NONEXISTENCE OF ARITHMETIC FAKE COMPACT HERMITIAN SYMMETRIC SPACES OF TYPES $\mathbf{B}_n$, $\mathbf{C}_n$, $\mathbf{E}_6$ AND $\mathbf{E}_7$

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1. Introduction

1.1. Let $\overline{\mathcal{G}}$ be a noncompact connected real semi-simple Lie group with trivial center and with no nontrivial compact connected normal subgroups, and $\mathfrak{g}$ be its Lie algebra. The group $\mathrm{Aut}(\overline{\mathcal{G}})$ (=$\mathrm{Aut}(\mathfrak{g})$) of automorphisms of $\overline{\mathcal{G}}$ is a Lie group with finitely many connected components, and $\overline{\mathcal{G}}$ is its identity component. We will denote the identity component of $\mathrm{Aut}(\overline{\mathcal{G}})$ in the *Zariski-topology* by $\mathrm{Int}(\overline{\mathcal{G}})$. Let X be the symmetric space of $\overline{\mathcal{G}}$ (X is the space of maximal compact subgroups of $\overline{\mathcal{G}}$), and X_u be the compact dual of X. There is a natural identification of the group of isometries of X with $\mathrm{Aut}(\overline{\mathcal{G}})$. We assume in this paper that X (and hence X_u) is hermitian. Then every holomorphic automorphism of X is an isometry. The group $\mathrm{Hol}(X)$ of holomorphic automorphisms of X is a subgroup of finite index of the group $\mathrm{Aut}(\overline{\mathcal{G}})$ of isometries, and it is known (see [Ta], the remark in §5) that $\mathrm{Hol}(X)\cap\mathrm{Int}(\overline{\mathcal{G}}) = \overline{\mathcal{G}}$.

1.2. We will say that the quotient X/Π of X by a *torsion-free* cocompact discrete subgroup Π of $\overline{\mathcal{G}}$ is a *fake compact hermitian symmetric space* if its Betti numbers are same as that of X_u; X/Π is an *irreducible arithmetic fake compact hermitian symmetric space* if, further, Π is irreducible (i.e., no subgroup of Π of finite index is a direct product of two infinite normal subgroups) and it is an arithmetic subgroup of $\overline{\mathcal{G}}$. Any such space can be endowed with the structure of a smooth complex projective variety. Several such spaces have been constructed in our two earlier papers [PY1] and [PY2]. In [PY1] we have given a classification of "fake projective planes", the first of which was constructed by David Mumford in [Mu] using p-adic uniformization. In [PY2] we have constructed four arithmetic fake $\mathbf{P}^4_{\mathbb{C}}$, four arithmetic fake Grassmannians $\mathrm{Gr}_{2,5}$, and five irreducible arithmetic fake $\mathbf{P}^{\mathbf{2}}_{\mathbb{C}}\times\mathbf{P}^{\mathbf{2}}_{\mathbb{C}}$. All these are Shimura varieties.

We note that if $\overline{\mathcal{G}}$ contains an irreducible arithmetic subgroup, then the simple factors of its complexification are isomorphic to each other, see [Ma], Corollary 4.5 in Ch. IX. Also, if the real rank of $\overline{\mathcal{G}}$ is at least 2, then by Margulis' arithmeticity theorem ([Ma], Ch. IX]), any irreducible discrete cocompact subgroup of $\overline{\mathcal{G}}$ (in fact, any irreducible lattice) is arithmetic.

If Π is a torsion-free cocompact discrete subgroup of $\overline{\mathcal{G}}$, then there is a natural embedding of $H^*(X_u,\mathbb{C})$ in $H^*(X/\Pi,\mathbb{C})$, see [B], 3.1 and 10.2, and hence X/Π is a fake compact hermitian symmetric space if and only if the natural homomorphism $H^*(X_u,\mathbb{C})\to H^*(X/\Pi,\mathbb{C})$ is an isomorphism.

1.3. Let $\overline{\mathcal{G}}$, X and X_u be as above, and let Π be a torsion-free cocompact discrete subgroup of $\overline{\mathcal{G}}$. Let $Z = X/\Pi$. If Z is a fake compact hermitian symmetric space, then the Euler-Poincaré characteristic $\chi(Z)$ of $Z = X/\Pi$, and so the Euler-Poincaré characteristic $\chi(\Pi)$ of Π equals $\chi(X_u)$. As X has been assumed to be hermitian, the Euler-Poincaré characteristic of X_u is positive. On the other hand, it follows from Hirzebruch proportionality principle, see [S], Proposition 23, that the Euler-Poincaré characteristic of X/Π is positive if and only if the complex dimension of X is even. Using the results of [BP], we can easily conclude that there are only finitely many irreducible arithmetic fake compact hermitian symmetric spaces. It is of interest to determine them all.

1.4. Hermitian symmetric spaces have been classified by Élie Cartan; see [H], Ch. IX. We recall that the irreducible hermitian symmetric spaces are the symmetric spaces of Lie groups $\mathrm{SU}(n+1-m, m)$, $\mathrm{SO}(2, 2n-1)$, $\mathrm{Sp}(2n)$, $\mathrm{SO}(2, 2n-2)$, $\mathrm{SO}^*(2n)$, an absolutely simple real Lie group of type E_6 with Tits index ${}^2E_{6,2}^{16'}$, and an absolutely simple real Lie group of type E_7 with Tits index $E_{7,3}^{28}$ respectively (for Tits indices see Table II in [Ti1]). The complex dimensions of these spaces are $(n+1-m)m$, $2n-1$, $n(n+1)/2$, $2n-2$, $n(n-1)/2$, 16 and 27 respectively. The Lie groups listed above are of type A_n, B_n, C_n, D_n, D_n, E_6 and E_7 respectively. We will say that an irreducible symmetric space is one of these types if it is the symmetric space of a simple Lie group of that type, and say that a locally hermitian symmetric space is of one of these types if its simply connected cover is a product of irreducible hermitian symmetric spaces of that type.

The following is the main theorem of this paper.

Theorem. *There does not exist an irreducible arithmetic fake compact hermitian symmetric space of type B_n, C_n, E_6 or E_7.*

In the following subsection we will explain the strategy of the proof, and fix notation which will be used throughout the paper.

1.5. Let $\overline{\mathcal{G}}$, X, X_u be as above. X will be assumed to be a hermitian symmetric space of one of the following types B_n, C_n, E_6 and E_7. Assume, if possible, that $\overline{\mathcal{G}}$ contains a cocompact irreducible arithmetic subgroup Π whose orbifold Euler-Poincaré characteristic $\chi(\Pi)$ equals $\chi(X_u)$. Then there exist a totally real number field k, a connected adjoint absolutely simple algebraic k-group $\overline{G}$ of same type as X, $\overline{G}$ of k-rank 0 (by Godement criterion since Π is cocompact), real places $v_1, \cdots, v_r$ of k such that $\overline{G}(k_v)$ is compact for every real place v different from $v_1, \cdots, v_r$, $\overline{\mathcal{G}}$ is isomorphic to $\prod_{j=1}^{r} \overline{G}(k_{v_j})^o$ (and will be identified with it), and Π is commensurable with an arithmetic subgroup of $\overline{G}(k)$. Let $\pi : G \to \overline{G}$ be the simply connected covering of $\overline{G}$ defined over k. The kernel of the isogeny π is the center C of the simply connected k-group G. If G is of type E_6, then it is an "outer k-form" of a split group (i.e., it is of type ${}^2E_6$) since X is hermitian. In this case let ℓ be the quadratic extension of k over which G is an "inner k-form". Then ℓ is totally complex.

Description of C: For a positive integer s, let μ_s be the kernel of the endomorphism $x \mapsto x^s$ of GL_1. Then if G is of type ${}^2E_6$, its center C is k-isomorphic to the kernel of the norm map $N_{\ell/k}$ from the algebraic group $R_{\ell/k}(\mu_3)$, obtained from μ_3 by Weil's restriction of scalars, to μ_3. If G is of type B_n, C_n or E_7, then C is isomorphic to μ_2.

It is known, and easy to see using the above description of C, that for any real place v of k, the order of the kernel of the induced homomorphism $G(k_v) \to \overline{G}(k_v)$ is 2 if G is not of type ${}^2E_6$, and of order 3 if it is of type ${}^2E_6$. Moreover, as $G(k_v)$ is connected, $\pi(G(k_v)) = \overline{G}(k_v)^o$. Let $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$, and let $\widetilde{\Pi}$ be the inverse image of Π in $\mathcal{G}$. Then the kernel of the homomorphism $\pi : \mathcal{G} \to \overline{\mathcal{G}}$ is of order s^r, and hence the orbifold Euler-Poincaré characteristic $\chi(\widetilde{\Pi})$ of $\widetilde{\Pi}$ equals $\chi(\Pi)/s^r = \chi(X_u)/s^r$, where, here and in the sequel, $s = 2$ if G is not of type E_6, and $s = 3$ if G is of type E_6. Now let Γ be a maximal discrete subgroup of $\mathcal{G}$ containing $\widetilde{\Pi}$. Then the orbifold Euler-Poincaré characteristic $\chi(\Gamma)$ of Γ is a submultiple of $\chi(\widetilde{\Pi}) = \chi(X_u)/s^r$. Using the volume formula of [P], some number theoretic estimates, the Bruhat-Tits theory, and the Hasse principle for semi-simple groups (Proposition 7.1 of [PR]), we will prove that $\mathcal{G}$ does not contain such a subgroup.

2. Preliminaries

2.1. We will use the notations introduced in 1.5. Thus k will be a totally real number field, G an absolutely simple simply connected algebraic k-group (of one of the following four types: B_n, C_n, ${}^2E_6$, and E_7), C its center, $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$. We will think of $G(k)$ as a subgroup of $\mathcal{G}$ in terms of its diagonal embedding. Γ is a maximal arithmetic subgroup of $\mathcal{G}$ (arithmetic with respect to the k-structure on G) whose orbifold Euler-Poincaré characteristic is a submultiple of $\chi(X_u)/s^r$. Then $\Lambda := \Gamma \cap G(k)$ is a "principal" arithmetic subgroup, i.e., for every nonarchimedean place v of k, the closure P_v of Λ in $G(k_v)$ is a parahoric subgroup, $\Lambda = G(k) \cap \prod_{v \in V_f} P_v$, and Γ is the normalizer of Λ in $\mathcal{G}$, see Proposition 1.4(iv) of [BP]. Let the "type" Θ_v of P_v be as in [BP], 2.2, and Ξ_{Θ_v} be as in 2.8 there. If either P_v is hyperspecial, or G is of type ${}^2E_6$ and it does not split over k_v, then Ξ_{Θ_v} is trivial. The order of Ξ_{Θ_v} is always a divisor of s.

In terms of the normalized Haar-measure μ on $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$ used in [P] and [BP], and to be used in this paper, $\chi(\Gamma) = \chi(X_u)\mu(\mathcal{G}/\Gamma)$ (see [BP], 4.2). Thus the condition that $\chi(\Gamma)$ is a submultiple of $\chi(X_u)/s^r$ is equivalent to the condition that $\mu(\mathcal{G}/\Gamma)$ is a submultiple of $1/s^r$. We will prove that there does not exist such a Γ.

A comprehensive survey of the basic notions and the main results of the Bruhat-Tits theory of reductive groups over nonarchimedean local fields is given in [Ti2].

2.2. All unexplained notations are as in [BP] and [P]. Thus for a number field K, D_K will denote the absolute value of its discriminant, h_K its class number, i.e., the order of its class group $Cl(K)$. We will denote by $h_{K,s}$ the order of the subgroup of $Cl(K)$ consisting of the elements of order dividing s, where, as in 1.5, $s = 2$ if G is not of type E_6, and $s = 3$ if G is of type E_6. Then $h_{K,s} \leqslant h_K$. We will denote by U_K

the multiplicative-group of units of K, and by K_s the subgroup of $K^\times$ consisting of the elements x such that for every normalized valuation v of K, $v(x) \in s\mathbb{Z}$.

V_f (resp. V_∞) will denote the set of nonarchimedean (resp. archimedean) places of k. As k admits at least r distinct real places, see 1.5, $d := [k : \mathbb{Q}] \geqslant r$. For $v \in V_f$, q_v will denote the cardinality of the residue field $\mathfrak{f}_v$ of k_v.

2.3. For a parahoric subgroup P_v of $G(k_v)$, we define $e(P_v)$ and $e'(P_v)$ by the following formulae (cf. Theorem 3.7 of [P]):

$$e(P_v) = \frac{q_v^{(\dim \overline{M}_v + \dim \overline{\mathcal{M}}_v)/2}}{\#\overline{M}_v(\mathfrak{f}_v)}. \tag{1}$$

$$e'(P_v) = e(P_v)\cdot \frac{\#\overline{\mathcal{M}}_v(\mathfrak{f}_v)}{q_v^{\dim \overline{\mathcal{M}}_v}} = q_v^{(\dim \overline{M}_v - \dim \overline{\mathcal{M}}_v)/2}\cdot \frac{\#\overline{\mathcal{M}}_v(\mathfrak{f}_v)}{\#\overline{M}_v(\mathfrak{f}_v)}. \tag{2}$$

2.4. Let $m_1, \ldots, m_n$ $(m_1 < \cdots < m_n)$, where n is the absolute rank of G, be the exponents of the Weyl group of G. For type B_n and C_n, $m_j = 2j - 1$; for type E_6, the exponents are 1, 4, 5, 7, 8 and 11; and for type E_7, the exponents are 1, 5, 7, 9, 11, 13 and 17. Then

- if either G is *not* of type ${}^2E_6$, or v splits in ℓ,

$$e'(P_v) = e(P_v)\prod_{j=1}^{n}\Big(1 - \frac{1}{q_v^{m_j+1}}\Big);$$

- if G is of type ${}^2E_6$ and v does not split in ℓ,

$$e'(P_v) = e(P_v)\Big(1-\frac{1}{q_v^2}\Big)\Big(1+\frac{1}{q_v^5}\Big)\Big(1-\frac{1}{q_v^6}\Big)\Big(1-\frac{1}{q_v^8}\Big)\Big(1+\frac{1}{q_v^9}\Big)\Big(1-\frac{1}{q_v^{12}}\Big),$$

or

$$e'(P_v) = e(P_v)\Big(1-\frac{1}{q_v^2}\Big)\Big(1-\frac{1}{q_v^6}\Big)\Big(1-\frac{1}{q_v^8}\Big)\Big(1-\frac{1}{q_v^{12}}\Big)$$

according as v *does not* or *does* ramify in ℓ.

2.5. It is obvious that $e'(P_v) < e(P_v)$. It is not difficult to check using (2) that *for all $v \in V_f$, and an arbitrary parahoric subgroup P_v of $G(k_v)$, $e'(P_v)$ is an integer.*

2.6. Now we will use the volume formula of [P] to write down the precise value of $\mu(\mathcal{G}/\Lambda)$. As the Tamagawa number $\tau_k(G)$ of G equals 1, Theorem 3.7 of [P] (recalled in 3.7 of [BP]), for $S = V_\infty$, provides us the following if G is *not* of type ${}^2E_6$,

$$\mu(\mathcal{G}/\Lambda) = D_k^{\frac{1}{2}\dim G}\Big(\prod_{j=1}^{n}\frac{m_j!}{(2\pi)^{m_j+1}}\Big)^d \mathcal{E}, \tag{3}$$

and if G is of type ${}^2E_6$,

$$\mu(\mathcal{G}/\Lambda) = (D_k D_\ell)^{13}\Big(\frac{4!5!7!8!11!}{(2\pi)^{42}}\Big)^d \mathcal{E}, \tag{4}$$

where n is the absolute rank of G, and $\mathcal{E} = \prod_{v\in V_f} e(P_v)$, with $e(P_v)$ as in 2.1.

2.7. Let ζ_k be the Dedekind zeta-function of k, and $L_{\ell|k}$ be the Hecke L-function associated to the quadratic Dirichlet character of ℓ/k. Then

$$\zeta_k(a) = \prod_{v\in V_f}(1-\frac{1}{q_v^a})^{-1};$$

$$L_{\ell|k}(a) = {\prod}'(1-\frac{1}{q_v^a})^{-1}{\prod}''(1+\frac{1}{q_v^a})^{-1},$$

where $\prod'$ is the product over the nonarchimedean places v of k which split in ℓ, and $\prod''$ is the product over all the other nonarchimedean places v which do not ramify in ℓ. Hence the Euler product $\mathcal{E}$ appearing in (3) can be rewritten as

$$\mathcal{E} = \prod_{v\in V_f} e'(P_v)\prod_{j=1}^{n}\zeta_k(m_j+1), \tag{5}$$

and the one appearing in (4) can be rewritten as

$$\mathcal{E} = \prod_{v\in V_f} e'(P_v)\cdot\zeta_k(2)L_{\ell|k}(5)\zeta_k(6)\zeta_k(8)L_{\ell|k}(9)\zeta_k(12). \tag{6}$$

2.8. If G is *not* of type ${}^2E_6$, let $\mathcal{T}$ be the set of all $v\in V_f$ such that P_v is not a hyperspecial parahoric subgroup of $G(k_v)$, and $\mathcal{T}'$ be the empty set. If G is of type ${}^2E_6$, let $\mathcal{T}$ be the set of all $v\in V_f$ which splits in ℓ and P_v is not a hyperspecial parahoric subgroup, and let $\mathcal{T}'$ be the set of v ($\in V_f$) which does not split in ℓ, and either P_v is not a hyperspecial parahoric subgroup of $G(k_v)$ but v is unramified over ℓ, or v is ramified in ℓ and P_v is not a special parahoric subgroup. Then for all nonarchimedean $v\notin\mathcal{T}$, Ξ_{Θ_v} is trivial; if $v\notin\mathcal{T}\cup\mathcal{T}'$, $e'(P_v)=1$, and $e'(P_v)>s$ if $v\in\mathcal{T}$. Therefore, $1\leqslant e'(P_v)<e(P_v)$, and $\mathcal{E}=\prod_{v\in V_f}e(P_v)>s^{\#\mathcal{T}}$. Hence, if G is not of type ${}^2E_6$,

$$\mu(\mathcal{G}/\Lambda) > D_k^{\frac{1}{2}\dim G}\Big(\prod_{j=1}^{n}\frac{m_j!}{(2\pi)^{m_j+1}}\Big)^d 2^{\#\mathcal{T}}, \tag{7}$$

and if G is of type ${}^2E_6$,

$$\mu(\mathcal{G}/\Lambda) > (D_kD_\ell)^{13}\Big(\frac{4!5!7!8!11!}{(2\pi)^{42}}\Big)^d 3^{\#\mathcal{T}}. \tag{8}$$

2.9. If G is not of type ${}^2E_6$, let

$$\mathcal{R} = D_k^{\frac{1}{2}\dim G}\Big(\prod_{j=1}^{n}\frac{m_j!}{(2\pi)^{m_j+1}}\Big)^d\prod_{j=1}^{n}\zeta_k(m_j+1), \tag{9}$$

and if G is of type ${}^2E_6$, let

$$\mathcal{R} = (D_kD_\ell)^{13}\Big(\frac{4!5!7!8!11!}{(2\pi)^{42}}\Big)^d\zeta_k(2)L_{\ell|k}(5)\zeta_k(6)\zeta_k(8)L_{\ell|k}(9)\zeta_k(12). \tag{10}$$

Then

$$\mu(\mathcal{G}/\Lambda) = \mathcal{R} \prod_{v\in\mathcal{T}\cup\mathcal{T}'} e'(P_v). \tag{11}$$

As $e'(P_v)$ is an integer for all v (see 2.5), we conclude that $\mu(\mathcal{G}/\Lambda)$ is an integral multiple of $\mathcal{R}$.

Using the functional equations

$$\zeta_k(2a) = D_k^{\frac{1}{2}-2a}\big(\frac{(-1)^a 2^{2a-1}\pi^{2a}}{(2a-1)!}\big)^d \zeta_k(1-2a),$$

and

$$L_{\ell|k}(2a+1) = \big(\frac{D_k}{D_\ell}\big)^{2a+\frac{1}{2}}\big(\frac{(-1)^a 2^{2a}\pi^{2a+1}}{(2a)!}\big)^d L_{\ell|k}(-2a),$$

for every positive integer a, and the fact that $\dim G = n + 2\sum m_j$, we find that if G is not of type ${}^2E_6$,

$$\mathcal{R} = 2^{-dn}|\prod_{j=1}^{n}\zeta_k(-m_j)|, \tag{12}$$

and if G is of type ${}^2E_6$,

$$\mathcal{R} = 2^{-6d}\zeta_k(-1)L_{\ell|k}(-4)\zeta_k(-5)\zeta_k(-7)L_{\ell|k}(-8)\zeta_k(-11). \tag{13}$$

2.10. As $\chi(\Lambda) = \chi(X_u)\mu(\mathcal{G}/\Lambda)$ ([BP], 4.2), we have the following

$$\chi(\Gamma) = \frac{\chi(\Lambda)}{[\Gamma:\Lambda]} = \frac{\chi(X_u)\mu(\mathcal{G}/\Lambda)}{[\Gamma:\Lambda]}. \tag{14}$$

Proposition 2.9 of [BP] applied to $G' = G$ and $\Gamma' = \Gamma$ implies that $[\Gamma:\Lambda]$ is a power of the prime number s. Now since $\mu(\mathcal{G}/\Lambda)$ is an integral multiple of $\mathcal{R}$, we conclude from (10) that if $\chi(\Gamma)$ is a submultiple of $\chi(X_u)$, then the numerator of the rational number $\mathcal{R}$ is a power of s. We state this as the following proposition.

Proposition 1. *If the orbifold Euler-Poincaré characteristic of* Γ *is a submultiple of* $\chi(X_u)$*, then the numerator of the rational number* $\mathcal{R}$ *is a power of* s*.*

2.11. In this paragraph we assume that G is not of type ${}^2E_6$. Then $C \cong \mu_2$, and the Galois cohomology $H^1(k,C) \cong k^\times/k^{\times 2}$. The order of the first term of the short exact sequence of Proposition 2.9 of [BP], for $G' = G$ and $S = V_\infty$, is 2^{r-1}. From the proof of Proposition 0.12 of [BP], we easily conclude that $\#k_2/k^{\times 2} \leqslant h_{k,2}2^d$. Now we can adapt the argument used to prove Proposition 5.1, and the argument in 5.5, of [BP], for $S = V_\infty$ and $G' = G$, to derive the following bound:

$$[\Gamma:\Lambda] \leqslant 2^{d+r-1+\#\mathcal{T}}h_{k,2}. \tag{15}$$

2.12. We shall assume now that G is of type ${}^2E_6$. As the norm map $N_{\ell/k}:\mu_3(\ell)\to\mu_3(k)$ is onto, the Galois cohomology group $H^1(k,C)$ is isomorphic to the kernel of the homomorphism $\ell^\times/\ell^{\times 3} \to k^\times/k^{\times 3}$ induced by the norm map. We shall denote this kernel by $(\ell^\times/\ell^{\times 3})_\bullet$.

By Dirichlet's unit theorem, $U_k \cong \{\pm 1\} \times \mathbb{Z}^{d-1}$, and $U_\ell \cong \mu(\ell) \times \mathbb{Z}^{d-1}$, where $\mu(\ell)$ is the finite cyclic group of roots of unity in ℓ. Hence, $U_k/U_k^3 \cong (\mathbb{Z}/3\mathbb{Z})^{d-1}$, and $U_\ell/U_\ell^3 \cong \mu(\ell)_3 \times (\mathbb{Z}/3\mathbb{Z})^{d-1}$, where $\mu(\ell)_3$ is the group of cube-roots of unity in ℓ. Now we observe that $N_{\ell/k}(U_\ell) \supset N_{\ell/k}(U_k) = U_k^2$, which implies that the homomorphism $U_\ell/U_\ell^3 \to U_k/U_k^3$, induced by the norm map, is onto. Therefore, the order of the kernel $(U_\ell/U_\ell^3)_\bullet$ of this homomorphism equals $\#\mu(\ell)_3$.

The short exact sequence (4) in the proof of Proposition 0.12 of [BP] gives us the following exact sequence:

$$1 \to (U_\ell/U_\ell^3)_\bullet \to (\ell_3/\ell^{\times 3})_\bullet \to (\mathcal{P} \cap \mathcal{I}^3)/\mathcal{P}^3,$$

where $(\ell_3/\ell^{\times 3})_\bullet = (\ell_3/\ell^{\times 3}) \cap (\ell^\times/\ell^{\times 3})_\bullet$, $\mathcal{P}$ is the group of all fractional principal ideals of ℓ, and $\mathcal{I}$ the group of all fractional ideals (we use multiplicative notation for the group operation in both $\mathcal{I}$ and $\mathcal{P}$). Since the order of the last group of the above exact sequence is $h_{\ell,3}$, see (5) in the proof of Proposition 0.12 of [BP], we conclude that

$$\#(\ell_3/\ell^{\times 3})_\bullet \leqslant \#\mu(\ell)_3 \cdot h_{\ell,3}.$$

Now we note that the order of the first term of the short exact sequence of Proposition 2.9 of [BP], for $G' = G$ and $S = V_\infty$, is $3^r/\#\mu(\ell)_3$.

Using the above observations, together with Proposition 2.9 and Lemma 5.4 of [BP], and a close look at the arguments in 5.3 and 5.5 of [BP] for $S = V_\infty$ and G as above, we can derive the following upper bound:

$$[\Gamma : \Lambda] \leqslant 3^{r+\#\mathcal{T}} h_{\ell,3}. \tag{16}$$

2.13. Since $\mu(\mathcal{G}/\Gamma) = \mu(\mathcal{G}/\Lambda)/[\Gamma : \Lambda]$ is a submultiple of $1/s^r$ (see 2.1), we conclude that $\mu(\mathcal{G}/\Lambda) \leqslant [\Gamma : \Lambda]/s^r$. From the bound for $[\Gamma : \Lambda]$ derived in 2.11 and 2.12 we obtain that if G is not of type ${}^2E_6$, then

$$\mu(\mathcal{G}/\Lambda) \leqslant 2^{d-1+\#\mathcal{T}} h_{k,2}, \tag{17}$$

and if G is of type ${}^2E_6$,

$$\mu(\mathcal{G}/\Lambda) \leqslant 3^{\#\mathcal{T}} h_{\ell,3}. \tag{18}$$

Now combining these with (7) and (8) respectively, we obtain

$$D_k^{\frac{1}{2}\dim G} < 2^{d-1} h_{k,2} \Big(\prod_{j=1}^{n} \frac{(2\pi)^{m_j+1}}{m_j!} \Big)^d, \tag{19}$$

if G is not of type ${}^2E_6$, and

$$(D_k D_\ell)^{13} < h_{\ell,3} \Big(\frac{(2\pi)^{42}}{4!5!7!8!11!} \Big)^d \tag{20}$$

if G is of type ${}^2E_6$.

3. Discriminant bounds

We will recall discriminant bounds required in later discussions. We define $M_r(d) = \min_K D_K^{1/d}$, where the minimum is taken over all totally real number fields K of degree d. Similarly, we define $M_c(d) = \min_K D_K^{1/d}$, by taking the minimum over all totally complex number fields K of degree d.

The precise values of $M_r(d), M_c(d)$ for low values of d are given in the following table (cf. [N]).

d	2	3	4	5	6	7	8
$M_r(d)^d$	5	49	725	14641	300125	20134393	282300416
$M_c(d)^d$	3		117		9747		1257728

The following proposition can be proved in the same way as Proposition 2 in [PY2] has been proved.

Proposition 2. Let k and ℓ be a totally real number field and a totally imaginary number field of degree d and $2d$ respectively.

$\forall d \geqslant$	2	3	4	5	6	7	8
$D_k^{1/d} >$	2.23	3.65	5.18	6.8	8.18	11.05	11.38
$D_\ell^{1/2d} >$	1.73		3.28		4.62		5.78

4. G of type B_n or C_n

4.1. In this section we assume that G is either of type B_n or C_n. Then its dimension is $n(2n+1)$. The j-th exponent $m_j = 2j-1$, and the complex dimension of the symmetric space X of $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$ is $r(2n-1)$ if G is of type B_n, and is $rn(n+1)/2$ if G is of type C_n. From (19) we obtain

$$D_k^{1/d} < f_1(n,d,h_{k,2}) := \big[\{2\prod_{j=1}^n \frac{(2\pi)^{2j}}{(2j-1)!}\}^d \cdot \frac{h_{k,2}}{2}\big]^{2/dn(2n+1)}. \tag{21}$$

According to the Brauer-Siegel Theorem, for a totally real number field k, and all real $\delta > 0$,

$$h_k R_k \leqslant 2^{1-d}\delta(1+\delta)\Gamma((1+\delta)/2)^d(\pi^{-d}D_k)^{(1+\delta)/2}\zeta_k(1+\delta),$$

where R_k is the regulator of k. Now from (21) we get the following bound:

$$\begin{aligned} D_k^{1/d} &< f_2(n,d,R_k,\delta) \\ &:= \big[\{\frac{\Gamma((1+\delta)/2)\zeta(1+\delta)}{\pi^{(1+\delta)/2}}\prod_{j=1}^n \frac{(2\pi)^{2j}}{(2j-1)!}\}\cdot\{\frac{\delta(1+\delta)}{R_k}\}^{1/d}\big]^{2/(2n^2+n-1-\delta)}, \end{aligned} \tag{22}$$

since $\zeta_k(1+\delta) \leqslant \zeta(1+\delta)^d$, where $\zeta = \zeta_{\mathbb{Q}}$. Using the lower bound $R_k \geqslant 0.04\, e^{0.46d}$, for a totally real number field k, due to R. Zimmert [Z], we get

$$D_k^{1/d} < f_3(n,d,\delta) \tag{23}$$

$$:= \big[\{\frac{\Gamma((1+\delta)/2)\zeta(1+\delta)}{\pi^{(1+\delta)/2}e^{0.46}}\prod_{j=1}^{n}\frac{(2\pi)^{2j}}{(2j-1)!}\}\cdot\{25\delta(1+\delta)\}^{1/d}\big]^{2/(2n^2+n-1-\delta)}.$$

4.2. It is obvious that for fixed n and $\delta \in [0.04, 9]]$, $f_3(n,d,\delta)$ decreases as d increases. Now we observe that for $n \geqslant 9$, $(2n-1)! > (2\pi)^{2n}$. From this it is easy to see that if for a given d, δ, and $n \geqslant 8$, $f_3(n,d,\delta) \geqslant 1$, then $f_3(n+1,d,\delta) < f_3(n,d,\delta)$, and if $f_3(n,d,\delta) < 1$, then $f_3(n+1,d,\delta) < 1$. In particular, if for given d, and $\delta \in [0.04, 9]$, $f_3(8,d,\delta) < c$, with $c \geqslant 1$, then $f_3(n,d',\delta) < c$ for all $n \geqslant 8$ and $d' \geqslant d$.

We obtain by a direct computation the following upper bound for the value of $f_3(n,2,3)$ for $6 \leqslant n \leqslant 14$.

n	14	13	12	11	10	9	8	7	6
$f_3(n,2,3) <$	1	1.1	1.2	1.3	1.4	1.6	1.8	2.1	2.4

From the bounds provided by the above table and the properties of f_3 we conclude that $f_3(n,d,3) < 2.1$ for all $n \geqslant 7$, and $d \geqslant 2$, and we conclude from Proposition 2 that unless $k = \mathbb{Q}$ (i.e., $d = 1$), $n \leqslant 6$.

We assert now that $n \leqslant 13$. To prove this, we can assume, in view of the result established in the preceding paragraph, that $k = \mathbb{Q}$. By a direct computation we see that $f_1(14,1,1) < 1$. Hence, $f_1(n,1,1) < 1$ for all $n \geqslant 14$. As $D_{\mathbb{Q}} = 1$, from bound (21) we conclude that $n \leqslant 13$.

We will now assume that $d \geqslant 2$ and consider each of the possible cases $2 \leqslant n \leqslant 6$ separately.

• $n = 6$: As $D_k^{1/d} < f_3(6,3,2.1) < 2.5$, from Proposition 2 we conclude that if $n = 6$, $d < 3$. If $d = 2$, $D_k^{1/d} < f_3(6,2,1.54) < 2.36$. Therefore, $D_k < 6$, which implies that $k = \mathbb{Q}(\sqrt{5})$ is the only possibility.

• $n = 5$: As $D_k^{1/d} < f_3(5,3,1.8) < 2.9$, from Proposition 2 we conclude that if $n = 5$, $d < 3$. If $d = 2$, $D_k^{1/d} < f_3(5,2,1.3) < 2.9$. Therefore, $D_k < 9$. So there are two possible real quadratic fields k, their discriminants are 5 and 8. Both the fields have class number 1, and we use the bound (21) to obtain $D_k^{1/d} < f_1(5,2,1) < 2.73$. So only $D_k = 5$ can occur.

• $n = 4$: As $D_k^{1/d} < f_3(4,3,1.3) < 3.61$, from Proposition 2 we conclude that if $n = 4$, $d < 3$. Let us assume that $d = 2$. Then since $D_k^{1/d} < f_3(4,2,1.1) < 3.76$, $D_k < 15$ and so the possible values of D_k are $5, 8, 12$ or 13. The quadratic fields with these D_k have class number 1. Now from bound (21) we obtain $D_k^{1/d} < f_1(4,2,1) < 3.4$. Hence, $D_k < 12$, and only $D_k = 5, 8$ can occur.

• $n = 3$: As $D_k^{1/d} < f_3(3,4,1.3) < 5.1$, from Proposition 2 we conclude that if $n = 3$, $d < 4$. Now since $D_k^{1/d} < f_3(3,3,1.13) < 5.21$, if $d = 3 = n$, $D_k < 142$ from which we infer that $D_k = 49$ or 81. Since $D_k^{1/d} < f_3(3,2,0.8) < 5.6$, if $d = 2$ (and $n = 3$), $D_k < 32$, and in this case the possible values of D_k are $5, 8, 12, 13, 17, 21, 24, 28$ or 29. The quadratic fields with these discriminants have class number 1, and we use bound (21) to obtain $D_k^{1/d} < f_1(3,2,1) < 4.52$. Hence, $D_k < 21$ and only $D_k = 5, 8, 12, 13, 17$ can occur.

• $n = 2$: As $D_k^{1/d} < f_3(2,7,1.1) < 9$, Proposition 2 implies that $d \leqslant 6$.

$n = 2$ and $d = 6$: As $D_k^{1/d} < f_3(2,6,1) < 9$, $D_k < 531441$. One can check from the table in [1] that $h_k = 1$ for all the five number fields satisfying this bound. We now use bound (21) to obtain $D_k^{1/d} < f_1(2,6,1) < 7.2$, which contradicts Proposition 2.

$n = 2$ and $d = 5$: As $D_k^{1/d} < f_3(2,5,1) < 9.3$, $D_k < 69569$. Again, one can check from the table in [1] that there are five such number fields and the class number of each of them is 1. Now we use bound (21) to obtain $D_k^{1/d} < f_1(2,5,1) < 7.1$. Hence, $D_k < 18043$. From [1] we find that $D_k = 14641$ is the only possibility.

$n = 2$ and $d = 4$: As $D_k^{1/d} < f_3(2,4,0.92) < 9.74$, $D_k < 9000$. According to [1], there are 45 totally real quartic number fields with discriminant < 9000, all of them have class number 1. We use bound (21) to obtain $D_k^{1/d} < f_1(2,4,1) < 7.037$. Hence, $D_k < 2453$. We find from [1] that there are eight totally real quartic number fields k with $D_k < 2453$. Their discriminants are

$$725, 1125, 1600, 1957, 2000, 2048, 2225, 2304.$$

$n = 2$ and $d = 3$: As $D_k^{1/d} < f_3(2,3,0.78) < 10.5$, $D_k < 1158$. From table B.4 of [C] we find that there are altogether 31 totally real cubics satisfying this discriminant bound. Each of these fields have class number 1. We use bound (21) to obtain $D_k^{1/d} < f_1(2,3,1) < 6.96$, which implies that $D_k < 338$. There are eight real cubic number fields satisfying this bound. The values of D_k are

$$49, 81, 148, 169, 229, 257, 316, 321.$$

$n = 2$ and $d = 2$: As $D_k^{1/d} < f_3(2,2,0.52) < 12$, $D_k < 144$. From table B.2 of totally real quadratic number fields given in [C], we check that the class number of all these fields are bounded from above by 2. Hence, $D_k^{1/d} < f_1(2,2,2) < 7.285$. So $D_k < 53$. Among the real quadratic fields with $D_k < 53$, there is only one field whose class number is 2, it is the field with $D_k = 40$. All the rest have class number 1, and from bound (21) we conclude that $D_k^{1/d} < f_1(2,2,1) < 6.8$, i.e., $D_k < 47$. Therefore, the following is the list of the possible values of D_k:

$$5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 40, 41, 44.$$

To summarize, for G of type B_n or C_n, the possible n, d and D_k are given in the following table.

n	d	D_k
$2,\ldots,13$	1	1
6	2	5
5	2	5
4	2	$5, 8$
3	3	$49, 81$
3	2	$5, 8, 12, 13, 17$
2	5	14641
2	4	$725, 1125, 1600, 1957, 2000, 2048, 2225, 2304$
2	3	$49, 81, 148, 169, 229, 257, 316, 321$
2	2	$5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 40, 41, 44.$

4.3. We will show that none of the possibilities listed in the above table actually give rise to an arithmetic fake compact hermitian symmetric space of type B_n or C_n. For this we recall first of all that $\overline{G}$, and so also G, are anisotropic over k (1.5). Now we observe that if G is a group of type B_n ($n \geqslant 2$), then it is k-isotropic if and only if it is isotropic at all real places of k (this is an immediate consequence of the classical Hasse principle for quadratic forms which says that a quadratic form over k is isotropic if and only if it is isotropic at every place of k, and the well-known fact that a quadratic form of dimension > 4 is isotropic at every nonarchimedean place). Also, a k-group of type C_n ($n \geqslant 2$) is isotropic if it is isotropic at all the real places of k (this is known, and follows, for example, from Proposition 7.1 of [PR]). These results imply that if $d = 1$, i.e., if $k = \mathbb{Q}$, then G is isotropic, and so $k = \mathbb{Q}$ is not possible.

Now let us take up the case where $d = 2$, i.e., k is a real quadratic field, and $n = 2$, 5 or 6. Then for any real place v of k where G is isotropic, the complex dimension of the symmetric space of $G(k_v)$ is odd (recall from 1.4 that the complex dimension of the symmetric space of $G(k_v)$ is $2n - 1$ if G is of type B_n, and it is $n(n+1)/2$ if G is of type C_n). But as the complex dimension of the hermitian symmetric space X is even (since the orbifold Euler-Poincaré characteristic of Γ is positive, see 1.3), we conclude that G must be isotropic at both the real places of k (note that G is anisotropic at a place v of k if and only if $G(k_v)$ is compact). From this observation we conclude that G is k-isotropic also in case $d = 2$, and $n = 2$, 5 or 6. Therefore these possibilities do not occur.

Now we will rule out the case where $n = 2$, $d = 5$, and $D_k = 14641$. In this case, $k = \mathbb{Q}[x]/(x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1)$. The class number of k is 1. It is easy to see that the cardinality q_v of the residue field of k at any nonarchimedean place v is at least 4.

Since $\mu(\mathcal{G}/\Gamma)$ is a submultiple of $1/2^r$ (2.1), and $[\Gamma : \Lambda] = 2^m$, where $m \leqslant 4 + r + \#\mathcal{T}$, see (15), using (3) we obtain

(24)

$$1 \geqslant 2^r\mu(\mathcal{G}/\Gamma) = 2^r\frac{\mu(\mathcal{G}/\Lambda)}{[\Gamma:\Lambda]} > \frac{2^r}{[\Gamma:\Lambda]}\Big(\frac{6D_k}{(2\pi)^6}\Big)^5\mathcal{E} > 2^{-4-\#\mathcal{T}}\Big(\frac{87846}{(2\pi)^6}\Big)^5\prod_{v\in\mathcal{T}}e(P_v),$$

where $\mathcal{E} = \prod_{v\in V_f} e(P_v)$, with $e(P_v)$ as in 2.1.

The group G is a simply connected k-anisotropic group of type C_2. Such a group is described in terms of a quaternion division algebra D, with center k (see [Ti1]). Since the symmetric space X is a complex analytic space of even complex dimension, and $d = 5$, we conclude that G is anisotropic at an odd number of real places of k. Hence the quaternion division algebra D ramifies at an odd number of real places of k. Since a quaternion division algebra ramifies at an even number of places, we infer that there is at least one nonarchimedean place where D ramifies. At such a place, G is of rank 1, so such a place lies in $\mathcal{T}$. This shows that $\mathcal{T}$ is nonempty. For $v \in \mathcal{T}$, $e(P_v)$ equals either $q_v^6/(q_v^4-1)$, or $q_v^6/(q_v+1)(q_v^2-1)$, or $q_v^6/(q_v^2-1)$. All these numbers are larger than q_v^2. Now as $q_v \geqslant 4$, and $\mathcal{T}$ is nonempty, from (24) we conclude that

$$1 \geqslant 2^r\mu(\mathcal{G}/\Gamma) > 2^{-4-\#\mathcal{T}}\Big(\frac{87846}{(2\pi)^6}\Big)^5 16^{\#\mathcal{T}} > \frac{1}{2}\Big(\frac{87846}{(2\pi)^6}\Big)^5 > 1,$$

which is absurd.

4.4. To rule out the remaining cases listed in the table in 4.2, we compute the value of $\mathcal{R}$ ($\mathcal{R}$ as in (12)) in each case. These values are given below. It turns out that in none of the remaining cases the numerator of $\mathcal{R}$ is a power of 2 and Proposition 1 then eliminates these cases.

n	d	k	D_k	$\zeta_k(-1)$	$\zeta_k(-3)$	$\zeta_k(-5)$	$\zeta(-7)$	$\mathcal{R}$
4	2	x^2-5	5	1/30	1/60	67/630	361/120	24187/34836480000
4	2	x^2-2	8	1/12	11/120	361/252	24611/240	97730281/22295347200.

n	d	k	D_k	$\zeta_k(-1)$	$\zeta_k(-3)$	$\zeta_k(-5)$	$\mathcal{R}$
3	3	x^3-x^2-2x+1	49	$-1/21$	79/210	$-7393/63$	584047/142248960
3	3	x^3-3x-1	81	$-1/9$	199/90	$-50353/27$	10020247/11197440
3	2	x^2-17	17	1/3	41/30	5791/63	237431/362880
3	2	x^2-13	13	1/6	29/60	33463/1638	970427/37739520
3	2	x^2-3	12	1/6	23/60	1681/126	38663/2903040
3	2	x^2-2	8	1/12	11/120	361/252	3971/23224320
3	2	x^2-5	5	1/30	1/60	67/630	67/72576000

n	d	k	D_k	$\zeta_k(-1)$	$\zeta_k(-3)$	$\mathcal{R}$
2	4	x^4-4x^2+1	2304	1	22011/10	22011/2560
2	4	$x^4-x^3-5x^2+2x+4$	2304	4/5	9202/5	4601/800
2	4	x^4-4x^2+2	2048	5/6	87439/60	87439/18432
2	4	x^4-5x^2+5	2000	2/3	3793/3	3793/1152
2	4	x^4-4x^2-x+1	1957	2/3	3541/3	3541/1152
2	4	x^4-6x^2+4	1600	7/15	17347/30	121429/115200
2	4	$x^4-x^3-4x^2+4x+1$	1125	4/15	2522/15	1261/7200
2	4	$x^4-x^3-3x^2+x+1$	725	2/15	541/15	541/28800
2	3	x^3-x^2-4x+1	321	−1	555/2	555/128
2	3	x^3-x^2-4x+2	316	−4/3	874/3	437/72
2	3	x^3-x^2-4x+3	257	−2/3	1891/15	1891/1440
2	3	x^3-4x-1	229	−2/3	1333/15	1333/1440
2	3	x^3-x^2-4x-1	169	−1/3	11227/390	11227/74880
2	3	x^3-x^2-3x+1	148	−1/3	577/30	577/5760
2	3	x^3-3x-1	81	−1/9	199/90	199/51840
2	3	x^3-x^2-2x+1	49	1/21	79/210	79/282240.

5. G of type ${}^2E_6$

5.1. In this section G is of type ${}^2E_6$. Its dimension is 78 and the complex dimension of the symmetric space of $\mathcal{G}=\prod_{j=1}^r G(k_{v_j})$ is $16r$. Let

$$A=\frac{(2\pi)^{42}}{4!5!7!8!11!}.$$

The Brauer-Siegel Theorem for the totally complex number field ℓ asserts that for all real $\delta>0$,

$$h_\ell R_\ell \leqslant w_\ell \delta(1+\delta)\Gamma(1+\delta)^d((2\pi)^{-2d}D_\ell)^{(1+\delta)/2}\zeta_\ell(1+\delta), \tag{25}$$

where R_ℓ is the regulator of ℓ and w_ℓ is the number of roots of unity contained in ℓ. Using this, from bound (20) we obtain

$$(D_kD_\ell)^{13}<h_\ell A^d\leqslant \frac{\delta(1+\delta)A^d\Gamma(1+\delta)^dD_\ell^{(1+\delta)/2}\zeta_\ell(1+\delta)}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.$$

Hence,

$$D_k^{13}D_\ell^{13-\frac{1+\delta}{2}}<\frac{\delta(1+\delta)A^d\Gamma(1+\delta)^d\zeta_\ell(1+\delta)}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.$$

As $D_k^2\leqslant D_\ell$, and $\zeta_\ell(1+\delta)\leqslant\zeta(1+\delta)^{2d}$, we conclude that

$$D_k^{38-\delta}<\frac{\delta(1+\delta)A^d\Gamma(1+\delta)^d\zeta(1+\delta)^{2d}}{(R_\ell/w_\ell)(2\pi)^{d(1+\delta)}}.$$

Therefore,

$$D_k^{1/d} < \big[\{A\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}}\}\cdot\{\frac{\delta(1+\delta)}{R_\ell/w_\ell}\}^{1/d}\big]^{1/(38-\delta)}.$$

Using the lower bound $R_\ell \geqslant 0.02w_\ell\, e^{0.1d}$ due to R. Zimmert [Z], we obtain from this the following

$$D_k^{1/d} < f(d,\delta) := \big[\{A\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}e^{0.1}}\}\cdot\{50\delta(1+\delta)\}^{1/d}\big]^{1/(38-\delta)}. \tag{26}$$

From bound (20) we also obtain,

$$D_\ell/D_k^2 < \big[A^d\frac{h_\ell}{D_k^{39}}\big]^{1/13}. \tag{27}$$

Furthermore, using (25) and Zimmert's bound $R_\ell \geqslant 0.02w_\ell e^{0.1d}$, we get from this that

$$D_\ell/D_k^2 < \mathfrak{p}(d,D_k,\delta) := \big[\{A\frac{\Gamma(1+\delta)\zeta(1+\delta)^2}{(2\pi)^{1+\delta}e^{0.1}}\}\cdot\{\frac{50\delta(1+\delta)}{D_k^{38-\delta}}\}^{1/d}\big]^{2d/(25-\delta)}. \tag{28}$$

5.2. For a fixed δ, $f(d,\delta)$ clearly decreases as d increases. By a direct computation we find that $f(3,2) < 2.3$, and hence for all $d \geqslant 3$,

$$D_k^{1/d} < f(d,2) \leqslant f(3,2) < 2.3.$$

But according to Proposition 2, for totally real number fields of degree $d \geqslant 3$, $D_k^{1/d} > 3.65$, so we conclude that $d \leqslant 2$.

Assume now that $d = 2$. Then $D_k^{1/2} \leqslant f(2,1.94) < 2.4$. Therefore $D_k < 6$ and hence $D_k = 5$. It follows from bound (28) with $\delta = 1.9$ that $D_\ell/D_k^2 < \mathfrak{p}(2,5,1.9) < 1.4$. Hence $D_\ell/D_k^2 = 1$ and $D_\ell = 25$, which contradicts the bound given by Proposition 2. Hence, $d = 1$ and $k = \mathbb{Q}$.

It is known, and follows, for example, from Proposition 7.1 of [PR], that a $\mathbb{Q}$-group G of type ${}^2E_6$, which at the unique real place of $\mathbb{Q}$ is the outer form of rank 2 (this is the form ${}^2E_{6,2}^{16'}$ which gives rise to a hermitian symmetric space), is isotropic over $\mathbb{Q}$. This contradicts the fact that G is anisotropic over $\mathbb{Q}$ (1.5), and hence we conclude that groups of type ${}^2E_6$ do not give rise to arithmetic fake compact hermitian symmetric spaces.

6. G of type E_7

6.1. In this section G is assumed to be of type E_7. Its dimension is 133, the exponents are 1, 5, 7, 9, 11, 13 and 17. The dimension of the symmetric space X of $\mathcal{G} = \prod_{j=1}^r G(k_{v_j})$ is $27r$.

Let

$$B = \prod_{j=1}^{7} \frac{(2\pi)^{m_j+1}}{m_j!}.$$

From (19) we obtain the following:

$$D_k^{1/d} < \left[2B(h_{k,2}/2)^{1/d}\right]^{2/133}.$$

Using the Brauer-Siegel Theorem for totally real number fields (see 4.1), and the obvious bound $\zeta_k(1+\delta) \leqslant \zeta(1+\delta)^d$, we obtain

$$D_k^{1/d} < \left[\{B\frac{\Gamma((1+\delta)/2)\zeta(1+\delta)}{\pi^{(1+\delta)/2}}\} \cdot \{\frac{\delta(1+\delta)}{R_k}\}^{1/d}\right]^{2/(132-\delta)}. \tag{29}$$

Now using the lower bound $R_k \geqslant 0.04\, e^{0.46d}$ due to R. Zimmert [Z] again, we get

$$D_k^{1/d} < \phi(d,\delta) := \left[\{B\frac{\Gamma((1+\delta)/2)\zeta(1+\delta)}{\pi^{(1+\delta)/2}e^{0.46}}\} \cdot \{25\delta(1+\delta)\}^{1/d}\right]^{2/(132-\delta)}. \tag{30}$$

6.2. For a fixed $\delta \geqslant 0.04$, $\phi(d,\delta)$ clearly decreases as d increases. By a direct computation we see that $\phi(2,4) < 2$, and hence for all totally real number field k of degree $d \geqslant 2$,

$$D_k^{1/d} < \phi(d,4) \leqslant \phi(2,4) < 2.$$

From this bound and Proposition 2 we conclude that d can only be 1, i.e., $k = \mathbb{Q}$. But then $r = 1$ and the complex dimension of the associated symmetric space X is 27. Then the Euler-Poincaré characteristic of any quotient of X by a cocompact torsion-free discrete subgroup of $\overline{\mathcal{G}}$ is negative (1.3), and hence it cannot be a fake compact hermitian symmetric space. Another way to eliminate this case is to observe that an absolutely simple $\mathbb{Q}$-group of type E_7 is isotropic if it is isotropic over $\mathbb{R}$ (this result follows, for example, from Proposition 7.1 of [PR]).

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