Second fundamental forms of holomorphic isometries of the Poincaré disk into bounded symmetric domains and their boundary behavior along the unit circle

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Motivated by problems arising from Arithmetic Geometry in Clozel-Ullmo [CU], in Mok [Mk3] one of the authors studied germs of holomorphic isometries between bounded domains with respect to the Bergman metric. In general, under assumptions that the Bergman kernels are rational functions, which is in particular the case for bounded symmetric domains, the graph of any germ of holomorphic isometry $f : (D, x_0) \to (\Omega, y_0)$ extends to an irreducible affine-algebraic variety $V \supset \mathrm{Graph}(f)$, and $V \cap (D \times \Omega)$ is the graph of a proper holomorphic isometric embedding whenever both domains D and Ω are complete with respect to the Bergman metric. These extension results apply especially in the case where $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ are bounded symmetric domains in their Harish-Chandra realizations. While holomorphic isometries are necessarily totally geodesic whenever D is irreducible and of rank ≥ 2 due to Hermitian metric rigidity [Mk1], in [Mk3] examples of non-standard holomorphic isometric embeddings from the Poincaré disk into the polydisk were constructed, and they arise primarily from the p-th root map $\rho_p : \mathcal{H} \to \mathcal{H}^p$, given by $f(\tau) = \left(\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \cdots, \gamma^{p-1}\tau^{\frac{1}{p}}\right)$, where $\gamma = e^{\frac{\pi i}{p}}$. In the same article an example of a non-standard holomorphic isometric embedding was also constructed from the upper half-plane to the Siegel upper half-plane $\mathcal{H}_3$ of genus 3.

In Mok [Mk2] we studied the asymptotic behavior of holomorphic isometries of the Poincaré disk into bounded symmetric domains. Denoting by σ the second fundamental form of the holomorphic isometry, we showed that $\varphi = \|\sigma\|^2$ extends across a general boundary point $b \in \partial\Delta$ of the unit circle as a real-analytic function. Among other things we checked that any holomorphic isometry of the Poincaré disk into the polydisk must be asymptotically geodesic at a general boundary point $b \in \partial\Delta$, and that furthermore the vanishing order of $\varphi = \|\sigma\|^2$ is exactly equal to 2. A variant of the example into the Siegel upper half-plane mentioned in the above was examined, and the asymptotic behavior of the second fundamental form of the map exhibit similarities with non-standard holomorphic isometries into polydisks as discussed in the above.

Because of the existence of non-standard holomorphic isometries of the Poincaré disk into certain bounded symmetric domains, it is natural to consider the question of

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classification of such maps. (In this direction the second author [Ng1] considered the question of classification of holomorphic isometries of the Poincaré disk into the polydisk Δ^p, and rigidity theorems were obtained for $p = 2, 3$ and for certain extremal cases.) A related and less difficult question is to characterize known examples of such isometries. For such problems the second fundamental form of non-standard holomorphic isometries of the Poincaré disk are expected to play an important role. Beyond Mok [Mk2] the current article is an attempt to study boundary behavior of holomorphic isometries of the Poincaré disk into bounded symmetric domains in terms of the second fundamental form. The purpose of our article is two-fold. First of all we wish to examine second fundamental forms of known examples of holomorphic isometries of the Poincaré disk. For this purpose we will compute explicitly $\varphi = \|\sigma\|^2$ both for the p-th root maps into a Cartesian product of upper half-planes (equivalently a polydisk), and also for the mentioned example in the case of the Siegel upper half-plane. Secondly, on the theoretical side we wish to study further asymptotic properties of the second fundamental form. In this article for theoretical issues we will focus on the special case where the target space is a polydisk, for which the computations are much easier to handle. In that case we know unconditionally that a non-standard holomorphic isometry of the Poincaré disk into a polydisk is necessarily of the first kind, viz., expressing via the Cayley transform in terms of the Euclidean coordinate $\tau = s + it$ on the upper half-plane $\mathcal{H}$, for $\varphi(\tau) = \|\sigma\|^2$ we have $\varphi(\tau) = t^2 u(\tau)$, where $u\big|_{t=0} \not\equiv 0$. We show that u must satisfy the first order differential equation $\dfrac{\partial u}{\partial t}\Big|_{t=0} \equiv 0$ on the real axis outside a finite number of points at which u is singular. Equivalently, knowing that the non-standard holomorphic isometry is of the first kind, the differential equation is a condition on the third-order jet of $\varphi(\tau)$ at a general point of the real axis given by $\dfrac{\partial^3 \varphi}{\partial t^3}\Big|_{t=0} \equiv 0$, an equation which is invariant under $\mathrm{SL}(2, \mathbb{R})$ for functions φ vanishing to the order ≥ 2 on the real axis at general points.

In the presentation of the article we will rather start in §1 with the theoretical aspect. To start with, we will recall the result showing that the vanishing order of $\varphi = \|\sigma\|^2$ at a general boundary point is at most 2, giving a streamlining of the proof and slightly generalizing the result. Then, we consider the special case of holomorphic isometries of the Poincaré disk into polydisks, verifying $\dfrac{\partial u}{\partial t}\Big|_{t=0} \equiv 0$ for $\varphi(\tau) = t^2 u(\tau)$ mentioned in the above. This is established using the very special property that the extension of Graph(f) beyond the unit circle gives actually holomorphic isometries outside of the unit circle, when we equip the exterior of the unit circle in $\mathbb{P}^1$ also with the Poincaré metric. In §2 we will compute explicitly $\varphi = \|\sigma\|^2$ both for the p-th map into a Cartesian product of the upper half-plane, and also for the example into $\mathcal{H}_3$. Such computations will then be related to the theoretical discussion in §1. As an example of applications of the computations, we give a proof that the image of the afore-mentioned map into $\mathcal{H}_3$ cannot be contained in a totally geodesic polydisk by showing that the corresponding function $\varphi = \|\sigma\|^2$ fails to satisfy the boundary differential equation necessary for the image of a holomorphic isometry to be contained in a maximal polydisk.

With an intention to construct further examples of holomorphic isometries of the Poincaré disk into $\mathcal{H}_3$, we write down in (2.2) a continuous family of holomorphic maps from $\mathcal{H}$ into $\mathcal{H}_3$ which are shown to be holomorphic isometries by a comparison of potentials for Kähler metrics very much in the spirit of Mok [Mk3]. It turns out somewhat unexpectedly that members of this family are equivalent to each other under symplectic transformations on $\mathcal{H}_3$. For the purpose of computation of $\varphi = \|\sigma\|^2 = t^2u$ it suffices to consider any member in the family, and the explicit computation shows, as mentioned above, that this map can be distinguished from holomorphic isometries of the Poincaré disk into polydisks on the basis of the differences exhibited by $\left.\frac{\partial u}{\partial t}\right|_{t=0}$. Another by-product of our discussion is the proof of the fact that any non-standard holomorphic isometry of the Poincaré disk into a polydisk must develop singularities along the unit circle. We have incorporated the latter result into the current article since its proof is based on the same principle underlying the proof of the identity $\left.\frac{\partial u}{\partial t}\right|_{t=0} \equiv 0$ for any such map, viz., based on the fact that the extended map beyond the unit circle remains an isometry. The question whether singularities must develop for non-standard holomorphic isometries of the Poincaré disk into arbitrary bounded symmetric domains remains unanswered.

Toward the end of the article we formulate problems motivated both by the theoretical results and by the computational results in the article. These include the characterization of the p-th root map among holomorphic isometric embeddings of the Poincaré disk into polydisks in terms of partial differential equations satisfied by $\varphi = \|\sigma\|^2$, the characterization of embeddings with images contained in maximal polydisks among holomorphic isometric embeddings of the Poincaré disk into bounded symmetric domains, and the problem of classifying holomorphic isometric embeddings of the Poincaré disk into bounded symmetric domains of rank 2. The computational results in the article can be taken to be experimental in nature. It is however hoped that the combination of theoretical and computational aspects on holomorphic isometries of the Poincaré disk into bounded symmetric domains can serve to motivate further studies on such mappings especially in terms of partial differential equations or inequalities satisfied by quantities arising from the second fundamental form.

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versity in honor of Professor Tongde Zhong on the occasion of his 80th birthday. He wishes to thank the organizers, especially Professor Chunhui Qiu, for their kind invitation. The current article is on a topic closely related to the theme of the talk at the Conference, and the authors wish to dedicate the article to Professor Zhong to celebrate his many contributions to the field of Several Complex Variables in China.

§1 The second fundamental form of holomorphic isometries of the Poincaré disk into bounded symmetric domains

(1.1) *Asymptotic behavior of the second fundamental form for holomorphic isometries into bounded symmetric domains* In what follows we equip the unit disk Δ with the Bergman metric $ds^2_\Delta = \frac{4\mathrm{Re}(dz\otimes d\overline{z})}{(1-|z|^2)^2}$, which is equivalently the Poincaré metric of constant curvature -1. Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization. We denote by g_Ω the Bergman metric on Ω. (In [Mk2] we considered only bounded symmetric domains Ω which are either irreducible or a Cartesian product of the same irreducible bounded symmetric domain, and we denote by ds^2_Ω an invariant Kähler metric which is equal to a multiple of the Bergman metric with the normalization that a minimal disk is of constant holomorphic curvature equal to -1.) Let λ be any positive real number and $f : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, g_\Omega)$ be a holomorphic isometry. In Mok [Mk3] we proved that f is a proper holomorphic isometric embedding and that $\mathrm{Graph}(f) \subset \Delta \times \Omega$ extends to an affine-algebraic subvariety $V \subset \mathbb{C} \times \mathbb{C}^N$. We state here a slight strengthening of [Mk2, (1.1), Theorem 1] by allowing $\Omega \Subset \mathbb{C}^N$ to be an arbitrary bounded symmetric domain in its Harish-Chandra realization. Although the following slightly strengthened statement follows readily from an adaptation of the the proof given in [Mk2, *loc. cit.*], we will give here a proof assuming only the curvature formula for the second fundamental form analogous to [Mk2, (1.2), Eqn.(5)] by a streamlined deduction of the statement on vanishing orders of $\varphi = \|\sigma\|^2$ which is applicable in the general case of an arbitrary bounded symmetric domain Ω.

Theorem 1. *Let λ be a positive real number and let $f : (\Delta, \lambda ds^2_\Delta) \to (\Omega, g_\Omega)$ be a holomorphic isometry of the Poincaré disk into a bounded symmetric domain Ω. Suppose f is not totally geodesic and it is asymptotically geodesic at a general boundary point. Then, the length of the second fundamental form $\|\sigma\|$ must vanish to the order* 1 *or* $\frac{1}{2}$ *at a general boundary point b of Δ,* i.e., *at a general point of the unit circle $S^1 = \partial\Delta$. In other words, the real-analytic function φ defined on a neighborhood of b must vanish either to the order* 2 resp. 1.

Proof. Identify Δ with the upper half-plane $\mathcal{H}$. We consider f as a map from $\mathcal{H}$ into Ω and write $S = f(\mathcal{H}) \subset \Omega$. In what follows, all the equations or inequalities are evaluated at a point $x \in S$ and α is the coordinate vector field with respect to a complex geodesic coordinate at x and $\|\alpha(x)\| = 1$.

For $\tau \in \mathcal{H}$, we write $\tau = s + it$. We recall Mok [Mk2, (1.2), Eqn.(5)].

$$\frac{2\lambda}{t^2} R_{\mu\overline{\mu}\alpha\overline{\alpha}} - \frac{\partial^2 \varphi}{\partial \tau \partial \overline{\tau}} = \frac{\lambda}{t^2} \left(R(R_{\alpha\overline{\alpha}\alpha}, \overline{\alpha}; \alpha, \overline{\alpha}) - \frac{1}{\lambda^2} + \frac{\varphi}{\lambda} \right). \tag{1}$$

Let g be the (Riemannian) metric at x, first note that

$$\begin{aligned} R(R_{\alpha\overline{\alpha}\alpha}, \overline{\alpha}; \alpha, \overline{\alpha}) &= -R(\overline{\alpha}, R_{\alpha\overline{\alpha}\alpha}; \alpha, \overline{\alpha}) = -g\left(R_{\alpha\overline{\alpha}\,\overline{\alpha}}, R_{\alpha\overline{\alpha}\alpha}\right) \\ &= -g\left(\overline{R_{\overline{\alpha}\alpha\alpha}}, R_{\alpha\overline{\alpha}\alpha}\right) = g\left(\overline{R_{\alpha\overline{\alpha}\alpha}}, R_{\alpha\overline{\alpha}\alpha}\right) = \|R_{\alpha\overline{\alpha}\alpha}\|^2. \end{aligned} \tag{2}$$

Now as $\|\alpha(x)\| = 1$, we have

$$\|R_{\alpha\overline{\alpha}\alpha}\|^2 \geq |g\left(R_{\alpha\overline{\alpha}\alpha}, \overline{\alpha}\right)|^2 = |R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}|^2. \tag{3}$$

Therefore,

$$\begin{aligned} \frac{2\lambda}{t^2} R_{\mu\overline{\mu}\alpha\overline{\alpha}} - \frac{\partial^2 \varphi}{\partial \tau \partial \overline{\tau}} &\geq \frac{\lambda}{t^2} \left(|R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}|^2 - \frac{1}{\lambda^2} + \frac{\varphi}{\lambda} \right) \\ = \frac{\lambda}{t^2} \left(\left(-\frac{1}{\lambda} + \varphi \right)^2 - \frac{1}{\lambda^2} + \frac{\varphi}{\lambda} \right) &= \frac{\lambda}{t^2} \left(-\frac{\varphi}{\lambda} + \varphi^2 \right). \end{aligned} \tag{4}$$

$$\Longleftrightarrow \frac{2\lambda}{t^2} R_{\mu\overline{\mu}\alpha\overline{\alpha}} \geq \frac{\partial^2 \varphi}{\partial \tau \partial \overline{\tau}} - \frac{\varphi}{t^2} + \frac{\lambda \varphi^2}{t^2}. \tag{5}$$

As f is assumed to be asymptotically totally geodesic and φ is real-analytic on a neighborhood of a general boundary point $b \in \partial\mathcal{H}$ (cf. [Mk2, (1.1), Proposition 1]), we can write $\varphi = t^q u$, where q is a positive integer, and $u\big|_{t=0} \not\equiv 0$. We have

$$\frac{\partial^2 \varphi}{\partial \tau \partial \overline{\tau}} = \frac{1}{4} \left((q(q-1) t^{q-2} u + 2q t^{q-1} \frac{\partial u}{\partial t} + t^q \frac{\partial^2 u}{\partial t^2} + t^q \frac{\partial^2 u}{\partial s^2} \right) \tag{6}$$

and

$$\frac{\varphi}{t^2} = t^{q-2} u. \tag{7}$$

Hence,

$$\frac{2\lambda}{t^2} R_{\mu\overline{\mu}\alpha\overline{\alpha}} \geq \frac{1}{4}(q^2 - q - 4) t^{q-2} u + \frac{1}{4} \left(2q t^{q-1} \frac{\partial u}{\partial t} + t^q \frac{\partial^2 u}{\partial t^2} + t^q \frac{\partial^2 u}{\partial s^2} \right) + \frac{\lambda \varphi^2}{t^2}. \tag{8}$$

Since $(q^2 - q - 4) > 0$ for $q > \dfrac{1 + \sqrt{17}}{2} \approx 2.6$ and we have nonpositive bisectional curvature in the target, the above inequality cannot hold true for arbitrarily small t unless $q = 1, 2$. $\square$

(1.2) *Asymptotic behavior of the second fundamental form for holomorphic isometries into polydisks and an associated differential equation on the boundary* We study in this article holomorphic isometries of the Poincaré disk into bounded symmetric domains. By Mok ([Mk3, Theorem]) they are necessarily proper holomorphic isometric embeddings. Such a holomorphic isometry is said to be non-standard if and only if it is not totally geodesic. It is said to be asymptotically totally geodesic at a general boundary point if and only if the following holds true. For any point $b \in \partial\Delta$ outside of a finite set of points, the norm of the second fundamental form $\|\sigma(z)\|$ of the embedding converges to 0 as z approaches b. We have the following result for holomorphic isometries from the Poincaré disk into a Cartesian product of complex unit balls.

Theorem (Mok [Mk2, Theorem 2]). *On the unit disk Δ denote by ds^2_Δ the Poincaré metric of constant Gaussian curvature -1. In general, for $n \geq 1$ on the unit ball B^n denote by $ds^2_{B^n}$ the canonical Kähler-Einstein metric of constant holomorphic sectional curvature -1. Let $\lambda > 0$ and $f : (\Delta, \lambda ds^2_\Delta) \to (B^n, ds^2_{B^n}) \times \cdots \times (B^n, ds^2_{B^n})$ be a non-standard holomorphic isometry. Then, F is asymptotically totally geodesic at a general boundary point $b \in \partial\mathcal{H}$. Moreover, F is a holomorphic isometric embedding of the first kind of the Poincaré disk.*

We will apply the above result to the special case where the target space is the polydisk. In this case we have $f : (\Delta, k\, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$, where $ds^2_{\Delta^p}$ denotes the Bergman metric on Δ^p. We identify the unit disk Δ with the upper half-plane via the Cayley transform. In terms of the Euclidean coordinate $\tau = s + it; s = \mathrm{Re}(\tau), t = \mathrm{Im}(\tau)$ on $\mathcal{H}$, writing $\varphi = \|\sigma\|^2$ as in the above at a general point $b \in \partial\mathcal{H}$ we have $\varphi(\tau) = t^2 u(\tau)$ on some neighborhood of b in the τ-plane $\mathbb{C}$. Our main result for non-standard holomorphic isometries of the Poincaré disk into polydisks is the following result giving an equation satisfied by the the first order jet of u along ∂H.

Theorem 2. *Let $p \geq 2$ be an integer, k be a positive integer, and $f : (\Delta, k\, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ be a non-standard holomorphic isometry, necessarily of the first kind. Write $S := f(\Delta) \subset \Delta^p$, and denote by φ the square of the norm of the second fundamental form σ of $\left(S, ds^2_{\Delta^p}\big|_S\right) \hookrightarrow (\Delta^p, ds^2_{\Delta^p})$ as a Kähler submanifold. Identify Δ with the upper half-plane $\mathcal{H}$, with Euclidean coordinate τ, via a Cayley transform, and write $\varphi(f(\tau)) = \|\sigma(f(\tau))\|^2 = t^2 u(\tau)$, where $t = \mathrm{Im}(\tau)$. Then, we have $\dfrac{\partial u}{\partial t} = 0$ over $U_b \cap \partial\mathcal{H}$ for an open neighborhood U_b of a general point $b \in \partial\mathcal{H}$.*

We start with some discussion in preparation for the proof of Theorem 2 and to put the statement of the theorem in perspective. First of all, while *a priori* the conclusion in Theorem 2 depends on the choice of a conformal equivalence between the unit disk and the upper half-plane, we show in the following lemma that in fact it does not. In other words, the statement in the conclusion of Theorem 2 is invariant under fractional linear transformations on the upper half-plane.

Lemma 1. *Let $\mathcal{H} = \{\tau : Im(\tau) > 0\}$ be the upper half-plane, $p \in \partial\mathcal{H}$, and φ be a real-analytic function defined on a neighborhood U_p of p in $\mathbb{C}$. Write $\tau = s + it$, $s, t \in \mathbb{R}$. Let $\tau = \Psi(z) = \dfrac{az+b}{cz+d}$ be a fractional linear transformation, where $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ and $q := \Psi^{-1}(p) \neq \infty$. Write $z = x + iy$; $x, y \in \mathbb{R}$. Suppose φ vanishes to the order ≥ 2 on $\partial\mathcal{H} \cap U_p$. Then $\dfrac{\partial^3\varphi}{\partial y^3}(q) = \left(\dfrac{\partial t}{\partial y}(q)\right)^3 \dfrac{\partial^3\varphi}{\partial t^3}(p)$. Equivalently, writing $\varphi = t^2 u$ on a neighborhood of p and $\varphi = y^2 v$ in a neighborhood of q we have $\dfrac{\partial v}{\partial y}(q) = \left(\dfrac{\partial t}{\partial y}(q)\right)^3 \dfrac{\partial u}{\partial t}(p)$. In particular, $\dfrac{\partial v}{\partial y}(q) = 0$ if and only if $\dfrac{\partial u}{\partial t}(p) = 0$.*

Proof. By assumption φ vanishes to the order ≥ 2 on $U_p \cap \partial\mathcal{H}$ for a neighborhood U_p of $p \in \partial H$. Since φ is real-analytic on U_p, we have $\varphi = t^2 u$ for some real-analytic function u on U_p. For $\tau = \Psi(z)$, the imaginary part $t = \mathrm{Im}(\tau)$ can be regarded as a real-analytic function $t(z) = \mathrm{Im}(\Psi(z))$ vanishing along $U_p \cap \partial\mathcal{H}$ exactly to the order 1. We have $\varphi = y^2 v$, where v can be considered as a real-analytic function on the neighborhood $\widetilde{U}_q$ of q in the z-plane $\mathbb{C}$ or by composition with Ψ as a real-analytic function on the neighborhood U_p of p in the τ-plane $\mathbb{C}$. From $\varphi = t^2 u = y^2 v$, it follows that $\dfrac{\partial^3\varphi}{\partial t^3} = 2\dfrac{\partial u}{\partial t}$ on $U_p \cap \partial H$ and $\dfrac{\partial^3\varphi}{\partial y^3} = 2\dfrac{\partial v}{\partial y}$ on $\widetilde{U}_q \cap \partial H$. To prove the lemma, it suffices to show that $\dfrac{\partial v}{\partial y} = \left(\dfrac{\partial t}{\partial y}\right)^3 \dfrac{\partial u}{\partial t}$ on $\widetilde{U}_q \cap \partial H$. From $t^2 u = y^2 v$, taking partial derivatives against y we have

$$2t\frac{\partial t}{\partial y}u + t^2\left(\frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\right) = 2yv + y^2\frac{\partial v}{\partial y}\,. \tag{1}$$

Taking partial derivatives against y again we have

$$2\left(\frac{\partial t}{\partial y}\right)^2 u + 2t\frac{\partial^2 t}{\partial y^2}u + 4t\frac{\partial t}{\partial y}\left(\frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\right) + O(t^2) = 2v + 4y\frac{\partial v}{\partial y} + y^2\frac{\partial^2 v}{\partial y^2}\,. \tag{2}$$

Here and henceforth for an integer $k \geq 1$, $O(t^k)$ stands for a function of the form $t^k\xi$ in which ξ is a real-analytic function on U_p, or equivalently of the form $y^2\eta$ in which η is a real-analytic function on $\widetilde{U}_q$, when we make a change of variable $z = \Psi(\tau)$.

We proceed to equate terms of order ≤ 1 in t (and equivalently in y) on both sides of (2). By the Cauchy-Riemann equations we have $\dfrac{\partial s}{\partial y} = -\dfrac{\partial t}{\partial x}$, giving $\left.\dfrac{\partial s}{\partial y}\right|_{y=0} \equiv 0$. On the other hand, $\left.\dfrac{\partial^2 t}{\partial y^2}\right|_{y=0} = -\left.\dfrac{\partial^2 t}{\partial x^2}\right|_{y=0} = 0$ since $t = \mathrm{Im}(\tau)$ is harmonic on $\widetilde{U}_q$ and $t = 0$

on U_p whenever $y = 0$ on $\widetilde{U}_q$. It follows from (2) that

$$2\left(\frac{\partial t}{\partial y}\right)^2 u + 4t\left(\frac{\partial t}{\partial y}\right)^2 \frac{\partial u}{\partial t} + O(t^2) = 2v + 4y\frac{\partial v}{\partial y}\,, \tag{3}$$

where the last term on the right-hand side of (2) is replaced by $O(y^2) = O(t^2)$ and hence absorbed by the term $O(t^2)$ on the left-hand side of (3). Writing on $\widetilde{U}_q$

$$t = \frac{\partial t}{\partial y}(x,0)y + \frac{\partial^2 t}{\partial y^2}(x,0)y^2 + O(t^3) = \frac{\partial t}{\partial y}(x,0)y + O(t^3)\,, \tag{4}$$

it follows from (4) and the equality $t^2u = y^2v$ that

$$2\left(\frac{\partial t}{\partial y}\right)^2 u - 2v = 2\left(\frac{t+O(t^3)}{y}\right)^2 u - 2v = \frac{2(t^2u - y^2v) + O(t^4)}{y^2} = O(t^2)\,, \tag{5}$$

and hence from (3) and (4) we conclude that

$$\left.\frac{\partial v}{\partial y}\right|_{y=0} = \left(\frac{\partial t}{\partial y}\right)^3 \left.\frac{\partial u}{\partial t}\right|_{t=0}, \tag{6}$$

as desired. The proof of Lemma 1 is complete. $\square$

In the study of holomorphic isometries of the Poincaré disk into bounded symmetric domains, the case where the target spaces are polydisks is distinguished by the fact that, properly interpreted, algebraic extensions of holomorphic isometries actually give also holomorphic isometries outside of the unit disk. More precisely, we have

Proposition 1. *Let $\Delta \subset \mathbb{C} \subset \mathbb{P}^1$ be the unit disk, p and k be positive integers, $f : (\Delta, k\, ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$ be a holomorphic isometry, and $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$ be the irreducible projective-algebraic variety which contains* Graph(f) *as an open subset. Equip $\mathcal{O} := \mathbb{P}^1 - \overline{\Delta}$ with the Hermitian metric $ds_{\mathcal{O}}^2$ which on $\mathbb{C}^1 - \overline{\Delta}$ is given by $ds_{\mathcal{O}}^2 = \frac{4\mathrm{Re}(dz\otimes d\overline{z})}{(|z|^2-1)^2}$. Let G be any connected component of $(\mathbb{P}^1 - \partial\Delta) \times (\mathbb{P}^1 - \partial\Delta)^p$, and write $G = W \times G'$, where W is either Δ or $\mathcal{O} = \mathbb{P}^1 - \overline{\Delta}$ and G' is a Cartesian product of p domains each of which is either Δ or $\mathcal{O}$. Equip G' with the Kähler metric $ds_{G'}^2$ which is the product metric of the Hermitian metrics ds_Δ^2 resp. $ds_{\mathcal{O}}^2$ for each Cartesian factor equal to Δ resp. $\mathcal{O}$. Then, either $V \cap G = \emptyset$ or $V \cap G \subset G$ is a complex submanifold which is the graph of a holomorphic isometric embedding $f_G : (\Delta, k\, ds_\Delta^2) \to (G', ds_{G'}^2)$.*

We note that, in terms of the holomorphic coordinate $w = \frac{1}{z}$ the Hermitian metric is given by

$$ds_{\mathcal{O}}^2 = \frac{4\mathrm{Re}\,(dw \otimes d\overline{w})}{|w|^4\left(\left|\frac{1}{w}\right|^2 - 1\right)^2} = \frac{4\mathrm{Re}\,(dw \otimes d\overline{w})}{(1-|w|^2)^2}\,,$$

which is the Poincaré metric of constant Gaussian curvature -1 on D, i.e., the unit disk in the w-coordinate. The Kähler form $\omega_{\mathcal{O}}$ of $ds^2_{\mathcal{O}}$ is given by

$$\omega_{\mathcal{O}} = -2\sqrt{-1}\partial\overline{\partial}\log(1-|w|^2)\,.$$

Proof of Proposition 1. By Clozel-Ullmo [CU, (2.2.1), Eqn.(9)], the holomorphic isometry $f:(\Delta, k\,ds^2_{\Delta}) \to (\Delta^p, ds^2_{\Delta^p})$ satisfies the functional identity

$$(\dagger)_1 \qquad \prod_{i=1}^{p}\left(1-|f_i(z)|^2\right) = (1-|z|^2)^k\,.$$

In what follows let $W \subset \mathbb{C}$ denote either the unit disk Δ or the complement of the closed unit disk, i.e., $\mathbb{C}-\overline{\Delta}$. For a general point z_0 on W and for a branch $f=(f_1,\cdots,f_p)$ on a neighborhood D of z_0 in W such that each f_i is a holomorphic function on D, it follows from $(\dagger)_1$ that $|f_i(z)| \neq 1$ on D. Thus, the set of indexes $\{1,\cdots,p\}$ divides into two subsets, one consisting of those i such that $|f_i(z)|<1$ on D, and the other consisting of those i for which $|f_i(z)|>1$ on D. Without loss of generality we assume that the former situation occurs for $1\le i\le s$ and the latter occurs for $s+1\le i\le p$. Rewrite $(\dagger)_1$ further in the form

$$(\dagger)_2 \qquad \left(\prod_{i=1}^{s}\left(1-|f_i(z)|^2\right)\right)\left(\prod_{i=s+1}^{p}\left(1-|f_i(z)|^2\right)\right) = (1-|z|^2)^k\,.$$

Suppose $z\in\Delta$. We rewrite $(\dagger)_2$ in the form

$$(\dagger)_3 \qquad \left(\prod_{i=1}^{s}\left(1-|f_i(z)|^2\right)\right)\left(\prod_{i=s+1}^{p}\left(|f_i(z)|^2\Big(1-\Big|\frac{1}{f_i(z)}\Big|^2\Big)\right)\right) = (1-|z|^2)^k\,.$$

Applying the Poincaré-Lelong operator to -2 times of the logarithms of both sides and observing that each f_i is a nowhere zero holomorphic function, we conclude that

$$\sum_{i=1}^{s} f_i^{\star}\omega_{\Delta} + \sum_{i=s+1}^{p} f_i^{\star}\omega_{\mathcal{O}} = k\,\omega_{\Delta}\,, \tag{1}$$

which says precisely that $f_G:\big(D, k\,ds^2_{\Delta}|_D\big) \to (G', ds^2_{G'})$ is a holomorphic isometry. In the case of a connected component $G=\mathcal{O}\times G'$, we rewrite the right-hand side of $(\dagger)_3$ as

$$(1-|z|^2)^k = |z|^k\left(1-\Big|\frac{1}{z}\Big|^2\right)^k\,, \tag{2}$$

and, noting that z is holomorphic and nowhere vanishing on $\mathbb{C}-\overline{\Delta} = \mathcal{O}-\{\infty\}$, exactly the same argument shows that

$$\sum_{i=1}^{s} f_i^{\star}\omega_{\Delta} + \sum_{i=s+1}^{p} f_i^{\star}\omega_{\mathcal{O}} = k\,\omega_{\mathcal{O}}\,, \tag{3}$$

over $\mathcal{O}-\{\infty\}$ and hence everywhere on $\mathcal{O}$. The proof of Proposition 1 is complete. □

We are now ready to give a proof of Theorem 2.

Proof of Theorem 2. Recall in the statement of Theorem 2 that $f : (\Delta, k\, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ is a non-standard holomorphic isometry, necessarily of the first kind. The mapping f is necessarily a proper embedding by Mok [Mk3]. Since f is asymptotically geodesic at a general point $b \in \partial\Delta$ the isometric constant k must be an integer in the range $1 \le k \le p$. One can rule out the case $k = p$. In fact in this case we have $ds^2_{\Delta^p} = p\, ds^2_\Delta$, and f has to be a totally geodesic embedding by the equality case of the Ahlfors-Schwarz Lemma. Write $S := f(\Delta) \subset \Delta^p$ for the image of $f : \Delta \to \Delta^p$ and σ for the second fundamental form of $S := f(\Delta) \subset \Delta^p$, identifying Δ with the upper half-plane $\mathcal{H}$ with Euclidean coordinate $\tau = s + it$, and choosing $b \in \partial\mathcal{H}$ for a general point, we have $\varphi = \|\sigma\|^2 = t^2 u$ on a neighborhood U_b of b on the τ-plane $\mathbb{C}$. Denoting at $x \in S$ by α a unit vector of type (1,0) at $x \in S$, by μ the lifting of $\sigma(\alpha, \alpha)$ to $T_x(\Delta^p)$ by orthogonal lifting. In the case $p = 2$ we have $k = 1$ and by Mok [Mk2, (1.3), Eqn.(12)] we have the curvature formula

$$R_{\mu\overline{\mu}\alpha\overline{\alpha}} = \frac{t^2}{2}\Big(\frac{\partial^2\varphi}{\partial\tau\partial\overline{\tau}} - \frac{\varphi}{2t^2}\Big) = \frac{1}{8}\big(q(q-1) - 2\big)t^q u + \frac{qt^{q+1}}{4}\,\frac{\partial u}{\partial t} + \frac{t^{q+2}}{8}\Big(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial s^2}\Big). \tag{1}$$

We note that (1) is valid for holomorphic isometries into an irreducible bounded symmetric domain of rank 2 or a Cartesian product of identical bounded symmetric domains when we use the normalization that the Gaussian curvature of a minimal disk is -1. The identity (1) is based on the same identity as stated in [(1.1), Eqn.(1)] of the current article, with algebraic simplifications in the rank-2 case. Although the case of holomorphic isometries of the Poincaré disk into the bidisk is completely classified by Ng [Ng1], we will ignore the classification. In fact the argument below will be applied to the general case of holomorphic isometries into polydisks. By [(1.1), Theorem 1], $q = 2$ in the case of a holomorphic isometry into the polydisk. Hence, we deduce from (1) that

$$R_{\mu\overline{\mu}\alpha\overline{\alpha}} = \frac{t^3}{2}\frac{\partial u}{\partial t} + \frac{t^4}{8}\Big(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial s^2}\Big)\,. \tag{2}$$

For a general point $b \in \partial\mathcal{H}$, there is a neighborhood U_b on the τ-plane such that $f|_{U_b \cap \mathcal{H}}$ extends holomorphically to U_b. Denoting the extended map on U_b again by f, by [(1.2), Proposition 1] the holomorphic map $f|_{U_b - \overline{\mathcal{H}}}$ is again a holomorphic isometry with image lying in some connected component G' of $(\mathbb{P}^1 - \partial\Delta)^p$. Representing the left-hand side on U_b as the quotient of two real-analytic functions the formal expression $R_{\mu\overline{\mu}\alpha\overline{\alpha}}$ at a point of $f(U_b)$ over a point $\tau \in U_b - \overline{\mathcal{H}}$ is geometrically the same as the curvature term arising from holomorphic map $f|_{U_b - \overline{\mathcal{H}}} \to G'$, which is a holomorphic isometry with respect to the Poincaré metric on $\mathbb{P}^1 - \overline{\Delta}$ and with respect to the Bergman metric on $G' \cong \Delta^p$. It follows that the left-hand side must remain nonnegative on $U_b - \overline{\mathcal{H}}$. Without loss of generality we may choose U_b convex so that $U_b \cap \partial H$ is connected. We would have a contradiction to (2) if $\frac{\partial u}{\partial t}\Big|_{U_b \cap \partial\mathcal{H}} \not\equiv 0$, since the right-hand side would be of the form $t^3 \frac{\partial u}{\partial t}(s,0) + O(t^4)$ which changes sign as one crosses a general point of $U_b \cap \partial\mathcal{H}$. Thus, by argument by contradiction we have proved that $\frac{\partial u}{\partial t}\Big|_{U_b \cap \partial\mathcal{H}} \equiv 0$, in the case where $p = 2$.

In the general case of a polydisk Δ^p, again basing on the identity in [(1.1), Theorem 1, Eqn.(1)] we have a curvature formula analogous to (2) with error terms, as follows. We recall the discussion in Mok [Mk2, (1.4)] on principal directions on the image of a holomorphic isometry into an irreducible bounded symmetric domain or a Cartesian product of identical irreducible bounded symmetric domains, under the normalization that the Gaussian curvature of the minimal disk is -1. This will then be applied to the special case where the target space is the polydisk equipped with the Bergman metric.

Denote the target bounded symmetric domain by Ω and write $p := \mathrm{rank}(\Omega)$. Choosing a general point $b \in \partial\mathcal{H}$ where f extends holomorphically to a neighborhood U_b of b in the τ-plane and it is asymptotically totally geodesic at b, the following discussion applies to $\tau \in U_b$ sufficiently close to b. At $\tau \in \mathcal{H}$ for a unit vector α of type (1,0) tangent to $S = f(\Delta)$ at $f(\tau)$, we write

$$\alpha = \frac{1}{\sqrt{k}} \sum_{i=1}^{p} \beta_i e_i \ , \quad \beta_1 \geq \beta_2 \geq \cdots \geq \beta_p \geq 0 \ , \tag{3}$$

where $\beta_i = 1 + \gamma_i$ for $1 \leq i \leq k$ and $\beta_j = \delta_{j-k}$ for $k+1 \leq j \leq p$. We call $\{e_i\}_{i=1}^p$ a set of principal directions. For $1 \leq i \leq k$, $\lim_{\tau \to b} \beta_i = 1$, and we call e_i, $1 \leq i \leq k$, a stretching principal direction. For $k+1 \leq j \leq p$, $\lim_{\tau \to b} \delta_j = 0$ and we call e_j a contracting principal direction. For each integer $\ell \geq 1$ we denote by $\Gamma_\ell := |\gamma_1|^\ell + \cdots + |\gamma_k|^\ell$, $\Delta_\ell := |\delta_1|^\ell + \cdots + |\delta_{r-k}|^\ell$. By Mok [Mk2, (1.4), Eqns.(20) & (21)]

$$R_{\mu\overline{\mu}\alpha\overline{\alpha}} = \frac{t^3}{2k}\frac{\partial u}{\partial t} + \frac{t^4}{8k}\Big(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial s^2}\Big) + \frac{3}{k^3}\Gamma_2 - \frac{3}{4k^3}\Delta^4 + O(\Gamma_3) + O(\Delta_6) \ . \tag{4}$$

In the special case where $\Omega = \Delta^p$, the calculations in Mok [Mk2, (2.1), Eqns.(4) & (5)] give $\|\beta_i\|^2 = 1 + a_i t^2 + O(t^3)$ for $1 \leq i \leq k$ and $\|\beta_j\|^2 = c_j t^2 + O(t^3)$ $k+1 \leq j \leq p$ as $\tau \to b$, for some constants a_i and b_j. It follows that

$$\gamma_i(\tau) = O(t^2) \ ; \qquad \delta_j(\tau) = O(t) \tag{5}$$

as $\tau \to b$ for $1 \leq i \leq k$ and $k+1 \leq j \leq p$. Hence we deduce from (4) the formula

$$R_{\mu\overline{\mu}\alpha\overline{\alpha}} = \frac{t^3}{2k}\frac{\partial u}{\partial t} + \frac{t^4}{8k}\Big(\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial s^2}\Big) + O(t^4) \tag{6}$$

The argument for the special case of $p = 2$ basing on (2) now applies *verbatim* to show that $\frac{\partial u}{\partial t}\Big|_{U_b \cap \partial \mathcal{H}} \equiv 0$, as desired. The proof of Theorem 2 is complete. $\square$

(1.3) *Existence of singularities of holomorphic isometries of the Poincaré disk into polydisks* It is expected that holomorphic isometries of the Poincaré metric into a bounded symmetric domain must develop singularities along the unit circle unless the mappings are totally geodesic (cf. Mok [Mk2]). We give here a proof of this in the special case where the target space is a polydisk. The proof is included in the current article as it relies on [(1.2), Proposition 1]. We have

Theorem 3. *Let $f : (\Delta, \lambda ds^2_\Delta; 0) \to (\Delta^p, ds^2_\Delta; 0)$ be a germ of holomorphic isometry. Suppose f extends holomorphically to some neighborhood of $\overline{\Delta}$. Then, f is totally geodesic.*

Proof. By Mok [Mk3, Theorem], $\mathrm{Graph}(f) \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$ extends holomorphically to a projective-algebraic subvariety $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$, and λ is a positive integer k, $1 \le k \le p$. Each local branch $f = (f_1, \cdots, f_p)$ of f satisfies the real-analytic functional identity

$$(\dagger) \qquad \prod_{i=1}^{p} \frac{1}{1 - |f_i(z)|^2} = \frac{1}{(1 - |z|^2)^k} \ .$$

Now by assumption f extends holomorphically to some neighborhood U of $\overline{\Delta}$. In what follows, we will write $f = (f_1, \cdots, f_p) : U \to \mathbb{C}$ and regard the domain of definition of each f_i to be exactly U. For each i, $1 \le i \le p$, the function $h(z) = |f_i(z)|^2$ is real-analytic on U. If $|f_i(z)|^2 = 1$ for z belonging to a non-empty open subset of the unit circle $\partial\Delta$, by the Identity Theorem for real-analytic functions we must have $|f_i(z)| = 1$ for any $z \in \partial\Delta$. It follows from ($\dagger$) that exactly k of the component functions satisfy the latter property. Without loss of generality we assume that $f_i(\partial\Delta) \subset \partial\Delta$ exactly for $i = 1, \cdots, k$. For $k + 1 \le i \le p$ we observe that $|f_i(z)| \ne 1$ whenever $z \in \partial\Delta$. In fact, for such an index i either $f_i \equiv 0$, or by the Open Mapping Theorem $f_i(U)$ is open. In the latter case suppose $|f_i(b)| = 1$ for some $b \in \partial\Delta$, then $f_i^{-1}(\partial\Delta)$ must contain some nonempty smooth real-analytic curve. However, by the functional identity ($\dagger$) we must have $|f_i(z)| \ne 1$ whenever $z \notin \partial\Delta$, and it follows that $|f_i(z)| = 1$ for z belonging to some non-empty open subset of $\partial\Delta$, contradicting with the choice of f_i.

Consider now any base point $x_0 \in U - \overline{\Delta}$. The holomorphic map f defines at x_0 a germ of holomorphic map into $G := \mathcal{O}^k \times \Delta^{p-k}$, where $\mathcal{O} := \mathbb{P}^1 - \overline{\Delta}$. By Proposition, the germ of f at x_0 is a germ of holomorphic isometry when we equip $\mathcal{O}$ with the Poincaré metric $ds^2_{\mathcal{O}} = \frac{4\mathrm{Re}(dz \otimes d\overline{z})}{(1 - |z|^2)^2}$ and G with the product metric for the i-th Cartesian factor, $1 \le i \le p$, is equal to $\frac{4\mathrm{Re}(dz^i \otimes d\overline{z^i})}{(1 - |z_i|^2)^2}$, irrespective of whether $1 \le i \le k$ (in which case

$|f_i(x_0)| > 1$) or $k+1 \le i \le p$ (in which case $|f_i(x_0)| < 1$). Now by Mok [Mk3, Theorem], the holomorphic isometry $f : (\mathcal{O}, k\, ds^2_{\mathcal{O}}; x_0) \to (G, ds^2_G; f(x_0))$ extends to a holomorphic isometric embedding f_∞ from $\mathcal{O}$ into G, so that pasting the graph of f over U and the graph of $f^\infty = (f_1^\infty, \cdots, f_p^\infty)$ over $\mathcal{O}$ we obtain the graph of a holomorphic map $F : \mathbb{P}^1 \to (\mathbb{P}^1)^p$, $F = (F_1, \cdots, F_p)$. However, for $k+1 \le i \le p$ we have $f(\overline{\Delta}) \subset \Delta$, and also $f_i^\infty(\mathcal{O}) \subset \Delta$, so that $F_i : \mathbb{P}^1 \to \Delta$, which contradicts the Maximum Principle unless F_i is a constant functions, hence $F_i \equiv 0$. It follows now $(f_1, \cdots, f_k) : \Delta \to \Delta^k$ is a holomorphic isometric with isometric constant k. On the other hand over Δ by the Schwarz Lemma we have

$$f^* ds_{\Delta^k} = f_1^* ds^2_\Delta + \cdots + f_k^* ds^2_\Delta \le k\, ds^2_\Delta \,,$$

and equality holds if and only if each component map f_i, $1 \le i \le k$, is a *bona fide* holomorphic isometry. It follows therefore that $f = (f_1, \cdots, f_k; 0, \cdots, 0) : \Delta \to \Delta^p$ is a totally geodesic holomorphic embedding, as desired. $\square$

§2 Computational results on second fundamental forms of examples of holomorphic isometries of the Poincaré disk into bounded symmetric domains

(2.1) *Second fundamental forms of the p-th root maps* In this section we study specifically holomorphic isometries of the Poincaré disk into polydisks equipped with the Bergman metric $ds^2_{\Delta^p}$. In this case $f^* ds^2_{\Delta^p} = k\, ds^2_\Delta$, where the isometric constant k is necessarily a positive integer $1 \le k \le p$, and f is necessarily a proper holomorphic isometric embedding. Let k be such a positive integer and $f : (\Delta, k\, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ be a proper holomorphic isometric embedding. By the Proper Mapping Theorem $S := f(\Delta) \subset \Delta^p$ is a complex submanifold. Denote by $\sigma(z)$ the second fundamental form of S in the polydisk Δ^p. Denoting by $\alpha(z) \in T_{f(z)}(S)$ a unit tangent vector of type (1,0), we have by the Gauss equation

$$\|\sigma(z)\|^2 = \frac{1}{k} + R\big(\alpha(z), \overline{\alpha(z)}; \alpha(z), \overline{\alpha(z)}\big) \,. \qquad (1)$$

From now on we will drop the reference to $z \in \Delta$. Thus, writing $f = (f_1, \cdots, f_p)$ we have

$$\|\sigma\|^2 = \frac{1}{k} + R_{\alpha\overline{\alpha}\alpha\overline{\alpha}} = \|\sigma\|^2 = \frac{1}{k} - \frac{1}{k^2} \sum_{i=1}^{p} \left| \frac{df_i}{dz} \right|^4 \left(\frac{1 - |z|^2}{1 - |f_i|^2} \right)^4 . \qquad (2)$$

Identify the unit disk Δ with the upper half-plane $\mathcal{H}$ by the Cayley transform, and equip the latter with the Poincaré metric $ds^2_{\mathcal{H}}$ of constant curvature -1 given by $ds^2_{\mathcal{H}} = \frac{\mathrm{Re}(d\tau \otimes d\overline{\tau})}{(\mathrm{Im}\,\tau)^2}$. Writing $ds^2_{\mathcal{H}^p}$ for the product metric on the Cartesian product $\mathcal{H}^p$, we will also write $f(\tau) = (f_1(\tau), \ldots, f_p(\tau))$ for a holomorphic isometry $f : (\mathcal{H}, k\, ds^2_{\mathcal{H}}) \to$

$(\mathcal{H}^p, ds^2_{\mathcal{H}^p})$. In terms of coordinates on the upper half-plane we have

$$\|\sigma\|^2 = \frac{1}{k} - \frac{1}{k^2}\sum_{i=1}^{p}\left|\frac{df_i}{d\tau}\right|^4 \left(\frac{\mathrm{Im}\,(\tau)}{\mathrm{Im}\,(f_i)}\right)^4 . \tag{3}$$

In [Mk3, (3.2)], Mok has constructed a non standard holomorphic isometric embedding of $\mathcal{H}$ into the $\mathcal{H}^p$ with $k = 1$ for each $p \geq 2$, called the $p-$th root map, given by $f(\tau) = (\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \ldots, \gamma^{p-1}\tau^{\frac{1}{p}})$, where $\gamma = e^{\frac{i\pi}{p}}$. We are going to compute the second fundamental form σ_p of the $p-$th root map using the formula (3). We have

$$\|\sigma\|^2 = \frac{1}{k} - \frac{1}{k^2}\sum_{i=1}^{p}\left|\frac{df_i}{d\tau}\right|^4 \left(\frac{\mathrm{Im}\,\tau}{\mathrm{Im}\,f_i}\right)^4 . \tag{4}$$

When $k = 1$, we have

$$\|\sigma\|^2 = \sum_{i\neq j}\left|\frac{df_i}{d\tau}\right|^2 \left|\frac{df_j}{d\tau}\right|^2 \left(\frac{\mathrm{Im}\,\tau}{\mathrm{Im} f_i}\right)^2 \left(\frac{\mathrm{Im}\,\tau}{\mathrm{Im}\,f_j}\right)^2 . \tag{5}$$

For the p-th root map, we have $f(\tau) = (\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \ldots, \gamma^{p-1}\tau^{\frac{1}{p}})$. If $\tau = |\tau|e^{i\theta}$ and $\gamma = e^{\frac{\pi i}{p}}$, then $\mathrm{Im} f_j = |\tau|^{\frac{1}{p}}\sin\theta_j$, where $\theta_j = \frac{(j-1)\pi}{p} + \frac{\theta}{p}$

$$\begin{aligned}
\|\sigma_p\|^2 = & \sum_{i\neq j}\frac{1}{p^4}\left|\frac{1}{\tau^{\frac{p-1}{p}}}\right|^4 \left(\frac{|\tau|^2\sin^2\theta}{|\tau|^{\frac{2}{p}}\sin\theta_i\sin\theta_j}\right)^2 \\
= & \frac{\sin^4\theta}{p^4}\sum_{i\neq j}\frac{1}{\sin^2\theta_i\sin^2\theta_j} .
\end{aligned} \tag{6}$$

Recall $\theta_j = \dfrac{(j-1)\pi}{p} + \dfrac{\theta}{p}$, $1 \leq j \leq 2p$, are the $2p$ solutions of the equation $\sin^2(p\lambda) = \sin^2\theta$ in λ, or equivalently, $\cos(2p\lambda) = \cos 2\theta$. By expanding $\cos(2p\lambda)$, we rewrite the equation as

$$\begin{aligned}
\cos 2\theta = & (-1)^p(\sin\lambda)^{2p} + (-1)^{p-1}\binom{2p}{2}(1-\sin^2\lambda)(\sin\lambda)^{2p-2} + \cdots \\
& - \binom{2p}{2p-2}(1-\sin^2\lambda)^{p-1}(\sin\lambda)^2 + (1-\sin^2\lambda)^p .
\end{aligned} \tag{7}$$

Let $y = \dfrac{1}{\sin^2\lambda}$, then we have

$$y^p\cos 2\theta = (-1)^p + (-1)^{p-1}\binom{2p}{2}(y-1) + \cdots - \binom{2p}{2p-2}(y-1)^{p-1} + (y-1)^p . \tag{8}$$

Now, $y_j = \dfrac{1}{\sin^2\theta_j}$, $1 \le j \le p$, are the p solutions of the above equation. Therefore

$$
\begin{aligned}
&\sum_{i \neq j} \frac{1}{\sin^2\theta_i \sin^2\theta_j} \\
&= 2 \cdot \frac{1}{1-\cos 2\theta}\left[\binom{2p}{2p-4} + (p-1)\binom{2p}{2p-2} + \frac{p(p-1)}{2}\right] \\
&= \frac{1}{\sin^2\theta}\left[\frac{2p(2p-1)(2p-2)(2p-3)}{4!} + (p-1)\frac{2p(2p-1)}{2} + \frac{p(p-1)}{2}\right] \\
&= \frac{1}{\sin^2\theta}\left[\frac{2}{3}p^2(p^2-1)\right] . \qquad (9)
\end{aligned}
$$

Finally, we have

$$\|\sigma_p\|^2 = \frac{2(p^2-1)}{3p^2}\sin^2\theta \, . \qquad (10)$$

We conclude our discussion on second fundamental forms of p-th root maps with the following proposition.

Theorem 4. *Let p be a positive integer. Denote by $ds^2_{\mathcal{H}}$ the Poincaré metric on the upper half-plane $\mathcal{H}$ of constant Gaussian curvature -1 and correspondingly by $ds^2_{\mathcal{H}_p}$ the product metric on the Cartesian product $\mathcal{H}^p$ of p copies of the upper half-plane. Write $\rho_p : (\mathcal{H}, ds^2_{\mathcal{H}}) \to (\mathcal{H}_p, ds^2_{\mathcal{H}_p})$ for the p-th root map given by $\rho_p(\tau) = \left(\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \ldots, \gamma^{p-1}\tau^{\frac{1}{p}}\right)$, where $\gamma = e^{\frac{\pi i}{p}}$. Then, the second fundamental form σ_p of ρ_p is given by $\|\sigma_p\|^2 = \frac{2(p^2-1)}{3p^2}\sin^2\theta$. In particular, $-\log\|\sigma_p\|^2$ is a potential function for $ds^2_{\mathcal{H}}$. In other words, denoting by $\omega_{\mathcal{H}}$ stands for the Kähler form of $ds^2_{\mathcal{H}}$, we have $\sqrt{-1}\partial\overline{\partial}(-\log\|\sigma_p\|^2) = \omega_{\mathcal{H}}$.*

Proof. From the discussion preceding the statement of the Proposition it remains to prove that $\sqrt{-1}\partial\overline{\partial}(-2\log(\sin\theta)) = \omega_{\mathcal{H}}$. Writing $\tau = s + it$ with $s = \mathrm{Re}(\tau); t = \mathrm{Im}(\tau)$ and $r := \sqrt{s^2+t^2}$ we have $\omega_{\mathcal{H}} = \sqrt{-1}\partial\overline{\partial}(-2\log t)$. On the other hand

$$\log(\sin\theta) = \log\left(\frac{t}{r}\right) = \log t - \log r \, . \qquad (1)$$

Since $2\log r = \log(s^2 + t^2 = \log|\tau|^2$ is harmonic, we conclude that

$$\sqrt{-1}\partial\overline{\partial}(-2\log(\sin\theta)) = \sqrt{-1}\partial\overline{\partial}(-2\log t) - \sqrt{-1}\partial\overline{\partial}(-2\log r) = \omega_{\mathcal{H}} \, , \qquad (2)$$

as desired, and the proof of Proposition is complete $\square$

(2.2) *On holomorphic isometries into the Siegel upper half-plane* For an integer $g \ge 1$ let $\mathcal{H}_g = \{g\text{-by-}g$ matrices τ with complex coefficients : $\tau^t = \tau; \mathrm{Im}\,\tau > 0\}$ be the Siegel upper half-plane of genus g, and denote by $ds^2_{\mathcal{H}_g}$ the Bergman metric on $\mathcal{H}_g$. In Mok

[Mk3, (3.3)] we constructed an example of a proper holomorphic isometric embedding $G : \left(\mathcal{H}, 2ds^2_{\mathcal{H}}\right) \to \left(\mathcal{H}_3, ds^2_{\mathcal{H}_3}\right)$, given by

$$G(\tau) = \begin{bmatrix} e^{\frac{\pi i}{6}}\tau^{\frac{2}{3}} & \sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}} & 0 \\ \sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}} & i & 0 \\ 0 & 0 & e^{\frac{\pi i}{3}}\tau^{\frac{1}{3}} \end{bmatrix}$$

In this section, proceeding along the arguments of Mok [Mk3] of verifying algebraic identities for potential functions, we exhibit a continuous family of holomorphic isometric embeddings of $\mathcal{H}$ into $\mathcal{H}_3$ which contains the embedding $G : \mathcal{H} \to \mathcal{H}_3$ given in the above. As it turn out, all the embeddings of this family are equivalent to one and other up to symplectic transformations, i.e., up to automorphisms of $\mathcal{H}_3$. We have

Proposition 2. *Let $\mu, \nu \in \mathbb{R}$ such that $\mu^2 + \nu^2 \neq 0$. Define $\omega = \mu + i\nu$ and $\zeta = \left[\frac{\sqrt{3}}{2}(\mu^2 + \nu^2) + \mu\nu\right] + i\nu^2$. Then*

$$H(\tau) = \begin{bmatrix} \zeta\tau^{\frac{2}{3}} & \omega\tau^{\frac{1}{3}} & 0 \\ \omega\tau^{\frac{1}{3}} & i & 0 \\ 0 & 0 & e^{\frac{i\pi}{3}}\tau^{\frac{1}{3}} \end{bmatrix}$$

is a holomorphic isometric embedding of $\mathcal{H}$ into $\mathcal{H}_3$.

Proof. Let $\tau^{\frac{1}{3}} = \alpha + i\beta$. Then $\tau^{\frac{2}{3}} = (\alpha^2 - \beta^2) + 2\alpha\beta i$.

$$\mathrm{Im}\,(H(\tau)) = \begin{bmatrix} \nu^2(\alpha^2 - \beta^2) + \alpha\beta[\sqrt{3}(\mu^2 + \nu^2) + 2\mu\nu] & \nu\alpha + \mu\beta & 0 \\ \nu\alpha + \mu\beta & 1 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2}\alpha + \frac{\beta}{2} \end{bmatrix}. \quad (1)$$

$$\det(\mathrm{Im}\,(H(\tau))) = \left[\nu^2(\alpha^2 - \beta^2) + \alpha\beta[\sqrt{3}(\mu^2 + \nu^2) + 2\mu\nu] - (\nu\alpha + \mu\beta)^2\right]\left(\frac{\sqrt{3}}{2}\alpha + \frac{\beta}{2}\right)$$
$$= (\mu^2 + \nu^2)(\sqrt{3}\alpha\beta - \beta^2)\left(\frac{\sqrt{3}}{2}\alpha + \frac{\beta}{2}\right) = \frac{\mu^2 + \nu^2}{2}(3\alpha^2\beta - \beta^3) = \frac{\mu^2 + \nu^2}{2}\mathrm{Im}\,(\tau)\,. \quad (2)$$

We can similarly check the positivity of the trace of the first 2×2 block of $\mathrm{Im}(H(\tau))$ and hence $\mathrm{Im}(H(\tau)) > 0$. Therefore H maps $\mathcal{H}$ into $\mathcal{H}_3$.

We are going to show that the above embeddings are equivalent. Recall the automorphisms of $\mathcal{H}_3$ are given by $Z \mapsto (AZ+B)(CZ+D)^{-1}$, $Z \in \mathcal{H}_3$, where $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is a 6-by-6 real symplectic matrix, i.e. M satisfies $M^T J M = J$, where $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$.

By considering the automorphisms given by $A = D^{-T} = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = C = 0$, where $a > 0$, we just need to focus on those embeddings with $\mu^2 + \nu^2 = 1$, i.e. $|\omega| = 1$.

When $\mu = 1$, $\nu = 0$, we write

$$H_{1,0} = \begin{bmatrix} \frac{\sqrt{3}}{2}\tau^{\frac{3}{2}} & \tau^{\frac{1}{3}} & 0 \\ \tau^{\frac{1}{3}} & i & 0 \\ 0 & 0 & e^{\frac{i\pi}{3}}\tau^{\frac{1}{3}} \end{bmatrix}. \tag{3}$$

Now for $\mu^2 + \nu^2 = 1$, consider the automorphism given by $A = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = -C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 0 \end{bmatrix}$. By direct calculation, we see that $(AH_{1,0}+B)(CH_{1,0}+D)^{-1} = H_{\mu,\nu}$, where

$$H_{\mu,\nu} = \begin{bmatrix} \zeta\tau^{\frac{2}{3}} & \omega\tau^{\frac{1}{3}} & 0 \\ \omega\tau^{\frac{1}{3}} & i & 0 \\ 0 & 0 & e^{\frac{i\pi}{3}}\tau^{\frac{1}{3}} \end{bmatrix} \tag{4}$$

and $\omega = \mu + i\nu$, $\zeta = \left[\frac{\sqrt{3}}{2} + \mu\nu\right] + i\nu^2$. Therefore all embeddings in the family are equivalent. $\square$

(2.3) *Computation of the second fundamental form in the Siegel case* In (2.2), we have constructed a family of holomorphic isometric embeddings $H_{\mu,\nu}$. The example of Mok [Mk3, (3.3)] is given by $G = H_{\frac{\sqrt{6}}{2},-\frac{\sqrt{2}}{2}}$. By [(2.2), Proposition 2] all members of the family $H_{\mu,\nu}$ are equivalent. For the computation of second fundamental forms we will choose instead the embedding corresponding to $F = H_{1,0}$ given by

$$F(\tau) = \begin{bmatrix} \frac{\sqrt{3}}{2}\tau^{\frac{2}{3}} & \tau^{\frac{1}{3}} & 0 \\ \tau^{\frac{1}{3}} & i & 0 \\ 0 & 0 & \gamma\tau^{\frac{1}{3}} \end{bmatrix}, \tag{1}$$

where $\gamma = e^{\frac{i\pi}{3}}$. We will compute the second fundamental form σ by the Gauss equation, for instance, see Mok [Mk2], basing on curvature formulas from Siegel [Si]. We have

$$\|\sigma\|^2 = 1 - \frac{\mathrm{Tr}(E\overline{E}E\overline{E})}{\left[\mathrm{Tr}(E\overline{E})\right]^2}, \tag{2}$$

where $E = F'(\operatorname{Im} F)^{-1}$.

$$F' = \frac{1}{3\tau^{\frac{2}{3}}} \begin{bmatrix} \sqrt{3}\tau^{\frac{1}{3}} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \gamma \end{bmatrix}; \tag{3}$$

$$\operatorname{Im} F = \frac{1}{2i} \begin{bmatrix} \frac{\sqrt{3}}{2}\left(\tau^{\frac{2}{3}} - \overline{\tau^{\frac{2}{3}}}\right) & \tau^{\frac{1}{3}} - \overline{\tau^{\frac{1}{3}}} & 0 \\ \tau^{\frac{1}{3}} - \overline{\tau^{\frac{1}{3}}} & 2i & 0 \\ 0 & 0 & \gamma\tau^{\frac{1}{3}} - \overline{\gamma}\overline{\tau^{\frac{1}{3}}} \end{bmatrix}; \tag{4}$$

$$(\operatorname{Im} F)^{-1} = \frac{2i}{\Delta} \begin{bmatrix} 2i & \overline{\tau^{\frac{1}{3}}} - \tau^{\frac{1}{3}} & 0 \\ \overline{\tau^{\frac{1}{3}}} - \tau^{\frac{1}{3}} & \frac{\sqrt{3}}{2}\left(\tau^{\frac{1}{3}} - \overline{\tau^{\frac{1}{3}}}\right) & 0 \\ 0 & 0 & \frac{\Delta}{\gamma\tau^{\frac{1}{3}} - \overline{\gamma}\overline{\tau^{\frac{1}{3}}}} \end{bmatrix}; \tag{5}$$

where

$$\Delta = \sqrt{3}i\left(\tau^{\frac{2}{3}} - \overline{\tau^{\frac{2}{3}}}\right) - \left(\tau^{\frac{1}{3}} - \overline{\tau^{\frac{1}{3}}}\right)^2 = 2\left(|\tau|^{\frac{2}{3}} - \overline{\gamma}\tau^{\frac{2}{3}} - \gamma\overline{\tau^{\frac{2}{3}}}\right). \tag{6}$$

Write

$$F' = \frac{1}{3\tau^{\frac{2}{3}}} \begin{bmatrix} \begin{matrix} & \\ & M \\ & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \gamma \end{bmatrix} \quad \text{and} \quad (\operatorname{Im} F)^{-1} = \frac{2i}{\Delta} \begin{bmatrix} N & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \frac{\Delta}{\gamma\tau^{\frac{1}{3}} - \overline{\gamma}\overline{\tau^{\frac{1}{3}}}} \end{bmatrix}. \tag{7}$$

We then have

$$E = \frac{2i}{3\tau^{\frac{2}{3}}\Delta} \begin{bmatrix} MN & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \zeta \end{bmatrix}, \tag{8}$$

where $\zeta = \dfrac{\Delta\gamma}{\gamma\tau^{\frac{1}{3}} - \overline{\gamma}\overline{\tau^{\frac{1}{3}}}}$ and

$$\frac{\operatorname{Tr}(E\overline{E}E\overline{E})}{\left[\operatorname{Tr}(E\overline{E})\right]^2} = \frac{\operatorname{Tr}(MN\overline{MN}MN\overline{MN}) + |\zeta|^4}{\left[\operatorname{Tr}(MN\overline{MN}) + |\zeta|^2\right]^2} = \frac{\left|\gamma\tau^{\frac{1}{3}} - \overline{\gamma}\overline{\tau^{\frac{1}{3}}}\right|^4 \operatorname{Tr}(MN\overline{MN}MN\overline{MN}) + |\Delta|^4}{\left[\left|\gamma\tau^{\frac{1}{3}} - \overline{\gamma}\overline{\tau^{\frac{1}{3}}}\right|^2 \operatorname{Tr}(MN\overline{MN}) + |\Delta|^2\right]^2}. \tag{9}$$

Now,

$$MN = \begin{bmatrix} \sqrt{3}\tau^{\frac{1}{3}} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2i & \overline{\tau^{\frac{1}{3}}} - \tau^{\frac{1}{3}} \\ \overline{\tau^{\frac{1}{3}}} - \tau^{\frac{1}{3}} & \frac{\sqrt{3}}{2}\left(\tau^{\frac{1}{3}} - \overline{\tau^{\frac{1}{3}}}\right) \end{bmatrix} = \begin{bmatrix} (-1+2\sqrt{3}i)\tau^{\frac{1}{3}} + \overline{\tau^{\frac{1}{3}}} & \frac{\sqrt{3}}{2}\left(2|\tau|^{\frac{2}{3}} - \tau^{\frac{2}{3}} - \overline{\tau^{\frac{2}{3}}}\right) \\ 2i & \overline{\tau^{\frac{1}{3}}} - \tau^{\frac{1}{3}} \end{bmatrix} \tag{10}$$

and

$$MN\overline{MN} =: \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} |\tau|^{\frac{2}{3}}(14-2\sqrt{3}i)+\tau^{\frac{2}{3}}(-1+3\sqrt{3}i)+\overline{\tau^{\frac{2}{3}}}(-1-\sqrt{3}i) & 6i|\tau|^{\frac{2}{3}}\tau^{\frac{1}{3}}-3i|\tau|^{\frac{2}{3}}\overline{\tau^{\frac{1}{3}}}-3i\tau \\ \tau^{\frac{1}{3}}(4i)+\overline{\tau^{\frac{1}{3}}}(4\sqrt{3}-4i) & |\tau|^{\frac{2}{3}}(2+2\sqrt{3}i)+\tau^{\frac{2}{3}}(-1-i\sqrt{3})+\overline{\tau^{\frac{2}{3}}}(-1-i\sqrt{3}) \end{bmatrix}. \tag{11}$$

Then,

$$\operatorname{Tr}(MN\overline{MN}) = a + d = 4\left(4|\tau|^{\frac{2}{3}} - \overline{\gamma}\tau^{\frac{2}{3}} - \gamma\overline{\tau^{\frac{2}{3}}}\right), \tag{12}$$

and

$$
\begin{aligned}
\operatorname{Tr}(MN\overline{M}\overline{N}MN\overline{M}\overline{N}) &= a^2 + 2bc + d^2 \\
&= 8\left[33|\tau|^{\frac{4}{3}} - \gamma\tau^{\frac{4}{3}} - \overline{\gamma}\overline{\tau^{\frac{4}{3}}} - 14|\tau|^{\frac{2}{3}}(\overline{\gamma}\tau^{\frac{2}{3}} + \gamma\overline{\tau^{\frac{2}{3}}})\right] .
\end{aligned} \tag{13}
$$

And hence,

$$
\begin{aligned}
\frac{\operatorname{Tr}(E\overline{E}E\overline{E})}{\left[\operatorname{Tr}(E\overline{E})\right]^2} &= \frac{\left|\gamma\tau^{\frac{1}{3}} - \overline{\gamma}\overline{\tau^{\frac{1}{3}}}\right|^4 \operatorname{Tr}(MN\overline{M}\overline{N}MN\overline{M}\overline{N}) + |\Delta|^4}{\left[\left|\gamma\tau^{\frac{1}{3}} - \overline{\gamma}\overline{\tau^{\frac{1}{3}}}\right|^2 \operatorname{Tr}(MN\overline{M}\overline{N}) + |\Delta|^2\right]^2} \\
= \frac{8\left[126|\tau|^{\frac{8}{3}} - 6|\tau|^{\frac{4}{3}}\operatorname{Re}(\gamma\tau^{\frac{4}{3}}) + 12|\tau|^{\frac{6}{3}}\operatorname{Re}(\overline{\gamma}\tau^{\frac{2}{3}}) + 36|\tau|^{\frac{2}{3}}\operatorname{Re}(\tau^{\frac{6}{3}}) - 6\operatorname{Re}(\overline{\gamma}\tau^{\frac{8}{3}})\right]}{(36|\tau|^{\frac{4}{3}})^2} .
\end{aligned} \tag{14}
$$

If we let $\dfrac{\tau^{\frac{1}{3}}}{\overline{\tau^{\frac{1}{3}}}} = e^{i\varphi}$, then we have

$$
\frac{\operatorname{Tr}(E\overline{E}E\overline{E})}{\left[\operatorname{Tr}(E\overline{E})\right]^2} = \frac{7}{9} + \frac{2}{27}\cos\left(\varphi - \frac{\pi}{3}\right) - \frac{1}{27}\cos\left(2\varphi + \frac{\pi}{3}\right) + \frac{2}{9}\cos 3\varphi - \frac{1}{27}\cos\left(4\varphi - \frac{\pi}{3}\right) , \tag{15}
$$

and

$$
\begin{aligned}
\|\sigma\|^2 &= 1 - \frac{\operatorname{Tr}(E\overline{E}E\overline{E})}{\left[\operatorname{Tr}(E\overline{E})\right]^2} \\
&= \frac{2}{27}\left[3 - \cos\left(\varphi - \frac{\pi}{3}\right) + \frac{1}{2}\cos\left(2\varphi + \frac{\pi}{3}\right) - 3\cos 3\varphi + \frac{1}{2}\cos\left(4\varphi - \frac{\pi}{3}\right)\right] \\
&= \frac{2}{27}\left[3 - \cos\left(\varphi - \frac{\pi}{3}\right) - 3\cos 3\varphi + \cos 3\varphi \cos\left(\varphi - \frac{\pi}{3}\right)\right] \\
&= \frac{2}{27}\left[1 - \cos 3\varphi\right]\left[3 - \cos\left(\varphi - \frac{\pi}{3}\right)\right] .
\end{aligned} \tag{16}
$$

For $\tau \in \mathcal{H}$, if we write $\tau = |\tau|e^{i\theta}$, $0 < \theta < \pi$, then $\dfrac{2\theta}{3} = \varphi$ and therefore

$$
\begin{aligned}
\|\sigma\|^2 &= \frac{2}{27}\left[1 - \cos 2\theta\right]\left[3 - \cos\left(\frac{2\theta}{3} - \frac{\pi}{3}\right)\right] \\
&= \frac{4}{27}\sin^2\theta\left[3 - \cos\left(\frac{2\theta}{3} - \frac{\pi}{3}\right)\right] .
\end{aligned} \tag{17}
$$

From the preceding computation we can contrast the boundary values of the square of the norm of the second fundamental from for holomorphic isometries of the Poincaré disk into polydisks and for the example above of such a holomorphic isometry into a Siegel upper half-plane. More specifically, we have

Proposition 3. *$F : \left(\mathcal{H}, 2ds^2_{\mathcal{H}}\right) \to \left(\mathcal{H}_3, ds^2_{\mathcal{H}_3}\right)$ be the holomorphic isometry of the Poincaré disk into the Siegel upper half-plane $\mathcal{H}_3$ of genus 3 given by*

$$F(\tau) = \begin{bmatrix} \frac{\sqrt{3}}{2}\tau^{\frac{2}{3}} & \tau^{\frac{1}{3}} & 0 \\ \tau^{\frac{1}{3}} & i & 0 \\ 0 & 0 & \gamma\tau^{\frac{1}{3}} \end{bmatrix},$$

Let m be a positive integer, and let $F_m : \left(\mathcal{H}, 2m\, ds^2_{\mathcal{H}}\right) \to \left((\mathcal{H}_3)^m, ds^2_{(\mathcal{H}_3)^m}\right)$ be the holomorphic isometry given by $F_m(\tau) = (F(\tau), \cdots, F(\tau))$. Identifying the Cartesian product $(\mathcal{H}_3)^m$ in the standard way as a totally geodesic complex submanifold of $\mathcal{H}_{3m}$, the image $F_m(\mathcal{H}) \subset \mathcal{H}_{3m}$ is not contained in any maximal polydisk of $\mathcal{H}_{3m}$.

Proof. In what follows we denote by σ_F the second fundamental form of the holomorphic embedding F and write $\|\sigma_F\|^2 = t^2 u_F$. From the definition of the second fundamental form it follows readily that the second fundamental form σ_{F_m} is given by $\sigma_{F_m} = (\sigma_G, \cdots, \sigma_G)$. Writing $\|\sigma_{F_m}\|^2 = t^2 u_{F_m}$ we have $u_m = mu$. Thus, $\frac{\partial u_{F_m}}{\partial t} = m\frac{\partial u_F}{\partial t}$. The computation in the preceding paragraphs gives

$$\|\sigma_F(\tau)\|^2 = \frac{4}{27}\sin^2\theta\left(3 - \cos\left(\frac{2\theta}{3} - \frac{\pi}{3}\right)\right), \tag{1}$$

so that

$$u_F(\tau) = \frac{4}{27\,(s^2+t^2)}\left(3 - \cos\left(\frac{2\theta}{3} - \frac{\pi}{3}\right)\right). \tag{2}$$

Since $\frac{\partial}{\partial t}\left(s^2+t^2\right)\Big|_{t=0} \equiv 0$, we conclude that

$$\begin{aligned} \frac{\partial u_F}{\partial t}\Big|_{t=0} &= \frac{4}{27s^2}\,\frac{\partial}{\partial t}\left(3 - \cos\left(\frac{2\theta}{3} - \frac{\pi}{3}\right)\right)\Big|_{t=0} \\ = \frac{8}{81s^2}\left(\sin\left(\frac{2\theta}{3} - \frac{\pi}{3}\right)\right)\Big|_{\theta=0,\pi} \frac{\partial\theta}{\partial t}\Big|_{t=0} &= \frac{8}{81s^3}\sin\left(\mp\frac{\pi}{3}\right) = \mp\frac{8\sqrt{3}}{162s^3} \neq 0\,. \end{aligned} \tag{3}$$

On the other hand, by [(1.1), Theorem 2], for any non-standard holomorphic isometry $f : (\Delta, k\, ds^2_{\mathcal{H}}) \to (\Delta^p, ds^2_{\Delta^p})$ into the polydisk, writing σ for the second fundamental form we must have $\|\sigma\|^2 = t^2 u$ where u is not identically zero on $\partial\mathcal{H}$ but it satisfies the differential equation $\frac{\partial u}{\partial t}\Big|_{t=0} \equiv 0$. Thus, by (3) the image $F_m(\mathcal{H}) \subset \mathcal{H}_{3m}$ of the holomorphic isometric embedding $F_m : \mathcal{H} \to \mathcal{H}_{3m}$ cannot be contained in any totally geodesic polydisk, *a fortiori* not in any maximal polydisk, proving Proposition 3. □

Proposition 3 in the special case $m = 1$ was proved in Mok [Mk3], but the proof there relies on a classification of holomorphic isometries of the Poincaré disk into a polydisk Δ^p for $p \leq 3$ given by Ng [Ng1].

(2.4) The computational results on second fundamental forms of examples of non-standard holomorphic isometric embedding of the Poincaré disk provide some motivation for the formulation of problems on the class of such mappings into bounded symmetric domains, as follows.

Problem 1. *Among holomorphic isometric embeddings of the Poincaré disk into polydisks characterize in terms of second fundamental forms of the embeddings those that are equivalent to the p-th root map up to re-parametrization on the Poincaré disk and up to automorphisms of the target polydisk.*

Given a holomorphic isometric embedding $F : (\Delta, \lambda\, ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ into a bounded symmetric domain Ω equipped with the Bergman metric and denoting by σ the second fundamental form of the holomorphic isometric embedding F, clearly the function $\|\sigma\|^2$ is unchanged when F is replaced by $\Psi \circ F$, where Ψ is an automorphism of Ω. In the case of a non-standard holomorphic isometric embedding of the Poincaré disk into a polydisk, identifying the unit disk with the upper half-plane via the Cayley transform, it is known (Mok [Mk2]) that $\varphi(\tau) := \|\sigma(\tau)\|^2$ vanishes to the order 2 at a general point of ∂H. By a re-parametrization $\tau' = \Psi(\tau)$ of the upper half-plane by a fractional linear transformation Ψ, we have $\varphi = t^2 u = t'^2 u'$ where $t^2 u(\tau) = t'^2 u'(\tau')$, so that $\log u'(\tau') = \log u(\tau) + 2 \log\left(\frac{t}{t'}\right)$. The last term is a harmonic function, and thus $\log u$ is harmonic if and only if $\log u'$ is harmonic. By the computations in (2.1), denoting by σ_p the second fundamental form of the p-th root map $\rho_p : \mathcal{H} \to \mathcal{H}^p$, the function u_p defined by $t^2 u_p = \|\sigma_p\|^2$ satisfies the differential equation $\sqrt{-1}\partial\overline{\partial} \log u_p = 0$. Thus, if a holomorphic isometric embedding $f : \Delta \to \Delta^p$ is equivalent to the p-th root map ρ_p up to re-parametrization of the Poincaré disk and up to an automorphism of the target disk, then the corresponding function $\log u$ defined by f is harmonic, and one may ask whether the log harmonicity of $u = t^{-2}\|\sigma\|^2$ characterizes such maps among non-standard holomorphic isometries of the Poincaré disk into polydisks. Equivalently, in view of the statement of [(2.1), Theorem 4], one may ask whether such maps f are characterized by the fact that $-\log\|\sigma\|^2$ is a potential function for the Poincaré metric $ds^2_{\mathcal{H}}$.

Problem 2. *Among holomorphic isometric embeddings of the Poincaré disk into bounded symmetric domains characterize in terms of second fundamental forms of the embeddings those that are given by holomorphic isometric embeddings into polydisks.*

For holomorphic isometries of the Poincaré disk into a polydisk and regarding the unit disk equivalently as the upper half-plane $\mathcal{H}$, in the notation above at a general point $b \in \partial\mathcal{H}$ we have $\varphi = \|\sigma\|^2 = t^2 u$ where $u(b) \neq 0$ and $\left.\dfrac{\partial u}{\partial t}\right|_{t=0} \equiv 0$, as given in [(1.2), Theorem 2]. The proof of the latter relies on the very special fact when the holomorphic isometry extends to a neighborhood of $b \in \partial H$, it remains a holomorphic isometry in the lower half-plane. Calculations in (2.3) on the example of holomorphic maps into the Siegel upper half-plane show that the analogue of the identity $\frac{\partial u}{\partial t}\Big|_{t=0} \equiv 0$ fails. It appears likely that all non-standard holomorphic isometries into a bounded symmetric domain Ω equipped with the Bergman metric are of the first kind, i.e., φ vanishes along $\partial\mathcal{H}$ to the order 2 at a general point $b \in \partial\mathcal{H}$ (cf. the discussion in [Mk2]), so that $\varphi = \|\sigma\|^2$ can be written as $t^2 u$ with $u\big|_{t=0} \not\equiv 0$. Assuming this one can ask whether the identity $\frac{\partial u}{\partial t}\Big|_{t=0} \equiv 0$ characterizes those non-standard isometries $f : \mathcal{H} \to \Omega$ which are holomorphic isometries into polydisks, i.e., $f(\mathcal{H}) \subset P \subset \Omega$ for some maximal polydisk $P \subset \Omega$. (At least one can raise the question for non-standard holomorphic isometries of the Poincaré disk into Ω of the first kind.)

Problem 3. *Classify all holomorphic isometric embeddings of the Poincaré disk into bounded symmetric domains of rank equal to 2.*

So far there are no examples of a holomorphic isometry of the Poincaré disk into a bounded symmetric domain Ω of rank 2 other than the square-root map given by $\rho_2(\tau) = \big(\sqrt{\tau}, i\sqrt{\tau}\big)$ in terms of coordinates on the upper half-plane. In the case of $\Omega = \Delta^2 \cong \mathcal{H} \times \mathcal{H}$, it was proven by Ng [Ng1] that the only non-standard holomorphic isometry $f : (\Delta, k\, ds^2_\Delta) \to \big(\Delta^2, ds^2_{\Delta^2}\big)$ is the square-root map up to re-parametrization and up to automorphisms of the target domain. The basic example of a non-standard holomorphic isometry of the Poincaré disk not contained in a polydisk is one given in Mok [Mk3], thus one into $\mathcal{H}_3$, the Siegel upper half-plane of genus 3, and it is natural to ask whether there exists one into $B^p \times B^q$, or one into an irreducible bounded symmetric domain (such as $D_2^{III} \cong \mathcal{H}_2$). With regard to curvature formulas derived from the basic structural equation for holomorphic isometries $f : \Delta \to \Omega$ (given by the Gauss equation) of the Poincaré disk into an irreducible bounded symmetric domain, the cases where Ω is rank 2 differs from those of rank ≥ 3 in one essential point, as follows. For a (1,0-vector α tangent to the the image $S = f(\Delta)$ of the Poincaré disk at $x \in S$, and for μ denoting the orthogonal lifting of a normal vector $\sigma(\alpha, \alpha)$ at x to $T_x(S)$, σ denoting the second fundamental form of S in Ω, in the case where the image is an irreducible bounded symmetric domain of rank 2 we have a precise formula for $R_{\mu\overline{\mu}\alpha\overline{\alpha}}$ in place of a formula with an error term (cf. Mok [Mk2, (1.3)] or [(1.2), proof of Theorem 2, Eqns.(1) & (4)] of the current article). This is the case, because the curvature identity involves in general 3 invariants arising from the normal form $(\alpha_1, \cdots, \alpha_r)$ of α, $r = \mathrm{rank}(\Omega)$, expressed as sums of α_i^2, α_i^4 and α_i^6. These are algebraically independent invariants when $r \geq 3$ but related to each when $r = 2$. In the latter case, the last invariant (involving α_i^6) can be expressed in terms of the first two invariants, which relates to the length of α and to curvature, thus giving a precise curvature formula for $R_{\mu\overline{\mu}\alpha\overline{\alpha}}$. For holomorphic isometries of the Poincaré disk into an irreducible bounded symmetric domain of rank

2 it is perceivable that boundary values of u and their derivatives satisfy differential equations which can be used to study the structure of such maps.

For the reducible case it should be noted that Ng [Ng2] has proved that for $n \geq 2$ there does not exist any non-standard holomorphic isometry of the complex n-ball B^n into $B^{2n-1} \times B^{2n-1}$ up to a normalizing constant, when both the domain and target space are equipped with the Bergman metric. The case for maps of the Poincaré disk into a a product of two unit balls is harder, since in place of proving total geodesy one has to ask whether the image is contained in a totally geodesic bidisk.

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