Relationship between the zeros of two polynomials

BY W.S. CHEUNG AND T.W. NG*

Department of Mathematics,

The University of Hong Kong,

Pokfulam, Hong Kong.

E-mail address: wshun@graduate.hku.hk, ntw@maths.hku.hk

Abstract

In this paper, we shall follow a companion matrix approach to study the relationship between zeros of a wide range of pairs of complex polynomials, for example, a polynomial and its polar derivative or Sz.-Nagy's generalized derivative. We shall introduce some new companion matrices and obtain a generalization of the Weinstein-Aronszajn Formula which will then be used to prove some inequalities similar to Sendov conjecture and Schoenberg conjecture and to study the distribution of equilibrium points of logarithmic potentials for finitely many discrete charges. Our method can also be used to produce, in an easy and systematic way, a lot of identities relating the sums of powers of zeros of a polynomial to that of the other polynomial.

1 Introduction and Preliminaries

The concept of differentiators, first introduced by Davis [2], is used to study the relationship between the zeros of a polynomial and the zeros of its derivative. In [10] Rajesh Pereira has further developed this idea and applied it

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successfully to solve several long standing conjectures, including the conjecture of Schoenberg. Similar ideas were also used independently at the same time by Semen Mark Malamud in [7] and [8] to solve these conjectures. Recently, Cheung and Ng [1] have introduced the D-companion matrix, a matrix form of the differentiator and applied it to solve de Bruin and Sharma's conjecture. In this paper, we will introduce a generalization of the D-companion matrix to study the relationship between zeros of two complex polynomials. Unlike the differentiator, this new tool could be applied to a wide range of pairs of polynomials, not just a polynomial and its derivative. In particular, we shall apply our results to study the relationship between zeros of a polynomial and its polar derivative or Sz.-Nagy's generalized derivative. Our starting point is to construct a matrix similar to the D-companion matrix when the two polynomials are related in certain ways. In fact, we have the following

Theorem 1.1 *Let A be an $n \times n$ matrix with characteristic polynomial $p(z) = \Pi_{j=1}^{n}(z - z_i)$ and $q(z)$ be a monic polynomial of degree n given by*

$$\frac{q(z)}{p(z)} = 1 + \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}.$$

Then, there exists a rank one matrix H such that the characteristic polynomial of the matrix $A-H$ is $q(z)$. In particular, if A is the diagonal matrix D formed by $z_1, \ldots, z_n$, then H can be chosen to be the matrix $\Lambda J = \begin{pmatrix} \lambda_1 & \cdots & \lambda_1 \\ \vdots & & \vdots \\ \lambda_n & \cdots & \lambda_n \end{pmatrix}$, where Λ is the diagonal matrix formed by $\lambda_1, \ldots, \lambda_n$ and J is the $n \times n$ all one matrix.

Theorem 1.2 *Let A be an $n \times n$ matrix with characteristic polynomial $p(z) = \Pi_{j=1}^{n}(z - z_i)$ and $q(z)$ be a monic polynomial of degree $n-1$ given by*

$$\frac{q(z)}{p(z)} = \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}.$$

There exists a rank one matrix H such that $H^2 = H$ and the characteristic polynomial of the matrix $A - AH$ is $zq(z)$. In particular, if A is the diagonal *matrix D formed by $z_1,\ldots,z_n$, then H can be chosen to be the matrix* $\Lambda J = \begin{pmatrix} \lambda_1 & \cdots & \lambda_1 \\ \vdots & & \vdots \\ \lambda_n & \cdots & \lambda_n \end{pmatrix}$, *where Λ is the diagonal matrix formed by $\lambda_1,\ldots,\lambda_n$ and J is the $n\times n$ all one matrix.*

Theorem 1.1 can be considered as a generalization of the famous Weinstein-Aronszajn Formula. The Weinstein-Aronszajn Formula, a tool to study the rank one perturbation of Hermitian matrices, is given by

$$\frac{\det(zI - A + r\mathbf{w}\mathbf{w}^*)}{\det(zI - A)} = 1 - r\sum_{j=1}^{n} \frac{|a_j|^2}{z - z_i}$$

where A is a Hermitian matrix with eigenvalues $z_1,\ldots,z_n$ and the corresponding orthonormal basis $\mathbf{u_1},\ldots,\mathbf{u_n}$, $r\in\mathbb{R}$, and $\mathbf{w} = a_1\mathbf{u_1} + \cdots + a_n\mathbf{u_n}$ is a unit vector (see [4, p.134]).

On the other hand, Theorem 1.1 says that for any order n matrix A with eigenvalues $z_1, ..., z_n$ and any $\lambda_1, ..., \lambda_n \in \mathbb{C}$,

$$\frac{\det(zI - A + H)}{\det(zI - A)} = 1 + \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j},$$

where H is some rank one matrix. If A is Hermitian, then A is unitarily diagonalizable and all the eigenvalues z_i of it are real. Hence, one may assume that A is the diagonal matrix formed by $z_1, ..., z_n$ and H can then be taken as ΛJ which can be written in the form $r\mathbf{w}\mathbf{w}^*$ easily if $\lambda_i = -r|a_j|^2$.

In Theorem 1.2, the polynomial

$$q(z) = p(z)\sum_{j=1}^{n}\frac{\lambda_j}{z - z_j}$$

is called Sz.-Nagy's generalized derivative if $\lambda_1, ..., \lambda_n$ are positive real numbers such that $\sum_{j=1}^{n}\lambda_j = n$ [9]. In particular, if $\lambda_j = \frac{1}{n}$, then $q = \frac{1}{n}p'$. In this case, if we take A to be the diagonal matrix D formed by $z_1,\ldots,z_n$, then H can be taken as the matrix $\frac{1}{n}J$ and $A - AH = D(I - \frac{1}{n}J)$ so that the D-companion matrix of $p'(z)$ introduced in [1] will be the principal submatrix of $U(D(I - \frac{1}{n}J))U^*$ for some unitary matrix U.

When q_1 is the polar derivative of p (see [11], p.97), then $q_1(z) = np(z) - (z-\alpha)p'(z)$ for some $\alpha \in \mathbb{C}$ and hence

$$q_1(z) = p(z) \sum_{j=1}^{n} \frac{\alpha - z_j}{z - z_j}.$$

If $\sum_{i=1}^n z_i \neq 0$, then $q(z) = \frac{1}{-(n-1)(\sum_{i=1}^n z_i)} q_1(z)$ is monic and

$$\frac{q(z)}{p(z)} = \frac{1}{-(n-1)(\sum\limits_{i=1}^{n} z_i)} \sum_{j=1}^{n} \frac{\alpha - z_j}{z - z_j}.$$

So we can apply Theorem 1.2 and its corollaries to both Sz.-Nagy's generalized derivatives and polar derivatives of polynomials. In general, a generic monic polynomial p will have only distinct zeros $z_1, ..., z_n$. If q is a monic polynomial with $\deg(q) < \deg(p) = n$, then by partial fraction decomposition, we have

$$\frac{q(z)}{p(z)} = \sum_{j=1}^{n} \frac{q(z_j)}{p'(z_j)} \frac{1}{z - z_j}. \tag{1}$$

If $\deg(q) = \deg(p) = n$, then $\deg(q-p) < \deg(p)$ and apply the above formula, we have

$$\frac{q(z)}{p(z)} = 1 + \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}$$

for some suitable λ_j. Therefore with the partial fraction decomposition formula (1), Theorem 1.1 and 1.2 can be applied to a wide range of pairs of polynomials (when $\deg(q) < n$, one can consider $z^k q(z)$ instead where $k = n - 1 - \deg(q)$). For example, consider the Dunkl operator Λ_α on $\mathbb{R}$ of index $\alpha + \frac{1}{2}$ associated with the reflection group $\mathbb{Z}_2$ [3] give by

$$\Lambda_\alpha(p)(x) = p'(x) + (\alpha + \frac{1}{2}) \frac{p(x) - p(-x)}{x}, \quad \alpha \geq -\frac{1}{2}.$$

Take q to be $\frac{1}{n+2\alpha+1}\Lambda_\alpha(p)$ which is a monic polynomial of degree $n-1$ and apply the partial fraction decomposition formula (1) to obtain the λ_i when p has distinct zeros only.

Theorem 1.1 and 1.2 allow us to apply results in matrix theory to deduce results concerning the zeros of a pair of polynomials in Section 3 and 4. For example, by applying Theorem 1.1. and 1.2, we prove some results (Corollary 4.2 and Corollary 4.1) in a style similar to the famous Gauss-Lucas Theorem :

Gauss-Lucas Theorem. *The zeros of the derivative of a polynomial are located inside the convex hull of the zeros of the polynomial.*

We shall also prove results (Corollary 3.3 and Corollary 3.7) similar to the Schoenberg conjecture, now a theorem—after it was proved independently by Pereira and Malamud in [10]and [8] respectively:

Malamud-Pereira Theorem. Let $z_1, \cdots, z_n$ be the zeros of a polynomial p of degree $n \geq 2$ and $w_1, \cdots, w_{n-1}$ be the zeros of p'. Then

$$\sum_{i=1}^{n-1} |w_i|^2 \leq \frac{1}{n^2} |\sum_{i=1}^{n} z_i|^2 + \frac{n-2}{n} \sum_{i=1}^{n} |z_i|^2$$

where equality holds if and only if all z_i lie on a straight line.

Theorem 1.1 and 1.2 also allow us to obtain some inequalities about the zeros of polynomials (Corollary 3.4 and Corollary 3.8), and deduce some minmax-maxmin inequalities (Corollary 3.5 and Corollary 3.9) similar to Sendov conjecture:

Sendov conjecture. *Let $z_1, ..., z_n$ be the zeros of a polynomial p of degree $n \geq 2$ and $w_1, ..., w_{n-1}$ be the zeros of p', the derivative of p. Then,*

$$\max_{1 \leq k \leq n} \min_{1 \leq i \leq n-1} |w_i - z_k| \leq \max_{1 \leq k \leq n} |z_k| .$$

Moreover, we will obtain, in a simple and systematic way, the sum of powers of the zeros of a polynomial in terms of that of the other polynomial. Those results (Corollary 3.6 and Corollary 3.10) could be obtained using the Newton's formulas, but in a more clumsy way.

Finally, we shall apply Theorem 1.1 to study the distribution of equilibrium points of logarithmic potentials for finitely many discrete charges.

2 Proof of Theorem 1.1 and 1.2

Proof of Theorem 1.1. For each eigenvalue z_i of A, we choose a corresponding eigenvector $\mathbf{v_i}$. Our choice should be that the vector $\mathbf{v} = \sum_{j=1}^{n} \lambda_j \mathbf{v_j}$ is nonzero.

Construct a matrix H such that $H\mathbf{v_i} = \mathbf{v}$ and $H\mathbf{x}$ is a scalar multiple of $\mathbf{v}$ for all $\mathbf{x}$. We claim that the characteristic polynomial of $A - H$ is $q(z)$.

We have

$$\begin{aligned} H(A - zI)^{-1}\mathbf{v} &= \sum_{j=1}^{n} \lambda_j H(A - zI)^{-1}\mathbf{v_j} \\ &= \sum_{j=1}^{n} \lambda_j (z_j - z)^{-1}\mathbf{v} \\ &= \left(\frac{q(z)}{p(z)} - 1\right)\mathbf{v}. \end{aligned}$$

As $H(A - zI)^{-1}$ is of rank one, $q(z)p(z)^{-1} - 1$ is its unique nonzero eigenvalue. Hence

$$\begin{aligned} \det(zI - (A - H)) &= \det(zI - A)\det(I + H(zI - A)^{-1}) \\ &= p(z)(1 + (q(z)p(z)^{-1} - 1)) \\ &= q(z) \end{aligned}$$

as desired. ■

Proof of Theorem 1.2. As q is monic, we have $\sum_{j=1}^{n} \lambda_j = 1$ and therefore

$$\frac{zq(z)}{p(z)} = 1 + \sum_{j=1}^{n} \frac{\lambda_j z_j}{z - z_j}. \tag{2}$$

Choose H to be a matrix corresponding to $p(z) + q(z)$ in Theorem 1.1, then $H_1 = AH$ is a matrix corresponding to $zq(z)$ in Theorem 1.1. It is straightforward to show that $H^2 = H$ as $H\mathbf{v} = (\sum_{j=1}^{n} \lambda_j)\mathbf{v} = \mathbf{v}$. ■

3 Schoenberg and Sendov type results

The introduction of the two matrices in Theorem 1.1 and 1.2 allows one to apply results in matrix theory to prove the above mentioned corollaries. For example, to obtain results similar to Schoenberg's theorem, we need the following result of Schur[12]:

Theorem 3.1 . *If A is an $n \times n$ matrix with eigenvalues $w_1, \ldots, w_n$, then we have*

$$\sum_{i=1}^{n} |w_i|^2 \leq \text{sum of entries of } AA^* = \text{sum of entries of } A^*A.$$

Equality holds if A is normal, i.e. $A^*A = AA^*$.

To achieve the minmax-maxmin inequalities, we need the following result in matrix theory:

Theorem 3.2 (**Gerschgorin's Theorem** ([6, p.344])) The eigenvalues of any square matrix $A = (a_{ij})$ of order $n \geq 2$, lie in the union $G = \bigcup_{i=1}^{n} G_i$ of the Gerschgorin disks

$$G_i = \{z \in \mathbb{C} : |z - a_{ii}| \leq R_i(A)\} \ ,$$

where $R_i(A) = \sum_{j=1\, j\neq i}^{n} |a_{ij}| \ , \ i = 1, \ldots, n.$

Finally we shall also make use of the simple fact that the sum of powers related to the fact that the sum of the n-th power of eigenvalues of a matrix A is equal to the trace of A^n.

Now we are ready to prove quite a number of corollaries of Theorem 1.1 and 1.2. We shall divide our results for the zeros of polynomials p and q into two cases: a) $deg(p) = deg(q)$ and b) $deg(p) = deg(q) + 1$.

Case a : $deg(p) = deg(q)$

For this case, we take $A - H$ to be $D - \Lambda J$.

Corollary 3.3 *Let p be a monic polynomial of degree n with zeros $z_1, \ldots, z_n$. Suppose q is a monic polynomial given by*

$$\frac{q(z)}{p(z)} = 1 + \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}.$$

If $w_1, \ldots, w_n$ are zeros of q, then

$$\sum_{i=1}^{n} |w_i|^2 \leq \sum_{i=1}^{n} |z_i|^2 - 2\sum_{i=1}^{n} Re(z_i\bar{\lambda}_i) + (\sum_{i=1}^{n} |\lambda_i|)^2.$$

Proof. We apply Theorem 3.1 to $D - \Lambda^{1/2} J \Lambda^{1/2}$ which is similar to $D - \Lambda J$. ∎

Corollary 3.4 *Let p be a monic polynomial of degree n with zeros $z_1, \ldots, z_n$. Suppose q is a monic polynomial given by*

$$\frac{q(z)}{p(z)} = 1 + \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}.$$

Then for any zero w of q, there exists z_k and z_l such that

$$|w - z_k - \lambda_k| \leq \sum_{j \neq k} |\lambda_j|$$

and

$$|w - z_l - \lambda_l| \leq (n-1)|\lambda_l|.$$

Proof. Apply Gerschgorin's theorem to $D - \Lambda J$. ∎

A direct consequence of Corollary 3.4 minmax-maxmin inequality similar to Sendov conjecture is the following:

Corollary 3.5 *Let p be a monic polynomial of degree n with zeros $z_1, \ldots, z_n$. Suppose q is a monic polynomial given by*

$$\frac{q(z)}{p(z)} = 1 + \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}.$$

If $w_1, \ldots, w_n$ are zeros of q, then

$$\max_{1 \leq i \leq n} \min_{1 \leq k \leq n} |w_i - z_k| \leq \sum_{j=1}^{n} |\lambda_j| \leq n \max_{1 \leq j \leq n} |\lambda_j| .$$

Corollary 3.5 can be proved directly: Let w_i be a root of q which is not a root of p, then

$$1 = \left| \sum_{j=1}^{n} \frac{\lambda_j}{w_i - z_j} \right| \leq \sum_{j=1}^{n} \frac{|\lambda_j|}{|w_i - z_j|} \leq n \frac{\max\limits_{1 \leq j \leq n} |\lambda_j|}{\min\limits_{1 \leq k \leq n} |w_i - z_k|}.$$

The trace of a matrix of A, denoted by $tr(A)$, is the sum of eigenvalues of A. By the fact that $tr(AB) = tr(BA)$ and that $JEJ = tr(E)J$ for any diagonal matrix E, we can obtain relations among the sum of powers of zeros of p and the zeros of q. For instance, we have

Corollary 3.6 *Let p be a monic polynomial of degree n with zeros $z_1, \dots, z_n$. Suppose q is a monic polynomial given by*

$$\frac{q(z)}{p(z)} = 1 + \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}.$$

If $w_1, \dots, w_n$ are zeros of q, then

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} \lambda_i,$$

$$\sum_{i=1}^{n} w_i^2 = \sum_{i=1}^{n} z_i^2 - 2\sum_{i=1}^{n} \lambda_i z_i + \left(\sum_{i=1}^{n} \lambda_i\right)^2,$$

and

$$\sum_{i=1}^{n} w_i^3 = \sum_{i=1}^{n} z_i^3 - 3\sum_{i=1}^{n} \lambda_i z_i^2 + 3\left(\sum_{i=1}^{n} \lambda_i\right)\sum_{j=1}^{n} \lambda_j z_j - \left(\sum_{i=1}^{n} \lambda_i\right)^3.$$

Proof. The three equalities follows from $\sum_{i=1}^{n} w_i^r = tr(D - \Lambda J)^r$, $r = 1, 2, 3$.

■

Case b : $deg(p) = deg(q) + 1$

Corollary 3.7 *Let p be a monic polynomial of degree n with zeros $z_1, \dots, z_n$. Suppose q is a monic polynomial of degree $n-1$ given by*

$$\frac{q(z)}{p(z)} = \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}.$$

If $w_1, \dots, w_{n-1}$ are zeros of q, then

$$\sum_{i=1}^{n} |w_i|^2 \leq \sum_{i=1}^{n} (1 - 2Re\lambda_i)|z_i|^2 + (\sum_{i=1}^{n} |\lambda_i z_i|)^2.$$

Proof. It follows from equation (2) and Corollary 3.3. ■

Corollary 3.8 *Let p be a monic polynomial of degree n with zeros $z_1, \dots, z_n$. Suppose q is a monic polynomial of degree $n-1$ given by*

$$\frac{q(z)}{p(z)} = \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}.$$

Then for any zeros w of q, there exists z_k and z_l such that

$$|w - z_k - \lambda_k z_k| \leq \sum_{j \neq k} |\lambda_j z_j|$$

and

$$|w - z_l - \lambda_l z_l| \leq (n-1)|\lambda_l z_l|.$$

Proof. It follows from equation (2) and Corollary 3.4. ■

By applying Corollary 3.8 for polynomials $p(z+a)$ and $q(z+a)$ where a is any complex number, we obtain the next minmax-maxmin inequality.

Corollary 3.9 *Let p be a monic polynomial of degree n with zeros $z_1, \ldots, z_n$. Suppose q is a monic polynomial of degree $n-1$ given by*

$$\frac{q(z)}{p(z)} = \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}.$$

If $w_1, \ldots, w_{n-1}$ are zeros of q, then

$$\max_{1 \le i \le n-1} \min_{1 \le k \le n} |w_i - z_k| \le \min_{a \in \mathbb{C}} \sum_{j=1}^{n} |\lambda_j| \, |z_j - a| \le n \max_{1 \le j \le n} |\lambda_j| \min_{a \in \mathbb{C}} \max_{1 \le k \le n} |z_j - a| \, .$$

Again, we have the relations involving the sum of powers of q and the zeros of p and we list the first three below.

Corollary 3.10 *Let p be a monic polynomial of degree n with zeros $z_1, \ldots, z_n$. Suppose q is a monic polynomial given by*

$$\frac{q(z)}{p(z)} = \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}.$$

If $w_1, \ldots, w_{n-1}$ are zeros of q, then

$$\sum_{i=1}^{n-1} w_i = \sum_{i=1}^{n} (1 - \lambda_i) z_i,$$

$$\sum_{i=1}^{n-1} w_i^2 = \sum_{i=1}^{n} (1 - 2\lambda_i) z_i^2 + \left(\sum_{i=1}^{n} \lambda_i z_i \right)^2,$$

and

$$\sum_{i=1}^{n-1} w_i^3 = \sum_{i=1}^{n} (1 - 3\lambda_i) z_i^3 + 3 \left(\sum_{i=1}^{n} \lambda_i z_i \right) \sum_{j=1}^{n} \lambda_j z_j^2 - \left(\sum_{i=1}^{n} \lambda_i z_i \right)^3.$$

4 Distribution of equilibrium points

So far we make no restrictions on λ_i. In this section, we will mainly consider real λ_i in order to study the distribution of equilibrium points of logarithmic potentials for finitely many discrete charges. In fact, when λ_i are positive real numbers, the zeros of functions of the form

$$f(z) = \sum_{j=1}^{n} \frac{a_k}{z - z_j}, \quad a_j > 0, z_j \in \mathbb{C}$$

are called equilibrium points of logarithmic potential U generated by the charged particles with charges $a_j > 0$ at z_j where

$$U(z) = \sum_{j=1}^{n} a_j \log |1 - \frac{z}{z_j}|.$$

The critical points of U (which are the zeros of f) coincide with equilibrium points of electrostatic field. It would therefore be interesting to locate the zeros of f in terms of the poles z_j. The following results are well-known (see [11, p.76] and [4, p.134]) but we shall give a matrix theoretical proof here.

Corollary 4.1 *Let p be a monic polynomial of degree n with zeros $z_1, \ldots, z_n$. Suppose q is a monic polynomial given by*

$$\frac{q(z)}{p(z)} = \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}.$$

If $\lambda_1, \ldots, \lambda_n \geq 0$, then the zeros of q are located inside the convex hull of the zeros of p. If furthermore, $z_1 \geq z_2 \geq \cdots \geq z_n$ and the zeros of q are $w_1 \geq w_2 \geq \cdots \geq w_{n-1}$, then $z_1 \geq w_1 \geq z_2 \geq w_2 \geq \cdots \geq w_{n-1} \geq z_n$.

Proof. We first recall that the eigenvalues of a matrix lie inside the numerical range of a matrix, and the numerical range of a normal matrix is exactly the convex hull of its eigenvalues. Now suppose that $z_i = 0$ for some i. Let $\mathbf{e}$ be the all one vector and

$$V = \begin{pmatrix} I - \Lambda^{1/2} J \Lambda^{1/2} & \Lambda^{1/2}\mathbf{e} \\ \mathbf{e}^T \Lambda^{1/2} & 0 \end{pmatrix}.$$

We have $V^2 = I$ and that V is Hermitian and hence unitary. Since $zq(z)$ is the characteristic polynomial of $D(I - \Lambda)$ which is similar to $(I - \Lambda^{1/2} J \Lambda^{1/2}) D (I - \Lambda^{1/2} J \Lambda^{1/2})$, a principal submatrix of the normal matrix $V(D \oplus 0)V$, we have the zeros of $zq(z)$ lying inside the numerical range of $D \oplus 0$ which is the convex hull of the zeros of p. Furthermore if the zeros of p are real, then $V(D \oplus 0)V$ is Hermitian and the interlacing property holds (see [6, p.185]).

For general p and q, we have the zeros of $q(z+z_1)$ lying inside the convex hull of the zeros of $p(z+z_1)$, thus the conclusion follows. ■

Finally, we have

Corollary 4.2 *Let A be an $n \times n$ matrix with characteristic polynomial $p(z) = \Pi_{j=1}^n (z - z_i)$ and $q(z)$ be a monic polynomial of degree n given by*

$$\frac{q(z)}{p(z)} = 1 + \sum_{j=1}^n \frac{\lambda_j}{z - z_j}.$$

We have

(i) If there exists a zero v of q such that $\frac{\lambda_j}{z_i - v} \geq 0$ for all j, then the other zeros of q are located inside the convex hull of the zeros of p.

(ii) If $z_1 \geq \cdots \geq z_n$ and $\lambda_1, \ldots, \lambda_n$ are all nonnegative or all nonpositive, then the zeros of q are also real. Furthermore, suppose that $w_1 \geq \cdots \geq w_n$, if λ_j's are nonnegative then $w_1 \geq z_1 \geq w_2 \geq z_2 \geq \cdots \geq w_n \geq z_n$ and if λ_j's are nonpositive then $z_1 \geq w_1 \geq z_2 \geq w_2 \geq \cdots \geq z_n \geq w_n$.

Proof.

(i) Without loss of generality, we may assume that $v = 0$.Then $q(z) = zq_1(z)$ and $\sum_{j=1}^n \frac{\lambda_j}{z_j} = 1$. Hence we have

$$\frac{q_1(z)}{p(z)} = \sum_{j=1}^n \frac{\lambda_j}{z_j(z - z_j)}.$$

By Corollary 4.1, the zeros of q_1 are located inside the convex hull of the zeros of p.

(ii) Suppose λ_j's are nonnegative. For z from z_n down to $-\infty$, we have $\frac{q}{p}$ increasing from $-\infty$ to 1, and hence there exists a zero $v < z_n$ of q. Thus $\frac{\lambda_j}{z_j - v} \geq 0$ for all j. Apply part (i) and Corollary 4.1 again. The case that all λ_j's are nonpositive is similar.

■

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