max(S_T - K, 0) I_{\{\min_{0 \le t \le T} S_t > L\}}, \tag{3.22}$$

where I_E is the indicator of an event E. Finally, we deal with a European-style lookback floating-strike call option with maturity at time T. The payoff of the lookback option is:

$$P_{LB}(T) = \max(S_T - M_{0,T}, 0) \ , \tag{3.23}$$

where $M_{0,T} := \min_{0 \le t \le T} S_t$.

4 Simulation Experiments

In this section, we give some simulation experiments to investigate the effect of the order of the HHMM on the pricing of the following options: Asian option, barrier option and lookback option described in the last section. In particular, we shall investigate the behaviors of the option prices implied by the second-order HHMM (Model I), the first-order HMM (Model II) and the model without switching regimes (Model III). For illustration, we assume that the hidden Markov chain has two states in each of the three models. That is, the economy has two states with State "1" and State "2" representing "Good" economic state and "Bad" economic state, respectively. We employ Monte Carlo simulation to compute the option prices and generate 5,000 simulation runs for computing each option price. All computations were done in a standard PC with C++ codes.

We specify some specimen values of the model parameters in the sequel. First, we specify these values for Model I. Let r_{ij} be the daily market interest rate when the economy in the current period is in the j^{th} state and the economy in the past period is in the i^{th} state, for $i, j = 1, 2$. We suppose that

$$r_{11} = \frac{0.06}{252} = 0.0238\%, \quad r_{12} = \frac{0.02}{252} = 0.00794\%,$$

$$r_{21} = \frac{0.04}{252} = 0.0159\%, \quad r_{22} = \frac{0.01}{252} = 0.00397\%.$$

Here, we assume that one year has 252 trading days. In other words, the corresponding annual market interest rates are 6%, 2%, 4% and 1%, respectively. Let σ_{ij} denote the daily volatility when the economy in the current period is in the j^{th} state and the economy in the past period is in the i^{th} state. We assume that

$$\sigma_{11} = \frac{0.1}{\sqrt{252}} = 0.63\%, \quad \sigma_{12} = \frac{0.3}{\sqrt{252}} = 1.89\%,$$

$$\sigma_{21} = \frac{0.2}{\sqrt{252}} = 1.26\%, \quad \sigma_{22} = \frac{0.4}{\sqrt{252}} = 2.52\%.$$

In other words, the corresponding annual volatilities are 10%, 30%, 20% and 40%, respectively. Let

$$\pi_{ijk} := P(X_t = k | X_{t-1} = i, X_{t-2} = j), \quad \text{for } i, j, k = 1, 2.$$

We suppose that

$$\pi_{111} = 0.7, \quad \pi_{121} = 0.3, \quad \pi_{211} = 0.6 \quad \text{and} \quad \pi_{221} = 0.2.$$

We assume that the two initial states of the second-order HHMM $X_0 = 1$ and $X_1 = 2$. Then, we specify the values of the model parameters for the Model II. For each $i = 1, 2$, let r_i and σ_i denote the daily market interest rate and the daily volatility when the current economy is in the i^{th} state, respectively. We suppose that

$$r_1 = r_{11} = 0.0238\%, \quad r_2 = r_{12} = 0.00794\%,$$

and

$$\sigma_1 = \sigma_{11} = 0.63\% \quad \sigma_2 = \sigma_{12} = 1.89\%.$$

Let

$$\pi_{ij} := P(X_t = j | X_{t-1} = i), \quad \text{for } i, j = 1, 2.$$

We assume that

$$\pi_{11} = \pi_{111} = 0.7 \quad \text{and} \quad \pi_{21} = \pi_{121} = 0.3.$$

We further assume that the initial state $X_0 = 1$. For Model III, we assume that the daily market interest rate

$$r = r_{11} = 0.0238\%$$

and the daily volatility

$$\sigma = \sigma_{11} = 0.63\%.$$

In all cases, we assume that the current price of the underlying stock $S_0 = 100$ and that the time to maturity ranges from 21 trading days (one month) to 126 trading days (six months), with an increment of 21 trading days. Table 1 displays the prices of the Asian options implied by Model I, Model II and Model III for various maturities.

Table 1: Prices of arithmetic average floating-strike Asian call options

Maturity (Days)	Model I	Model II	Model III
21	0.355948	0.185336	0.00134111
42	1.62487	1.19555	0.248206
63	2.59119	2.05436	0.738202
84	3.27272	2.68161	1.19598
105	3.98376	3.21456	1.58142
126	4.49409	3.77677	1.94598

Assume the barrier level $L = 80$ and the strike price $K = 100$. Table 2 displays the prices of the barrier options implied by the three models for various maturities.

Table 2: Prices of down-and-out European call options

Maturity (Days)	Model I	Model II	Model III
21	3.26916	2.60273	1.36266
42	4.75085	3.87316	2.05369
63	5.97215	4.96323	2.69594
84	6.97627	5.82885	3.30289
105	7.88732	6.64117	3.89378
126	8.57693	7.34083	4.42853

Table 3 presents the prices of the lookback options implied by the three models for various maturities.

Table 3.: Prices of lookback floating-strike call options

Maturity (Days)	Model I	Model II	Model III
21	5.28862	4.22157	2.1354
42	7.85881	6.41523	3.27736
63	9.91941	8.16278	4.25719
84	11.6199	9.64562	5.12792
105	13.1429	10.9298	5.94646
126	14.4624	12.1277	6.67989

We can regard Model III (i.e. the no-regime-switching case) as a zero-order HHMM and Model I as a first-order HHMM. Then, from Tables 1, 2 and 3, we can see that the prices of the Asian options, the barrier options and the lookback options, respectively, increase substantially as the order of the HHMM does. These prices are sensitive to the order of the HHMM. This is true for the options with various maturities. In other words, the long-range dependence in the states of economy has significant impact on the prices of these path-dependent exotic options. The differences between the prices implied by the first-order HHMM and those implied by the zero-order HHMM are more substantial than the difference between the prices obtained from the second-order HHMM and those obtained from the first-order HHMM.

5 Conclusion

We investigated the pricing of exotic options under a discrete-time Markov-modulated process driven by a HHMM, which can incorporate the long-range dependence of the states of an economy. We supposed that the market interest rate, the stock appreciation rate and the stock volatility switch over time according to the states of the economy. The Esscher transform has been employed to select a pricing measure under the incomplete market setting. We investigated the impact of the long-range dependence in the states of the economy on the prices of the path-dependent exotic options, Asian options, lookback options and barrier options via some simulation experiments. We found that the presence of the long-range dependence in the states of the economy has significant impact on the prices of the path-dependent exotic options with various maturities.

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