Classification of fake projective planes

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Abstract

The purpose of this article is to explain recent developments on classification and construction of fake projective planes. The main goal is to give an expository on the formulation and results of Gopal Prasad and the author in [PY1]. We also mention the recent results of Donald Cartwright and Tim Steger in [CS]. Earlier geometric setup and further developments are discussed as well.

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1 Introduction

1.1 A fake projective plane is a smooth complex surface which has the same Betti numbers as $P_{\mathbb{C}}^2$ but which is not biholomorphic to $P_{\mathbb{C}}^2$. It is special in the sense that it has the smallest Euler number among surfaces of general type. Furthermore, a fake projective plane turns out to be a quotient of the complex two ball by a torsion-free discrete subgroup of $PU(2,1)$. Hence it is either a Shimura surface or a finite cover of a Shimura surface and carries rich geometric and arithmetic structures. The purpose of this article is to survey recent results on fake projective planes, leading to the classification and construction of such surfaces by Gopal Prasad and the author in [PY1], and the completion of the program by Cartwright and Steger in [CS]. It is also our purpose to give a reasonable coherent outline of proof. The article is an expansion of the summary given in [Ye4].

The complete study requires the input of differential and algebraic geometry, non-linear analysis, algebraic groups, Bruhat-Tits theory, algebraic and analytic number theory. As a chapter in the Handbook in Geometric Analysis, our goal here is to introduce the ideas and explain the results from a more geometric point of view. For a survey on the subject from a more algebraic point of view, we refer the readers to the Bourbaki Seminar report of Rémy in [Rem].

1.2 To explain the origin of the topic, we begin with a well known problem of Severi, who asked whether there exists a smooth complex surface homeomorphic

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to but not isomorphic to $P^2_{\mathbb{C}}$. The problem was finally settled in the negative in the paper of Yau [Ya1], after the solution of Calabi Conjecture in the case of negative Ricci curvature by Aubin [A] and Yau [Ya2].

The result of Yau shows that there is no exotic structure on $P^2_{\mathbb{C}}$. Mumford asked how well one can mimic $P^2_{\mathbb{C}}$. In particular, he asked if there was a smooth complex surface with the same Betti numbers as $P^2_{\mathbb{C}}$ but which is not biholomorphic to $P^2_{\mathbb{C}}$. Mumford provided the answer himself by constructing in [Mu] an example with the same Betti numbers as $P^2_{\mathbb{C}}$ but has ample canonical line bundle and hence is not biholomorphic to $P^2_{\mathbb{C}}$. Such a surface is now known as fake projective plane.

Algebraic geometers have been interested in fake projective planes since the appearance of Mumford's example, as they play special roles in the geography of smooth surfaces of general type, by being the ones with smallest Euler-Poincaré characteristic. It is hence meaningful to classify such surfaces.

It follows from the result of Aubin and Yau that a fake projective plane is uniformized by the complex two ball $B^2_{\mathbb{C}}$ in $\mathbb{C}^2$. Hence it can be written as the quotient of $B^2_{\mathbb{C}}$ by a cocompact lattice Π of $PU(2,1)$. It turns out that Π is an arithmetic lattice in $PU(2,1)$ according to the result of Klingler [Kl] and Yeung [Ye1] on geometric rigidity. The arithmeticity of the lattice involved allows us to apply the volume formula of Prasad [Pra] to reduce classification of fake projective planes to classification of arithmetic lattices.

Near the end of this article (Theorem 10 in §7), we will state a strong characterization of $P^2_{\mathbb{C}}$ in terms of homology groups which generalize the original problem of Severi, as a result of the project on the classification of fake projective planes.

1.3 The example of Mumford was constructed from p-adic uniformization introduced by Kurihara and Mustafin. His method was different from the classical methods of construction of surfaces. Two more examples of fake projective planes were later found by Ishida and Kato in [IK], utilizing the examples of discrete subgroups of $GL_3(\mathbb{Q}_v)$ which act transitively on the vertices of the Bruhat-Tits building constructed by Cartwright, Mantero, Steger and Zappa in [CMSZ]. The construction of Ishida and Kato was related to the original construction of Mumford. More recently, Keum constructed a fake projective plane with an order 7 automorphism in [Ke1], starting with Ishida's analysis on Mumford's example in [I]. As explained in the introduction in [Ke1], it was not clear whether the example in [Ke1] was different from the earlier three examples in the construction of [Ke1]. The example turned out to be different from the earlier three using the classification of [PY1].

As a result of the work of [PY1] and its addendum, there are twenty-eight non-empty classes of fake projective planes, and there can be at most five more classes which are conjectured not to exist. In the addendum some corrections were made and the final classification is a more refined one compared to the original scheme proposed in [PY1]. As a result, the final count of classes in the addendum is slightly different from those mentioned earlier in [Rem] and [Ye4]. Each of the examples known earlier lies in one of the twenty-eight classes mentioned. Very recently, Cartwright and Steger [CS], using sophisticated computer-assisted group theoretical argument, have shown that the twenty-eight classes of fake projective planes exhaust all possibilities by eliminating the five putative classes. Moreover, there are precisely one hundred distinct fake projective planes up to biholomorphism in the twenty-eight classes. This concludes the scheme of classification and listing of fake projective planes. We refer the readers to §7 for more details.

At the end of this article, we will explain an application of the work of Cartwright and Steger [CS] to geography of smooth algebraic surfaces of general type. We will also formulate the corresponding classification problem in higher dimensions and explain results of Prasad and Yeung in [PY2] and [PY3].

1.4 Here is the table of content for this article.

1.5 It is a great pleasure for the author to thank his collaborator Gopal Prasad for his detailed comments and invaluable help in polishing the survey, Jiu-Kang Yu for his comments related to buildings, and Donald Cartwright and Tim Steger for explaining their results and helpful comments.

2 Uniformization of fake projective planes

2.1 The purpose of this section is to show that any fake projective plane is the quotient of the unit ball $B^2_{\mathbb{C}}$ in $\mathbb{C}^2$ by a torsion-free cocompact lattice in $PU(2,1)$. This is a classical example of how results in non-linear analysis can lead to advance in algebraic and complex geometry.

Let us begin with the following classical result as stated in [Ya1].

Theorem 1. *a. On a surface M of general type, $c_1^2 \leqslant 3c_2$. (Bogomolov-Miyaoka-Yau Inequality).*
b. The inequality becomes equality if and only if the universal covering $\widetilde{M}$ of M is biholomorphic to the unit ball $B^2_{\mathbb{C}}$ in $\mathbb{C}^2$.

Bogomolov proved in [Bo] a Chern number inequality for any stable vector bundle on a surface, which in the case of tangent bundle of a surface of general type leads to $c_1^2 \leqslant 4c_2$. For the tangent bundle on a surface of general type Miyaoka [Mi] proved that $c_1^2 \leqslant 3c_2$. The proofs of Bogomolov and Miyaoka are algebraic geometric in nature. Yau proved in [Ya1] that $c_1^2 \leqslant 3c_2$ holds for smooth surfaces with an ample canonical line bundle as a consequence of the solution of the Calabi Conjecture in the case of negative scalar curvature [Ya2]. The proof of Yau uses non-linear analysis in differential geometry, and the method has the advantage that uniformization result in Theorem 1b follows from the argument. The role of existence of Kähler-Einstein metric in uniformization as stated in Theorem 1b has been observed by H. Guggenheimer [Gug], as explained in [Ya1].

2.2 The solutions of the Calabi Conjecture in the case of negative scalar curvature by Aubin and Yau ([A], [Ya2]) and the Ricci flat case by Yau [Ya2] are fundamental results in complex differential geometry. The conjecture is a consequence of insight of Calabi ([Cal1], [Cal2]) in his quest for canonical metrics on a complex manifold. An example of such a canonical metric is Kähler-Einstein metric. Recall that a Kähler metric g with a Kähler form ω is Kähler-Einstein if it satisfies the Einstein equation

$$\mathrm{Ric}(\omega) = c\omega,$$

where $\mathrm{Ric}(\omega)$ represents the Ricci curvature form of ω. For our purpose we only state the result in the case of negative scalar curvature.

Theorem 2. *(Aubin [A] and Yau [Ya2]) Let M be a compact complex manifold equipped with a Kähler metric ω of negative Ricci curvature. Then there exists a Kähler-Einstein metric of negative Ricci curvature in the cohomology class of ω.*

The solution of the problem is reduced to solving a Monge-Ampere equation, a non-linear elliptic partial differential equation.

2.3 Assuming Theorem 2, we will explain the proof of Theorem 1 in the case that the canonical line bundle of M is ample, which is sufficient for our study of fake projective planes. From Chern-Weil Theory, we can express the Chern numbers as integrals of expressions invariant in terms of the curvature tensor R of a given Kähler metric g over M. Let Θ be the curvature $(1,1)$-form of a Kähler metric g with Kähler form $\omega = \frac{\sqrt{-1}}{2}\sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ in terms of local holomorphic coordinates, which can be chosen so that $g_{i\bar{j}} = \delta_{ij}$ at the origin. Then the chern forms are given by

$$C_1 = \frac{\sqrt{-1}}{2\pi}\mathrm{tr}(\Theta)$$
$$C_2 = \frac{1}{8\pi^2}\mathrm{tr}(\Theta \wedge \Theta).$$

where $\Theta_{k\bar{l}} = \frac{1}{2}\sum_{i,j} R_{i\bar{j}k\bar{l}} dz^i \wedge \bar{z}^j$ in terms of local coordinates. The first two Chern numbers are given by $c_1^2 = \int_M C_1^2$ and $c_2 = \int_M C_2$. Hence $c_1^2 - 3c_2 = \int_M (C_1^2 - 3C_2)$, where the integrand can be expressed explicitly in terms of the curvature tensor R, the Ricci curvature tensor $\mathrm{Ric}_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}}$ and the scalar curvature $s = 2g^{i\bar{j}}\mathrm{Ric}_{i\bar{j}}$. From direct calculation,

$$C_1^2 - 3C_2 = \frac{1}{4\pi^2}(\frac{s^2}{4} - 4|\mathrm{Ric}|^2 + 3|R|^2) = -\frac{1}{4\pi^2}(-\frac{s^2}{4} + 3|R|^2),$$

where in the second equality we use the fact that $|\mathrm{Ric}|^2 = \frac{s^2}{8}$ since ω is a Kähler-Einstein metric. On the other hand, with respect to the Kähler metric, the curvature tensor has an orthogonal decomposition $R = S + S^{\perp}$, where $S = \frac{s}{12}(R_o)$,

R_o is a four tensor given by $(R_o)_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}$, and $S^\perp$ is orthogonal to S with respect to g. Hence

$$|R|^2 \geqslant |S|^2 = \frac{s^2}{12^2}|R_o|^2 = \frac{s^2}{12}.$$

This immediately implies that $c_1^2 - 3c_2 \leqslant 0$.

Moreover, when the inequality becomes an equality, the integrand $C_1^2 - 3C_2$ vanishes identically on M and hence $R = S = \frac{s}{12}(R_o)$, which up to a constant is the curvature of the Poincaré metric of $B^2_{\mathbb{C}}$ since $s < 0$. After scaling by a constant if necessary, this implies that M is isometric to a complex ball quotient with the Poincaré metric. This completes the proof of Theorem 1. For more elaborate discussions, we refer the readers to [Be] or [Zh].

2.4 Theorem 1 allows us to put fake projective plane in proper perspective.

Theorem 3. *a. A fake projective plane is the quotient of $B^2_{\mathbb{C}}$ by a cocompact torsion free lattice in $PU(2,1)$.*
b. A fake projective plane achieves the minimal Euler-Poincaré number among all smooth projective algebraic surfaces of general type.

Proof In the following we refer the readers to [BHPV] for terminology and basic facts about algebraic surfaces. Let M be a fake projective plane. Denote by $b_i = b_i(M)$ the i-th Betti number of M. Since M is a fake projective plane,

$$b_0 = 1,\ b_1 = 0,\ b_2 = 1,\ b_3 = 0,\ b_4 = 1.$$

Hence the second Chern number $c_2(M)$, which is also the Euler-Poincaré characteristic of M, is 3.

A conjecture of Kodaira, which is a theorem by a result of Siu [Siu1], states that a complex surface with even b_1 is Kähler. Since $b_1(M) = 0$, we conclude that M is Kähler. Poincaré Duality and the Hodge decomposition implies that $b_p = \sum_{i+j=p} h^{i,j}$, where $h^{i,j}$ are the Hodge numbers of M. As all the numbers involved are non-negative integers, and $h^{i,j} = h^{j,i}$ from Hodge identity, we conclude that

$$h^{0,0} = h^{1,1} = h^{2,2} = 1$$

are the only non-zero Hodge numbers. Recall that the arithmetic genus of M is defined by $\chi(\mathcal{O}) = \sum_{i=0}^{3}(-1)^i h^{0,i}$. It follows that $\chi(\mathcal{O}) = 1$ for a fake projective plane. Applying Noether's Formula $\chi(\mathcal{O}) = \frac{1}{12}(c_1^2 + c_2)$ again, we conclude that $c_1^2 + c_2 = 12$. Since $c_2 = 3$, this implies that $c_1^2 = 9$. Hence $c_1^2 = 3c_2$.

Note that $h^{1,1} = 1$ implies that the Picard number of M is 1. Hence the canonical line bundle K_M is a multiple of a generator of the Neron-Severi group modulo torsion. We conclude that K_M is either positive or negative.

Suppose K_M is negative, it follows from a result of Hirzebruch-Kodaira [HK] that M is biholomorphic to $P^2_{\mathbb{C}}$. A more elementary argument for the biholomorphism of M and $P^2_{\mathbb{C}}$ is as follows. We know that M has Picard number $\rho(M) = 1$.

Let H be a generator of the Neron-Severi group $H^{1,1} \cap H^2(M, \mathbb{Z})$ modulo torsion. We choose H to be ample. From Poincaré Duality, the assumption that K_M is negative and the facts that $K_M^2 = 9$ and $\rho(M) = 1$, we know that $K_M = -3H$ modulo torsion. Observe that the linear system $|H|$ has dimension at least 2 from Riemann-Roch Theorem and Kodaira Vanishing Theorem. H has no base locus of dimension 1, for otherwise from Picard number 1, the component has to be homologous to aH modulo torsion for some $a \leqslant 0$, which is not possible. Furthermore, the numerical condition $H^2 = 1, K \cdot H = 1$ implies that any section D in $|H|$ is a rational curve and that the restriction of H to D is positive. It follows that $|H|$ restricted to D is base point free and hence there is no base locus of dimension 0 on M. Hence $|H|$ gives rise to a birational morphism of M on $P_{\mathbb{C}}^2$, which has to be a biholomorphism since no curve can be contracted in view of the fact that the Picard number is 1.

Since we assume that M is not biholomorphic to $P_{\mathbb{C}}^2$, we conclude that K_M is positive and hence M is of general type. It follows from Theorem 1b that the universal covering of M is $B_{\mathbb{C}}^2$. Hence Theorem 3a follows.

The Bogomolov-Miyaoka-Yau Inequality in Theorem 1a and the Noether's Formula also imply that for any smooth surface N of general type,

$$\chi(\mathcal{O}_N) = \frac{1}{12}(c_1^2(N) + c_2(N)) \leqslant \frac{1}{3}c_2(N).$$

It is well-known that $\chi(\mathcal{O}_N) > 0$ for any minimal surface of general type and hence for any smooth surface of general type. As $\chi(\mathcal{O}_N)$ is an integer, $\chi(\mathcal{O}_N) \geqslant 1$. We conclude that $c_2(N) \geqslant 3\chi(\mathcal{O}_N) \geqslant 3$.

Furthermore, the only situation that $c_2(N) = 3$ can occur is when all the above inequalities become identities. This happens only if $c_1^2(N) = 3c_2(N)$ and $\chi(\mathcal{O}_N) = 1$. In other words, $c_1^2(N) = 9, c_2(N) = 3$. These are precisely the characteristic numbers shared by fake projective planes. This concludes the proof of Theorem 3b.

3 Geometric estimates on the number of fake projective planes

3.1 Before the work of [PY1], we only know three distinct fake projective planes, namely, the example of Mumford in [Mu] and the two examples of Ishida-Kato [IK]. The example of Keum [Ke1] appeared when the paper of [PY1] was being completed and it was not clear at that time whether it was different from the other three. On the other hand, it was already known by Mumford in [Mu] that there were only a finite number of fake projective planes. It was hence a natural question whether there exists an effective estimate for the number of fake projective planes. In this section, we are going to explain a rough but nevertheless effective estimates obtained in [Ye2]. The estimates are obtained by completely geometric method. In later sections, we will give much finer classification results obtained in [PY1].

To understand the restriction on the cardinality of fake projective planes and also for later purpose, let us recall the following results on rigidity.

Theorem 4. *(a) (Calabi-Vesentini [CV]) A fake projective plane is locally rigid as a complex analytic manifold.*
(b) (Siu [Siu2]) Any Kähler surface with the same fundamental group as a fake projective plane is actually biholomorphic or conjugate biholomorphic to it.
(c) (Mostow [Mo1]) Any locally symmetric space with the same fundamental group as a fake projective plane is isometric to it with respect to the Killing metric.

3.2 The theorem has the following immediate consequence.

Lemma 1. *The moduli of fake projective planes consists of a finite set of points.*

Proof Any part of the above theorem shows that the set $\mathcal{M}$ of all fake projective planes consists of points. It suffices to show that $\mathcal{M}$ is bounded. There are at least two approaches to this problem. The first is algebraic geometric in nature. Observe that the Chern numbers of a fake projective plane are all bounded. It follows from Matsusaka's Big Theorem that $\mathcal{M}$ is a bounded set as well. The second method is Lie theoretical. A result of H.C. Wang [W] states that the set of lattices Γ in $PU(2,1)$ with covolume in a Haar measure bounded by a constant consists of a finite number of conjugacy classes. The description of $\mathcal{M}$ as a moduli follows either from a result of Gieseker [Ge] or Viehweg [V], since we know that M is a surface of general type.

3.3 The algebraic geometric argument in the above proof can be make effective to give the following effective estimates in [Ye2].

Theorem 5. *The cardinality of fake projective planes is bounded from above by* $(\frac{294!}{5!289!})^{44100} \leqslant 10^{451925}$.

Proof Let M be a fake projective plane. Let H be a generator of the Neron-Severi group $H^{1,1} \cap H^2(M, \mathbb{Z})$ modulo torsion. From the proof Theorem 3, we know that K_M is positive and $K_M = 3H$ modulo torsion. Since the torsion part does not affect intersection numbers, we may just assume that $K_M = 3H$ for all practical purposes in the remaining argument of the theorem.

From the solution of I. Reider [Rei] on Fujita Conjecture in dimension two, we conclude that $K + 4H = 7H$ is very ample. Let $\Phi : M \to P^N_{\mathbb{C}}$ be the projective embedding associated to $7H$. From Riemann-Roch and Kodaira Vanishing Theorem,

$$N = \frac{1}{2}(c_1(7H)(c_1(7H) - c_1(3H)) + \frac{1}{12}(c_1^2(3H) + c_2(M)) - 1 = 14.$$

The degree of the image is given by

$$\deg_\Phi(M) = \int_{\Phi(M)} C_1^2(H_{P^{14}_{\mathbb{C}}}) = \int_M C_1^2(\Phi^* H_{P^{14}_{\mathbb{C}}}) = c_1^2(7H) = 49.$$

Here by abuse of language, we denote by C_1 is the first Chern form. We may choose a generic projection of $\pi : P^{14}_{\mathbb{C}} \to P^5_{\mathbb{C}}$ so that $\pi \circ \Phi : M \to P^5_{\mathbb{C}}$ is still an embedding with image of degree 49.

We now consider the Chow varieties parameterizing cycles of degree d on $P^5_{\mathbb{C}}$. First we note that for each irreducible component C of the Chow variety containing a fake projective space a, each cycle b in C is actually biholomorphic to a. This is a consequence of either Theorem 4a or Theorem 4b. Hence it suffices for us to bound the number of components of the appropriate Chow variety.

Catanese [Cat] gave the first effective bound for the number of components. This bound was improved through the work of Kollár [Ko1] and others. Denote by $\mathrm{Chow}'_{k,d}(P^N_{\mathbb{C}}))$ the union of irreducible components of the Chow variety of dimension k and degree d in $P^N_{\mathbb{C}}$ whose general element is irreducible. According to an expression of Guerra ([Gue]) which was made effective by Heier in [Hei], the number of irreducible components of $\mathrm{Chow}'_{k,d}(P^N_{\mathbb{C}}))$ is bounded from above by

$$\{\frac{[(N+1)d]!}{N![(N+1)d-N]!}\}^{(N+1)(k+1)\frac{(d+k-1)!}{(k-1)!(d-1)!}}.$$

In our case of fake projective space, with $N = 5, k = 2$ and $d = 49$, we get a upper bound of

$$\{\frac{[6\cdot 49]!}{5![(6)49-5]!}\}^{6\cdot 3\frac{(49+1)!}{48!}} = 17689173558^{44100} < 10^{451925}.$$

This concludes the proof of Theorem 5.

3.4 We would like to make a few remarks. First of all we define a fake projective space of complex dimension n to be a Kähler manifold with the same Betti numbers as the projective space of the same dimension, a notion that we will explain more in later sections. Then similar argument as above yields the following results in [Ye2].

Theorem 6. *(a). The cardinality of fake projective spaces of complex dimension n is bounded from above by $N(n)$, where*

$$N(n) = \{\frac{[(2n+2)A]!}{(2n+1)![(2n+2)A-(2n+1)]!}\}^{(2n+2)(n+1)\frac{(A+n-1)!}{(n-1)!(A-1)!}}$$

and

$$A = (n+1)^n[2(2n+3+\frac{(3n+1)!}{(n-1)!(2n+1)!})]^n.$$

(b). Let us equip a locally Hermitian symmetric space of non-positive and non-trivial curvature with a Kähler-Einstein metric whose Kähler form ω satisfies $Ric(\omega) = -\omega$. The resulting Kähler metric differs from the Bergman metric by a normalizing constant. Let $\rho(V)$ be the number of compact locally Hermitian symmetric spaces of non-compact type of complex dimension n which has volume bounded from above by V with respect to the above metric. Then $\rho(V) \leqslant (aV)^{bV^n}$, where $a = \frac{(2n+2)B^n}{[(2n+1)!]^{1/(2n+1)}}$, $b = \frac{2(n+1)^2B^{n^2}}{(n-1)!}$, and $B = 2(n+3+\frac{(3n+1)!}{(2n+1)!(n-1)!})$.

The second remark is that finiteness results as stated in Theorem 3 and 4 are related to the results of Prasad [Pra], Borel and Prasad [BP], stating that the set of all arithmetic locally symmetric spaces with volume bounded by any constant is finite, including all possible dimensions and the Lie groups G involved. We will refer to the papers [Pra] and [BP] in greater details later on, which are crucial for the argument in [PY1]. In another direction, finiteness result could also be established by a differential geometric argument as developed in the papers [BGLM] and [Ge].

4 Arithmeticity of lattices associated to fake projective planes

4.1 In the last section we have explained that any fake projective plane M is a smooth complex two ball quotient $B^2_{\mathbb{C}}/\Pi$ for some torsion free cocompact discrete subgroup Π in $PU(2,1)$. Hence one way to understand and classify M is to determine Π. For this purpose, let us recall some terminology in the study of lattices in semisimple Lie groups.

Semi-simple Lie groups can be studied geometrically through symmetric spaces. Take a semi-simple Lie group and a maximal compact subgroup K of G. The quotient space G/K is a symmetric space, on which there is a natural invariant metric given by the Killing form. G/K is a Hermitian symmetric space if there is a G-invariant complex structure on it. Symmetric spaces are higher dimensional geometric models which to a certain extent play the role of the upper half plane, the Riemann sphere and the Euclidean plane in real two dimensions. In particular, in the special case of $G = PU(n,1)$ and $K = P(U(n) \times U(1))$, we get a complex n ball $B^n_{\mathbb{C}}$, the unit ball in complex Euclidean space of complex dimension n. Symmetric spaces and semi-simple Lie groups have been classified by E. Cartan (cf. [Hel]). In the case that G is non-compact, we obtain a non-compact symmetric space on which the Killing metric has non-positive sectional curvatures. According to Cartan-Hadamard Theorem, the non-compact symmetric spaces are diffeomorphic to a Euclidean space. Symmetric spaces are the universal covering of the locally symmetric space $\Gamma\backslash G/K$ if Γ is torsion-free. Hence the topology of a locally symmetric space $\Gamma\backslash G/K$ with torsion-free Γ is mostly determined by Γ. We will be mainly interested in complex ball quotients $B^n_{\mathbb{C}}/\Gamma$ where the corresponding lattices Γ involved are cocompact.

4.2 We recall that a lattice Γ of a semi-simple Lie group G is arithmetic if there exists a semisimple algebraic $\mathbb{Q}$-group H and a surjective homomorphism $\varphi : H^0_{\mathbb{R}} \to G$ with compact kernel such that $\varphi(H^0_{\mathbb{R}} \cap H_{\mathbb{Z}})$ is commensurable to Γ. A fundamental result of Margulis [Mag] states that in the case that the real rank of G is at least 2, any irreducible lattice Γ in G is arithmetic. In the case that the symmetric space is either a quaternionic hyperbolic space or the Cayley hyperbolic plane, it follows from the results of Corlette [Co] and Gromov-Schoen [GS] that any lattice in the corresponding G are arithmetic as well. From the classification of E. Cartan, the only symmetric spaces left are the real and complex hyperbolic spaces. It has been known for sometime that lattices in the real hyperbolic spaces are not as rigid as in the higher rank case and in particular, there are lots of non-arithmetic lattices. Hence the remaining class of lattices, those in $PU(n,1)$, the automorphism group of a complex hyperbolic space, are particularly interesting from this point of view. Not much is known about the lattices in $PU(n,1)$. It has long been observed by Mostow [Mo1] and Deligne-Mostow [DM] that there are non-arithmetic lattices in $PU(2,1)$ and $PU(3,1)$, which can be traced to the construction of Picard [Pi] and Le Vavasseur [Le]. Hence an interesting problem is to find conditions to characterize arithmetic lattices in the complex hyperbolic spaces.

We note that the problem of arithmeticity or superrigidity can be formulated and generalized to the setting of geometric superrigidity, such that the non-linear version of a Bochner type formula gives rise to superrigidity or arithmeticity, while the linear version leads to cohomology vanishing results, namely Matsushima's vanishing theorem. From this point of view, topological conditions may be related to arithmeticity. It turns out that such a conjecture was formulated by Rogawski, where the motivation seems to come instead from study of a concrete series of examples in Langlands program.

Conjecture 1. *(cf. [Rez]) Let Γ be a torsion-free cocompact lattice of $PU(2,1)$ so that the corresponding ball quotient $M = \Gamma\backslash B^2_{\mathbb{C}}$ satisfies the conditions that the first Betti number $b_1(M) = 0$ and the Neron-Severi group $H^{1,1}(M) \cap H^2(M,\mathbb{Z})$ modulo torsion is $\mathbb{Z}$. Then Γ is arithmetic of second type (cf. §4.8).*

We refer the reader to Section 3 for more details about the descriptions and terminology of arithmetic lattices of complex hyperbolic spaces. In [Ro], Rogawski shows that the congruence subgroups coming from arithmetic lattices of second type satisfy the above cohomological properties.

The papers [Kl] and [Ye1] are some independent attempts to address the conjecture. We have the following result as a corollary.

Theorem 7. *([Kl] [Ye1]) The lattice associated to a fake projective plane is arithmetic.*

In the following we give an outline of proof of the above statement, following the expository in [Ye1].

4.3 Let M be a fake projective space. As mentioned earlier, the universal covering $\widetilde{M}$ of M is biholomorphic to the complex ball $B^2_{\mathbb{C}}$ in $\mathbb{C}^2$ and M is the quotient of $\widetilde{M}$ by a torsion free cocompact subgroup of $PU(2,1)$.

A result of Weil tells us that any cocompact lattice Γ of $PU(2,1)$ is locally rigid, from which it follows that Γ can be defined over a number field (cf. [Ra]), that is, there exists an injective homomorphism $\rho : \Gamma \to G(k)$, G an algebraic group defined over a number field k with a real place v_o such that $G(k_{v_o}) \cong PU(2,1)$. We call Γ integral if there exists a subgroup Γ' of finite index in Γ so that $\rho(\Gamma') \subset G(\mathcal{O}_k)$.

In the setting of the above discussion, the proof of Theorem 7 is then separated into two steps, namely, Step A, the analogue of non-Archimedean superrigidity, that the lattice Γ is integral at any non-Archimedean place; and Step B, the analogue of Archimedean superrigidity at any Archimedean place in the sense of §22 in [Mo2] or Appendix C(3.5) in [Mag]

4.4 Step A: Integrality

As above, let $\Pi \subset PU(2,1)$ be the fundamental group of a fake projective plane M. We assume for the sake of proof by contradiction that Π is not integral. Denote by k_v the completion of k at a finite place v. Non-integrality of Π implies that there exists a finite place v of k such that the induced homomorphism $\rho_v : \Gamma \to G(k_v)$ is unbounded. This is a situation where techniques of non-linear analysis in geometry can be brought in. We also need some basic properties of Bruhat-Tits buildings, for which we refer the readers to [Br].

Let X be the Bruhat-Tits building associated to $G(k_v)$. X is endowed with a metric such that the restriction of the metric to an apartment Σ of X is isometric to the Euclidean metric on $\mathbb{R}^r$, where r is the dimension of the building, which is also the k_v-rank of G. The metric is non-positively curved in the sense of Alexandrov spaces. According to Theorem 4.4, Theorem 7.1 of Gromov and Schoen [GS] on harmonic maps into singular spaces which are simplicial complexes, there exists a energy minimizing ρ_v-equivariant harmonic map $f : \widetilde{M} \to X$. The mapping f is said to be regular at $x \in \widetilde{M}$ if a neighborhood of x is mapped to an apartment of X. The complement of the set of regular points is denote by $\mathcal{S}(f)$ and is called the singular set of f. According to Theorem 6.4 of [GS], f is sufficiently regular in the sense that f is Lipschitz continuous and the singular set $\mathcal{S}(f)$ of f has Hausdorff codimension at least 2. The regularity of the map is crucial for our argument in more than one place. In particular, it is required for the purpose of integration by parts so that Bochner formula can be applied.

Since the absolute rank of G is 2, $r = \mathrm{rank}_{k_v}(G)$ is either 1 or 2. An apartment Σ of the building X is isometric to $\mathbb{R}^r$. In the case of $\mathrm{rank}_{k_v}(G) = 2$, $G(k_v) = PGL(3, k_v)$. As in page 147 of [Br], we write an apartment Σ as $\mathbb{R}^2 \cong \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + x_2 + x_3 = 0\}$, on which the affine Weyl group $W \cong A \rtimes \overline{W}$ acts by translations in $A \cong \mathbb{Z}^2$ and $\overline{W} \cong S_3$, symmetric group of three elements. In the case that $\mathrm{rank}_{k_v}(G) = 1$, X is a tree and the affine Weyl group $W \cong A \rtimes \overline{W}$ with $A \cong \mathbb{Z}$ and $\overline{W} \cong \mathbb{Z}_2$.

The harmonic map $f : \widetilde{M} \to X$ into the building obtained by Gromov-Schoen [GS] is actually pluriharmonic by the Bochner formula of Siu [Siu2]. At a regular point of the harmonic map f so that $f(B_\delta(x))$ lies in an apartment Σ for a small $\delta > 0$, the pull-back f^*dx gives rise to a locally defined harmonic one form for any linear function x on X. To have a globally defined one, let $x = x_1 - x_2$ be a root vector and observe that $s_i x$, $s_i \in \overline{W}$, give a root system of the Lie algebra of G. Consider an m-tuple of one forms $(f^*(s_i(dx)))_\Sigma := (f^*(s_1(dx)), \cdots, f^*(s_m(dx)))$, where $m := |\overline{W}|$ denotes the cardinality of $\overline{W}$. At the intersection of two apartments Σ_1 and Σ_2, two such ordered sets $(f^*(s_i(dx))_{\Sigma_1}$ and $(f^*(s_i(dx)))_{\Sigma_2}$ agree up to the permutation given by an element of $\overline{W}$. Hence we see that they glue together to form a m-valued 1-form on $\widetilde{M} \backslash \mathcal{S}(f)$. From the regularity of f given by §6 of [GS], we may apply the Bochner formula of Siu (cf. [Siu2]) to conclude that, $(f^*(s_i(dx)) \otimes \mathbb{C})^{1,0})_\Sigma$, the $(1,0)$-part of the complexification of the multivalues one forms, can be regarded as multi-valued holomorphic one forms on M.

Let σ_i be the elementary symmetric polynomials of degree i constructed from $(f^*(s_j(dx))\otimes\mathbb{C})^{1,0}, j=1,\dots,m$ so that $\sigma_i\in H^0(M,S^i(T_M^*))$ is independent of Σ, where $S^i(T_M^*)$ is the symmetric product of the holomorphic cotangent bundle. The equation

$$x^m-\sigma_1x^{m-1}+\cdots+(-1)^m\sigma_m=0$$

then defines a m-fold cover $\widetilde{N}$ of $\widetilde{M}$ on the cotangent bundle of $\widetilde{M}$. Since f is $\pi_1(M)$-equivariant, the covering descends to give a covering $\pi_o:N\to M$, which is proper according to [Sim3]. On a Zariski open set U of M on which the image of f does not hit the wall of a building, it is clear that N is a m-sheeted cover of U. The roots of the above equation lead to m holomorphic 1-forms on $\pi_o^{-1}(U)$, which can be extended to N from Lipschitz continuity of f.

The covering N by our construction may have several connected isomorphic components, corresponding to a subgroup of the Weyl group $\overline{W}$. Let N_1 be a connected component of N. Let M_1 be a desingularization of N_1 by Hironaka's resolution of singularities. In the following, we call M_1 the spectral covering of M for simplicity in terminology. The covering maps $\pi:M_1\to M$ and $\widetilde{\pi}:\widetilde{M_1}\to M$ are possibly ramified, with the covering group $\overline{W}_1$ a subgroup of $\overline{W}$. We still denote $\omega_i=(\widetilde{\pi}^*f^*(s_i(dx))\otimes\mathbb{C})^{1,0}, j=1,\dots,m$, which are also regarded as forms on M_1. From construction, the singularity of $f\circ\widetilde{\pi}$ can only occur at the ramification locus of $\widetilde{\pi}$ where $\omega_i=\omega_j$ for some $i\neq j$, and the image in X has to lie in a wall of an apartment in X.

We refer the reader to [Sim3], [JZ], or [Ka] for more discussions on the formulations related to spectral coverings.

To illustrate the argument for integrality, let us assume first that $\mathrm{rank}_{k_v}(G)=1$. In this case, X is a tree. $\omega=(\pi^*f^*dx\otimes\mathbb{C})^{1,0}$ becomes a holomorphic 1-form on M_1, where one uses Siu's Bochner formula and the regularity of f mentioned above to obtain the holomorphic form from a harmonic one. Hence the Albanese map $\alpha:M_1\to\mathrm{Alb}(M_1)$ is non-trivial. We may assume that $|\overline{W}_1|\neq1$ for otherwise $M=M_1$ supports non-trivial holomorphic one forms, contradicting $h^0(M)=0$. In case of $|\overline{W}_1|=2$, we apply a result of Simpson [Sim1] on Lefschetz Theorem, to reduce to the case where the image of α is a curve, and α descends to a non-trivial map $\alpha_o:M\to C_o$, an orbicurve. This however contradicts the assumption that $\rho(M)=1$, where ρ is the Picard number of M, since a generic fiber of α_o and the pull-back of an integral $(1,1)$ class on C_o give rise to two linear independent classes in the Neron-Severi group. The proof for $\mathrm{rank}_{k_v}(G)=1$ follows more or less from an unpublished result of Simpson.

The case $\mathrm{rank}_{k_v}(G)=2$ is similar, except that it is technically more difficult. In this case, an apartment in X can be written as $\Sigma=\{(x_1,x_2,x_3)\in\mathbb{R}^3|x_1+x_2+x_3=0\}\cong\mathbb{R}^2$. The covering group $\overline{W}_1$ of the spectral covering $p:M_1\to M$ is a subgroup of $\overline{W}\cong S_3$, the Weyl of the root system of X. The key point of our argument is to show that π is unramified. Instead of the set of holomorphic one forms $\omega_i, i\in|\overline{W}|$, on M described earlier, sometimes it is more convenient to consider holomorphic one forms $\kappa_i=(\pi^*f^*dx_i\otimes\mathbb{C})^{1,0}$, $i=1,2,3$, in terms of the standard coordinates on an apartment described earlier. There is the obvious relation $\sum_{i=1}^3\kappa_i=0$. Let $\mathrm{Alb}_{\overline{W}_1,\{\omega_i\}}(M_1)$ be the abelian variety defined as the quotient of the albanese variety by the $\overline{W}_1$-invariant abelian subvariety annihilated by $\omega_i, i = 1, 2, 3$, and $\alpha : M_1 \to \mathrm{Alb}_{\overline{W}_1,\{\omega_i\}}(M_1)$ be the corresponding albanese mapping. For simplicity, we denote $\mathrm{Alb}_{\overline{W}_1,\{\omega_i\}}(M_1)$ by $\mathrm{Alb}(M_1)$ when there is no danger of confusion.

We claim that the image of α has complex dimension 2. Otherwise $\alpha : M \to C := \alpha(M_1)$ is of complex dimension 1. Since α is $\overline{W}_1$-invariant, we conclude that $h(F)$ is contracted by α for each element $h \in \overline{W}_1$ and each fiber F of α. Hence the fibration $\alpha : M_1 \to C$ descends to a mapping $\alpha_o : M \to C_o$ onto an orbicurve C_o. As argued earlier, this contradicts $\rho(M) = 1$. The claim is proved.

4.5 We would like to understand the ramification locus R_π of the spectral covering. Let R^o_π be union of irreducible components in R_π that are not contracted by π to a point on M. Recall that any component of R^o_π is defined by $\omega_i = \omega_j$ for some $i \neq j$. On the other hand, the singularity of the harmonic map f has image lying in the walls of an apartment of the building. Writing ω_i as $\kappa_{i_1} - \kappa_{i_2}$, it means that only those pairs of (i, j) with $x_{i_1} - x_{i_2} = x_{j_1} - x_{j_2}$ defining a wall in an apartment as described earlier can be possible candidates in R^o_π. Suppose we write $\omega_1 = \kappa_1 - \kappa_2$, $\omega_2 = \kappa_1 - \kappa_3$, $\omega_3 = \kappa_2 - \kappa_3$ and $\omega_i = -\omega_{i-3}$ for $i = 4, 5, 6$. We can check easily which pair of (i, j) can be ruled out as a candidate to be in R^o_π. In particular $\omega_1 - \omega_3 = 0$ cannot be a candidate since $(x_1 - x_2) - (x_2 - x_3) = 0$ does not lie in the wall of an apartment of X, but $\omega_1 - \omega_2$ is a possible candidate since $(x_1 - x_2) - (x_1 - x_3) = 0$ defines a wall on an apartment. Clearly the ramification index of π at such a possible ramification divisor in R^o_π is 2, corresponding to the element $(1)(23) \in S_3$ which fixes x_1 but permutes x_2 and x_3. This is clearly the case for all possible candidates in R^o_π. We claim the following.

Lemma 2. *$\pi : M_1 \to M$ is unramified.*

Proof The key point is to show that R^o_π is empty. We argue by proof by contradiction. Assume that D is an irreducible component in R^o_π, the ramification divisor of π. First we observe that D cannot be contracted by the Albanese map α, for otherwise similar argument as in the last paragraph shows that its image $\pi(D)$ in M would be contracted by a mapping α_o onto an orbifold of complex dimension 2, which already supports an invariant Kähler form. Again, this would contradicts the assumption that $\rho(M) = 1$. Hence $\alpha(D)$ is actually a divisor on $\mathrm{Alb}(M_1)$. From definition of the spectral mapping $\pi : M_1 \to M$, D corresponds to the set on M_1 on which two of those one forms ω_i's are identified. Hence without loss of generality, we may assume that on D, $\omega_1 = \omega_2$ and hence $\alpha(D)$ lies in the kernel of a non-trivial holomorphic one form of the type $dz_1 - dz_2$ in terms of some Euclidean coordinates on the universal covering $\widetilde{A} \cong \mathbb{C}^2$ of $\mathrm{Alb}(M_1)$. Hence a lift of $\alpha(D)$ to $\widetilde{A}$ is defined by a linear equation $z_1 - z_2 = c$ for some constant c. The projection of such a linear space in the abelian variety $\mathrm{Alb}(M_1)$ can be a divisor only if the divisor is an abelian subvariety. Hence $\alpha(D)$ is an Abelian subvariety of $\mathrm{Alb}(M_1)$.

Let C be a generic fiber of $M_1 \xrightarrow{\alpha} \mathrm{Alb}(M_1) \xrightarrow{\beta} \mathrm{Alb}(M_1)/\alpha(D) = E$. In the next few paragraphs, we reach a contradiction by observing that on one hand, $\overline{p_*(\pi(C))}$, the Zariski closure of $p_*(\pi(C))$, is a non-trivial normal subgroup of the simple group G and hence is equal to G, and on the other hand $f(\pi(C))$ is a tree lying in X with the stabilizer given by a proper subgroup of G.

Here are the details of the argument. Let $h = \beta \circ \alpha : M_1 \to E$. The set of regular values of h is a Zariski open set E_o of E. E_o is an affine curve. Consider the homotopy sequence of the fibration,

$$1 = \pi_2(E_o) \to \pi_1(h^{-1}(a)) \xrightarrow{i_{1*}} \pi_1(h^{-1}(E_o))) \to \pi_1(E_o) \to \cdots,$$

where i_{1*} is the homomorphism induced by the embedding. As $\pi_1(h^{-1}(a))$ is a normal subgroup of $\pi_1(h^{-1}(E_o))$. $h^{-1}(E_o)$ is a Zariski-open subset of M_1 and the homomorphism $i_{2*} : \pi_1(h^{-1}(E_o)) \to \pi_1(M_1)$ is surjective. Pulling back the homomorphism $\rho : \pi_1(M) \to G(k_p)$ by the projection $\pi : M_1 \to M$ and restricting to $h^{-1}(a)$, we get a homomorphism $\sigma = \rho \circ \pi_* \circ i_{2*} \circ i_{1*} : \pi_1(h^{-1}(a)) \to G(k_p)$. The Zariski closure $\overline{\sigma(\pi_1((h^{-1}(a)))}$ of the image of σ is a normal algebraic subgroup of G. Since G is simple, $\overline{\sigma\pi_1(h^{-1}(a)))}$ is either trivial or G. Let $\widetilde{h^{-1}(a)}$ be the universal covering of $h^{-1}(a)$. Let $\widetilde{\pi} : \widetilde{h^{-1}(a)} \to \widetilde{M}$ be a lift of $p|_{h^{-1}(a)} : h^{-1}(a) \to M$ to the corresponding universal covering spaces. The holomorphic one forms ω_i on M_1 are constructed from $(\pi^* f^* dx_i \otimes \mathbb{C})^{1,0}$.

From construction, it follows that locally around a regular point $x \in \widetilde{M}_1$ of f where $f(x)$ lies in the interior of a chamber in an apartment in X, dh is locally the same as the real part of $d\alpha$. Hence as α is unbounded on $\widetilde{h^{-1}(a)}$, so is $f \circ \widetilde{\pi}$ on $\widetilde{h^{-1}(a)}$. As a fundamental domain of $h^{-1}(a)$ is compact, we conclude that σ and hence the Zariski closure $\overline{\sigma(\pi_1(h^{-1}(a)))}$ of $\sigma(\pi_1(h^{-1}(a)))$ in G is non-trivial. Since $\overline{\sigma(\pi_1(h^{-1}(a)))}$ is a normal subgroup of G, a simple algebraic group, this implies that $\overline{\sigma(\pi_1(h^{-1}(a)))} = G$.

On the other hand, we are going to show in the following that for a generic choice of a, the set $\overline{\sigma(\pi_1(h^{-1}(a)))}$ is a proper subgroup of $G(k_p)$.

We need the following sublemma. We say that a subset S of the building X is convex if the line segment joining any two points $x_1, x_2 \in S$ has to line in S.

Sublemma 1. *The set $f \circ \widetilde{\pi}(H_a)$ is convex.*

Proof Suppose Σ is an apartment of the building such that $f \circ \widetilde{\pi}(H_a) \cap \Sigma$ is unbounded. We need to show that the image $f \circ \widetilde{\pi}(H_a) \cap \Sigma$ is isometric to $\mathbb{R} \cap \Sigma$ as a set with the Euclidean metric, if $\mathbb{R} \cap \Sigma \neq \emptyset$, where $H_a = \widetilde{h^{-1}(a)}$. The earlier discussions show that it is true for each chamber in Σ. Without loss of generality, we may assume that the image of the ramification divisor R by f in an apartment Σ containing an open set of L_a is defined by $x_2 - x_3 = 0$, so that $L_a \cap C$ for some chamber C in Σ is a line segment defined by $x_2 - x_3 = c_a$, a constant. Hence R is defined by $\omega_2 - \omega_3 = 0$ on M_1. We are done if $f|_{\widetilde{\pi}(H_a)}$ is non-singular, which implies that $f \circ \widetilde{\pi}(H_a)$ is isometric to $\mathbb{R}$. $f \circ \widetilde{\pi}|_{H_a}$ has singularity only along another ramification divisor R_1 on $\widetilde{M}_1$, which is the stabilizer of an element $\iota \in \overline{W}$ of order 2 since its image lies in the wall of a building. Hence we may assume that the image of R_1 in Σ is defined by $x_1 - x_2 = 0$. As the local covering group generated by ι switches dx_1 and dx_2, we observe that $f \circ \widetilde{\pi}|_{H_a}$ is extended beyond

$f\circ\widetilde{\pi}|_{H_a}\cap f\circ\widetilde{\pi}(R_1)$ in a unique way as a line segment in the adjacent chamber of C in Σ defined by $x_1-x_3=c_a'$ for some constant c_a' determined by continuity.

Let τ be the global one form on M_1 annihilating R_1, defined locally by $1/2(\kappa_1+\kappa_2)$, where $\kappa_i=(p^*f^*(dx_i)\otimes\mathbb{C})^{1,0}$. τ is the pull back of a holomorphic one form τ_o on $E:=A/\ker(\tau)$ by $\alpha_o: M_1\xrightarrow{\alpha}A\xrightarrow{q}E$. Let $\eta=\alpha_o^*\Re(\tau_o)$. Fix $z_o\in H_a$ so that $\widetilde{\pi}(z_o)$ is a regular point of f. Define $\Phi:\widetilde{M_1}\to\mathbb{R}$ by $\Phi(z)=\int_{z_o}^z\eta$.

We claim that for each apartment Σ for which $L_a\cap\Sigma$ is unbounded, there is a covering map $\Psi:\mathbb{R}=\Phi(\widetilde{M_1})\to L_a\cap\Sigma$ which is a local isometry, so that $\Psi\circ\Phi(z)=f\circ\widetilde{\pi}(z)$. Suppose $\Phi(z_1)=\Phi(z_2)$. Join z_i to z_o by a unique geodesic γ_i on H_a for $i=1,2$. It follows that for all t on γ_1, there exists $w(t)\in\gamma_2$ varying continuously with respect to t such that $\Phi(w(t))=\Phi(t)\in\mathbb{R}$. It suffices for us to show that $f\circ\widetilde{\pi}(t)=f\circ\widetilde{\pi}(w(t))$ for all $t\in\gamma_1$ by continuity argument. This is clearly so for t in a small neighborhood of z_o or a regular point of f from definition of τ. Hence we only need to make sure that the argument can be extended beyond the singularity set $\mathcal{S}$ of f. Observe that the spectral covering is defined equivariantly on $\widetilde{M}$ and it suffices for us to discuss on M_1. As formulated in §2 of the paper, M_1 is the desingularization of M_{1o}, a connected component of M_1' defined by the single equation $\sum_{i=0}^{\ell}\alpha_i(x)t^{l-i}=0$ in T^*M, where $l=6$. Let $p:\widetilde{M_1}\to M_1$ be the universal covering. Then $\alpha_o(p(t))=\alpha_o(p(w(t)))=Q\in\alpha_o(\mathcal{S})$. It follows that $t,w(t)\in\alpha^{-1}(E_Q)$, where $E_Q:=q^{-1}(Q)$ is connected in A. Here $E_Q\cap\alpha(R_{23})=\emptyset$ from construction, where R_{23} is defined by $\kappa_2-\kappa_3=0$ on M_1. The Albanese map $\alpha:M_1\to A$ descends to $\alpha_o:M_{1o}\to A$, since from definition it contracts the fibers of the desingularization map $M_1\to M_{1o}$. For the claim, it suffices for us to make the observation, to be proved below, that $\alpha_o^{-1}(E_Q)$ is irreducible and is a component of the ramification divisor containing both $p(t)$, and $p(w(t))$. Let U_t and $U_{w(t)}$ be neighborhoods of t and $w(t)$ on H_a. From the discussions in the first paragraph above, the extension of $f\circ\widetilde{\pi}|_{U(t)}$ and $f\circ\widetilde{\pi}|_{U(w(t))}$ through $f\circ\widetilde{\pi}(t)=f\circ\widetilde{\pi}(w(t))$ is uniquely determined by the order two stabilizer of $\alpha^{-1}(q^{-1}(Q))$ in M_1. The claim follows from continuity.

To explain the observation, suppose there are two different components D_{1o} and D_{2o} of $\alpha_o^{-1}(E_Q)$, in which D_{1o} is a ramification divisor of π and is defined by $\kappa_1-\kappa_2=0$. M_{1o} does not support any divisor with negative self-intersection contracted by α_o. In fact, an irreducible divisor D' of M_1 contracted by α to the abelian surface $\alpha(M_1)$ has to be a rational curve. As M is hyperbolic, $\pi(D')$ is a single point. Since the spectral covering $M_{1o}\to M$ is (everywhere) finite, It follows that the image of D' in M_{1o} is finite as well. With no curves of negative self-intersection, by writing $\alpha^*E_Q=D_{1o}+D_{2o}+R'$, where R' is the rest of the components in α^*E_Q, it follows from $E_Q\cdot E_Q=0$ that actually $D_{1o}\cdot D_{2o}=0$ and hence $D_{1o}\cap D_{2o}=\emptyset$. On the other hand, their images as effective divisors on M intersect at least at a point $P\in M$, which follows from the fact that the images of D_{1o} and D_{2o} on the quotient building $X/\rho(\Gamma)$ by our mapping are the same as a set, or from the assumption that the Picard number of M is one. Hence there are at least two distinct points X_1 and X_2 at $\pi^{-1}(P)\cap(D_{1o}\cup D_{2o})$. Since $\overline{W}=S_3$ acts on M_1', locally above a point $y\in\pi(D_{1o})$ on M_1, there are components of ramification divisors $\epsilon_1(D_{1o})=R_{23}$ and $\epsilon_2(D_{1o})=R_{13}$ given by $\kappa_3-\kappa_2=0$ and $\kappa_1-\kappa_3=0$, where $\epsilon_i=(i3)\in S_3, i=1,2$. If D_{1o} meets neither $\epsilon_1(D_{1o})$ nor $\epsilon_2(D_{1o})$, $\alpha^{-1}(E_Q)$

is not meeting any ramification divisor and hence is regular everywhere, which implies that image $f(\pi(\alpha^{-1}(E_{Q'})))$ for a point Q' near Q is a straight line in Σ and we are done. Hence assume that D_o meets, say $\epsilon_1(D_{1o})$ on M_1' at X_1. As $\omega_1 = \omega_2 = \omega_3$ at X_1 from definition, $\epsilon_2(D_o)$ passes through X_1 as well. It follows that for a generic point X' in a small neighborhood U of X in M_1, the cardinality of $U \cap \pi^{-1}(\pi(X'))$ is at least 6 and hence $U \cap \pi^{-1}(\pi(X')) = \pi^{-1}(\pi(X'))$. We conclude that there cannot be another divisor D_{2o} passing through another point in $X_2 \in (\pi^{-1}(P) - \{X_1\}) \subset M_1$. This also follows from the fact that the orbit of X_1 by S_3 is just X_1 itself, implying that $\{X_1\} = \pi^{-1}(P)$ as a set.

Hence the claim is valid. From the claim, a connected unbounded component of $L_a \cap \Sigma$ is covered isometrically by $\mathbb{R}$. Recall also that $H_a \cap R = \emptyset$ from definition, since their image in A are disjoint fibers of $q : A \to E$. For a connected component $\widetilde{R}$ of the pull back of R to $\widetilde{M}_1$, it follows that $(L_a \cap \sigma(\pi_*(\pi_1(M_1)))f(\widetilde{R})) \cap \Sigma = \emptyset$, here $\sigma(\pi_*(\pi_1(M_1)))f(\widetilde{R}) \cap \Sigma$ forms an infinite number of equally spaced parallel straight lines given by $x_2 = x_3 + k$ for different integers k when we represent Σ as $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. Hence the connected set L_a lies in a strip confined by two such parallel lines as above. From the proof of the claim, L_a can bend only at intersection with some lines of form $x_1 = x_2 + c$ or $x_1 = x_3 + d$ for some integers c and d in a unique way as described earlier. Recall that the stabilizer of G acts on Σ by the affine Weyl group $W := \mathbb{Z}^2 \rtimes S_3$. Bending along $x_1 = x_2 + c$ or $x_1 = x_3 + d$ correspond to reflections $(12), (23) \in S_3$ respectively. The fact that no bending occurs along any line $x_2 = x_3 + e$ for some integer e means that $\sigma(\pi_1(h^{-1}(a))) \cap W$ does not contain element of type $(a, (13)) \in \mathbb{Z}^2 \rtimes S_3$. Hence only one of (12) of (23) may occur as element h for some element $(b, h) \in (\mathbb{Z}^2 \rtimes S_3) \cap \sigma(\pi_1(h^{-1}(a)))$, for otherwise their product gives rise to some element $(b', (13)) \in (\mathbb{Z}^2 \rtimes S_3) \cap \sigma(\pi_1(h^{-1}(a)))$. Suppose $(b, (12))$ does appear as an element in $(\mathbb{Z}^2 \rtimes S_3) \cap \sigma(\pi_1(h^{-1}(a)))$. Then for all elements $(b, h) \in (\mathbb{Z}^2 \rtimes S_3) \cap \sigma(\pi_1(h^{-1}(a)))$, h can only take the value of (1) or (12). We now show that the latter cannot happen. Suppose on the contrary that $(b_1, (12))$ belongs to $(\mathbb{Z}^2 \rtimes S_3) \cap \sigma(\pi_1(h^{-1}(a)))$ for some $b_1 \in \mathbb{Z}_2$. It implies that a bend of l_a does occur at some $x_1 = x_2 + c$. By taking composition with another element in the infinite group $(\mathbb{Z}^2 \rtimes S_3) \cap \sigma(\pi_1(h^{-1}(a)))$ if necessary, we may assume that a_1 is non-trivial. Taking powers of such elements shows that L_a, which is connected and unbounded in Σ, cannot be confined to a strip between $x_2 = x_3 + k_1$ and $x_2 = x_3 + k_2$ for some fixed integers k_1 and k_2, by considering a sequence of line segments in the apartment which can only be parallel either to $x_2 = x_3$ or $x_1 = x_2$ and bend only at intersection with lines of form $x_1 = x_2 + c_i$, $c_i \in \mathbb{Z}$ in the way determined by the claim. Hence $h = 1$ and $\sigma(\pi_1(h^{-1}(a))) \cap W$ can only act by translation, corresponding to the first factor in $\mathbb{Z}^2 \rtimes S_3$. Since L_a is isometric to $\mathbb{R}$ from the claim, it means that L_a sits as a straight line in Σ. This concludes the proof of the Sublemma.

Let us now continue with the proof that the stabilizer of $\overline{\sigma(\pi_1(h^{-1}(a)))}$ is a proper subgroup of $G(k_p)$. Let us now fix such a generic value a. On each fixed apartment, clearly we may assume that $c = 0$ after moving by some isometry on the apartment. From the second remark above, it follows that the isometry can be extended to a global isometry on X so that $c = 0$ for all apartments. As a result, $f \circ \pi(h^{-1}(a)) \cap \Sigma$ is contained in a linear subspace Δ of codimension one defined on Σ by $x_i - x_j = 0$ on each apartment, where x_i, x_j are consistently defined throughout X and is independent of the apartments. For our building X associated to $G(k_p) = SL(3, k_p)$, vertices of an apartment Σ correspond to equivalence classes $[L]$ of lattices of the form

$$L = < p^{r_1}e_1, p^{r_2}e_2, p^{r_3}e_3 >_{\mathcal{O}_{k_p}}$$

with integer values r_i in a 3 dimensional vector space $V =< e_1, e_2, e_3 >$. Without loss of generality, we assume that $\omega_o = \omega_1 - \omega_3$ and the image $f \circ \pi(h^{-1}(a)) \cap \Sigma$ is described by $x_1 - x_3 = 0$. Since $x_i = r_i - \frac{\sum_{j=1}^3 r_j}{3}$, this corresponds to $r_1 = r_3$ in all the lattices corresponding to the image $f \circ \pi(h^{-1}(a)) \cap \Sigma$. Consider a two dimensional vector subspace $V'_\Sigma =< e_1 + e_3, e_2 >$ of V. The set of lattices in V'_Σ with $e'_1 = e_1 + e_3$ and $e'_2 = e_2$ as a basis contributes to an apartment Σ' of a rank 2 simplicial complex X'_Σ associated to V'_Σ. From the earlier remarks, $V'_{\Sigma_1} = V'_{\Sigma_2}$ for two apartments Σ_1, Σ_2 sharing a common chamber and hence that $V' = V'_\Sigma$ is independent of Σ for which $f_1(M_1) \cap \Sigma$ contains at least a chamber. In this way, V' is a well-defined two dimensional subspace of V since it is independent of the chambers chosen. The image of f_1 is a subset on which $G'(k_p) = SL(V', k_p) \cong SL(2, k_p)$ as a subgroup of $G(k_p) = SL(V, k_p) \cong SL(3, k_p)$ acts. Hence the stabilizer of $\overline{\sigma(\pi_1(h^{-1}(a)))}$ lies in $G'(k_p)$, a proper subgroup of $G(k_p)$. This contradicts the fact that G involved is absolutely simple as mentioned earlier. Hence R^o_π is empty.

Since $R^o_\pi = \emptyset$, all codimension one components in R_π are contracted by π. It means that $\widetilde{\pi}_o : N \to M$ in §4.4 is unramified except at a finite number of points. Since the ramification divisor of $\widetilde{\pi}_o$ is obtained by identifying two roots of the equation $\sum_{i=0}^m (-1)^i \sigma_i x^{m-i} = 0$ in the cotangent bundle, its image in $\widetilde{M}$ is the zero set of the discriminant of the above polynomial on M. Hence the ramification locus of $\widetilde{\pi}_o$ has to be of codimension 1 if it is non-trivial. This implies that π is an unramified covering. This completes the proof of the lemma.

4.6

Lemma 3. *There is a real two torus T^2 and a real analytic mapping of q : $Alb(M_1) \to T^2$, so that the lift of q to the universal covering $\widetilde{q} : \mathbb{C}^2 \to \mathbb{R}^2$ is the projection into the real coordinates.*

Proof The lift of the Albanese map to the universal covering of $\widetilde{M}_1$ is given by $\widetilde{\alpha} : \widetilde{M}_1 \to \mathbb{C}^2$. Consider the mapping $\widetilde{h}_R : \widetilde{M}_1 \to \mathbb{R}^2$ defined by

$$\begin{aligned} \widetilde{h}_R(z) &= (\int_{z_o}^z (f \circ \pi)^* dx_1, \int_{z_o}^z (f \circ \pi)^* dx_2, \int_{z_o}^z (f \circ \pi)^* dx_3) \\ &\in \{(w_1, w_2, w_3) \in \mathbb{R}^3 | \sum_{i=1}^3 w_i = 0\} \cong \mathbb{R}^2, \end{aligned}$$

where f is the harmonic map into the building, and x_i's are the affine functions in defining an apartment of X as discussed in §4.4. Clearly $\widetilde{h}_R(z)$ is just the projection of $\widetilde{\alpha}$ onto the real part of $\mathbb{C}^2$. We need to show that the projection of the lattice points involved is discrete on $\mathbb{R}^2$.

From construction, at a regular point $\pi(z)$ of the harmonic map f so that $f(\pi(z))$ lies on an apartment $\Sigma \subset X$, df is the same as $d\widetilde{h}_R$ on identifying Σ with $\mathbb{R}$ isometrically. When Σ is equipped with the Euclidean metric, the stabilizer of Σ in $G(k_v)$ acts discretely on Σ, since the action moves any vertex of Σ into another vertex. Here we recall that the topology is with respect to the standard metric on X which on each apartment is isometric to $\mathbb{R}^2$. We conclude that $(\widetilde{h}_R)_*\pi_1(M_1)(\widetilde{h}_R(z)) = \widetilde{h}_R(\pi_1(M_1)z)$ is a discrete set of points on the $\widetilde{h}_R(\widetilde{M_1})$.

Suppose $z \in \circ f^{-1}\mathcal{S}$, where $\mathcal{S}$ is the set of singular points of the harmonic map $\widetilde{f}_1$. After translating by some element $w \in \widetilde{h}_R(\widetilde{M_1})$, we may assume that $w+\widetilde{h}_R(z)$ lies in the image $\widetilde{h}_R(z_o)$ of some regular point z_o of $\widetilde{f}_1$. Then $(\widetilde{h}_R)_*\pi_1(M_1)(w + \widetilde{h}_R(z))$ is discrete on $\widetilde{h}_R(\widetilde{M_1}) \cong \mathbb{R}^2$ from the discussions in the last paragraph. Clearly it also means that $(\widetilde{h}_R)_*\pi_1(M_1)(\widetilde{h}_R(z))$ is discrete since $(\widetilde{h}_R)_*\pi_1(M_1)$ by definition of the Albanese map is abelian and commutes with the translations. Hence $\widetilde{h}_R(\pi_1(M_1)z)$ is discrete in $\mathbb{R}^2$.

It follows that $\widetilde{h}_R$ induces a pluriharmonic map of real rank 2, $h_R : M_1 \to T = \mathbb{R}^2/(\widetilde{h}_R)_*(\pi_1(M_1))$, a real 2-torus. This concludes the proof for the lemma.

We can now finish the proof for integrality of the lattice. Let R be the ramification divisor of α and $B = \alpha(R)$. B is a complex analytic subvariety of $\mathrm{Alb}(M_1)$. For a generic p, $q^{-1}(p) \cap B$ is an isolated set of points on the two real dimensional set $q^{-1}(p) \cong T^2$. On the real two subtorus torus $q^{-1}(p)$ of $\mathrm{Alb}(M_1)$, we may choose standard curves γ and η representing $H_1(q^{-1}(p))$, so that γ and η and $-\gamma-\eta$ meet pairwise at a single point. By moving the curves slightly if necessary, we may choose such three curves to be avoiding B, so that $\alpha : \alpha^{-1}(D) \to D$ is unramified, where $D = \gamma \cup \eta \cup (-\gamma-\eta)$. A connected component of $\alpha^{-1}(D)$ consists of the three components, which we name as $\gamma_1, \tau\gamma_1$ and $\tau^2\gamma_1$ under deck transformation corresponding to A_3 of S_3.

Let $\gamma_o = \pi(\gamma_1)$. Now from definition, $\pi : \alpha^{-1}(D) \to \gamma_o$ is an unramified covering as π is unramified, and hence $\alpha^{-1}D$ is a smooth (one dimenisonal real) manifold. On the other hand, $\alpha^{-1}D$ contains non-empty singular set since $\alpha : \alpha^{-1}D \to D$ is unramified and D as the union of the three curves has cross-overs at the pairwise intersections of the three curves γ, η and $-\gamma-\eta$. We reach a contradiction.

We remark that the proof given here does not need the assumption that the Picard number is 1 in an essential way.

4.7 Step B: Archimedean superrigidity

From Step A, we know that there is totally real number field k and an Archimedean place v_o such that $G(k_{v_o}) \cong PU(2,1)$ and $\Pi \cap G(\mathcal{O}_k)$ has finite index in Π, where Π is the fundamental group of the given fake projective plane.

Let $v_i, i = 2, \ldots, n$ be the other Archimedean places of k so that we may consider $R_{k/\mathbb{Q}}(G)(\mathbb{R}) = G(k_{v_o}) \times G(k_{v_2}) \times \cdots G(k_{v_n})$. From the type of Lie algebra under consideration, we conclude that $G(k_{v_i}) \cong PGL(3,\mathbb{R}), PGL(3,\mathbb{C})$ or $PU(p, 3-p)$ for $p = 0, 1$ or 2. Note that $\Pi := \pi_1(M)$ is the fundamental group of a fake projective plane and hence is a Kähler group. Let us consider $\rho : \Pi \to G = G(k_{v_o})$ as the identity embedding and identify Π with $\rho(\Gamma)$. The homomorphism $\rho_v : \Pi \to G(k_v)$ induced by another Archimedean place is a faithful rigid representation and is Zariski dense. From Lemma 4.5 and the list of Hodge groups on page 50 of Simpson [Sim2], we know that $PSL(3,\mathbb{R})$ and $PSL(3,\mathbb{C})$ cannot be the Zariski closure of the image of a faithful rigid representation from the fundamental group of a Kähler manifold. Hence we are reduced to the cases that $G(k_{v_i})(\mathbb{R}) \cong PU(3,0)$ or $PU(2,1)$.

Let us analyze the situation that one of them, say, $G_2 := G(k_{v_2})(\mathbb{R})$ is $\cong PU(2,1)$. Let K_2 be a maximal compact subgroup of G_2 and $\widetilde{M_2} = G_2/K_2$. $\widetilde{M_2}$ is naturally equipped with an invariant complex structure and is biholomorphic to $B^2_{\mathbb{C}}$. There exists a Π-equivariant holomorphic map $\Phi : \widetilde{M} \to \widetilde{M_2}$ obtained as follows. Lemma 4.5 of [Sim2] implies that ρ_v has to come from a complex variation of Hodge structure, which means that ρ_v induces an equivariant mapping $\Phi : \widetilde{M} \to \widetilde{M_2}$ such that Φ can be lifted to a holomorphic map $\widetilde{M}$ to a Griffiths' Period domain $\widetilde{N}$ above $\widetilde{M_2} \cong B^2_{\mathbb{C}}$. Since $B^2_{\mathbb{C}}$ is the only Griffiths' Period domain above $B^2_{\mathbb{C}}$, we conclude that Φ is holomorphic. Alternatively, existence of such a holomorphic map also follows from existence of harmonic maps (cf. [ES], [Co] and [Lab]), and holomorphicity of harmonic maps (cf. [Siu2] and §7 of [CT]) into complex hyperbolic spaces. From an observation of Kollár ([Ko2], Lemma 8.3), $K_{\widetilde{M_2}} = 3H$ modulo torsion as Π^σ-equivariant bundles. On the other hand, as explained earlier, $K_M = 3H$ modulo torsion for some line bundle H, where $H \cdot H = 1$. Hence modulo torsion, H can be considered as a generator of the Neron-Severi group modulo torsion. From Hurwitz Formula, we know that $K_{\widetilde{M}} = n\Phi^* K_{\widetilde{M_2}} + R$ regarded as equivariant line bundles, where n is the degree of the mapping and R is the ramification divisor. Since K_M is known to be three times a generator of the Neron-Severi group modulo torsion in the case of fake projective planes as mentioned above, it follows that ramification divisor $R = 0$ and $n = 1$. Hence Φ is a biholomorphism. The invariant metric on $B^2_{\mathbb{C}}$ is up to a constant the Bergman metric, which is a biholomorphic invariant. This implies that Φ induces an isometry from $\widetilde{M}$ to $\widetilde{M_2}$. As $G_o := G(k_{v_o})(\mathbb{R})$ and G_2 are the automorphism groups of $\widetilde{M}$ and $\widetilde{M_2}$ as Riemannian manifolds equipped with the Bergman metrics respectively, we conclude that Φ induces an isomorphism between the two real Lie groups G_o and G_2.

Hence we conclude that for $i \geqslant 2$, either $G(k_{v_i})(\mathbb{R}) \cong PU(3,0)$ and is hence compact, or ρ_v induces an isomorphism between G_o and $G(k_{v_i})$. From Step A, we also know that $\rho(\Pi)$ is integral. According to Lemma 6.1.6 and 6.1.7 and their proofs in [Zimmer], we conclude that G can be defined over a real number field k so that $\Pi \subset G_k$ and Π is integral. With a real number field k, the properties of ρ_v proved above implies that ρ_v is standard in the sense of Appendices C 3.5 of [Mag] or §22 of [Mo2]. Applying a criterion of arithmeticity for lattices in the same references, see also Cor 12.2.8 of [DM], we conclude that Π is arithmetic.

This concludes the proof of Step B.

4.8 Theorem 7 shows that the fundamental group of any fake projective plane is an arithmetic subgroup of $PU(2,1)$. Its inverse image in $SU(2,1)$ (under the natural surjective homomorphism $\varphi : SU(2,1) \to PU(2,1)$) is an arithmetic subgroup of $SU(2,1)$. Note that $SU(2,1)$ is a group of type ${}^2A_2$ (Tits [Ti1]). Given an arithmetic subgroup Γ of $SU(2,1)$, let k be the associated number field and G the associated k-form of $SU(2,1)$. Let ℓ be the quadratic extension of k over which G is an inner form. Then k is a totally real number field of degree d and ℓ a totally complex quadratic extension of k. G can be described as follows. There is a division algebra $\mathcal{D}$ with center ℓ and of degree $s := \sqrt{[\mathcal{D} : \ell]}$, $s|3$, $\mathcal{D}$ given with an involution σ of the second kind such that $k = \{x \in \ell \,|\, x = \sigma(x)\}$, and there is a non-degenerate hermitian form h on $\mathcal{D}^{3/s}$ defined in terms of the involution σ such that G is the special unitary group $\mathrm{SU}(h)$ of h, and such that for a real place, say v_o, of k, $G(k_{v_o}) \cong \mathrm{SU}(2,1)$, and for all archimedean places $v \neq v_o$, $G(k_v)$ is isomorphic to the compact Lie group $\mathrm{SU}(3)$. The group Γ is commensurable with $G(\mathcal{O}_k)$.

For convenience of notation, we call Γ an arithmetic lattice of first type (respectively second type) if $\mathcal{D} = \ell$ (respectively $[D : \ell] > 1$) in the above formulation.

Theorem 7, the classification above, and the fact that the Euler-Poincaré characteristic of a fake projective plane is 3 point us to a classification of fake projective plane by listing all possible $k, \ell, \mathcal{D}$ and h. This is possible due to a fundamental result of Prasad [Pra] giving the covolume of any arithmetic subgroup. In the next section, we will introduce the covolume formula of Prasad.

5 Covolume formula of Prasad

5.1 The search for a precise covolume formula for an arithmetic group has a long history. For lattices beyond those of $PGL(2,\mathbb{R})$, it can be traced back to the work of Siegel. In [Pra], G. Prasad gave a very general volume formula which works for all principal arithmetic lattices in any semi-simple algebraic group. The statement and proof of the covolume formula relies deeply on the theory of algebraic groups and the Bruhat-Tits Theory. In the following we only go through the formulation of the formula needed for our later exposition. We refer the reader to the original paper of Prasad [Pra] for the proof. An expository account of the covolume formula can be found in the Bourbaki Seminar report of Rémy [Re].

5.2 In this section, we fix some notation to be used in later discussions. We refer the reader to [Pra] and [Ti2] for more elaborate discussion. Let k be a number field. Denote by D_k the absolute value of the discriminant of k. Let V_∞ and V_f be the set of archimedean and non-archimedean places of k respectively. For each $v \in V = V_\infty \cup V_f$, denote by $|\cdot|_v$ the absolute value associated to v and k_v the completion of k with respect to v. For $v \in V_f$, denote by $\mathcal{O}_v$ the ring of integers and $\mathfrak{f}_v$ the (finite) residue field of k_v. Let q_v be the order of $\mathfrak{f}_v$. We also denote by $\mathbb{A} = \mathbb{A}(k)$ the ring of adèles of k.

Let G be a simply connected, absolutely almost simple algebraic group. For $v \in V_f$, let X_v be the Bruhat-Tits building of $G(k_v)$. An Iwahori subgroup of $G(k_v)$ is the subgroup fixing a chamber in X_v. A parahoric subgroup of $G(k_v)$ is the stabilizer of a simplex of X_v. A parahoric subgroup is a compact subgroup of $G(k_v)$ containing an Iwahori subgroup of X_v. A vertex x of X_v is called special if the affine Weyl group is a semidirect product of the translation subgroup by the isotropy group of x. A parahoric subgroup P is special if it fixes a special vertex of X_v. Let $\hat{k}_v$ be the maximal unramified extension of k_v and $\hat{X}_v$ the Bruhat-Tits building associated to $G(\hat{k}_v)$. X_v can be identified with the fixed point set of $\mathrm{Gal}(\hat{k}_v/k_v)$ in $\hat{X}_v$. If G splits over $\hat{k}_v$, and is quasi-split over k_v, a special vertex of $\hat{X}_v$ lying in X_v is called hyperspecial, and the corresponding isotropy group is called a hyperspecial parahoric subgroup.

Let $P = (P_v)_{v\in V_f}$ be a collection of parahoric subgroups P_v for each finite place $v \in V_f$. P is called coherent if $\prod_{v\in V_\infty} G(k_v) \cdot \prod_{v\in V_f} P_v$ is an open subgroup of the adèle group $G(\mathbb{A})$. Let P be a coherent collection of parahoric subgroups. Let S be finite set of primes containing V_∞. We assume that $G(k_v)$ is non-compact (equivalently, G is isotropic over v) for some $v \in V_\infty$. Let $G_S = \prod_{v\in S} G(k_v)$ and Λ be the natural projection of $G(k) \cap \prod_{v\in S} P_v$ in G_S. Λ is then an arithmetic subgroup of G and is called a principal S-arithmetic determined by P.

5.3 We can now state the covolume formula of Prasad in the case of $SU(2,1)$. In this case, following the discussions in §4.8, k is a totally real number field and ℓ is a totally complex quadratic extension of k. Denote $d = [k : \mathbb{Q}]$. G is an outer anisotropic k-form of SL_3. Let Λ be a principal arithmetic subgroup. The formula of Prasad is given by

$$\mu(G(k_{v_o})/\Lambda) = D_k^4 (D_\ell/D_k^2)^{5/2} (16\pi^5)^{-d} \mathcal{E};$$

where $\mathcal{E} = \prod_{v\in V_f} e(P_v)$, and the value of $e(P_v)$ is given in §2.4 of [PY1].

It turns out that

$$\mathcal{E} = \zeta_k(2) L_{\ell|k}(3) \prod_{v\in\mathcal{T}} e'(P_v),$$

where $\mathcal{T}$ is the set of places $v \in V_f$ for which either P_v is not maximal, or P_v is not hyperspecial and v is not ramified in ℓ. The values of $e'(P_v)$ are given in §2.5 of [PY1]. Note that for all v, $e'(P_v)$ is an integer.

In conclusion,

$$\mu(G(k_{v_o})/\Lambda) = (D_\ell^{5/2}/D_k)(16\pi^5)^{-d} \zeta_k(2) L_{\ell|k}(3) \prod_{v\in\mathcal{T}} e'(P_v) \qquad (5.1)$$

$$= 2^{-2d} \zeta_k(-1) L_{\ell|k}(-2) \prod_{v\in\mathcal{T}} e'(P_v), \qquad (5.2)$$

where we used the functional equations for the Dedekind zeta function and the Dirichlet L-function.

6 Formulation of proof

6.1 The purpose of this section is to explain the idea of proof of the main result of [PY1]. The strategy used is as follows. Let M be a fake projective plane, with fundamental group given by a torsion-free cocompact lattice $\Pi \subset PU(2,1)$, the automorphism group of $B^2_{\mathbb{C}}$. Since Π is arithmetic according to Theorem 7, we can apply Prasad's Covolume Formula in Theorem 8 and the constraint $\chi(M) = 3$ to restrict the possibilities for k, ℓ and $\mathcal{D}$ appearing in §4.8 in the description of Π as an arithmetic lattice. Once we reduce the list of all possible $(k, \ell, \mathcal{D})$ to a sufficiently small one, we either construct an example in each class or use more refine method to eliminate it. The actual proof makes use of techniques in analytic number theory for the estimates, and Bruhat-Tits theory for the construction. Tables of number fields of low degree and relatively small discriminant, and computations with the help of computers are also employed.

6.2 We now go deeper into the details. Since $PU(2,1)$ is the quotient of $SU(2,1)$ by its center, which is the cyclic group of order 3, we have the following commutative diagram.

$$\begin{array}{ccccc} 0 \to \mathbb{Z}_3 \to & G(k_{\sigma_o}) = SU(2,1) & \xrightarrow{\varphi} & PU(2,1) \\ & \cup & & \cup \\ 0 \to \mathbb{Z}_3 \to & \widetilde{\Pi} = \varphi^{-1}(\Pi) & \to & \Pi = \pi_1(M) \end{array}$$

The group Π is arithmetic. Let k be the number field and G be the k-form of $\mathrm{SU}(2,1)$ associated with Π in §4.8. G is a simple simply connected algebraic k-group such that for a real place, say v_o, of k, $G(k_{v_o}) \cong \mathrm{SU}(2,1)$, and for all archimedean places $v \neq v_o$, $G(k_v)$ is isomorphic to the compact Lie group $\mathrm{SU}(3)$. $\widetilde{\Pi} := \varphi^{-1}(\Pi)$ is then an arithmetic subgroup of $G(k_{v_o})$.

Let V_f be the set of nonarchimedean places of k. For all $v \in V_f$, we fix a parahoric subgroup P_v of $G(k_v)$ which is minimal among the parahoric subgroups of $G(k_v)$ normalized by Π.

Let $\Lambda := G(k) \cap \prod_{v \in V_f} P_v$. Then Λ is a principal arithmetic subgroup ([Pra], 3.4) which is normalized by Π, and therefore also by $\widetilde{\Pi}$. Let Γ be the normalizer of Λ in $G(k_{v_o})$, and $\overline{\Gamma}$ be its image in $\overline{G}(k_{v_o})$. Then $\overline{\Gamma} \subset \overline{G}(k)$ ([BP], 1.2). As the normalizer of Λ in $G(k)$ equals Λ, $\Gamma \cap G(k) = \Lambda$.

Since the Euler-Poincaré characteristic $\chi(\Pi) = 3$, we conclude that the orbifold Euler-Poincaré characteristic $\chi(\widetilde{\Pi})$ of $\widetilde{\Pi}$ equals 1. It follows that the orbifold Euler-Poincaré characteristic $\chi(\Gamma)$ of Γ is a reciprocal integer

We observe that

$$\chi(\Gamma) = 3\mu(G(k_{v_o})/\Gamma) = \frac{3\mu(G(k_{v_o})/\Lambda)}{[\Gamma : \Lambda]}, \tag{6.1}$$

where the first equality was explained in §4 of [BP], following from Hirzebruch Proportionality Principle and the fact that the Euler-Poincaré characteristic of $P^2_{\mathbb{C}}$, the compact dual of the symmetric space $B^2_{\mathbb{C}}$, is 3. Following results of Borel-Prasad in [BP], the factor $[\Gamma : \Lambda]$ was bounded from above by an expression ([PY1], §2.3) involving $h_{\ell,3}$, where $h_{\ell,3}$ is the order of the subgroup of the class group of ℓ consisting of elements of order dividing 3. Hence $0 < h_{\ell,3} \leqslant h_\ell$, where h_ℓ is the class number of ℓ. Together with the explicit factors in the expression of e'_v in §2.5

of [PY1], we come up with

$$\begin{aligned}\frac{1}{3} &\geqslant \mu(G(k_{v_0})/\Gamma)\\ &= \frac{D_\ell^{5/2}\zeta_k(2)L_{\ell|k}(3)}{(16\pi^5)^d[\Gamma:\Lambda]D_k}\prod_{v\in\mathcal{T}} e'(P_v)\\ &\geqslant \frac{D_\ell^{5/2}\zeta_k(2)L_{\ell|k}(3)}{3(16\pi^5)^d h_{\ell,3}D_k}\prod_{v\in\mathcal{T}} e''(P_v),\end{aligned} \tag{6.2}$$

where $e''(P_v) \geqslant 1$ and $e'(P_v) \in \mathbb{Z}$ are given explicitly as follow, see [PY1], §2.5.
(i) $e''(P_v) = \frac{e'(P_v)}{3} = \frac{(q_v^2+q_v+1)(q_v+1)}{3}$, if v splits in ℓ, G splits at v and P_v is Iwahori.
(ii) $e''(P_v) = e'(P_v) = (q_v^2+q_v+1)$, if v splits in ℓ, G splits at v and P_v not Iwahori.
(iii) $e''(P_v) = \frac{e'(P_v)}{3} = \frac{(q_v-1)^2(q_v+1)}{3}$, if v splits in ℓ and G is anisotropic at v.
(iv) $e''(P_v) = e'(P_v) = (q_v^3+1)$, if v is inert in ℓ and P_v is Iwahori.
(v) $e''(P_v) = e'(P_v) = (q_v^2-q_v+1)$, if v is inert in ℓ and P_v is a non-hyperspecial maximal parahoric.

6.3 The strategy in [PY1] consists of five steps.
(A). We limit k for fake projective plane by (5.1), (6.2) using estimates from analytic number theory. and the fact that $\chi(\Gamma) \leqslant 1$.
(B). For each k left in step 1, we list all possible ℓ.
(C). For each of those possible (k,ℓ) left behind, we use the fact that $\chi(\Gamma)$ is a reciprocal integer and hence that the numerator of $2^{-2d}\zeta_k(-1)L_{\ell|k}(-2)$ is a power of 3 ([PY1], Proposition 2.12).
(D). We use local Euler factor $\mathcal{E}'$ to narrow down the possibilities of $\mathcal{D}$.
(E). For each of the few remaining cases of $(k,\ell,\mathcal{D})$, we either construct examples or make use of more specific number theoretic or geometric information to rule out the case.

In the following, we will elaborate on each of the above steps.

6.4 Step A We are going to apply estimates in (6.2). To obtain an estimate on $h_{\ell,3}(\leqslant h_\ell)$, we use the following estimates of Brauer-Siegel. For all $\delta > 0$,

$$h_\ell R_\ell \leqslant \delta(1+\delta)w_\ell\Gamma(1+\delta)^d\left(\frac{D_\ell}{(2\pi)^{2d}}\right)^{(1+\delta)/2}\zeta_\ell(1+\delta), \tag{6.3}$$

where R_ℓ is the regulator of ℓ and w_ℓ is the number of roots of unity in ℓ. These two number theoretic invariants can however be estimated by the results of Zimmert [Zimmert] and its variant due to Slavutskii [Sl],

$$\text{Zimmert:}\quad R_\ell \geqslant 0.02w_\ell e^{0.1d} \tag{6.4}$$

$$\text{Slavutskii:}\quad R_\ell \geqslant 0.00136w_\ell e^{0.57d}. \tag{6.5}$$

Substituting (6.3)(6.5) into (6.2),

$$D_k^{1/d} \leqslant D_\ell^{1/2d} < f(\delta, d)$$
$$:= \Big[\frac{\delta(1+\delta)}{0.00136}\Big]^{1/(3-\delta)d} \cdot \big[2^{3-\delta}\pi^{4-\delta}\Gamma(1+\delta)\zeta(1+\delta)^2 e^{-0.57}\big]^{1/(3-\delta)}$$

Here we used the fact that $\zeta_\ell(1+\delta) \leqslant \zeta(1+\delta)^{2d}$. The first key fact is that $f(\delta, d)$ is decreasing in d.

On the other hand, we let $M_r(d) := \min_K D_K^{1/d}$, over all totally real number fields K of degree d. The precise values of $M_r(d)$ is known only for small d listed below.

d	2	3	4	5	6	7	8
$M(d)^d$	5	49	725	14641	300125	20134393	282300416

It is useful to have a good lower bound for the root discriminant $M_r(d)$ in terms of the degree $d = [k : \mathbb{Q}]$. Using geometry of numbers, Minkowski (cf. [L]) proved that $M_r(d) > 1$ for each d, and gave the first non-trivial estimates that $M_r(d) \geqslant \sqrt{\frac{\pi}{4}}(d^2/(d!)^{2/d}$. From analytic properties of the Dedekind zeta function, Stark observed that D_k can be related to the zeroes of ζ_k. In this direction, Odlyzko [O] obtained the following interesting lower bound of $M_r(d)$.

Let $b(x) = [5 + (12x^2 - 5)^{1/2}]/6$. Define

$$g(x, d) = \exp\Big[\log(\pi) - \tfrac{\Gamma'}{\Gamma}(x/2) + \tfrac{(2x-1)}{4}\big(\tfrac{\Gamma'}{\Gamma}\big)'(b(x)/2) + \tfrac{1}{d}\big\{-\tfrac{2}{x} - \tfrac{2}{x-1} - \tfrac{2x-1}{b(x)^2} - \tfrac{2x-1}{(b(x)-1)^2}\big\}\Big].$$

Let $\alpha = \sqrt{\frac{14-\sqrt{128}}{34}}$. Then from [O], $M(d) \geqslant g(x, d)$ provided that $x > 1$ and $b(x) \geqslant 1 + \alpha x$.

Let x_o be the positive root of the quadratic equation $b(x) = 1 + \alpha x$.
$x_o = \frac{\alpha+\sqrt{2-5\alpha^2}}{2(1-3\alpha^2)} = 1.01....$
Define $\mathfrak{N}(d) = \limsup_{x\geqslant x_o} g(x, d)$. Then the second key fact is that for each $d > 1$, $M_r(d) \geqslant \mathfrak{N}(d)$ and $\mathfrak{N}(d)$ is an increasing function of d.

From the first key fact, for $d \geqslant 20$,

$$M_r(d) \leqslant f(0.9, 20) < 16.38.$$

From the second key fact,

$$M_r(d) \geqslant \mathfrak{N}(20) \geqslant g(1.43, 20) > 16.4$$

We reach a contradiction. Hence $d \leqslant 20$.

Refinement of the above estimates allows us to eliminate the cases $15 \leqslant d \leqslant 19$ as well. The argument becomes more sophisticated as we decrease the degree d of k. Nevertheless, using the theory of Hilbert class fields (i.e. maximal unramified abelian extension of ℓ) and the best estimates of $D_\ell^{1/2d}$ available for complex multiplication fields, we eliminate the cases of $8 \leqslant d \leqslant 14$.

The cases for $2 \leqslant d \leqslant 7$ are more delicate and involved. We devise an iterative scheme to decrease the upper bound for discriminant D_ℓ systematically.

There are several ingredients used. First of all we use more precise estimates for R_ℓ and w_ℓ for number fields of small degrees and small discriminants. Part of the information is provided by Friedman [F]. Then we need to have sharper estimates for $M(d)$ provided by the table in [Mat] and [O2]. Together with the use of Hilbert Class Fields, this allows us to reduce possible candidates to those with sufficiently small discriminants D_k and D_ℓ. When the size of the discriminants of the number fields are small enough to be listed in the tables of number fields in [1], we can read off the value of the class number h_ℓ, apply equation (6.2) directly and in turn get a better bound on discriminant bound. Iteration of the above arguments allows us to eliminate the cases of $d = 5, 6, 7$.

In the case of $2 \leqslant d \leqslant 4$, the above procedure allows us to show that the pair of number fields involved satisfy $D_k^{1/d} \leqslant D_\ell^{1/2d} \leqslant 10$, $D_\ell/D_k^2 \leqslant 104$ and $h_{\ell,3} \leqslant 3$. At this step, Gunter Malle [Mal] provided us the list of all number fields (k, l) satisfying the above constraints. There are altogether 40 such pairs, labelled as $\mathcal{C}_i, i = 1, \cdots, 40$ as in [PY1], §8.2.

As a brief summary, apart from the case of $k = \mathbb{Q}$, there are only forty pairs of (k, ℓ) available to be defining number fields of the arithmetic lattice associated to a fake projective plane.

6.5 Step B Let us consider the case of $k = \mathbb{Q}$. In this case $D_k = D_\mathbb{Q} = 1$. Estimates (6.2), (6.3) and (6.4) leads to

$$D_\ell < (2\pi)^2 \Big(\frac{5^2 \cdot \delta(1+\delta) \cdot \Gamma(1+\delta)\zeta(1+\delta)^2}{e^{0.1}\zeta(2)^{1/2}}\Big)^{2/(4-\delta)}. \tag{6.6}$$

Choosing $\delta = 0.34$, we find that $D_\ell \leqslant 460$.

From the table of imaginary quadratic number fields of discriminant bounded by 500 as in [BS], we know that the class number $h_\ell \leqslant 21$, and hence, $h_{\ell,3} \leqslant n_{\ell,3} \leqslant 9$. The precise bound of $h_{\ell,3}$ allows us to use (6.2) directly to yield $D_\ell \leqslant 63$. Iteration of the argument leads to $\ell = \mathbb{Q}(\sqrt{-a})$, where a is one of the following eleven integers, $1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31$.

6.6 Step C At this point, we have the forty pairs of (k, ℓ) in $\mathcal{C}_i, i = 1, \ldots, 40$ given at the end of Step A and the eleven pairs of $(\mathbb{Q}, \mathbb{Q}(\sqrt{-a}))$ given at the end of Step B as possible candidates for the defining number fields for the fundamental group of a fake projective plane. In this step, we use the constraint that $\mu := 2^{-2d}\zeta_k(-1)L_{\ell|k}(-2)$ is a power of 3 to eliminate some more cases from the above lists.

To compute the values of μ, starting from a classical result of Siegel [Sie], we know that both $\zeta_k(-1)$ and $L_{\ell|k}(-2)$ are rational numbers in $\mathbb{Q}$. For our purpose, we need to know the precise values of $\zeta_k(-1)$ and $L_{\ell|k}(-2)$. Explicit formula for computation of values of $\zeta_k(-1)$ were already available in Siegel [Sie] in terms of coefficients of appropriate Eisenstein series. For $L_{\ell|k}(-2)$, explicit formulae were available in Tsuyumine [Ts1]. The formula are involved. Nowadays one can use software such as Magma or PARI/GP to compute the zeta and L values to high order of accuracy. On the other hand, the theoretical formula of [Sie] allows one to bound the size of the denominator of the zeta values. Similarly, Tsuyumine [Ts2] explained to us that the denominators of all the L values we needed associated to the above pairs of (k,ℓ) can be found explicitly using [Ts1]. These theoretical results allow us to conclude that the zeta and L values we obtained from Magma with accuracy of forty digits are actually the precise values. Here is the list of μ for $k=\mathbb{Q}$.

a	1	2	3	5	6	7	11	15	19	23	31
μ	$\frac{1}{96}$	$\frac{1}{16}$	$\frac{1}{216}$	$\frac{5}{8}$	$\frac{23}{24}$	$\frac{1}{21}$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{11}{24}$	1	2

Hence we conclude from the requirement that the numerator of μ is a power of 3 that for $(k,\ell)=(\mathbb{Q},\mathbb{Q}(\sqrt{-a})$, a can only take the values of $1,2,3,7,11,15$ and 23.

In the case of $[k:\mathbb{Q}]>1$, we refer the readers to [PY1] §8.2 for the values of $\zeta_k(-1)$ and $L_{\ell|k}(-2)$ for $\mathcal{C}_i, i=1,\dots,40$ mentioned above. In particular, it shows that only fifteen of those $\mathcal{C}_i$ are possible candidates for defining number fields for Λ associated to a fake projective plane ([PY1] §8.4)

6.7 Step D Let us again illustrate the argument for the cases of $k=\mathbb{Q}$ among the lists of examples at the beginning of Step C. We are going to make use of the local Euler factor in the volume formula.

The reason behind Proposition 2.12 of [PY1] is that the numerators of the expression $\mu\prod_{p\in\mathcal{T}}e'(P_v)$ is a power of 3. Let $\mathcal{T}_o=\{v\in\mathcal{T}:G$ is anisotropic over $k_v\}$. Every prime $v\in\mathcal{T}_o$ splits in ℓ as mentioned in [PY1] §2.2. Note from Godement Criterion and compactness of a fake projective plane, we know that $\mathcal{D}\neq\ell$ (see §4.1 of [PY1]). Hence $\mathcal{T}_o$ is non-empty. Since we are considering $k=\mathbb{Q}$, we may just write v as p for some rational prime p. In this case, each $p\in\mathcal{T}_o$ would contribute a factor of $e'(P_v)=(p-1)^2(p+1)$ to $\mathcal{E}'$, according to the explicit formulae in §6.2. Hence going through the table in Step C again with factors of $e'(P_p)$ taken into account, we conclude as in [PY1], §4.4 and Addendum the following.

Case I: $k=\mathbb{Q}$, either
(i) $\mathcal{T}=\mathcal{T}_0$ consists of a single prime, and the pair (a,p), where $\ell=\mathbb{Q}(\sqrt{-a})$, and $\mathcal{T}_0=\{p\}$, belongs to the set $\{(1,5),(2,3),(7,2),(15,2),(23,2)\}$; or
(ii) $(a,\mathcal{T})=(1,\{2,5\})$, $(2,\{2,3\})$, $(7,\{2,3\})$ or $(7,\{2,5\})$.

The case of $[k:\mathbb{Q}]>1$, is more complicated. The general principle is the same, using the fact that the numerator of $2^{-2d}\zeta_k(-1)L_{\ell|k}(-2)\prod_{v\in\mathcal{T}}e'(P_v)$ are powers of 3, but technically more involved. We have to separate into two cases, where we used notations in [PY1]. Here is the conclusion.

Case II: k is a totally real number field of degree $\geqslant 2$.
(i) $[\mathcal{D},\ell]>1$: (k,ℓ) is given by one of the pairs $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_{10}$, $\mathcal{C}_{18}$, $\mathcal{C}_{20}$, $\mathcal{C}_{26}$, $\mathcal{C}_{31}$, $\mathcal{C}_{35}$ and $\mathcal{C}_{39}$. ([PY1] §8.6 and Addendum). Moreover, $\mathcal{T}_o=\{\mathfrak{v}\}$, where $\mathfrak{v}$ is the unique place of k lying over 2 except for the pairs $\mathcal{C}_3$ and $\mathcal{C}_{18}$, for $\mathcal{C}_3$ it is the unique place of k lying over 5, and for $\mathcal{C}_{18}$ it is the unique place of k lying over 3. Except for $\mathcal{C}_{18}$ and $\mathcal{C}_{20}$, $\mathcal{T}=\mathcal{T}_o$. If $(k,\ell)=\mathcal{C}_{18}$, then $\mathcal{T}$ either equals $\mathcal{T}_o=\{\mathfrak{v}\}$ or it is $\{\mathfrak{v},\mathfrak{v}_2\}$, where $\mathfrak{v}_2$ is the unique place of $k=\mathbb{Q}(\sqrt{6})$ lying over 2. In the case of $\mathcal{C}_{20}$, $\mathcal{T}$

either equals $\mathcal{T}_o = \{\mathfrak{v}\}$ or it is $\{\mathfrak{v}, \mathfrak{v}_3'\}$ or $\{\mathfrak{v}, \mathfrak{v}_3''\}$, where $\mathfrak{v}_3', \mathfrak{v}_3''$ are the places of $k = \mathbb{Q}(\sqrt{7})$ above 3.
(ii) $\mathcal{D} = \ell$: (k, ℓ) can only be one of the following five: $\mathcal{C}_1$, $\mathcal{C}_8$, $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$

For convenience of the readers, we include the list of the thirteen pairs (k, ℓ) appearing in (i) and (ii).

$\mathcal{C}_1$ $(\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\zeta_5))$,
$\mathcal{C}_2$ $(\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}, \zeta_3))$,
$\mathcal{C}_3$ $(\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}, \zeta_4))$
$\mathcal{C}_8$ $(\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\zeta_8))$
$\mathcal{C}_{10}$ $(\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{-7+4\sqrt{2}}))$,
$\mathcal{C}_{11}$ $(\mathbb{Q}(\sqrt{3}), \mathbb{Q}(\zeta_{12}))$
$\mathcal{C}_{18}$ $(\mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{6}, \zeta_3))$,
$\mathcal{C}_{20}$ $(\mathbb{Q}(\sqrt{7}), \mathbb{Q}(\sqrt{7}, \zeta_4))$,
$\mathcal{C}_{21}$ $(\mathbb{Q}(\sqrt{33}), \mathbb{Q}(\sqrt{33}, \zeta_3))$
$\mathcal{C}_{26}$ $(\mathbb{Q}(\sqrt{15}), \mathbb{Q}(\sqrt{15}, \zeta_4))$,
$\mathcal{C}_{31}$ $(\mathbb{Q}(\zeta_7 + \zeta_7^{-1}), \mathbb{Q}(\zeta_7))$,
$\mathcal{C}_{35}$ $(\mathbb{Q}(\zeta_{20} + \zeta_{20}^{-1}), \mathbb{Q}(\zeta_{20}))$,
$\mathcal{C}_{39}$ $(\mathbb{Q}(\sqrt{5+2\sqrt{2}}), \mathbb{Q}(\sqrt{5+2\sqrt{2}}, \zeta_4))$.

6.8 Step E As a result of discussions from Step A to Step D, we know that the arithmetic group associated to a fake projective plane should be determined by principal arithmetic groups Λ expressed in terms of $(k, \ell, \mathcal{D}, G, (P_v)_{v \in V_f})$ satisfying the constraints given in those steps. The purpose is either to construct examples of fake projective planes from each of this set of data, or to find some other means to rule them out.

Recall that Λ gives rise to the normalizer Γ in $G(k_{v_o})$. Denote by $\overline{\Lambda} = \varphi(\Lambda)$ and $\overline{\Gamma} = \varphi(\Gamma)$ in the notation of §6.2. Each Λ gives rise to a unique Γ. From definition, $\chi(\overline{\Lambda})$ is an integral multiple of $\chi(\overline{\Gamma})$. Hence finding the fundamental group of a fake projective plane amounts to finding a subgroup Π of $\overline{\Gamma}$ satisfying
(i) $[\overline{\Gamma} : \Pi] = 3/\chi(\overline{\Gamma})$,
(ii) Π is torsion-free,
(iii) $b_1(\Pi) = 0$.

Notice that (i) and (ii) implies that $M = B_{\mathbb{C}}^2/\Pi$ is a smooth projective variety with $\chi(M) = 3$. (iii) implies $b_3(\Pi) = 0$ from Poincaré Duality. As $\chi(M) = 3$, it follows that $b_2(M) = 1$ as well. This implies that M is a fake projective plane.

Let us now confine to the case of $k = \mathbb{Q}$ to explain the idea involved. In this case, as we have explained earlier, $[\mathcal{D} : \ell] > 1$ from compactness and Godement Criterion. Suppose we can find a congruence subgroup Π satisfying (i) and (ii), we can apply Rogawski's result in [Ro] to conclude (iii). Hence the problem is reduced to finding a torsion free subgroup of $\overline{\Gamma}$ satisfying (i), (ii) and (iii) for each of the cases listed at the end of Step D ($k = \mathbb{Q}$ cases).

We define

$$A_\Lambda := A_{(k, \ell, \mathcal{D}, G, (P_v)_{v \in V_f})} = \{\Pi : \Pi \text{ satisfies (i), (ii), (iii)}\}.$$

To find a lattice in A_Λ, we first need to have precise value of $[\Gamma : \Lambda]$ for the principal arithmetic group Λ specified in Step D. This is given by $[\Gamma : \Lambda] = 3^{1+\#\mathcal{T}_o}$ for all the cases in Step D ([PY1] §5.4). Here is the list of indices $[\overline{\Gamma} : \Pi]$ required by (i) for all the entries of Case I, $k = \mathbb{Q}$ listed.

a	1	2	7	15	23	7	7
$\mathcal{T}$	$\{5\}$	$\{3\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2,3\}$	$\{2,5\}$
$[\overline{\Gamma} : \Pi]$	3	3	21	3	1	3	1

The constraints set by (ii) and (iii) are more difficult to check. In [PY1] §5 and Addendum, It is proved that for all Λ associated to the cases of $k = \mathbb{Q}$ listed at the end of Step D, $A_\Lambda \neq \emptyset$. In fact, there exists a congruence subgroup of $\overline{\Gamma}$ of index given in the above table satisfying (i), (ii) and (iii), once the data $(k, \ell, \mathcal{D}, G, (P_v)_{v \in V_f})$ or equivalently Λ is fixed. For Congruence subgroups, Condition (iii) is satisfied automatically by a theorem of Rogawski [Ro] as mentioned above. Condition (ii) requires careful case by case checking making sure that the lattice involved is torsion free. Here we use algebraic number theory and results on strong approximation.

Let us now consider Case II, k is a totally real number field of degree 2 over $\mathbb{Q}$. Consider the subcase (i) that $[\mathcal{D}, \ell] > 1$. Among the pairs listed at the end of Step D, it is observed that $\chi(\overline{\Gamma}) = 3$ for $\mathcal{C}_3, \mathcal{C}_{26}, \mathcal{C}_{31}, \mathcal{C}_{35}$ and $\mathcal{C}_{39}$. Hence no subgroup of $\overline{\Gamma}$ can be the fundamental group of a fake projective plane except $\overline{\Gamma}$ itself. However it is shown using an intricate argument in [PY1] §9.7 and A10 in the Addendum that $\overline{\Gamma}$ always contains an element of order 3 in these three cases. Hence none of them is a candidate for the fundamental group of a fake projective plane.

For the remaining ones, here is the list of indices. Recall that $\mathcal{T}_o$ consists of $\mathfrak{v}_p$ as explained in §9 and Addendum of [PY1], where $\mathfrak{v}_p$ is the unique place above $p \in \mathbb{Q}$, and in the case of $\mathcal{C}_{20}$, $\mathfrak{v}_3', \mathfrak{v}_3''$ are the places of $k = \mathbb{Q}(\sqrt{7})$ above 3.

(k, ℓ)	$\mathcal{C}_2$	$\mathcal{C}_{10}$	$\mathcal{C}_{18}$	$\mathcal{C}_{20}$
$\mathcal{T}$	$\{\mathfrak{v}_2\}$	$\{\mathfrak{v}_2\}$	$\{\mathfrak{v}_3\}$ or $\{\mathfrak{v}_2, \mathfrak{v}_3\}$	$\{\mathfrak{v}_2\}$ or $\{\mathfrak{v}_2, \mathfrak{v}_3'\}$ or $\{\mathfrak{v}_2, \mathfrak{v}_3''\}$
$[\overline{\Gamma}, \Pi]$	9	3	9 or 1 or 3	21 or 3 or 3

For each of the above three pairs (k, ℓ), we show that there exists a torsion free congruence subgroup of appropriate index satisfying all the properties (i), (ii) and (iii), thereby providing examples of fake projective planes ([PY1], §9.3, 9.5). For $\mathcal{C}_{18}$, it turns out that the constraints given by the volume formula also allow us to choose for the place $\mathfrak{v}_2$ of k lying over 2, either the Iwahori subgroup or a nonhyperspecial maximal parahoric in $G(k_{\mathfrak{v}_2})$ to obtain Λ and Γ, with $[\overline{\Gamma}, \Pi]$ given by 1 or 3 for the case of iwahori subgroup or nonhyperspecial maximal parahoric subgroup respectively. In each of the three cases, a torsion-free subgroup satisfying conditions (i), (ii), (iii) exists, as shown in [PY1] and the addendum to the paper.

A natural strategy to find all subgroups of appropriate index satisfying (i), (ii) and (iii) is as follows. We try to find a generating set for $\overline{\Gamma}$. Then list all subgroups of small index as above. One then checks whether there is any torsion element in the group. For the first Betti number, one checks whether $\Pi/[\Pi, \Pi]$ is infinite. The procedure has been carried out successfully by Cartwright and Steger with sophisticated group theoretical arguments and computer programmings. In particular, they have eliminated the five cases in II(ii) for $\mathcal{D} = \ell$. We will mention their results in greater details in the next section.

Finally, we give two remarks regarding uniqueness of examples obtained in the above.

6.9 Remarks

1. First of all from Mostow's Strong Rigidity (cf. Theorem 4c in §3), we know that $\Pi_1 \cong \Pi_2$ implies that $B^2_{\mathbb{C}}/\Pi_1$ is isometric to $B^2_{\mathbb{C}}/\Pi_2$ as Riemannian manifolds equipped with the Killing metric.

2. As far as biholomorphic structures on a compact complex ball quotient are concerned, it follows from [Siu2] that there are at most two different complex surfaces in each class, corresponding to a complex structure and its complex conjugate. According to [KK], the conjugate complex structure does give rise to a different complex structure up to biholomorphism. Hence the underlying Riemannian manifold with respect to the Killing metric gives rise to two biholomorphically different fake projective planes.

3. Suppose v is a nonarchimedean place of k which ramifies in ℓ, there are two possible choices of a maximal parahoric subgroup P_v of $G(k_v)$ up to conjugation. Moreover, all the hyperspecial parahoric subgroups of $G(k_v)$ are conjugate to each other under $\overline{G}(k_v) := (\mathrm{Aut}G)(k_v)$ ([PY1] §2.2, 9.6). Furthermore, if $\mathcal{P} = (P_v)_{v\in V_f}$ and $\mathcal{P}' = (P'_v)_{v\in V_f}$ are two coherent collections of parahoric subgroups such that for every v, P'_v is conjugate to P_v under an element of $\overline{G}(k_v)$, then there is an element $\overline{G}(k)$ which conjugates $\mathcal{P}'$ to $\mathcal{P}$ if $h_{\ell,3} = 1$ or $(a,p) = (23,2)$ above ([PY1] Proposition 5.3). As a consequence, each of Case I and also $\mathcal{C}_2, \mathcal{C}_{10}$ in Case II(i) gives rise to two different classes of fake projective planes, while $\mathcal{C}_{18}$ of Case II(i) gives rise to three classes of fake projective planes by choosing $P_{\mathfrak{v}_3}$ to be either hyperspecial or a nonhyperspecial maximal parahoric, or an iwahori subgroup.

7 Statements of the results

7.1 Let us first recall our notation of class $A_\Lambda = A_{(k,\ell,\mathcal{D},G,(P_v)_{v\in V_f})}$ used in the last section. Here k is a totally real number field, ℓ is a totally imaginary quadratic extension of k, $\mathcal{D}$ is a cubic division algebra with center ℓ equipped with an involution of the second kind, $G(k) = \{z \in \mathcal{D}^\times \mid z\sigma(z) = 1 \text{ and } \mathrm{Nrd}(z) = 1\}$ and $(P_v)_{v\in V_f}$ is a coherent collection of parahoric subgroups of $G(k_v)$. Furthermore, there is an archimedean place v_o of k such that $G(k_o) \cong SU(2,1)$ and $G(k_v) \cong SU(3)$ for all $v \in V_\infty \backslash \{v_o\}$. Then $\Lambda = G(k) \cap \prod_{v\in V_f} P_v$ is a principal arithmetic subgroup. The normalizer $\Gamma = N_{G(k_{v_o})}(\Lambda) \subset SU(2,1)$ projects to a lattice $\overline{\Gamma} \subset PU(2,1)$. A class of fake projective planes in [PY1] is defined to be the ones with the fundamental group in the following set

$$A_\Lambda = \{\Pi \leq \overline{\Gamma} : [\overline{\Gamma} : \Pi] = \frac{3}{\chi(\overline{\Gamma})}, |\Pi/[\Pi,\Pi]| < \infty, \text{and } \Pi \text{ is torsion-free,}\}$$

where we use the notation $A \le B$ to denote that a group A is a subgroup of another group B.

Here is the main theorem of [PY1], see also the Addendum.

Theorem 8. *(a) There are twenty-eight non-empty classes A_Λ, eighteen coming from Case I with $k = \mathbb{Q}$, and ten coming from Case II(i) with real quadratic k and $[\mathcal{D} : \ell] > 1$, including two from $\mathcal{C}_2$, two from $\mathcal{C}_{10}$, three from $\mathcal{C}_{18}$, and three from $\mathcal{C}_{20}$.*
(b) There can at most be five more classes of fake projective planes, corresponding to Case II(ii), namely, $\mathcal{C}_1$, $\mathcal{C}_{11}$ and $\mathcal{C}_{18}$, two for each of the first two cases, and one for the last one.

7.2 Remark

1. The cases in Theorem 8(b) should not contain any fake projective plane according to Conjecture 1. These cases have been eliminated by Cartwright and Steger very recently, see Theorem 9 below.

2. The two cases of $\mathcal{T} \neq \mathcal{T}_o$ in Case I ($k = \mathbb{Q}$), the case of $\mathcal{C}_{18}, \mathcal{T} \neq \mathcal{T}_o$, and cases of $\mathcal{C}_{20}$ in Case II(i) were left out in the original paper [PY1]. This was pointed out to us by Steger, and was corrected in the Addendum to [PY1].

3. The fake projective plane of Mumford [Mu] corresponds to a subgroup of class with $(a, p) = (7, 2)$ in Case I. The two cases of Ishida-Kato [IK] correspond to $(a, p) = (15, 2)$ in Case I. The example of Keum [Ke] corresponds to $(a, p) = (7, 2)$ in Case I as well.

4. The scheme of proof in the last section actually provides a rough listing of all arithmetic lattices Π of $B^2_{\mathbb{C}}$ of volume bounded by 3 when the $B^2_{\mathbb{C}}$ is equipped with the $PU(2,1)$-invariant metric normalized in such a way that the total volume is the same as the Euler-Poincaré characteristic of $B^2_{\mathbb{C}}/\Pi$. In principle, the same scheme can be applied to find the list of all ball quotients, compact or non-compact with finite volume, of normalized volume bounded by some number V_o if V_o is sufficiently small.

5. Related to the above remark, the scheme of proof provides another verification of the fact that the smallest number among Euler-Poincaré characteristics of a cofinite non-cocompact arithmetic lattice of $PU(2,1)$ is 1, which is achieved by a Picard modular surface. The result was first proved by Holzapfel [H]. §3.6 of [PY1] provides another proof.

7.3 At this point, let us deduce some geometric consequences to the above theorem.

Corollary 1. *([PY1]) Let M be a fake projective plane belonging to Case I or Case II(i). Then the following hold.*
(a). The first homology group $H_1(M, \mathbb{Z})$ is always non-trivial.
(b). The orders of the automorphism groups of a fake projective planes are given by one of the following numbers, $1, 3, 7, 9$ or 21.
(c). For all the fake projective planes constructed, except for some of those coming

from $\mathcal{C}_{18}$, the lattice Λ can be embedded as a lattice in $SU(2,1)$.

The results are proved in [PY1] §10. Note that (a) means that there is always torsion in the homology class. This is proved in Theorem 10.1 in [PY1]. (b) follows rather easily from the table of indices in Step D of the proof in the last section. A consequence of (c) is that the canonical line bundle K_M of a fake projective plane (for $k = \mathbb{Q}$) can be written as $3H$ for a generator H of Neron-Severi group modulo torsion. Recall that we have observed in the proof of Theorem 5 that $K_M = 3H$ modulo torsion for all fake projective planes from Poincaré Duality. Hence the key point here is that the identity is true with no adjustment by a torsion line bundle.

We also remark that following the recent result of Cartwright and Steger [CS], see Theorem 9, the five cases $\mathcal{C}_1$, $\mathcal{C}_{11}$ and $\mathcal{C}_{18}$ of Case II(ii) listed in Theorem 8 can be eliminated as candidates to accommodate fake projective planes and hence (a) is true for all fake projective planes. For (c), Cartwright and Steger show that there exist examples of fake projective planes in the case of $\mathcal{C}_{18}$ for which Λ cannot be embedded as a lattice in $SU(2,1)$, see §7.5.

7.4 As mentioned in the last section, a natural way to construct all examples in a class A_Λ is to find a set of generators of $\overline{\Gamma}$ and try to see if all subgroups of finite index in $\overline{\Gamma}$ satisfying the three properties in the definition of A_Λ can be determined. Suppose $\chi(\overline{\Gamma}) = 3$ as for the case of $k = \mathbb{Q}$, $\ell = \mathbb{Q}(\sqrt{-23})$ and $\mathcal{T} = \mathcal{T}_0 = \{2\}$ listed as Case I in §6.7. It follows from definition that $A_\Lambda = \{\overline{\Gamma}\}$ is a singleton and there is only one fake projective plane in such class. In general we only need to enumerate all subgroups of small index $3, 7, 9$ or 21 for each of the classes with $\chi(\overline{\Gamma}) \neq 3$. This is however difficult and technically very involved, and is finally completed very recently by Cartwright and Steger [CS].

Cartwright and Steger [CS] developed powerful computational tools to study lattices associated to the classes described in Theorem 8. It requires sophisticated group theory, powerful computer programming and clever human intervention. Here are the main steps.

First of all, from the description of the maximal arithmetic lattice involved in terms of $(k, \ell, \mathcal{D}, G, (P_v)_{v \in V_f})$ as explained in §6, a matrix representation of $\overline{\Gamma}$ is found which gives a set of matrix generators and relations. To show that the generators exhaust $\overline{\Gamma}$, Cartwright and Steger constructed a fundamental domain and verify the necessary facts from covolume formula of Prasad as given in §5. Secondly, with the help of computer softwares, the set of generators is reduced to a sufficiently small size with manageable set of relations. Finally to exhaust the list of torsion free subgroup of index which is not too large but still beyond the current capability of computer power, they have to combine group theoretical knowledge and clever enumeration scheme with the help of computer softwares to achieve the goal.

As a consequence, Cartwright and Steger succeeded in listing all possible fake projective planes in each class in Theorem 8a and eliminate the five classes in Theorem 8b as candidates for fake projective planes. Here is the results they obtained.

Theorem 9. *(Cartwright-Steger [CS]) (a) There are altogether* 31 *examples of fake projective planes as a Riemannian manifold in Case I, the case of* $k = \mathbb{Q}$. *Here is the break down of the cardinality* $|A_\Lambda|$ *of a class* A_Λ *according to the earlier listing.*

a	1	2	7	15	23	1	2	7	7
$\mathcal{T}$	$\{5\}$	$\{3\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{2,5\}$	$\{2,3\}$	$\{2,3\}$	$\{2,5\}$
$\sum_\Lambda \vert A_\Lambda\vert$	2	2	7	10	2	1	1	4	2

(b) There are altogether seven examples of fake projective planes as a Riemannian manifold for $\mathcal{C}_2$, *two examples for* $\mathcal{C}_{10}$, *five examples for* $\mathcal{C}_{18}$ *and five examples for* $\mathcal{C}_{20}$ *in Case II(i).*
(c) Hence as complex surfaces, there are 100 *fake projective planes up to biholomorphism.*

Let us recall the remark about complex and conjugate complex structure mentioned in §6.9. In Theorem 8 (a), (b) and earlier discussions, we are classifying fake projective planes according to the semi-simple algebraic group G. Hence as a Riemannian manifold equipped with the Killing metric, according to Mostow's Strong Rigidity Theorem, each class consists of a single Riemannian manifold. Each such Riemannian manifold supports two non-biholomorphic complex structures according to §6.9(2). Hence each example in (a) and (b) gives rise to two different fake projective planes.

7.5 The table below gives a complete list of fake projective planes according to the scheme of [PY1] and [CS], where N_1 is the number of classes and N_2 is the number of fake projective spaces classified up to isometry for each pair of number fields. Hence there are $2N_2$ fake projective planes classified up to biholomorphism after considering complex conjugates.

(k,ℓ)	$\mathcal{T}$	N_1	N_2
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-1}))$	$\{5\}$	2	2
	$\{2,5\}$	1	1
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-2}))$	$\{3\}$	2	2
	$\{2,3\}$	1	1
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-7}))$	$\{2\}$	2	7
	$\{2,3\}$	2	4
	$\{2,5\}$	2	2
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-15}))$	$\{2\}$	4	10
$(\mathbb{Q}, \mathbb{Q}(\sqrt{-23}))$	$\{2\}$	2	2
$(\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}, \zeta_3))$	$\{\mathfrak{v}_2\}$	2	7
$(\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{-7+4\sqrt{2}}))$	$\{\mathfrak{v}_3\}$	2	2
$(\mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{6}, \zeta_3))$	$\{\mathfrak{v}_3\}$	1	1
	$\{\mathfrak{v}_2, \mathfrak{v}_3\}$	2	4
$(\mathbb{Q}(\sqrt{7}), \mathbb{Q}(\sqrt{7}, \zeta_4))$	$\{\mathfrak{v}_2\}$	1	1
	$\{\mathfrak{v}_2, \mathfrak{v}_3'\}$	1	2
	$\{\mathfrak{v}_2, \mathfrak{v}_3''\}$	1	2

7.6 Remark
Some of the examples found by Cartwright-Steger [CS] illustrate interesting geometric and group theoretical properties. We only mention the following two.

1. Some of the examples found correspond to non-congruence subgroups of the maximum arithmetic lattice $\overline{\Gamma}$, including some for $(k,\ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{7}))$ in Case I and some for $(k,\ell) = \mathcal{C}_2$ in Case II.

2. Some of the lattices obtained in the case of $(k,\ell) = \mathcal{C}_{18}$ as subgroup of $PU(2,1)$ cannot be lifted in a one-to-one manner to $SU(2,1)$.

7.7 Here we mention another geometric consequence of the above study of fake projective planes. Explicit geometric construction of complex hyperbolic surfaces was rare in the literature. A well-known general construction was the one initiated by Picard [Pi] related to monodromy groups associated to some appropriate hypergeometric series. The construction has been studied and clarified by Picard, Le Vavasseur, Terada and Deligne-Mostow. We refer the readers to [DM] for general references and call such a surface a Deligne-Mostow surface. Toledo has conjectured that a fake projective plane can never be commensurable with a Deligne-Mostow surface. This is now confirmed by the table in §7.5. The conjecture in fact would follow from Theorem 8 once the cases $\mathcal{C}_1, \mathcal{C}_8$ and $\mathcal{C}_{11}$ were eliminated, which was completed by Cartwright and Steger in [CS].

7.8 Finally, as a generalization of the Severi problem mentioned in the Introduction in §1, we may now mention the following strong topological characterization of $P^2_{\mathbb{C}}$.

Theorem 10. *Any smooth complex surface with the same singular homology groups as $P^2_{\mathbb{C}}$ is biholomorphic to $P^2_{\mathbb{C}}$.*

Proof Let M be smooth surface with the same singular homology groups as $P^2_{\mathbb{C}}$. Assume that M is not biholomorphic to $P^2_{\mathbb{C}}$. From definition, M is a fake projective plane. According to Theorem 3(a), M is the quotient of $B^2_{\mathbb{C}}$ by a lattice in $PU(2,1)$. The lattice is arithmetic by Theorem 7. From Theorem 8 and Theorem 9, we conclude that the lattice involved are all of type $[\mathcal{D} : \ell] > 1$. According to Corollary 1 to Theorem 8, $H_1(M, \mathbb{Z})$ is non-trivial. This contradicts the assumption that M has the same singular homology groups as $P^2_{\mathbb{C}}$.

8 Further studies

8.1 One of the reasons that a fake projective plane is interesting is that it provides a concrete geometric example with small numerical invariants and rich structure which is useful for various mathematical studies. In this section we will mention two directions for further study.

A potential application of the research of [PY1] is that it provides a list of projective algebraic surfaces equipped with a finite non-trivial automorphism group and small Chern numbers that may be useful in constructing new interesting surfaces in the uncharted parts of the geography of surfaces of general type. For example, it is mentioned in Corollary to Theorem 8 that the automorphism groups H of the fake projective planes M involved have orders given by one of $1,3,7,9$ and 21. As we explained earlier, M has the smallest Euler-Poincaré characteristics among smooth surfaces of general type. Suppose the order $|H|$ of H is greater than 1. This implies H cannot be torsion free, for otherwise the quotient would be a surface of general type of smaller Euler-Poincaré. According to results of Chevalley and Prill (cf. [Pri]), a fixed point of H maps to a smooth point on the quotient M/H if and only if H is generated by complex reflection at the point. Since the mirror of any complex reflection is a totally geodesic curve, which does not exist on arithmetic complex ball quotient of second type and in particular on the list of M in Theorem 8a, we conclude that M/H is singular at each of the image of the fixed point set of H. Desingularization of such surfaces would give interesting surfaces with relatively small numerical invariants such as the Chern numbers.

Let us illustrates the above discussion by considering the case that $\mathbb{Z}_3$ is a subgroup of the automorphism group of M. Let μ be a generator of $\mathbb{Z}_3$. First of all we observe that in our cases, M is defined by an arithmetic lattice of second type and hence there is no totally geodesic curves on M. This implies that the fixed point set of μ, which has to be totally geodesic, does not have dimension one components and hence consists of isolated points. By the usual Lefschetz Fixed point theorem, there are three fixed points, each of order 3 since the local index is a divisor of 3, the degree of the projection $M \to N = M/\mathbb{Z}_3$. Each singularity is hence a singularity of type $A_{3,2}$ or of type $A_{3,1}$ (cf. [BHPV]). However, from the Holomorphic Lefschetz Fixed Point Formula, we know that

$$1 = L(f,\mathcal{O}) = \sum_{\mu(p)=p} \frac{1}{\det(I - \mathcal{J}(\mu)(p))},$$

where $\mathcal{J}(\mu)(p)$ is the Jacobian of μ at p. From direct computation, singularity of type $A_{3,2}$ (resp. $A_{3,1}$) contributes $1/3$ (resp. $1/(3\omega)$ for some cube root of unity ω). Holomorphic Lefschetz Fixed Point Formula allows us to conclude that all the fixed points of $\mathbb{Z}_3$ are of type $A_{3,2}$.

Since $\mathbb{Z}_3$ acts on M with three isolated singularities, we may represent K_N as a Cartier divisor not passing through the singular points on N. The fact that Picard number $\rho(M) = 1$ implies $\rho(N) = 1$. Hence K_N is a positive rational multiple of an effective curve on N and $K_N \cdot K_N = K_M \cdot K_M/3 = 3$. Direct counting also gives Euler-Poincaré characteristic $e(N) = 3$. As a singularity of type $A_{3,2}$ has Milnor number 0, Laufer's formula as in [Lau] implies that the arithmetic genus of $\widetilde{N}$, a minimal resolution of N, is given by $\chi(\mathcal{O}_{\widetilde{N}}) = (e(N) + K_N^2)/12 = 1$. The self intersection $K_{\widetilde{N}} \cdot K_{\widetilde{N}} = 3$ follows from direct counting using the configuration of the resolution, or from the fact that the singularity is rational. The smooth surface $\widetilde{N}$ is of general type since the set of pluricanonical sections of $\widetilde{N}$ contains the pull-back of pluricanonical sections of N, on which K_N is a positive rational multiple of an ample divisor.

In conclusion, we obtain a smooth surface of general type $\widetilde{N}$ with $K_{\widetilde{N}} \cdot K_{\widetilde{N}} = 3$, first Betti number $b_1(\widetilde{N}) = 0$ and $h^{2,0}(\widetilde{N}) = 0$.

We remark that the singularities of M/H for a general fake projective plane M described in Theorem 9 and an automorphism group H have recently been studied and completed by Keum [Ke2] by different arguments. The results have also been verified independently by explicit computations by Cartwright-Steger [CS] as will be seen from the following discussions.

8.2 The above discussions do not give information about the fundamental group of N and hence $\widetilde{N}$. Since the lattices are explicitly defined in terms of the number fields and parahorics, one may hope to be able to determine the fundamental group and hence all the topological information about M/H and their desingularization, if for example, one has a good way to write down the generators of lattice. This happens to be a difficult task. As mentioned in the last section, Cartwright and Steger [CS] recently manage to find generators for the classes of fake projective planes in Case I as stated in Theorem 8 and Theorem 9. As a consequence, they have constructed many interesting surfaces with small topological invariants.

Theorem 11. *(Cartwright-Steger [CS]) There exists smooth surfaces S with $K_S^2 = 3, b_1(S) = 0$ and $p_g = 0$ and $\pi_1(S)$ one of the following groups obtained from resolving the singularities of appropriate quotients of the fake projective planes in Theorem 8 and Theorem 9 by its automorphism group.*

$$\{1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_7, \mathbb{Z}_2 \times \mathbb{Z}_7, \mathbb{Z}_{13}, \mathbb{Z}_{14}, S_3, Q_8, D_8\}$$

In the above, S_3 is the symmetry group of three elements, Q_8 is the quaternionic eight-group, and D_8 is the 8-element symmetry group of a square. We remark that a simply connected surface of general type with $p_g = 0$ and $K^2 = 3$ has been recently constructed by H. Park, J. Park and D. Shin [PPS1]. More recently, they have also constructed by similar method such an example with $\pi_1(S) = \mathbb{Z}_2$ in [PPS2]. The method used by Park et al is very different from the method of [CS]. It uses blow-down surgery and $\mathbb{Q}$-Gorenstein smoothing theory.

Cartwright and Steger [CS] listed five simply connected examples with trivial fundamental group, including three from $\mathcal{C}_2$ and two from $\mathcal{C}_{18}$. We are going to name these as Cartwright-Steger surfaces as in [PY1], Addendum. Let us explain the Cartwright-Steger surfaces in the case $\mathcal{C}_2 = (\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}, \zeta_3))$ with some details. As explained in Theorem 9, Cartwright and Steger found seven fake projective planes associated to $\mathcal{C}_2$, including the two constructed in [PY1]. If we take the one constructed in [PY1] with automorphism group is $\mathbb{Z}_3 \times \mathbb{Z}_3$, there are four subgroups of order 3. Cartwright and Steger show that a quotient N of M by any of the four subgroups of order 3 is a simply connected singular surface. N contains three singularities of type $A_{3,2}$, providing an alternate and direct verification of the argument in §8.1. The minimal resolution $\widetilde{N}$ of N is simply connected since the same is true for N. From the same reason as before, $K_{\widetilde{N}} \cdot K_{\widetilde{N}} = 3$, first Betti number $b_1(\widetilde{N}) = 0$ and $h^{2,0}(\widetilde{N}) = 0$. In this way, we get four Cartwright-Steger surfaces.

8.3 Another direction for further study is to consider similar problems for higher dimensions. For this purpose, let us make the following definition.

Definition Let $\overline{G}$ be a connected semi-simple real algebraic group of adjoint type. Let X be the symmetric space of $\overline{G}(\mathbb{R})$ and X_u be the compact dual of X. We shall say that the quotient X/Π of X by a cocompact torsion-free arithmetic subgroup Π of $\overline{G}(\mathbb{R})$ is an arithmetic fake X_u if its Betti numbers are same as that of X_u; X/Π is an irreducible arithmetic fake X_u if, further, Π is irreducible (i.e., no subgroup of Π of finite index is a direct product of two infinite normal subgroups).

In particular, a Kähler manifold M of complex dimension n is an arithmetic fake projective space if it is the quotient of $B^n_{\mathbb{C}}$ by a cocompact torsion-free arithmetic lattice and has the same Betti numbers as $P^n_{\mathbb{C}}$.

Theorem 12. *(Prasad-Yeung [PY2])*
(a) There exists no arithmetic fake projective space of dimension different from 2 *and* 4.
(b) There are at least four classes of arithmetic fake projective spaces in dimension 4*. Furthermore, they arise from* $k = \mathbb{Q}, \ell = \mathbb{Q}(\sqrt{-7})$.
(c) There are at least four distinct arithmetic fake $\mathbf{Gr}_{2,5}$ *and at least five irreducible arithmetic fake* $\mathbf{P}^2_{\mathbb{C}} \times \mathbf{P}^2_{\mathbb{C}}$

The examples in (b) and (c) provide non-trivial examples for geography of four and six dimensional projective algebraic manifolds of general type. The geometric and arithmetic properties of such 4-folds are yet to be studied.

In general, Hermitian symmetric spaces have been classified by Élie Cartan. We recall that the irreducible hermitian symmetric spaces are the symmetric spaces of Lie groups $\mathrm{SU}(n+1-m,m)$, $\mathrm{SO}(2,2n-1)$, $\mathrm{Sp}(2n)$, $\mathrm{SO}(2,2n-2)$, $\mathrm{SO}^*(2n)$, an absolutely simple real Lie group of type E_6 with Tits index ${}^2E^{16'}_{6,2}$, and an absolutely simple real Lie group of type E_7 with Tits index $E^{28}_{7,3}$ respectively (for Tits indices see Table II in [Ti1]). The complex dimensions of these spaces are $(n+1-m)m$, $2n-1$, $n(n+1)/2$, $2n-2$, $n(n-1)/2$, 16 and 27 respectively. The Lie groups listed above are of type A_n, B_n, C_n, D_n, D_n, E_6 and E_7 respectively. Our goal is to classify all fake arithmetic fake Hermtian symmetric spaces.

Very recently, the following result is obtained.

Theorem 13. *(Prasad-Yeung [PY3]) There is no arithmetic fake Hermitian symmetric space of type other than* A_n.

The analysis for fake compact Hermitian symmetric spaces of Type A is almost complete as well. As an application, we observe that if Π is a torsion-free cocompact discrete subgroup of $\overline{G}$, then there is a natural embedding of $H^*(X_u,\mathbb{C})$ in $H^*(X/\Pi,\mathbb{C})$, see [B], 3.1 and 10.2, and hence X/Π is a fake X_u if and only if the natural homomorphism $H^*(X_u,\mathbb{C}) \to H^*(X/\Pi,\mathbb{C})$ is an isomorphism. Hence Theorme 13 has the following corollary.

Corollary 2. *Apart from the case that* G *is of type* A, *there exists no associated locally Hermitian symmetric space* X/Γ, *where* Γ *is an arithmetic cocompact torsion-free lattice, such that* X/Γ *and its compact dual have isomorphic rational cohomology groups.*

Related to the discussions in §8.1 and §8.2, it is to be hopeful that quotients by appropriate finite groups of automorphisms of any arithmetic fake Hermitian symmetric spaces constructed will yield interesting concrete examples of higher dimensional algebraic manifolds.

8.4 Related to results in §8.3, we may define a fake compact hermitian symmetric space to be a Kähler manifold which has the same Betti numbers as a hermtian symmetric space of compact type of the same dimension. It comes naturally the geometric problem of deciding when a fake compact hermtian symmetric space is an arithmetic fake compact hermitian symmetric space. The two notions are the same for fake projective planes, but the problem is much more complicated and essentially open in higher dimensions. To understand the difficulty of the problem, let us restrict our discussions to fake projective spaces. First of all if the complex dimension of the Kähler manifold is odd, there exists fake projective three spaces given by hyperquardrics but there is no arithmetic fake projective three space according to Theorme 12a. Hence the two notions can possibly be identified only for even complex dimensions. Not much is known about the problem. The following is a positive result in this direction.

Theorem 14 (Ye3)**.** *A fake projective four space is uniformized by a complex hyperbolic four space if any of the following conditions is satisfied.*
(i) $c_1^4(M) \neq 225$,
(ii) $H^4(M, \mathbb{Z})$ *modulo torsion is generated by* $\theta \cup \theta$*, where* θ *is a generator of* $H^2(M, \mathbb{Z})$ *modulo torsion, or*
(iii) The cycle corresponding to the canonical line bundle K_M *is not a generator of the Neron-Severi group.*

The proof is achieved by characterizing all the possible Chern numbers for a homology complex projective four space. More precisely, we show that the Chern numbers $(c_1^4, c_1c_3, c_1^2c_2, c_2^2, c_4)$ of a smooth rational homology complex projective space of complex dimension 4 can only take one of the following two sets of values, (i). $(625, 50, 250, 100, 5)$, or, (ii). $(225, 50, 150, 100, 5)$. This is achieved by generalizing an argument of Libgober and Wood [LW]. The theorem is then proved by utilizing a Chern number inequality given by Yau (cf. [Be]).

Equipped with Theorem 14, we may now generalize the argument in §4, where the corresponding result in complex dimension two was known in the work of [Kl] and [Ye1], to conclude that a fake projective four space with one of the extra condition posted in Theorem 14 is an arithmetic fake projective four space. It is however not known if the extra condition can be removed. Equivalently, we do not know if the Chern numbers given by $(c_1^4, c_1c_3, c_1^2c_2, c_2^2, c_4) = (225, 50, 150, 100, 5)$ can be realized by any connected Kähler manifold. To illustrate the intricate nature of the problem, according to a result of Milnor, there exists disconnected algebraic manifolds with Chern numbers given as above (cf. [H1]).

The problem of uniformization in higher dimensions appears to be challenging.

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