A Unified Approach to Box-Mengerian Hypergraphs

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Abstract

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and let A be the $\mathcal{E}-V$ incidence matrix. We call $\mathcal{H}$ box-Mengerian if the linear system $A\boldsymbol{x} \geq \mathbf{1}$, $\boldsymbol{x} \geq \mathbf{0}$ is box-totally dual integral (box-TDI). As it is NP-hard in general to recognize box-Mengerian hypergraphs, a basic theme in combinatorial optimization is to identify such objects associated with various problems. In this paper we show that the so-called ESP (equitable subpartion) property, first introduced by Ding and Zang in their characterization of all graphs with the min-max relation on packing and covering cycles, turns out to be even sufficient for box-Mengerian hypergraphs. We also establish several new classes of box-Mengerian hypergraphs based on ESP property. This approach is of transparent combinatorial nature and hence is fairly easy to work with.

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1 Introduction

Many important combinatorial optimization problems can be naturally formulated as integer linear programs. One approach to getting around these problems is to consider corresponding linear programming (LP) relaxations and explore integrality properties satisfied by their constraints. Let $A\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ be a linear system and let P denote the polyhedron $\{\boldsymbol{x} : A\boldsymbol{x} \geq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$. We call P *integral* if each face of P contains integral vectors. It is well known that P is integral if and only if the minimum in the LP-duality equation

$$\min\{\boldsymbol{w}^T\boldsymbol{x} : A\boldsymbol{x} \geq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\} = \max\{\boldsymbol{y}^T\boldsymbol{b} : \boldsymbol{y}^T A \leq \boldsymbol{w}^T,\ \boldsymbol{y} \geq \boldsymbol{0}\} \tag{1.1}$$

has an integral optimal solution, for every integral vector $\boldsymbol{w}$ for which the optimum is finite. If, instead, the maximum in the equation enjoys this property, then the system $A\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is called *totally dual integral* (TDI). Furthermore, the system is called *box-totally dual integral* (box-TDI) if $A\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$, $\boldsymbol{u} \geq \boldsymbol{x} \geq \boldsymbol{l}$ is TDI for all rational vectors $\boldsymbol{u}$ and $\boldsymbol{l}$, where coordinates of $\boldsymbol{u}$ are allowed to be $+\infty$. The model of TDI systems plays a crucial role in combinatorial optimization, and serves as a general framework for establishing various min-max theorems because, as shown by Edmonds and Giles [11], total dual integrality implies primal integrality: if $A\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is TDI and $\boldsymbol{b}$ is integral, then P is integral. Under what conditions do such integrality properties hold? This question is of both great theoretical interest and practical value; it is also the major concern of polyhedral combinatorics.

The present paper is devoted to box-total dual integrality (box-TDI) property associated with hypergraphs. A *hypergraph* is a pair $\mathcal{H} = (V, \mathcal{E})$, where V is a finite set and $\mathcal{E}$ is a family of subsets of V. Elements of V and $\mathcal{E}$ are called the *vertices* and *edges* of $\mathcal{H}$, respectively. Let A be the $\mathcal{E} - V$ incidence matrix. We call $\mathcal{H}$ *ideal* if the system $A\boldsymbol{x} \geq \boldsymbol{1}$, $\boldsymbol{x} \geq \boldsymbol{0}$ defines an integral polyhedron, where $\boldsymbol{1}$ is the all-one vector. Let $\boldsymbol{w}$ be a nonnegative integral weight function defined on V. A family $\mathcal{F}$ of edges (with repetition allowed) of $\mathcal{H}$ is called a $\boldsymbol{w}$*-packing* of $\mathcal{H}$ if each $v \in V$ belongs to at most $w(v)$ members of $\mathcal{F}$. Let $\nu(\mathcal{H}, \boldsymbol{w})$ denote the maximum size of a $\boldsymbol{w}$-packing of $\mathcal{H}$, and let $\tau(\mathcal{H}, \boldsymbol{w})$ denote the minimum total weight of a vertex cover, which is a vertex subset that intersects all edges of $\mathcal{H}$. Obviously, $\nu(\mathcal{H}, \boldsymbol{w}) \leq \tau(\mathcal{H}, \boldsymbol{w})$; this inequality, however, need not hold equality in general. We call $\mathcal{H}$ *Mengerian* if the min-max relation $\nu(\mathcal{H}, \boldsymbol{w}) = \tau(\mathcal{H}, \boldsymbol{w})$ is satisfied by any nonnegative integral function $\boldsymbol{w}$ defined on V. From the aforementioned Edmonds-Giles theorem [11], it follows that $\mathcal{H}$ is Mengerian if and only if $A\boldsymbol{x} \geq \boldsymbol{1}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is a TDI system. We further call $\mathcal{H}$ *box-Mengerian* if $A\boldsymbol{x} \geq \boldsymbol{1}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is a box-TDI system. Observe that $\mathcal{H}$ is box-Mengerian if and only if, for any rational vectors $\boldsymbol{l}$ and $\boldsymbol{u}$, the maximum of the following LP-duality equation

$$\min\{\boldsymbol{w}^T\boldsymbol{x} : A\boldsymbol{x} \geq \boldsymbol{1}, \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{u} \geq \boldsymbol{x} \geq \boldsymbol{l}\} = \max\{\boldsymbol{\alpha}^T\boldsymbol{1} + \boldsymbol{\beta}^T\boldsymbol{l} - \boldsymbol{\gamma}^T\boldsymbol{u} : \boldsymbol{\alpha}^T A + \boldsymbol{\beta}^T - \boldsymbol{\gamma}^T \leq \boldsymbol{w}^T,\ \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \geq \boldsymbol{0}\}$$

has an integral optimal solution, for any integral vector $\boldsymbol{w}$ for which the optimum is finite, and so does the minimum provided both $\boldsymbol{l}$ and $\boldsymbol{u}$ are integral.

By Cook's characterization of box-TDI systems [7], a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is box-Mengerian if and only if $A\boldsymbol{x} \geq \boldsymbol{1}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is a TDI system and for any rational vector $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)^T$, where $n = |V|$, there exists an integral vector $\tilde{\boldsymbol{c}} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n)^T$ such that $\lfloor c_i \rfloor \leq \tilde{c}_i \leq \lceil c_i \rceil$, for all $1 \leq i \leq n$, and such that every optimal solution of $\min\{\boldsymbol{c}^T\boldsymbol{x} : A\boldsymbol{x} \geq \boldsymbol{1},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ is also an optimal solution of $\min\{\tilde{\boldsymbol{c}}^T\boldsymbol{x} : A\boldsymbol{x} \geq$

$\mathbf{1},\ \boldsymbol{x} \geq \mathbf{0}\}$. Nevertheless, this necessary and sufficient condition is not so "user friendly" and can hardly be verified in practice. In fact, due to NP-hardness of recognizing box-Mengerian hypergraphs [8], a basic theme in combinatorial optimization is to identify such objects associated with various problems. In [17], Schrijver analyzed proof techniques of a number of classical min-max theorems, such as the max-flow min-cut theorem, the Lucchesi-Younger theorem, and Fulkerson's optimal arborescence theorem, and observed that these proofs essentially proceed by showing that, first, the active constraints in the optimum of the LP relaxation of the problem in consideration can be chosen to be "nice", say "cross-free" or "laminar"; second, these nice constraint sets are totally unimodular. Based on this observation, Schrijver proved the following general theorem (see Theorem 5.35 in [19]), which implies that the above-mentioned min-max theorems can all be further strengthened with box-TDI properties.

Theorem 1.1 [19] *Let $A\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \geq \mathbf{0}$ be a linear system. Suppose that for any rational vector $\boldsymbol{c}$, the program $\min\{\boldsymbol{c}^T\boldsymbol{x} : A\boldsymbol{x} \geq \boldsymbol{b},\ \boldsymbol{x} \geq \mathbf{0}\}$ has (if finite) an optimal dual solution $\boldsymbol{y}$ such that the rows of A corresponding to positive components of $\boldsymbol{y}$ form a totally unimodular submatrix of A. Then $A\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \geq \mathbf{0}$ is box-TDI.*

When A is restricted to the edge-vertex incidence matrix of a hypergraph and $\boldsymbol{b}$ is set to $\mathbf{1}$, Theorem 1.1 yields a sufficient condition for box-Mengerian hypergraphs. For various classes of box-Mengerian hypergraphs resulted from this theorem, see [17, 19]. Owing to the demanding total unimodularity requirement, it is desirable to have other powerful approaches for establishing box-Mengerian hypergraphs. The purpose of this paper is to derive an analogue of Schrijver's theorem to fulfill such a need, and to give several interesting applications of this approach.

Let us introduce some notations and terminology before proceeding. As usual, we use $\mathbb{Q}$ and $\mathbb{Z}$ to denote the sets of rationals and integers, respectively, and use $\mathbb{Q}_+$ and $\mathbb{Z}_+$ to denote the sets of nonnegative numbers in the corresponding sets. For any two sets Ω and K, where Ω is always a set of numbers and K is always finite, we use Ω^K to denote the set of vectors $\boldsymbol{x} = (x(k) : k \in K)$ whose coordinates are members of Ω. Suppose $J \subseteq K$. The $|J|$-dimensional vector $\boldsymbol{x}|_J = (x(j) : j \in J)$ stands for the projection of $\boldsymbol{x}$ to Ω^J. In addition, $x(J)$ denotes the value $\sum_{j \in J} x(j)$. A vector $\boldsymbol{x}$ is called $\frac{1}{d}$-*integral*, where d is a positive integer, if all coordinates of $d\boldsymbol{x}$ are integral. A $\frac{1}{2}$-integral vector is also called *half-integral*.

In this paper, a *collection* is a synonym of a multiset in which elements may occur more than once. So if $X = \{x_1, x_2, \ldots, x_m\}$ is a collection, then possibly $x_i = x_j$ for some distinct i, j. In contrast, in a *set* and in a *subset* (of a collection), all its elements are distinct. The *size* $|X|$ of X is defined to be m. If $Y = \{y_1, y_2, \ldots, y_n\}$ is also a collection, then $X \cup Y$ is the collection $\{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}$. Note that the size of the union of two collections is always the sum of the sizes of the two collections, which is different from what happens to the union of two sets. Similarly, we can define $X \cap Y$ and $X - Y$ of these two collections. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and let Λ be a collection of its edges. We use $d_\Lambda(v)$ to denote the number of edges in Λ that contain v. An *equitable subpartition* of Λ consists of two collections Λ_1 and Λ_2 of edges in $\mathcal{E}$ (which are not necessarily in Λ) such that

(i) $|\Lambda_1| + |\Lambda_2| \geq |\Lambda|$;

(ii) $d_{\Lambda_1 \cup \Lambda_2}(v) \leq d_\Lambda(v)$ for all $v \in V$; and

(iii) $\max\{d_{\Lambda_1}(v), d_{\Lambda_2}(v)\} \le \lceil d_\Lambda(v)/2 \rceil$ for all $v \in V$.

We call $\mathcal{H}$ *equitably subpartitionable*, abbreviated ESP, if every collection of its edges admits an equitable subpartition. We refer to the above (i), (ii), and (iii) as *ESP property*, which was first introduced by Ding and Zang [9] in their characterization of all graphs with the min-max relation on packing and covering cycles, where they proved that every ESP hypergraph is Mengerian. One objective of this paper is to show that the ESP property turns out to be even sufficient for box-Mengerian hypergraphs.

Let $A\boldsymbol{x} \ge \boldsymbol{1}, \boldsymbol{u} \ge \boldsymbol{x} \ge \boldsymbol{l}, \boldsymbol{x} \ge \boldsymbol{0}$ be a linear system. With a slight abuse of notation, we write $\text{Max}(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ for both the linear program $\max\{\boldsymbol{\alpha}^T\boldsymbol{1} + \boldsymbol{\beta}^T\boldsymbol{l} - \boldsymbol{\gamma}^T\boldsymbol{u} : \boldsymbol{\alpha}^T A + \boldsymbol{\beta}^T - \boldsymbol{\gamma}^T \le \boldsymbol{w}^T, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \ge \boldsymbol{0}\}$ and its optimal value. When integrality is imposed on its solutions, we write $\text{Max}(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$ for the corresponding integer program and optimal value. When half-integrality is imposed, we write $\text{Max}(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z}/2)$ for the corresponding program and optimal value, where $\mathbb{Z}/2 = \{k/2 : k \in \mathbb{Z}\}$. Suppose A is the $\mathcal{E} - V$ incidence matrix of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and suppose $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ is an optimal solution to $\text{Max}(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$. Let Λ be the edge collection of $\mathcal{H}$ such that each $U \in \mathcal{E}$ appears exactly $\alpha^*(U)$ times in Λ. We call Λ the edge collection *corresponding* to $\boldsymbol{\alpha}^*$.

The following theorem constitutes our main tool for studying box-Mengerian hypergraphs.

Theorem 1.2 *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and let A be the $\mathcal{E} - V$ incidence matrix. Suppose that for any $\boldsymbol{l}, \boldsymbol{u} \in \mathbb{Q}^V$ and $\boldsymbol{w} \in \mathbb{Z}^V$ with finite $Max(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, there exists an optimal solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ to $Max(A, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$ such that the edge collection corresponding to $\boldsymbol{\alpha}^*$ admits an equitable subpartition. Then $\mathcal{H}$ is box-Mengerian.*

Corollary 1.3 *Every ESP hypergraph is box-Mengerian.*

A linear system $A\boldsymbol{x} \ge \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$ is called *totally dual half-integral* (TDI/2) if the maximum in the LP-duality equation (1.1) has a half-integral optimal solution, for every integral vector $\boldsymbol{w}$ for which the optimum is finite. Furthermore, the system is called *box-totally dual half-integral* (box-TDI/2) if $A\boldsymbol{x} \ge \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$, $\boldsymbol{u} \ge \boldsymbol{x} \ge \boldsymbol{l}$ is TDI/2 for all rational vectors $\boldsymbol{u}$ and $\boldsymbol{l}$, where coordinates of $\boldsymbol{u}$ are allowed to be $+\infty$. Similar to the above Edmonds-Giles theorem [11], we can prove that if $A\boldsymbol{x} \ge \boldsymbol{b}$, $\boldsymbol{x} \ge \boldsymbol{0}$ is TDI/2 and $\boldsymbol{b}$ is integral, then the minimum in equation (1.1) also has a half-integral optimal solution, for every integral vector $\boldsymbol{w}$ for which the optimum is finite. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and let A be the $\mathcal{E} - V$ incidence matrix. We call $\mathcal{H}$ *half-Mengerian* (resp. *box-half-Mengerian*) if $A\boldsymbol{x} \ge \boldsymbol{1}$, $\boldsymbol{x} \ge \boldsymbol{0}$ is a TDI/2 (resp. box-TDI/2) system. Let Λ be an edge collection of $\mathcal{H}$. A *pseudo-equitable subpartition* of Λ consists of two collections Λ_1 and Λ_2 of edges in $\mathcal{E}$ (which are not necessarily in Λ) such that

(i) $|\Lambda_1| + |\Lambda_2| \ge |\Lambda|$;

(ii) $d_{\Lambda_1 \cup \Lambda_2}(v) \le d_\Lambda(v)$ for all $v \in V$;

(iii') $\max\{d_{\Lambda_1}(v), d_{\Lambda_2}(v)\} \le 2\lceil d_\Lambda(v)/4 \rceil$ for all $v \in V$; and

(iv) $|d_{\Lambda_1}(v) - d_{\Lambda_2}(v)| \le 2$ for all $v \in V$.

We call $\mathcal{H}$ *pseudo-equitably subpartitionable*, abbreviated PESP, if every collection of its edges admits a pseudo-equitable subpartition. We refer to the above (i), (ii), (iii'), and (iv) as *PESP property.* Observe that (iii) specified in the ESP property implies (iii'). The following theorem is the counterpart of Theorem 1.2 for box-half-Mengerian hypergraphs.

Theorem 1.4 *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph and let A be the $\mathcal{E} - V$ incidence matrix. Suppose that for any $\boldsymbol{l}, \boldsymbol{u} \in \mathbb{Q}^V$ and $\boldsymbol{w} \in \mathbb{Z}^V$ with finite Max$(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, there exists an optimal solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ to Max$(A, \boldsymbol{l}, \boldsymbol{u}, 4\boldsymbol{w}; \mathbb{Z})$ such that the edge collection corresponding to $\boldsymbol{\alpha}^*$ admits a pseudo-equitable subpartition. Then $\mathcal{H}$ is box-half-Mengerian.*

Corollary 1.5 *Every PESP hypergraph is box-half-Mengerian.*

To facilitate better understanding of the PESP property, we remark that if for every edge collection Λ of $\mathcal{H}$, there exist two collections Λ_1 and Λ_2 of edges in $\mathcal{E}$ (which are not necessarily in Λ) satisfying (i), (ii), and (iii'), then $\mathcal{H}$ is half-Mengerian (see [10] for a proof), but it is not necessarily box-half-Mengerian.

Let $\mathcal{H}$ and A be as given in Theorem 1.4, and let Λ be an edge collection of $\mathcal{H}$. A *quasi-equitable subpartition* of Λ consists of two collections Λ_1 and Λ_2 of edges in $\mathcal{E}$ (which are not necessarily in Λ) such that the above (i), (ii), (iii'), and the following

(iv') $|d_{\Lambda_1}(v) - d_{\Lambda_2}(v)| \leq 2$ for all $v \in V$ with $d_{\Lambda_1 \cup \Lambda_2}(v) = d_\Lambda(v)$

hold simultaneously. We call $\mathcal{H}$ *quasi-equitably subpartitionable*, abbreviated QESP, if every collection of its edges admits a quasi-equitable subpartition. We refer to the above (i), (ii), (iii'), and (iv') as *QESP property.* Observe that (iii) specified in the ESP property implies both (iii') and (iv'). The following theorem is a generalization of Corollary 1.5 and is, we believe, much more useful in combinatorial applications.

Theorem 1.6 *Every QESP hypergraph is box-half-Mengerian.*

The proofs of the above three theorems will be given in Section 2. As applications of these theorems, several new classes of box-Mengerian and box-half-Mengerian hypergraphs will be established in Section 3. It is worthwhile pointing out that none of them can be derived from Theorem 1.1 directly. Our approach is of transparent combinatorial nature and hence is fairly easy to work with.

2 Proofs

As shown by Schrijver and Seymour, a linear system $A\boldsymbol{x} \geq \mathbf{1}, \boldsymbol{u} \geq \boldsymbol{x} \geq \boldsymbol{l}, \boldsymbol{x} \geq \mathbf{0}$ is TDI if and only if Max$(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z}/2) =$ Max$(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$ for any integral vector $\boldsymbol{w}$ for which Max$(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ is finite (see Theorem 22.13 in [18]). A corollary of this theorem is the following necessary and sufficient condition for total dual integrality, where TDI/1 is exactly the same as TDI.

Lemma 2.1 [18] *For $k = 1, 2$, the system $A\boldsymbol{x} \geq \mathbf{1}, \boldsymbol{u} \geq \boldsymbol{x} \geq \boldsymbol{l}, \boldsymbol{x} \geq \mathbf{0}$ is TDI/k if and only if*

$$\text{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 2k\boldsymbol{w}; \mathbb{Z}) \leq 2\,\text{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, k\boldsymbol{w}; \mathbb{Z})$$

for any integer vector $\boldsymbol{w}$ for which Max$(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$ is finite. ■

Let us now present the proofs of the first two theorems.

Proof of Theorem 1.2. In view of Lemma 2.1, to prove the theorem it suffices to show that for any $\boldsymbol{l}, \boldsymbol{u} \in \mathbb{Q}^V$ and $\boldsymbol{w} \in \mathbb{Z}^V$ with finite $\text{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, we have $\text{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z}) \leq 2\,\text{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$. By hypothesis, there exists an optimal solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ to $\text{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$ such that the edge collection Λ corresponding to $\boldsymbol{\alpha}^*$ admits an equitable subpartition (Λ_1, Λ_2). Our objective is to find a feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ to $\text{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w}; \mathbb{Z})$ such that $\boldsymbol{\alpha}^T\mathbf{1} + \boldsymbol{\beta}^T\boldsymbol{l} - \boldsymbol{\gamma}^T\boldsymbol{u} \geq [(\boldsymbol{\alpha}^*)^T\mathbf{1} + (\boldsymbol{\beta}^*)^T\boldsymbol{l} - (\boldsymbol{\gamma}^*)^T\boldsymbol{u}]/2$.

For ease of description, let us first impose some additional constraints on $\boldsymbol{\beta}^*$ and $\boldsymbol{\gamma}^*$. We may assume that

(1) $\beta^*(v)\gamma^*(v) = 0$ for all $v \in V$.

Otherwise, neither $\beta^*(v)$ nor $\gamma^*(v)$ is zero for some vertex v. Set $\delta = \min\{\beta^*(v), \gamma^*(v)\}$. Then $\delta > 0$. Let $\boldsymbol{\beta}'$ be the vector obtained from $\boldsymbol{\beta}^*$ by replacing $\beta^*(v)$ with $\beta^*(v) - \delta$, and let $\boldsymbol{\gamma}'$ be the vector obtained from $\boldsymbol{\gamma}^*$ by replacing $\gamma^*(v)$ with $\gamma^*(v) - \delta$. Clearly, $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}')$ is a feasible solution to $\text{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$. Note that $(\boldsymbol{\alpha}^*)^T\mathbf{1} + (\boldsymbol{\beta}')^T\boldsymbol{l} - (\boldsymbol{\gamma}')^T\boldsymbol{u} = (\boldsymbol{\alpha}^*)^T\mathbf{1} + (\boldsymbol{\beta}^*)^T\boldsymbol{l} - (\boldsymbol{\gamma}^*)^T\boldsymbol{u} + [u(v) - l(v)]\delta \geq (\boldsymbol{\alpha}^*)^T\mathbf{1} + (\boldsymbol{\beta}^*)^T\boldsymbol{l} - (\boldsymbol{\gamma}^*)^T\boldsymbol{u}$. So $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}')$ is also an optimal solution to $\text{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$. Hence (1) holds for otherwise we can replace $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ with $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}')$ and repeat the process.

(2) $\beta^*(v) = 0$ for all $v \in V$ with $l(v) < 0$.

Suppose the contrary: $\beta^*(v) > 0$ for some $v \in V$ with $l(v) < 0$. Let $\boldsymbol{\beta}'$ be the vector obtained from $\boldsymbol{\beta}^*$ by replacing $\beta^*(v)$ with 0. Then $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}^*)$ is a feasible solution to $\text{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$, with objective value greater than that of $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$, this contradiction justifies (2).

Observe that $d_\Lambda(v) + \beta^*(v) - \gamma^*(v) \leq 2w(v)$, the inequality constraint in $(\boldsymbol{\alpha}^*)^TA + (\boldsymbol{\beta}^*)^T - (\boldsymbol{\gamma}^*)^T \leq 2\boldsymbol{w}^T$ corresponding to a vertex v, can be strengthened as follows.

(3) $d_\Lambda(v) + \beta^*(v) - \gamma^*(v) = 2w(v)$ for all $v \in V$ with $\beta^*(v) + \gamma^*(v) > 0$.

Suppose not, $d_\Lambda(v) + \beta^*(v) - \gamma^*(v) < 2w(v)$ for some $v \in V$ with $\beta^*(v) + \gamma^*(v) > 0$. Set $\delta = 2w(v) - [d_\Lambda(v) + \beta^*(v) - \gamma^*(v)]$. Then $\delta > 0$. If $\beta^*(v) > 0$ then, by (1) and (2), we have $\gamma^*(v) = 0$ and $l(v) \geq 0$. In this case let $\boldsymbol{\beta}'$ be the vector obtained from $\boldsymbol{\beta}^*$ by replacing $\beta^*(v)$ with $\beta^*(v) + \delta$ and set $\boldsymbol{\gamma}' = \boldsymbol{\gamma}^*$. If $\gamma^*(v) > 0$ then, by (1), we have $\beta^*(v) = 0$. In this case let $\boldsymbol{\gamma}'$ be the vector obtained from $\boldsymbol{\gamma}^*$ by replacing $\gamma^*(v)$ with $\max\{0, \gamma^*(v) - \delta\}$ and set $\boldsymbol{\beta}' = \boldsymbol{\beta}^*$. It is easy to see that $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}')$ is also an optimal solution to $\text{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$. Let us replace $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$ with $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}', \boldsymbol{\gamma}')$ and repeat the process until we get stuck. Clearly, the resulting solution satisfies (1), (2), and (3) simultaneously.

For $i = 1, 2$, define a vector $\boldsymbol{\alpha}_i \in \mathbb{Z}_+^{\mathcal{E}}$, such that $\alpha_i(U)$ is precisely the multiplicity of the edge U in Λ_i for all $U \in \mathcal{E}$. By (i) of the ESP property, we have $|\Lambda_1| + |\Lambda_2| \geq |\Lambda|$. So

(4) $\boldsymbol{\alpha}_1^T\mathbf{1} + \boldsymbol{\alpha}_2^T\mathbf{1} \geq (\boldsymbol{\alpha}^*)^T\mathbf{1}$.

Consider an arbitrary vertex $v \in V$. Suppose $d_{\Lambda_s}(v) \leq d_{\Lambda_t}(v)$, where $\{s, t\} = \{1, 2\}$. Then (ii) and (iii) of the ESP property yield

(5) $d_{\Lambda_s}(v) \leq \lfloor d_\Lambda(v)/2 \rfloor$ and $d_{\Lambda_t}(v) \leq \lceil d_\Lambda(v)/2 \rceil$.

Set

- $\beta_s(v) = \lceil \beta^*(v)/2 \rceil$ and $\gamma_s(v) = \lfloor \gamma^*(v)/2 \rfloor$;
- $\beta_t(v) = \lfloor \beta^*(v)/2 \rfloor$ and $\gamma_t(v) = \lceil \gamma^*(v)/2 \rceil$.

Then

(6) $\beta_s(v)+\beta_t(v)=\beta^*(v)$ and $\gamma_s(v)+\gamma_t(v)=\gamma^*(v)$.

We propose to show that

(7) $d_{\Lambda_i}(v)+\beta_i(v)-\gamma_i(v)\le w(v)$ for $i=1,2$.

We distinguish between two cases according to the parity of $d_\Lambda(v)$. If $d_\Lambda(v)$ is even then, by (1) and (3), both $\beta^*(v)$ and $\gamma^*(v)$ are even. It follows from (5) that $d_{\Lambda_i}(v)+\beta_i(v)-\gamma_i(v)\le \frac{1}{2}(d_\Lambda(v)+\beta^*(v)-\gamma^*(v))\le w(v)$ for $i=1,2$, as deisred. It remains to consider the case when $d_\Lambda(v)$ is odd. If both $\beta^*(v)$ and $\gamma^*(v)$ are zero then, by (5), for $i=1,2$, we have $d_{\Lambda_i}(v)+\beta_i(v)-\gamma_i(v)=d_{\Lambda_i}(v)\le[d_\Lambda(v)+1]/2$, which is an integer bounded above by $[2w(v)+1]/2=w(v)+1/2$ and hence by $w(v)$. It follows that $d_{\Lambda_i}(v)+\beta_i(v)-\gamma_i(v)\le w(v)$. So we assume that $\beta^*(v)+\gamma^*(v)>0$. Consequently, $d_\Lambda(v)+\beta^*(v)-\gamma^*(v)=2w(v)$ by (3). Since $d_\Lambda(v)$ is odd, so is $\beta^*(v)-\gamma^*(v)$. In addition, $\beta^*(v)\gamma^*(v)=0$ by (1). From the definition we see that $\beta_s(v)-\gamma_s(v)=[\beta^*(v)-\gamma^*(v)+1]/2$ and $\beta_t(v)-\gamma_t(v)=[\beta^*(v)-\gamma^*(v)-1]/2$. Combining them with (5), we establish (7).

For $i=1,2$, set $\boldsymbol{\beta}_i=(\beta_i(v):v\in V)$ and $\boldsymbol{\gamma}_i=(\gamma_i(v):v\in V)$. By (7), we have $\boldsymbol{\alpha}_i^TA+\boldsymbol{\beta}_i^T-\boldsymbol{\gamma}_i^T\le\boldsymbol{w}^T$, so $(\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i,\boldsymbol{\gamma}_i)$ is a feasible solution to $\text{Max}\,(A,\boldsymbol{l},\boldsymbol{u},\boldsymbol{w};\mathbb{Z})$. From (6) it follows that $\boldsymbol{\beta}_1+\boldsymbol{\beta}_2=\boldsymbol{\beta}$ and $\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2=\boldsymbol{\gamma}$. Hence $\sum_{i=1}^2(\boldsymbol{\beta}_i^T\boldsymbol{l}-\boldsymbol{\gamma}_i^T\boldsymbol{u})=(\boldsymbol{\beta}^*)^T\boldsymbol{l}-(\boldsymbol{\gamma}^*)^T\boldsymbol{u}$. Using (1), we conclude that the inequality $\boldsymbol{\alpha}_i^T\mathbf{1}+\boldsymbol{\beta}_i^T\boldsymbol{l}-\boldsymbol{\gamma}_i^T\boldsymbol{u}\ge[(\boldsymbol{\alpha}^*)^T\mathbf{1}+(\boldsymbol{\beta}^*)^T\boldsymbol{l}-(\boldsymbol{\gamma}^*)^T\boldsymbol{u}]/2$ holds for $i=1$ or 2; the corresponding $(\boldsymbol{\alpha}_i,\boldsymbol{\beta}_i,\boldsymbol{\gamma}_i)$ is clearly a solution to $\text{Max}\,(A,\boldsymbol{l},\boldsymbol{u},\boldsymbol{w};\mathbb{Z})$ as desired, completing the proof. ■

Proof of Theorem 1.4. In view of Lemma 2.1, to prove the theorem it suffices to show that for any $\boldsymbol{l},\boldsymbol{u}\in\mathbb{Q}^V$ and $\boldsymbol{w}\in\mathbb{Z}^V$ with finite $\text{Max}\,(A,\boldsymbol{l},\boldsymbol{u},\boldsymbol{w})$, we have $\text{Max}\,(A,\boldsymbol{l},\boldsymbol{u},4\boldsymbol{w};\mathbb{Z})\le 2\,\text{Max}\,(A,\boldsymbol{l},\boldsymbol{u},2\boldsymbol{w};\mathbb{Z})$. By hypothesis, there exists an optimal solution $(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*,\boldsymbol{\gamma}^*)$ to $\text{Max}\,(A,\boldsymbol{l},\boldsymbol{u},4\boldsymbol{w};\mathbb{Z})$ such that the edge collection Λ corresponding to $\boldsymbol{\alpha}^*$ admits a pseudo-equitable subpartition (Λ_1,Λ_2). Our objective is to find a feasible solution $(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})$ to $\text{Max}\,(A,\boldsymbol{l},\boldsymbol{u},2\boldsymbol{w};\mathbb{Z})$ such that $\boldsymbol{\alpha}^T\mathbf{1}+\boldsymbol{\beta}^T\boldsymbol{l}-\boldsymbol{\gamma}^T\boldsymbol{u}\ge[(\boldsymbol{\alpha}^*)^T\mathbf{1}+(\boldsymbol{\beta}^*)^T\boldsymbol{l}-(\boldsymbol{\gamma}^*)^T\boldsymbol{u}]/2$.

Using the same arguments as employed in the proof of the preceding theorem, we may assume that

(1) $\beta^*(v)\gamma^*(v)=0$ for all $v\in V$.

(2) $\beta^*(v)=0$ for all $v\in V$ with $l(v)<0$.

(3) $d_\Lambda(v)+\beta^*(v)-\gamma^*(v)=4w(v)$ for all $v\in V$ with $\beta^*(v)+\gamma^*(v)>0$.

For $i=1,2$, define a vector $\boldsymbol{\alpha}_i\in\mathbb{Z}_+^{\mathcal{E}}$, such that $\alpha_i(U)$ is precisely the multiplicity of the edge U in Λ_i for all $U\in\mathcal{E}$. By (i) of the PESP property, we have $|\Lambda_1|+|\Lambda_2|\ge|\Lambda|$. So

(4) $\boldsymbol{\alpha}_1^T\mathbf{1}+\boldsymbol{\alpha}_2^T\mathbf{1}\ge(\boldsymbol{\alpha}^*)^T\mathbf{1}$.

We propose to show that for each $v\in V$, there exist nonnegative integers $\beta_1(v),\beta_2(v),\gamma_1(v)$, and $\gamma_2(v)$ such that

(5) $\beta_1(v)+\beta_2(v)=\beta^*(v)$ and $\gamma_1(v)+\gamma_2(v)=\gamma^*(v)$

and that

(6) $d_{\Lambda_i}(v)+\beta_i(v)-\gamma_i(v)\le 2w(v)$ for $i=1,2$.

To this end, suppose $d_{\Lambda_s}(v) \le d_{\Lambda_t}(v)$, where $\{s,t\} = \{1,2\}$. We may assume that

(7) $d_{\Lambda_t}(v) > \lceil d_\Lambda(v)/2 \rceil$.

Otherwise, $d_{\Lambda_t}(v) \le \lceil d_\Lambda(v)/2 \rceil$. Set

- $\beta_s(v) = \lceil \beta^*(v)/2 \rceil$ and $\gamma_s(v) = \lfloor \gamma^*(v)/2 \rfloor$;
- $\beta_t(v) = \lfloor \beta^*(v)/2 \rfloor$ and $\gamma_t(v) = \lceil \gamma^*(v)/2 \rceil$.

Clearly (5) holds. Since $d_\Lambda(v) + \beta^*(v) - \gamma^*(v) \le 4w(v)$, imitating (7) in the proof of the preceding theorem, we see that (6) is also satisfied. Thus (7) is established.

(8) $\beta^*(v) + \gamma^*(v) > 0$.

Otherwise, $\beta^*(v) = \gamma^*(v) = 0$. Set $\beta_1(v), \beta_2(v), \gamma_1(v)$, and $\gamma_2(v)$ all equal to zero. By (iii') of the PESP property, $\max\{d_{\Lambda_1}(v), d_{\Lambda_2}(v)\} \le 2\lceil d_\Lambda(v)/4 \rceil \le 2w(v)$. So we have (6), and hence (8) holds.

Using (7) and (iii') of the PESP property, we obtain $\lceil d_\Lambda(v)/2 \rceil + 1 \le d_{\Lambda_t}(v) \le 2\lceil d_\Lambda(v)/4 \rceil$. By considering possible congruence of $d_\Lambda(v)$ modulo four and comparing the lower and upper bounds of the preceding inequality, we find

(9) $\lceil d_\Lambda(v)/2 \rceil + 1 = d_{\Lambda_t}(v) = 2\lceil d_\Lambda(v)/4 \rceil$ and $d_\Lambda(v) \equiv 1$ or $2 \pmod 4$.

Observe further that

(10) If $\gamma^*(v) > 0$, then $\gamma^*(v) \ge 2$.

Otherwise, $\gamma^*(v) = 1$. By (1) and (3), we get $d_\Lambda(v) \equiv 1 \pmod 4$. It follows from (9) that $d_{\Lambda_t}(v) = [d_\Lambda(v)+3]/2$. In view of (ii) of the PESP property, we obtain $d_{\Lambda_s}(v) \le d_\Lambda(v) - [d_\Lambda(v)+3]/2 = [d_\Lambda(v)-3]/2$. So $d_{\Lambda_t}(v) - d_{\Lambda_s}(v) \ge 3$, contradicting (iv) of the PESP property. Thus (10) is established.

By (9), $d_\Lambda(v) \equiv k \pmod 4$, where $k = 1$ or 2, and $d_{\Lambda_t}(v) = [d_\Lambda(v) + 4 - k]/2$. Set $\beta_t(v) = 0$ if $\beta^*(v) = 0$ and $[\beta^*(v) - 4 + k]/2$ otherwise, and set $\gamma_t(v) = 0$ if $\gamma^*(v) = 0$ and $[\gamma^*(v) + 4 - k]/2$ otherwise. From (1), (8), (3), and (10), we deduce that

(11) $0 \le \beta_t(v) \le \beta^*(v)$, $0 \le \gamma_t(v) \le \gamma^*(v)$, and $d_{\Lambda_t}(v) + \beta_t(v) - \gamma_t(v) = 2w(v)$.

Set $\beta_s(v) = \beta^*(v) - \beta_t(v)$ and $\gamma_s(v) = \gamma^*(v) - \gamma_t(v)$. Then $0 \le \beta_s(v) \le \beta^*(v)$ and $0 \le \gamma_s(v) \le \gamma^*(v)$. If (6) were violated by $i = s$, then $d_\Lambda(v) + \beta^*(v) - \gamma^*(v) \ge \sum_{i=1}^2 [d_{\Lambda_i}(v) + \beta_i(v) - \gamma_i(v)] > 4w(v)$ by (11), contradicting (3). So (6) holds for $i = 1$ and 2.

For $i = 1, 2$, set $\boldsymbol{\beta}_i = (\beta_i(v) : v \in V)$ and $\boldsymbol{\gamma}_i = (\gamma_i(v) : v \in V)$. Similarly, we can prove that either $(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \boldsymbol{\gamma}_1)$ or $(\boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \boldsymbol{\gamma}_2)$ is a solution to Max $(A, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$ as desired. ∎

Let us define two terms and prove a simple lemma before presenting the proof of Theorem 1.6. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, and let Λ and Ω be two edge collections of $\mathcal{H}$. We say that Λ *dominates* Ω if $|\Lambda| \ge |\Omega|$ while $d_\Lambda(v) \le d_\Omega(v)$ for all $v \in V$. The domination relation is obviously reflexive and transitive. An edge collection Ω of $\mathcal{H}$ is called *atomic* if for every edge collection Π that dominates Ω, we have both $|\Pi| = |\Omega|$ and $d_\Pi(v) = d_\Omega(v)$ for all $v \in V$.

Lemma 2.2 *Every edge collection of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is dominated by an atomic edge collection.*

Proof. Let Ω be an arbitrary edge collection of $\mathcal{H}$ and let Λ be an edge collection that dominates Ω such that

(1) $|\Lambda|$ is maximized;

(2) subject to (1), $\sum_{v \in V} d_\Lambda(v)$ is minimized.

It is a routine matter to check that Λ is atomic. ■

Proof of Theorem 1.6. In view of Lemma 2.1, to prove the theorem it suffices to show that for any $\boldsymbol{l}, \boldsymbol{u} \in \mathbb{Q}^V$ and $\boldsymbol{w} \in \mathbb{Z}^V$ with finite $\mathrm{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, \boldsymbol{w})$, we have $\mathrm{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 4\boldsymbol{w}; \mathbb{Z}) \leq 2\,\mathrm{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$. We shall actually prove that for every optimal solution $(\boldsymbol{\alpha}_*, \boldsymbol{\beta}_*, \boldsymbol{\gamma}_*)$ to $\mathrm{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 4\boldsymbol{w}; \mathbb{Z})$, there exists a feasible solution $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ to $\mathrm{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 2\boldsymbol{w}; \mathbb{Z})$ such that $\boldsymbol{\alpha}^T\mathbf{1} + \boldsymbol{\beta}^T\boldsymbol{l} - \boldsymbol{\gamma}^T\boldsymbol{u} \geq [\boldsymbol{\alpha}_*^T\mathbf{1} + \boldsymbol{\beta}_*^T\boldsymbol{l} - \boldsymbol{\gamma}_*^T\boldsymbol{u}]/2$.

Let Ω be the edge collection of $\mathcal{H}$ corresponding to $\boldsymbol{\alpha}_*$. Then Lemma 2.2 guarantees the existence of an atomic edge collection Λ that dominates Ω. By definitions, we have

(a) $|\Lambda| \geq |\Omega|$ and $d_\Lambda(v) \leq d_\Omega(v)$ for all $v \in V$;

(b) for every edge collection Π that dominates Λ, $|\Pi| = |\Lambda|$ and $d_\Pi(v) = d_\Lambda(v)$ for all $v \in V$.

Define a vector $\boldsymbol{\alpha}^* \in \mathbb{Z}_+^{\mathcal{E}}$, such that $\alpha^*(U)$ is precisely the multiplicity of the edge U in Λ for all $U \in \mathcal{E}$. From (a) we deduce that $(\boldsymbol{\alpha}^*)^T\mathbf{1} \geq \boldsymbol{\alpha}_*^T\mathbf{1}$ and $(\boldsymbol{\alpha}^*)^T A \leq \boldsymbol{\alpha}_*^T A$ and hence $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*, \boldsymbol{\gamma}^*)$, with $\boldsymbol{\beta}^* = \boldsymbol{\beta}_*$ and $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}_*$, is also an optimal solution to $\mathrm{Max}\,(A, \boldsymbol{l}, \boldsymbol{u}, 4\boldsymbol{w}; \mathbb{Z})$. In particular, $(\boldsymbol{\alpha}^*)^T\mathbf{1} + (\boldsymbol{\beta}^*)^T\boldsymbol{l} - (\boldsymbol{\gamma}^*)^T\boldsymbol{u} = \boldsymbol{\alpha}_*^T\mathbf{1} + \boldsymbol{\beta}_*^T\boldsymbol{l} - \boldsymbol{\gamma}_*^T\boldsymbol{u}$.

Since $\mathcal{H}$ is QESP, Λ admits a quasi-equitable subpartition (Λ_1, Λ_2). From (i) and (ii) of the QESP property, it can be seen that $\Lambda_1 \cup \Lambda_2$ dominates Λ. As Λ is atomic, by (b) we get

(c) $|\Lambda_1| + |\Lambda_2| = |\Lambda|$ and $d_{\Lambda_1}(v) + d_{\Lambda_2}(v) = d_\Lambda(v)$ for all $v \in V$.

The remainder of the proof is exactly the same as that of Theorem 1.4, except that to establish (10), we have to apply both (c) and (iv') of the QESP property. ■

Let us digress to exhibit some properties enjoyed by equitable subpartions, which will be used repeatedly in the applications of the above theorems. The following is clear from the definitions.

Lemma 2.3 *Let Λ and Ω be two edge collections of a hypergraph $\mathcal{H} = (V, \mathcal{E})$. If Λ dominates Ω, then every equitable subpartition of Λ is an equitable subpartition of Ω.* ■

For each edge collection Λ of $\mathcal{H}$, let $\partial(\Lambda)$ denote the subset of $\mathcal{E}$, consisting of edges that appear an odd number of times in Λ. The following assertion was implicitly established by Ding and Zang in their proof of Theorem 2.1 in [9].

Lemma 2.4 *Let Λ be an edge collection of a hypergraph $\mathcal{H} = (V, \mathcal{E})$. If $\partial(\Lambda)$ admits an equitable subpartition, then so does Λ.*

Proof. For completeness, we give a sketch of Ding and Zang's proof [9] here. For each $U \in \mathcal{E}$, let $m(U)$ stand for its multiplicity in Λ. Let Λ_0 be the edge collection such that each $U \in \mathcal{E}$ appears $\lfloor m(U)/2 \rfloor$ times. Clearly, $\Lambda = \Lambda_0 \cup \Lambda_0 \cup \partial(\Lambda)$. By hypothesis, $\partial(\Lambda)$ admits an equitable subpartition (Ω_1, Ω_2). For $i = 1, 2$, set $\Lambda_i = \Lambda_0 \cup \Omega_i$. It is easy to verify that (Λ_1, Λ_2) is an equitable subpartition of Λ. ■

Lemma 2.5 *A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is ESP if and only if every atomic subset of $\mathcal{E}$ admits an equitable subpartition.*

Proof. The "only if" part follows instantly from the definition of ESP hypergraphs. To prove the "if" part, we assume the contrary: some edge collection Ω of $\mathcal{H}$ admits no equitable subpartition. By Lemma 2.2, Ω is dominated by an atomic edge collection Λ. Observe that $\partial(\Lambda)$ is also atomic, for otherwise there would exist an edge collection Π that dominates $\partial(\Lambda)$, with either $|\Pi| > |\partial(\Lambda)|$ or $\sum_{v\in V} d_{\Pi}(v) < \sum_{v\in V} d_{\partial(\Lambda)}(v)$. Set $\Lambda' = (\Lambda - \partial(\Lambda)) \cup \Pi$. It is easy to see that Λ' dominates Λ, with either $|\Lambda'| > |\Lambda|$ or $\sum_{v\in V} d_{\Lambda'}(v) < \sum_{v\in V} d_{\Lambda}(v)$, contradicting the atomic assumption on Λ. It follows that $\partial(\Lambda)$ admits an equitable subpartition; so does Λ by Lemma 2.4 and hence Ω by Lemma 2.3, this contradiction completes the proof. ∎

Given two hypergraphs $\mathcal{H}_i = (V_i, \mathcal{E}_i)$ for $i = 1, 2$ with $V_1 \cap V_2 = \emptyset$, the hypergraph $(V_1 \cup V_2, \mathcal{E}_1 \cup \mathcal{E}_2)$ is called the 0-*sum* of $\mathcal{H}_1$ and $\mathcal{H}_2$. The following lemma asserts that ESP property is preserved under this summing operation.

Lemma 2.6 [9] *The* 0*-sum of two ESP hypergraphs is also ESP.*

We shall appeal to this lemma to establish some inductive arguments on hypergraphs that can be decomposed into smaller ones under this summing operation.

3 Applications

The purpose of this section is to establish several new classes of box-Mengerian and box-half-Mengerian hypergraphs by using the preceding theorems.

3.1 Path hypergraphs

In this subsection we study hypergraphs arising from paths in undirected trees. Let T be a tree with edge set V, and let $\mathcal{E}$ be edge sets of some paths in T, such that each edge of T is contained in at least one of these paths. We call $\mathcal{H} = (V, \mathcal{E})$ the *edge path tree* (EPT) *hypergraph* supported by T, and call T a *supporting tree* of $\mathcal{H}$ (note that a supporting tree may not be unique). For characterizations of EPT hypergraphs, see Fournier [12]. The problem of recognizing EPT hypergraphs is closely related to the well-known graph realization problem [3, 4, 13] (see also Chapter 20 in Schrijver [18]).

A subset of edges $\{P_1, P_2, \dots, P_k\}$ in $\mathcal{H}$ is called a *pie* if T contains a vertex u and k edges $e_1, e_2, \dots, e_k$ all incident with u, such that $P_i \cap \{e_1, e_2, \dots, e_k\} = \{e_i, e_{i+1}\}$ for $i = 1, 2, \dots, k$, where $e_{k+1} = e_1$; the pie is called *odd* if k is odd. For any two disjoint subsets X and Y of V (possibly X or Y is empty), let $V' = V - (X \cup Y)$ and let $\mathcal{E}'$ be the set of all *minimal* members in $\{P - Y : P \cap X = \emptyset, P \in \mathcal{E}\}$, where the adjective *minimal* is meant with respect to set-inclusion rather than size. Then $\mathcal{H}' = (V', \mathcal{E}')$ is called the *minor* of $\mathcal{H}$ obtained by deleting X and contracting Y. Clearly, $\mathcal{H}'$ is supported by T', the tree obtained from T by contracting edges in $X \cup Y$. The hypergraph $\mathcal{H}$ is said to be *odd-pie-free* if it contains no odd pies, and is said to be *odd-M-pie-free* if none of its minors is an odd pie. As observed by Apollonio [1], an odd-M-pie-free hypergraph may contain an odd pie. A *clutter* is a hypergraph in which no edge is contained in another one.

Theorem 3.1 *Let $\mathcal{H} = (V, \mathcal{E})$ be an EPT clutter. Then the following statements are equivalent:*

(i) $\mathcal{H}$ is odd-M-pie-free;

(ii) $\mathcal{H}$ is ideal;

(iii) $\mathcal{H}$ is Mengerian;

(iv) $\mathcal{H}$ is box-Mengerian; and

(v) $\mathcal{H}$ is ESP.

The equivalence of the first three statements was established recently by Apollonio [1]. We aim to show that for the same combinatorial structures, the stronger statements (iv) and (v) hold. Our proof relies heavily on the following two lemmas due to Apollonio.

Lemma 3.2 [1] *Let $\mathcal{H} = (V, \mathcal{E})$ be an odd-pie-free EPT hypergraph and let A be the $\mathcal{E} - V$ incidence matrix. Then A is totally unimodular.*

Lemma 3.3 [1] *Let $\mathcal{H} = (V, \mathcal{E})$ be an odd-M-pie-free EPT clutter. If $\mathcal{H}$ contains an odd pie $\mathcal{P}$, then there exist four edges P_1, P_2, P_3, P_4 in $\mathcal{H}$, with $\{P_3, P_4\} \subseteq \mathcal{P}$, such that*

$$P_1 \cap P_2 = \emptyset, \;\; P_3 \cap P_4 \neq \emptyset, \;\; \textit{and} \;\; P_1 \cup P_2 \subseteq (P_3 - P_4) \cup (P_4 - P_3). \tag{3.1}$$

Proof of Theorem 3.1. In view of Corollary 1.3, we have (v)⇒(iv)⇒(iii)⇒(ii)⇒(i). It remains to establish the implication (i)⇒(v).

By Lemma 2.5, we only need to show that every atomic subset $\mathcal{F}$ of $\mathcal{E}$ admits an equitable subpartition. To this end, let T be a supporting tree of $\mathcal{H}$ and let W be the set of all edges of T contained in members of $\mathcal{F}$. Then $\mathcal{G} = (W, \mathcal{F})$ is also an EPT clutter. Observe that

(1) $\mathcal{H}$ contains no four edges P_1, P_2, P_3, P_4, with $\{P_3, P_4\} \subseteq \mathcal{F}$, as described in (3.1).

Otherwise, let $\mathcal{F}'$ be the edge collection obtained from $\mathcal{F}$ by replacing $\{P_3, P_4\}$ with $\{P_1, P_2\}$. Then $\mathcal{F}'$ dominates $\mathcal{F}$ and $\sum_{v \in W} d_{\mathcal{F}'}(v) < \sum_{v \in W} d_{\mathcal{F}}(v)$, contradicting the atomic assumption on $\mathcal{F}$.

(2) $\mathcal{G}$ contains no odd pie.

Suppose on the contrary that $\mathcal{P}$ is an odd pie in $\mathcal{G}$. Since $\mathcal{P}$ is fully contained in $\mathcal{H}$, Lemma 3.3 guarantees the existence of four edges P_1, P_2, P_3, P_4 of $\mathcal{H}$, with $\{P_3, P_4\} \subseteq \mathcal{P}$, as described in (3.1), contradicting (1) for $\mathcal{P} \subseteq \mathcal{F}$.

Let B be the $\mathcal{F} - W$ incidence matrix. By Lemma 3.2 (with respect to $\mathcal{G}$), matrix B is totally unimodular. From the Ghouila-Houri theorem (see Theorem 19.3 of [18]), it follows that the rows of B can be split into two parts so that the sum of rows in one part minus the sum of the rows in the other part is a vector with entries only 0, +1, and −1. Clearly, these two parts correspond to an equitable subpartition of $\mathcal{F}$, completing the proof. ■

3.2 Cycle hypergraphs

Throughout this subsection, by a cycle in a digraph we always mean a directed one. Let $G = (V, E)$ be a graph (undirected or directed) and let $\boldsymbol{w} \in \mathbb{Z}_+^V$. A *feedback vertex set* (FVS) of G is a vertex subset that intersects each cycle in G, and a ***w**-cycle packing* of G is a collection $\mathcal{C}$ of cycles (with repetition allowed) such that each vertex v is contained in at most $w(v)$ members of $\mathcal{C}$. The *feedback vertex set problem* is to find an FVS with minimum total weight (denoted by $\tau(G, \boldsymbol{w})$), while the *cycle packing problem* is to find a $\boldsymbol{w}$-cycle packing with maximum size (denoted by $\nu(G, \boldsymbol{w})$). It is well known that both of them are NP-hard, so neither can be solved in polynomial time unless $NP = P$. We call G *cycle Mengerian* (CM) if $\tau(G, \boldsymbol{w}) = \nu(G, \boldsymbol{w})$ for any $\boldsymbol{w} \in \mathbb{Z}_+^V$. Since a structural characterization of all CM graphs yields not only a beautiful min-max theorem but also polynomial-time algorithms for both the feedback vertex set and the cycle packing problems, this graph class has been a subject of extensive research. In [9], Ding and Zang obtained a characterization of all CM undirected graphs. Due to the long list of forbidden structures, to find a good characterization of all CM digraphs seems to be extremely difficult. While this characterization problem remains open in general, it was resolved completely on tournaments by Cai, Deng, and Zang [5]. (As usual, a *tournament* is an orientation of an undirected complete graph.) The purpose of this subsection is to give a strengthening of each of these two results.

Let us define three more terms before presenting our theorems. An *odd ring* is a graph obtained from an odd cycle by replacing each edge $e = xy$ with either a triangle containing e or two triangles xab, ycd together with two additional edges ac and bd (see Figure 1). A *wheel* is obtained from a cycle by adding a new vertex and making it adjacent to all vertices of the cycle. Let $G = (V, E)$ be a graph (undirected or directed) and let $\mathcal{E}$ consist of the vertex sets of all cycles in G. Then $\mathcal{H} = (V, \mathcal{E})$ is called the *cycle hypergraph* of G. For convenience, we use $\mathfrak{L}$ to denote the class of all simple undirected graphs containing no induced subgraph isomorphic to a subdivision of an odd ring, or $K_{2,3}$, or a wheel.

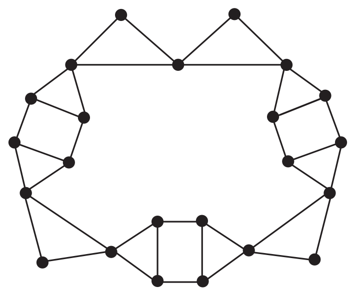

Figure 1: An odd ring obtained from a cycle of length 7.

Theorem 3.4 *Let $G = (V, E)$ be a simple undirected graph and let $\mathcal{H} = (V, \mathcal{E})$ be its cycle hypergraph. Then the following statements are equivalent:*

(i) $G \in \mathfrak{L}$;

(ii) $\mathcal{H}$ is ideal;

(iii) $\mathcal{H}$ is Mengerian;

(iv) $\mathcal{H}$ is box-Mengerian; and

(v) $\mathcal{H}$ is ESP.

The equivalence of (i), (ii), (iii), and (v) was established by Ding and Zang [9]; our contribution here is to strengthen the original total dual integrality as box-total dual integrality, see (iv), whose validity follows instantly from Corollary 1.3.

Theorem 3.5 *Let $G = (V, E)$ be a tournament and let $\mathcal{H} = (V, \mathcal{E})$ be its cycle hypergraph. Then the following statements are equivalent:*

(i) G contains neither F_1 nor F_2 as a subgraph (see Figure 2);

(ii) $\mathcal{H}$ is ideal;

(iii) $\mathcal{H}$ is Mengerian;

(iv) $\mathcal{H}$ is box-Mengerian; and

(v) $\mathcal{H}$ is ESP.

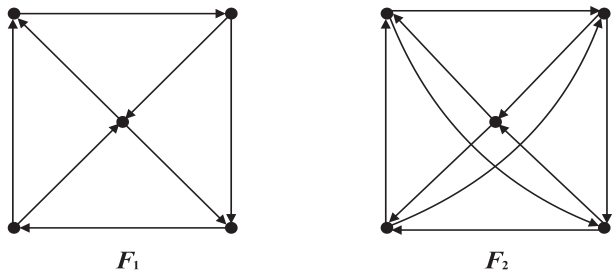

Figure 2: *Forbidden subgraphs F_1 and F_2.*

The equivalence of (i) and (iii) was derived by Cai, Deng, and Zang [5]. Our main objective here is to exhibit box-TDI and ESP properties associated with the same combinatorial structures.

A *triangle* is a directed cycle of length three. Note that a vertex subset of a tournament is an FVS if and only if it intersects all triangles, and that the cycle packing problem on tournaments actually reduces to the triangle packing problem. Our proof of the above theorem is based on the following structural description, which is a combination of Lemma 2.1, Corollary 2.1, and Corollary 2.2 in [5].

Lemma 3.6 *Let $G = (V, E)$ be a strongly connected tournament containing no F_1 nor F_2 as a subgraph. Then V can be partitioned into $V_1, V_2, \ldots, V_k$ for some k with $3 \le k \le |V|$, which have the following properties:*

(i) For each $i = 1, 2, \ldots, k$, V_i is acyclic and hence admits a linear order $\prec$ such that $x \prec y$ whenever (x, y) is an arc in V_i;

(ii) For each $i = 1, 2, \ldots, k-1$, if (u, v) is an arc from V_{i+1} to V_i, then

- *(x, v) is an arc for any $x \in V_{i+1}$ with $x \prec u$, and*
- *(u, y) is an arc for any $y \in V_i$ with $v \prec y$;*

(iii) For any i, j with $1 \le i \le j-2 \le k-2$, each arc between V_i and V_j is directed from V_i to V_j;

(iv) For any triangle xyz in G, there exists a subscript i with $1 \le i \le k-2$ such that $x \in V_{i+2}$, $y \in V_{i+1}$, and $z \in V_i$ (renaming x, y and z if necessary).

Proof. Statements (i), (iii) and the second half of (ii) are contained in Lemma 2.1 in [5], the first half of (ii) is established in Corollary 2.1, and (iv) is exactly the same as Corollary 2.2 in [5]. ∎

Lemma 3.7 *Let $G = (V, E)$ be a tournament and let $\mathcal{H} = (V, \mathcal{E})$ be its cycle hypergraph. If G contains F_1 or F_2 as a subgraph, then $\mathcal{H}$ is nonideal.*

Proof. By hypothesis, G contains F_i as a subgraph for $i = 1$ or 2. Let A be the $\mathcal{E} - V$ incidence matrix and let $\boldsymbol{w} \in \mathbb{Z}_+^V$ such that $w(v) = 1$ if v is a vertex in F_i and 0 otherwise. Let $\tau^*(\mathcal{H}, \boldsymbol{w})$ denote the optimal value of the linear program $\min\{\boldsymbol{w}^T\boldsymbol{x} : A\boldsymbol{x} \geq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0}\}$. Define $\boldsymbol{x} \in \mathbb{Q}_+^V$ such that $x(v) = 1/3$ if v is a vertex in F_i and 1 otherwise. Obviously, $\boldsymbol{x}$ is a feasible solution to the problem with objective value 5/3. So $\tau^*(\mathcal{H}, \boldsymbol{w}) \leq 5/3$.

It is easy to check that the vertices of the above F_i can be labeled as v_1, v_2, v_3, v_4, v_5 so that $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, and $\{v_1, v_4, v_5\}$ are the vertex sets of three triangles T_1, T_2, T_3 in G, respectively. Let B be the $3 \times |V|$ submatrix of A whose rows correspond to T_1, T_2, T_3. Consider the LP-duality relation

$$\min\{\boldsymbol{w}^T\boldsymbol{x} : B\boldsymbol{x} \geq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0}\} = \max\{\boldsymbol{y}^T\mathbf{1} : \boldsymbol{y}^T B \leq \boldsymbol{w}^T,\ \boldsymbol{y} \geq \mathbf{0}\}. \tag{3.2}$$

Define $\boldsymbol{x} \in \mathbb{Q}_+^V$ such that $x(v) = 1/2$ if $v \in \{v_1, v_2, v_4\}$, 0 if $v \in \{v_3, v_5\}$, and 1 otherwise, and define $\boldsymbol{y} \in \mathbb{Q}_+^{\mathcal{E}}$ such that $y(U) = 1/2$ if $U \in \{T_1, T_2, T_3\}$ and 0 otherwise. Clearly, $\boldsymbol{x}$ and $\boldsymbol{y}$ are feasible solutions to the above primal-dual pair in (3.2), respectively. Since $\boldsymbol{w}^T\boldsymbol{x} = \boldsymbol{y}^T\mathbf{1} = 3/2$, from the LP-duality theorem we conclude that $\boldsymbol{x}$ and $\boldsymbol{y}$ are actually optimal solutions and 3/2 is the optimal value. As B is a submatrix of A, the optimal value of the minimization problem in (3.2) is bounded above by $\tau^*(\mathcal{H}, \boldsymbol{w})$. So $\tau^*(\mathcal{H}, \boldsymbol{w}) \geq 3/2$.

Combining the above two inequalities, we get $3/2 \leq \tau^*(\mathcal{H}, \boldsymbol{w}) \leq 5/3$. Hence $\tau^*(\mathcal{H}, \boldsymbol{w})$ in not integral and therefore $\mathcal{H}$ is nonideal. ■

Suppose $G = (V, E)$ is a strongly connected tournament containing no F_1 nor F_2 as a subgraph. Then V admits a partition $\{V_1, V_2, \ldots, V_k\}$ as described in Lemma 3.6. Let D denote the digraph obtained from G by deleting all arcs in V_i for each i and deleting all arcs from V_i to V_j for any $i < j$, let P_3 denote a (directed) path with 3 vertices in D, and let $\mathcal{P}_3$ be the set of all P_3's in D.

Consider the order $\prec$ introduced in Lemma 3.6. Recall that this order does not apply to any two vertices in distinct V_i's. Let us now fill this gap by extending $\prec$ to the whole vertex set V. Define $u \prec v$ for any $u \in V_i$ and $v \in V_j$ with $i < j$. Note that if $v_1v_2v_3$ is a P_3 in D, then $v_3 \prec v_2 \prec v_1$. The order $\prec$ on V naturally yields a lexicographic order on $\mathcal{P}_3$. Let $Q_1 = u_1u_2u_3$ and $Q_2 = v_1v_2v_3$ be two P_3's in D. Define $Q_1 \prec Q_2$ if $u_j \prec v_j$ for the *largest* subscript j with $u_j \neq v_j$.

Two directed paths $P = u_1u_2u_3$ and $P' = u'_1u'_2u'_3$ in D are said to be *crossing* if some V_ℓ, $1 \leq \ell \leq k$, contains two vertices u_i, u'_j, $1 \leq i, j \leq 3$, such that $u'_j \prec u_i$ while $P \prec P'$ or such that $u_i \prec u'_j$ while $P' \prec P$. Suppose P and P' form a crossing pair with $P \prec P'$. In view of Lemma 3.6(iv), P is contained in $\cup_{h=s}^{s+2} V_h$ for some $s \leq k-2$, P' is contained in $\cup_{h=t}^{t+2} V_h$ for some t with $s \leq t \leq s+2$, and each V_h contains at least one and at most two vertices of P and P', where $s \leq h \leq t+2$. Let v_h and v'_h denote the vertices in $V_h \cap (V(P) \cup V(P'))$ with $v_h \preceq v'_h$ ($v_h = v'_h$ if V_h contains only one vertex of P and P'). Define

$$P \wedge P' = v_{s+2}v_{s+1}v_s \quad \text{and} \quad P \vee P' = v'_{t+2}v'_{t+1}v'_t.$$

We claim that

$$\{P \wedge P',\ P \vee P'\} \subseteq \mathcal{P}_3 \ \text{and}\ \ P \wedge P' \prec P \prec P' \prec P \vee P'. \tag{3.3}$$

To justify this, note that for $h = s+1$ and $s+2$, if neither P nor P' contains (v_h, v_{h-1}), then $v_h \prec v'_h$ and (v'_h, v_{h-1}) is an arc of P or P'. Thus by Lemma 3.6(ii), $(v_h, v_{h-1}) \in E$. It follows that $P \wedge P' \in \mathcal{P}_3$. Similarly, $P \vee P' \in \mathcal{P}_3$. By the definition of crossing paths, some V_ℓ, $1 \le \ell \le k$, contains two vertices u_i, u'_j, $1 \le i, j \le 3$, such that $u'_j \prec u_i$. Since $u'_j \in P \wedge P'$ and $u_i \in P \vee P'$, we have $P \wedge P' \prec P \prec P' \prec P \vee P'$. So (3.3) is established.

Let $\mathcal{C}_3$ stand for the set of all triangles in G. From Lemma 3.6(iv), we see that there is a one-to-one correspondence between $\mathcal{C}_3$ and $\mathcal{P}_3$: a triangle in $\mathcal{C}_3$ and a P_3 in $\mathcal{P}_3$ with the same vertex set correspond to each other. For $i = 1, 2$, let $T_i \in \mathcal{C}_3$ and let Q_i be the P_3 corresponding to T_i. We call T_1 and T_2 *crossing* if Q_1 and Q_2 are crossing, and define $T_1 \prec T_2$ if $Q_1 \prec Q_2$. Similarly, we can define $T_1 \wedge T_2$ and $T_1 \vee T_2$. Let $\Lambda = \{T_1, T_2, \ldots, T_m\}$ and $\Lambda' = \{T'_1, T'_2, \ldots, T'_m\}$ be two collections of triangles in G, whose members are arranged in *nondecreasing* order with respect to $\prec$. Define $\Lambda \prec \Lambda'$ if $T_i \prec T'_i$ for the smallest subscript i with $T_i \ne T'_i$.

Proof of Theorem 3.5. By Corollary 1.3 and Lemma 3.7, we have (v)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i). It remains to establish the implication (i)$\Rightarrow$(v).

Let $G = (V, E)$ be a tournament containing no F_1 nor F_2 as a subgraph and let $\mathcal{H} = (V, \mathcal{E})$ be its cycle hypergraph. To show that $\mathcal{H}$ is ESP, we apply induction on $|V|$. The statement holds trivially when $|V| = 1$, so we proceed to the induction step. We may assume that

(1) G is strongly connected. Thus V admits a partition $\{V_1, V_2, \ldots, V_k\}$ as described in Lemma 3.6.

Otherwise, let G_1 be a strongly connected component of G, let G_2 be the graph obtained from G by deleting all vertices in G_1, and let $\mathcal{H}_i$ be the cycle hypergraph of G_i for $i = 1, 2$. Then $\mathcal{H}$ is the 0-sum of $\mathcal{H}_1$ and $\mathcal{H}_2$. By induction hypothesis, $\mathcal{H}_i$ is ESP for $i = 1, 2$; so is $\mathcal{H}$ using Lemma 2.6. Hence (1) holds.

By Lemma 2.5, it suffices to show that every atomic subset Ω of $\mathcal{E}$ admits an equitable subpartition. For this purpose, let Λ be an arbitrary atomic edge collection that dominates Ω (see Lemma 2.2). Observe that

(2) Each member in Λ is a triangle and $|\Lambda| = |\Omega|$.

Suppose not, some C in Λ is a cycle of length at least four. Since G is a tournament, we can find a triangle T whose vertices are all contained in C. Let Λ' be obtained from Λ by replacing C with T. Clearly, Λ' dominates Λ and $\sum_{v \in V} d_{\Lambda'}(v) < \sum_{v \in V} d_\Lambda(v)$, contradicting the atomic assumption on Λ. Since Ω is atomic, by definition we have $|\Lambda| = |\Omega|$. This proves (2).

Recall that we have defined the lexicographic order $\prec$ for triangle collections of the same size. So (2) allows us to fix an *atomic* edge collection Λ that dominates Ω and

(3) Subject to this, Λ has the smallest lexicographic order.

(4) No two triangles in Λ are crossing.

Suppose the contrary: T_1 and T_2 in Λ form a crossing pair with $T_1 \prec T_2$. From (3.3), we deduce that $T_1 \wedge T_2 \prec T_1 \prec T_2 \prec T_1 \vee T_2$. Let Λ' be the edge collection obtained from Λ by replacing $\{T_1, T_2\}$ with $\{T_1 \wedge T_2, T_1 \vee T_2\}$. Then Λ' dominates Λ (and hence Ω). Since Λ is atomic, so is Λ'. Clearly, $\Lambda' \prec \Lambda$, this contradiction to (3) yields (4).

Write $\Lambda = \{T_1, T_2, \ldots, T_m\}$, where $m = |\Lambda|$ and $T_i \preceq T_{i+1}$ for $i = 1, 2, \ldots, m-1$.

(5) For any i, j with $1 \leq i < j \leq m$ and any $v \in V$, if both T_i and T_j contains v, then so does T_h for any h between i and j.

Assume on the contrary that v is not contained in T_h for some h with $i < h < j$. Then $T_i \prec T_h \prec T_j$. By (1), $v \in V_g$ for some g with $1 \leq g \leq k$. If T_h contains no vertex in V_g, then T_h is fully contained in $\cup_{t=1}^{g-1} V_t$ or fully contained in $\cup_{t=g+1}^{k} V_t$. It follows that $T_h \prec T_i$ or $T_j \prec T_h$, contradicting our previous observation. So T_h contains a vertex u in V_g. It is easy to see that Q_i and Q_h are crossing if $u \prec v$ and Q_h and Q_j are crossing if $v \prec u$, contradicting (4). So (5) is justified.

Set $\Lambda_1 = \{T_i : i \text{ is odd and } 1 \leq i \leq m\}$ and $\Lambda_2 = \{T_i : i \text{ is even and } 1 \leq i \leq m\}$. Clearly, we have

$$|\Lambda_1| + |\Lambda_2| = |\Lambda| \text{ and } d_{\Lambda_1 \cup \Lambda_2}(v) = d_\Lambda(v) \text{ for all } v \in V.$$

Consider an arbitrary vertex v of G. In view of (5), we may assume that $T_t, T_{t+1}, \ldots, T_{t+d_\Lambda(v)-1}$ are the $d_\Lambda(v)$ triangles in Λ containing v for some t. It follows from the definitions of Λ_1 and Λ_2 that

$$d_{\Lambda_i}(v) \leq \lceil d_\Lambda(v)/2 \rceil \text{ for } i = 1, 2.$$

So (Λ_1, Λ_2) is an equitable subpartition of Λ and hence of Ω by Lemma 2.3, completing the proof. ■

3.3 Matroid ports

As usual, let $U_{2,4}$ be the uniform matroid on four elements of rank two, let F_7 be the Fano matroid, let F_7^* be the dual of F_7, and let F_7^+ be the unique series extension of F_7. We refer to Oxley [16] for an in-depth account of matroid theory and undefined terms.

Let M be a matroid [16] on $E \cup \{\ell\}$, where $\ell \notin E$ is a distinguished element of M. A matroid obtained from M by deleting and contracting elements in E is called a *minor* of M *using* ℓ. The *ℓ-port* of M is the hypergraph $\mathcal{P}_{M,\ell} = (E, \mathcal{E})$, where $\mathcal{E} = \{P : P \subseteq E \text{ with } P \cup \{\ell\} \text{ a circuit of } M\}$. In [20], Seymour characterized all pairs (M, ℓ) for which $\mathcal{P}_{M,\ell}$ is Mengerian; this theorem yields many important min-max relations in combinatorial optimization and has attracted tremendous research efforts in matroid optimization.

Let A be the $\mathcal{E} - E$ incidence matrix. For any vectors $\boldsymbol{l}, \boldsymbol{u}$, let $Q(A, \boldsymbol{l}, \boldsymbol{u})$ denote the polytope $\{\boldsymbol{x} : A\boldsymbol{x} \geq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0}\} \cap \{\boldsymbol{x} : \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}\}$. Given a positive integer d, we call $\mathcal{P}_{M,\ell}$ *box $\frac{1}{d}$-integral* if all vertices of $Q(A, \boldsymbol{l}, \boldsymbol{u})$ are $\frac{1}{d}$-integral for all $\frac{1}{d}$-integral vectors $\boldsymbol{l}, \boldsymbol{u}$. In [14], Gerards and Laurent obtained a structural characterization of all pairs (M, ℓ) for which $\mathcal{P}_{M,\ell}$ is box $\frac{1}{d}$-integral for all positive integers d; this theorem can be found in several interesting applications [14, 15].

Recently, Chen, Ding, and Zang [6] managed to characterize all pairs (M, ℓ) for which $\mathcal{P}_{M,\ell}$ is box-Mengerian; this characterization also yields a number of interesting results in combinatorial optimization (see [6]). The purpose of this subsection is to strengthen this box-TDI property with ESP property and to present a much shorter proof than the one given in [6]. For convenience, let $\Im$ be the set of all pairs (M, ℓ), where M is a matroid on at least two elements, including ℓ, such that M has no $U_{2,4}$-minor using ℓ, no F_7^*-minor using ℓ, and no F_7^+-minor using ℓ as a series element.

Theorem 3.8 *Let M be a matroid on $E \cup \{\ell\}$ with $\ell \notin E$. Then the following statements are equivalent:*

(i) $(M, \ell) \in \Im$;

(ii) $\mathcal{P}_{M,\ell}$ *is box* $\frac{1}{d}$*-integral for all positive integers* d*;*

(iii) $\mathcal{P}_{M,\ell}$ *is box-Mengerian; and*

(iv) $\mathcal{P}_{M,\ell}$ *is ESP.*

The equivalence of (i) and (ii) was established by Gerards and Laurent [14], and that of (i) and (iii) was derived by Chen, Ding, and Zang [6]. Since (iv)⇒(iii)⇒(ii)⇒(i), we only need to show that (i)⇒(iv). We point out that our proof curtails many technical parts of Chen, Ding, and Zang's original proof and hence is much easier to follow.

For an edge collection Λ of the ℓ-port $\mathcal{P}_{M,\ell} = (E, \mathcal{E})$, its *incidence vector* is $\boldsymbol{x} \in \mathbb{Z}_+^{\mathcal{E}}$ such that $x(P)$ is the multiplicity of P in Λ for any $P \in \mathcal{E}$. For a set K and a vector $\boldsymbol{y} \in \mathbb{Q}^K$, we define $\bar{\boldsymbol{y}}$ to be the vector $(|y(k)| : k \in K) \in \mathbb{Q}_+^K$.

To prove the theorem, we first consider the case when M is a regular matroid. The following is a combination of Lemmas 4.1 and 4.2 in [6].

Lemma 3.9 [6] *Let M be a regular matroid on $E \cup \{\ell\}$, with $\ell \notin E$, represented by a totally unimodular matrix U, and let A be the $\mathcal{E} - E$ incidence matrix of $\mathcal{P}_{M,\ell}$. Then the following statements hold:*

(i) For any $\boldsymbol{\alpha} \in \mathbb{Z}_+^{\mathcal{E}}$, there exists $\boldsymbol{x} \in \mathbb{Z}^{E\cup\{\ell\}}$ such that $U\boldsymbol{x} = \boldsymbol{0}$, $x(\ell) = \boldsymbol{\alpha}^T\boldsymbol{1}$, and $\bar{\boldsymbol{x}}|_E \le A^T\boldsymbol{\alpha}$.

(ii) For any $\boldsymbol{x} \in \mathbb{Z}^{E\cup\{\ell\}}$ with $U\boldsymbol{x} = \boldsymbol{0}$, there exists $\boldsymbol{\alpha} \in \mathbb{Z}_+^{\mathcal{E}}$ such that $\boldsymbol{\alpha}^T\boldsymbol{1} \ge |x(\ell)|$ and $A^T\boldsymbol{\alpha} \le \bar{\boldsymbol{x}}|_E$.

For regular matroids, the assertion of Theorem 3.8 is established below.

Lemma 3.10 *Let M be a regular matroid on $E \cup \{\ell\}$ with $\ell \notin E$. Then $\mathcal{P}_{M,\ell}$ is ESP.*

Proof. By Lemma 2.5, it suffices to show that every subset Λ of $\mathcal{E}$ admits an equitable subpartition. To this end, let $\boldsymbol{\alpha}$ be the incidence vector of Λ. Then Lemma 3.9(i) guarantees the existence of a vector $\boldsymbol{x} \in \mathbb{Z}^{E\cup\{\ell\}}$ such that

(1) $U\boldsymbol{x} = \boldsymbol{0}$, $x(\ell) = \boldsymbol{\alpha}^T\boldsymbol{1}$, and $\bar{\boldsymbol{x}}|_E \le A^T\boldsymbol{\alpha}$.

Observe that

(2) $\bar{\boldsymbol{x}}|_E = A^T\boldsymbol{\alpha}$.

Otherwise, $\bar{\boldsymbol{x}}|_E \lneq A^T\boldsymbol{\alpha}$. By Lemma 3.9(ii), there exists $\boldsymbol{\beta} \in \mathbb{Z}_+^{\mathcal{E}}$ such that $\boldsymbol{\beta}^T\boldsymbol{1} \ge |x(\ell)|$ and $A^T\boldsymbol{\beta} \le \bar{\boldsymbol{x}}|_E$. It follows that $A^T\boldsymbol{\beta} \lneq A^T\boldsymbol{\alpha}$. Next, let Ω be the edge collection of $\mathcal{P}_{M,\ell}$ with incidence vector $\boldsymbol{\beta}$. Then $|\Omega| = \boldsymbol{\beta}^T\boldsymbol{1} \ge |x(\ell)|$. By (1), we have $|\Lambda| = \boldsymbol{\alpha}^T\boldsymbol{1} = x(\ell) \le |\Omega|$. Therefore Λ, dominated by Ω, is not atomic. This contradiction justifies (2).

Consider the polyhedron $Q = \{\boldsymbol{y} : U\boldsymbol{y} = \boldsymbol{0}, \lfloor \boldsymbol{x}/2 \rfloor \le \boldsymbol{y} \le \lceil \boldsymbol{x}/2 \rceil\}$. By (1), $\boldsymbol{x}/2 \in Q$, so $Q \ne \emptyset$. Since U is totally unimodular, Q contains an integral vector $\boldsymbol{y}_1 \in \mathbb{Z}^{E\cup\{\ell\}}$. Set $\boldsymbol{y}_2 = \boldsymbol{x} - \boldsymbol{y}_1$. From (1) and the definition of Q, we deduce that

(3) $\{\boldsymbol{y}_1, \boldsymbol{y}_2\} \subseteq Q \cap \mathbb{Z}^{E\cup\{\ell\}}$ and $\bar{\boldsymbol{y}}_1 + \bar{\boldsymbol{y}}_2 = \bar{\boldsymbol{x}}$.

By Lemma 3.9(ii), there exists $\boldsymbol{\alpha}_i \in \mathbb{Z}_+^{\mathcal{E}}$ such that $\boldsymbol{\alpha}_i^T\boldsymbol{1} \ge |y_i(\ell)|$ and $A^T\boldsymbol{\alpha}_i \le \bar{\boldsymbol{y}}_i|_E$ for $i = 1, 2$. It follows from (3) that

(4) $\boldsymbol{\alpha}_1^T\boldsymbol{1} + \boldsymbol{\alpha}_2^T\boldsymbol{1} \ge |y_1(\ell)| + |y_2(\ell)| = |x(\ell)|$ and $A^T\boldsymbol{\alpha}_1 + A^T\boldsymbol{\alpha}_2 \le \bar{\boldsymbol{y}}_1|_E + \bar{\boldsymbol{y}}_2|_E = \bar{\boldsymbol{x}}|_E$.

For $i = 1, 2$, let Λ_i be the edge collection of $\mathcal{P}_{M,\ell}$ with incidence vector $\boldsymbol{\alpha}_i$. Using (4) and (2), we obtain $|\Lambda_1|+|\Lambda_2| = \boldsymbol{\alpha}_1^T\mathbf{1}+\boldsymbol{\alpha}_2^T\mathbf{1} \geq |x(\ell)| = |\Lambda|$ and $A^T\boldsymbol{\alpha}_1+A^T\boldsymbol{\alpha}_2 \leq \bar{\boldsymbol{x}}|_E = A^T\boldsymbol{\alpha}$, which amounts to saying that $d_{\Lambda_1\cup\Lambda_2}(e) \leq d_\Lambda(e)$ for all $e \in E$. Recall that $A^T\boldsymbol{\alpha}_i \leq \bar{\boldsymbol{y}}_i|_E$ for $i = 1, 2$. Hence for each $e \in E$, we have $d_{\Lambda_i}(e) \leq |y_i(e)|$. It follows from (3), the definition of Q, and (2) that $d_{\Lambda_i}(e) \leq \lceil |x(e)|/2 \rceil = \lceil d_\Lambda(e)/2 \rceil$. Therefore (Λ_1, Λ_2) is an equitable subpartition of Λ. ■

To prove Theorem 3.8 for the general case, we shall appeal to a structural description of all (M, ℓ) in $\Im$. Let us define a few more terms before proceeding. Let M be a matroid on E. A partition (E_1, E_2) of E is called a k-*separation*, where k is a positive integer, if $\min\{|E_1|, |E_2|\} \geq k$ and $r(E_1)+r(E_2) \leq r(E)+k-1$. We say that the k-separation (E_1, E_2) is *strict* if $r(E_1) + r(E_2) = r(E) + k - 1$. A matroid is k-*connected* if it has no k'-separation, for any $k' < k$. As customary, 2-connected matroids are called *connected* and others are *disconnected*.

For $i = 1, 2$, let M_i be a matroid on E_i with $E_1 \cap E_2 = \emptyset$. The 1-*sum* of M_1 and M_2 is the matroid M on $E_1 \cup E_2$ such that C is a circuit of M if and only if it is a circuit of either M_1 or M_2. Next, for $i = 1, 2$, let M_i be a matroid on $E_i \cup \{p\}$, where $E_1 \cap E_2 = \emptyset$ and $p \notin E_1 \cup E_2$. The *parallel connection* of M_1 and M_2 is the matroid M on $E_1 \cup E_2 \cup \{p\}$ such that C is a circuit of M if and only if C is a circuit of M_1 or M_2, or $C = (C_1 - \{p\}) \cup (C_2 - \{p\})$, where, for $i = 1, 2$, C_i is a circuit of M_i that contains p. Matroid $M\backslash p$ is called the 2-*sum* of M_1 and M_2.

Let $\Im^c$ denote the set of all pairs (M, ℓ) in $\Im$ with M connected. The following structural theorem on $\Im^c$ is due to Gerards and Laurent.

Theorem 3.11 [14] *If $(M, \ell) \in \Im^c$, then either M is regular or M/ℓ is not* 3*-connected.*

Let M be a matroid on $E \cup \{\ell\}$ with $\ell \notin E$ and $(M, \ell) \in \Im$. Note that M is a binary matroid. To keep track of the information about M while working on M/ℓ, we need to consider a representation of M on M/ℓ. A *signed matroid* is a pair (N, Σ), where N is a binary matroid on E and Σ is a subset of E. A subset $X \subseteq E$ is called Σ-*odd* or Σ-*even* if $|X \cap \Sigma|$ is odd or even, respectively. For convenience, let $\mathcal{O}_{N,\Sigma}$ denote the hypergraph of all Σ-odd circuits in N, and let $\Re$ denote the family of all subsets Σ of E such that $\Sigma \cup \{\ell\}$ is a cocircuit of M.

In the remaining discussion we always assume that $N = M/\ell$. Since M is a binary matroid,

$$\mathcal{P}_{M,\ell} = \mathcal{O}_{N,\Sigma} \text{ for every } \Sigma \in \Re. \tag{3.4}$$

Thus we shall make no effort in distinguishing between them and use whichever is more convenient.

Proof of Theorem 3.8. To show that (i)⇒(iv), we assume the contrary: $\mathcal{P}_{M,\ell} = (E, \mathcal{E})$ is not ESP for some $(M, \ell) \in \Im$ and, subject to this, $|E(M)|$ is minimum. Then, by Lemma 2.5, some atomic subset Ω of $\mathcal{E}$ admits no equitable subpartition. It follows that $\mathcal{E} \neq \emptyset$ and $\mathcal{E} \neq \{\emptyset\}$, implying

(1) ℓ is neither a loop nor a coloop of M.

If M is not connected, then M is the 1-sum of some matroids M_1 and M_2 with $\ell \in E(M_1)$, and hence $\mathcal{P}_{M,\ell}$ is the 0-sum of $\mathcal{P}_{M_1,\ell}$ and the hypergraph $(E(M_2), \emptyset)$. From the minimality assumption on $\mathcal{P}_{M,\ell}$, we deduce that $\mathcal{P}_{M_1,\ell}$ is ESP for $|E(M_1)| < |E(M)|$. Trivially, $(E(M_2), \emptyset)$ is also ESP. Hence, by

Lemma 2.6, $\mathcal{P}_{M,\ell}$ is ESP; this contradiction implies that M is connected, which, together with (1), yields $(M,l) \in \Im^c$. So, by Lemma 3.10, M is not a regular matroid. Set $N = M/\ell$. It follows from Theorem 3.11 that

(2) N is not 3-connected.

If N has a 1-separation (E_1, E_2), let $\mathcal{H}_1$ (resp. $\mathcal{H}_2$) denote the ℓ-port of the matroid $M\backslash E_2$ (resp. $M\backslash E_1$), then $\mathcal{P}_{M,\ell}$ is the 0-sum of $\mathcal{H}_1$ and $\mathcal{H}_2$. By the minimality assumption on (M,ℓ), both $\mathcal{H}_1$ and $\mathcal{H}_2$ are ESP. From Lemma 2.6, we see that so is $\mathcal{P}_{M,\ell}$, a contradiction. Hence

(3) N is connected.

From (2) and (3), we deduce that N has a strict 2-separation (E_1, E_2) and thus (see Theorem 8.3.1 in [16])

(4) N is the 2-sum of its minors N_1 and N_2 such that $E(N_i) = E_i \cup \{p\}$ for $i = 1, 2$, where $p \notin E$.

Let us partition the edge set $\mathcal{E}$ of $\mathcal{P}_{M,\ell}$ into $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2$, such that $\mathcal{E}_i$ is the set of all edges contained in E_i for $i = 1, 2$ and $\mathcal{E}_0 = \mathcal{E} - \mathcal{E}_1 - \mathcal{E}_2$. For any $\Sigma \in \Re$, from (3.4) we see that $\mathcal{E}_i$ is the set of all Σ-odd circuits of N contained in E_i for $i = 1, 2$, and $\mathcal{E}_0$ consists of all Σ-odd circuits of N that meet both E_1 and E_2. Clearly, we have

(5) For any $P \in \mathcal{E}_0$ and $\Sigma \in \Re$, *either* $P \cap E_1$ is Σ-odd and $P \cap E_2$ is Σ-even *or* $P \cap E_1$ is Σ-even and $P \cap E_2$ is Σ-odd.

For the edge collection Ω specified at the beginning of our proof, let us choose an edge collection Λ of $\mathcal{P}_{M,\ell}$ such that

(6) Λ dominates Ω and, subject to this, $|\Lambda \cap \mathcal{E}_0|$ is minimum.

Two members P_1, P_2 of $\mathcal{E}_0$ are called Σ-*crossing* for some $\Sigma \in \Re$ if both $P_1 \cap E_1$ and $P_2 \cap E_2$ are Σ-odd. Observe that

(7) Λ contains no Σ-crossing pair for any $\Sigma \in \Re$.

Suppose the contrary: Λ contains an Σ-crossing pair P_1, P_2 for some $\Sigma \in \Re$. For $i, j \in \{1, 2\}$, recall that $P_{ij} = (P_i \cap E_j) \cup \{p\}$ is a circuit of the matroid N_j. By definition, both P_{11} and P_{22} are Σ-odd and hence, using (4), both P_{12} and P_{21} are Σ-even. It follows that the symmetric differences $P_{11}\Delta P_{21}$ and $P_{12}\Delta P_{22}$ are both Σ-odd. For $j = 1, 2$, since N_j is binary, $P_{1j}\Delta P_{2j}$ is a disjoint union of circuits of $N_j \backslash p$. Thus at least one of these circuits, denoted by C_j, is Σ-odd. Clearly, $C_j \in \mathcal{E}_j$ for $j = 1, 2$ and $d_{C_1}(e) + d_{C_2}(e) \le d_{P_1}(e) + d_{P_2}(e)$ for all $e \in E$. Let Λ' be obtained from Λ by replacing $\{P_1, P_2\}$ with $\{C_1, C_2\}$. Then Λ' dominates Λ and hence Ω. Moreover, $|\Lambda' \cap \mathcal{E}_0| < |\Lambda \cap \mathcal{E}_0|$, this contradiction to (6) yields (7).

By (3), N is connected, so $\mathcal{E}_0 \neq \emptyset$. Let us choose P^* in $\mathcal{E}_0$ such that $P^* \in \Lambda \cap \mathcal{E}_0$ provided this set is nonempty. Symmetry and (5) allow us to assume that $P^* \cap E_1$ is Σ-odd for some $\Sigma \in \Re$. In [14] (see page 201), Gerards and Laurent made the following observation:

(8) For any $e_1 \in P^* \cap E_2$ and $e_2 \in P^* \cap E_1$, there exists $\Sigma^* \in \Re$ with the following properties:

(a) $\Sigma^* \cup \{\ell\}$ is the fundamental cocircuit of ℓ in a base X^* of M containing $(P^* - \{e_2\}) \cup \{\ell\}$;

(b) $e_1 \notin \Sigma^*$ and $e_2 \in \Sigma^*$; and

(c) $P^* \cap E_1$ is Σ^*-odd.

For completeness, we include their proof here: Let X^* be a base of M containing $(P^* - \{e_2\}) \cup \{\ell\}$ and let $\Sigma^* \cup \{\ell\}$ be the fundamental cocircuit of ℓ in X^*. Then $e_2 \in \Sigma^*$ for $P^* \cup \{\ell\}$ is the fundamental circuit of e_2 in X^*, and $e_1 \notin \Sigma^*$ for $e_1 \in X^*$. Thus both (a) and (b) hold. Since Σ^* is disjoint from $P^* - \{e_2\}$, we also have (c). This proves (8).

Define $M_1 = M/((P^* \cap E_2) - \{e_1\})\backslash(E_2 - P^*)$ and $M_2 = M/((P^* \cap E_1) - \{e_2\})\backslash(E_1 - P^*)$. Note that $E(M_1) = E_1 \cup \{e_1, \ell\}$, $E(M_2) = E_2 \cup \{e_2, \ell\}$, and $(M_i, \ell) \in \Im$ for $i = 1, 2$. Let X^* and Σ^* be as described in (8). Set $X_i = X^* \cap E_i$ and $Y_i = E_i - X_i$ for $i = 1, 2$. Then $M_1 \cong M/(X_2 - \{e_1\})\backslash Y_2$ and $M_2 \cong M/X_1\backslash(Y_1 - \{e_2\})$. Moreover, $(P^* \cap E_1) \cup \{e_1, \ell\}$ is a circuit of M_1 and $(P^* \cap E_2) \cup \{e_2, \ell\}$ is a circuit of M_2. Rename the element e_i in $E(M_i)$ as p for $i = 1, 2$, and set $\Sigma_1 = \Sigma^* \cap E_1$ and $\Sigma_2 = (\Sigma^* \cap E_2) \cup \{p\}$. Then

(9) $N_i = M_i/\ell$, $\Sigma_i \subseteq E(M_i) - \{\ell\}$, and $\Sigma_i \cup \{\ell\}$ is a cocircuit of M_i for $i = 1, 2$.

From (9) and (3.4), it can be seen that

(10) $\mathcal{P}_{M_i,\ell} = \mathcal{O}_{N_i,\Sigma_i}$ for $i = 1, 2$.

By the minimality assumption on $\mathcal{P}_{M,\ell}$, we obtain

(11) $\mathcal{P}_{M_i,\ell}$ is ESP for $i = 1, 2$.

For convenience, set $\Lambda_i = \Lambda \cap \mathcal{E}_i$ for $i = 0, 1, 2$. Then $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$. By (c) of statement (8), $P^* \cap E_1$ is Σ^*-odd. By (5) and (7), $P \cap E_1$ is Σ^*-odd for any $P \in \Lambda_0$. It follows from (4) and the definition of Σ_1 and Σ_2 that

(12) For any $P \in \Lambda_0$ and $i = 1, 2$, $(P \cap E_i) \cup \{p\}$ is an Σ_i-odd circuit in N_i.

Set $\Pi_i = \Lambda_i \cup \{(P \cap E_i) \cup \{p\} : P \in \Lambda_0\}$ for $i = 1, 2$, Combining (9) and (12), we see that

(13) The following statements hold:

(a) Π_i is an edge collection of $\mathcal{P}_{M_i,\ell}$ for $i = 1, 2$; and

(b) $|\Pi_i| = |\Lambda_0| + |\Lambda_i|$ and $d_{\Pi_i}(p) = |\Lambda_0|$ for $i = 1, 2$.

It follows from (11) that

(14) For $i = 1, 2$, the edge collection Π_i admits an equitable subpartition (Π_i^1, Π_i^2) in $\mathcal{P}_{M_i,\ell}$ such that

(a) $|\Pi_i^1| + |\Pi_i^2| \geq |\Pi_i|$;

(b) $d_{\Pi_i^1 \cup \Pi_i^2}(e) \leq d_{\Pi_i}(e)$ for all $e \in E_i \cup \{p\}$; and

(c) $\max\{d_{\Pi_i^1}(e), d_{\Pi_i^2}(e)\} \leq \lceil d_{\Pi_i}(e)/2 \rceil$ for all $e \in E_i \cup \{p\}$.

Now let us partition $\Pi_i^1 \cup \Pi_i^2$ into two collections Γ_i^1 and Γ_i^2, such that each member of Γ_i^1 contains p and no member of Γ_i^2 contains p for $i = 1, 2$. Then $|\Gamma_i^1| + |\Gamma_i^2| = |\Pi_i^1| + |\Pi_i^2|$ for $i = 1, 2$. By (14), (13), and the definition of M_1 and M_2, we have

(15) $|\Gamma_i^1| + |\Gamma_i^2| \geq |\Lambda_0| + |\Lambda_i|$, and each member of Γ_i^2 is an edge of $\mathcal{P}_{M,\ell}$ for $i = 1, 2$.

Moreover,

(16) $|\Gamma_i^1| \leq |\Lambda_0|$ and $|\Gamma_i^2| \geq |\Lambda_i|$ for $i = 1, 2$.

Indeed, from (14) and (13) it can be seen that $|\Gamma_i^1| = d_{\Pi_i^1 \cup \Pi_i^2}(p) \leq d_{\Pi_i}(p) = |\Lambda_0|$ and that $|\Gamma_i^2| = |\Pi_i^1| + |\Pi_i^2| - |\Gamma_i^1| \geq |\Pi_i| - |\Lambda_0| = |\Lambda_i|$ for $i = 1, 2$. So (16) is true.

For $i = 1, 2$, write $\Gamma_i^1 = \{P_1^i, P_2^i, \ldots, P_{t_i}^i\}$, where $t_i = |\Gamma_i^1|$. Since each member of Γ_i^1 is an edge of

$\mathcal{P}_{M_i,\ell}$, it is Σ_i-odd (recall (3.4)). From the definition of Σ_1 and Σ_2, we see that $P_j^1 - \{p\}$ is $(\Sigma^* \cap E_1)$-odd for $j = 1, 2, \ldots, t_1$ and $P_j^2 - \{p\}$ is $(\Sigma^* \cap E_2)$-even for $j = 1, 2, \ldots, t_2$. Put $k = \min\{t_1, t_2\}$ and set $\Gamma_0 = \{(P_j^1 - \{p\}) \cup (P_j^2 - \{p\}) : j = 1, 2, \ldots, k\}$ (note that $\Gamma_0 = \emptyset$ if $k = 0$). From (4) we derive that

(17) Γ_0 is a collection of Σ^*-odd circuits of N and hence edges of $\mathcal{P}_{M,\ell}$.

Set $\Gamma = \Gamma_0 \cup \Gamma_1^2 \cup \Gamma_2^2$. By (15) and (17), Γ is an edge collection of $\mathcal{P}_{M,\ell}$. For any $e \in E$, we have $d_\Gamma(e) \le d_{\Pi_1^1 \cup \Pi_1^2 \cup \Pi_2^1 \cup \Pi_2^2}(e) = d_{\Pi_1^1 \cup \Pi_1^2}(e) + d_{\Pi_2^1 \cup \Pi_2^2}(e)$. Using (14), we obtain $d_\Gamma(e) \le d_{\Pi_1}(e) + d_{\Pi_2}(e) = d_\Lambda(e)$. Moreover, by definition, $|\Gamma| = |\Gamma_0| + |\Gamma_1^2| + |\Gamma_2^2|$. By symmetry, we may assume that $|\Gamma_1^1| \le |\Gamma_2^1|$. Thus $|\Gamma_0| = |\Gamma_1^1|$ and $|\Gamma| = |\Gamma_1^1| + |\Gamma_1^2| + |\Gamma_2^2|$. By (15) and (16), we have $|\Gamma| \ge |\Lambda_0| + |\Lambda_1| + |\Gamma_2^2| \ge |\Lambda_0| + |\Lambda_1| + |\Lambda_2| = |\Lambda|$. So Γ dominates Λ and hence Ω. By (6), we obtain $|\Lambda_0| \le |\Gamma_0| = |\Gamma_1^1| \le |\Gamma_2^1|$. Thus from (16) we conclude that

(18) $|\Lambda_0| = |\Gamma_0| = |\Gamma_1^1| = |\Gamma_2^1| = k$.

For $i = 1, 2$, by (13) and (18), we have $d_{\Pi_i}(p) = k$ and $d_{\Pi_i^1 \cup \Pi_i^2}(p) = |\Gamma_i^1| = k$. Switching labels 1 and 2 if necessary, we may assume that $d_{\Pi_i^1}(p) \ge d_{\Pi_i^2}(p)$. It follows from (14) that

(19) $d_{\Pi_1^1}(p) = d_{\Pi_2^1}(p) = \lceil k/2 \rceil$ and $d_{\Pi_1^2}(p) = d_{\Pi_2^2}(p) = \lfloor k/2 \rfloor$.

As $\Gamma_i^1 = \{P_1^i, P_2^i, \ldots, P_k^i\}$, we may further assume that

- $P_j^i \in \Pi_i^1$ for $j = 1, 2, \ldots, \lceil k/2 \rceil$, and
- $P_j^i \in \Pi_i^2$ for $j = \lceil k/2 \rceil + 1, \lceil k/2 \rceil + 1, \ldots, k$.

Similar to (17), we can guarantee that

(20) $(P_j^1 - \{p\}) \cup (P_j^2 - \{p\})$ is an Σ^*-odd circuit of N and hence an edge of $\mathcal{P}_{M,\ell}$ for $j = 1, 2, \ldots, k$.

Let Ω^1 (resp. Ω^2) denote the set consisting of $(P_j^1 - \{p\}) \cup (P_j^2 - \{p\})$ for all j with $1 \le j \le \lceil k/2 \rceil$ (resp. $\lceil k/2 \rceil + 1 \le j \le k$). Define

$$\Lambda^i = (\Pi_1^i \cap \Gamma_1^2) \cup (\Pi_2^i \cap \Gamma_2^2) \cup \Omega^i \quad \text{for } i = 1, 2.$$

By (15) and (20), both Λ^1 and Λ^2 are collections of edges in $\mathcal{P}_{M,\ell}$. We propose to show that

(21) (Λ^1, Λ^2) is an equitable subpartition of Λ in $\mathcal{P}_{M,\ell}$.

Observe that, first, $|\Lambda^1| + |\Lambda^2| = |(\Pi_1^1 \cup \Pi_1^2) \cap \Gamma_1^2| + |(\Pi_2^1 \cup \Pi_2^2) \cap \Gamma_2^2| + |\Omega^1 \cup \Omega^2| = |\Gamma_1^2| + |\Gamma_2^2| + k$. By (16) and (18), we have $|\Lambda^1| + |\Lambda^2| \ge |\Lambda_1| + |\Lambda_2| + |\Lambda_0| = |\Lambda|$; second, for any $e \in E$, we have $d_{\Lambda^1 \cup \Lambda^2}(e) = d_{\Pi_1^1 \cup \Pi_2^1 \cup \Pi_1^2 \cup \Pi_2^2}(e) = d_{\Pi_1^1 \cup \Pi_1^2}(e) + d_{\Pi_2^1 \cup \Pi_2^2}(e)$. Using (14), we get $d_{\Lambda^1 \cup \Lambda^2}(e) \le d_{\Pi_1}(e) + d_{\Pi_2}(e) = d_\Lambda(e)$; finally, for $i = 1, 2$ and any $e \in E$, say $e \in E_j$, by definition we have $d_{\Lambda^i}(e) = d_{\Pi_j^i \cap \Gamma_j^2}(e) + d_{\Omega^i}(e) = d_{\Pi_j^i}(e)$. It follows from (14) that $\max\{d_{\Lambda^1}(e), d_{\Lambda^2}(e)\} = \max\{d_{\Pi_j^1}(e), d_{\Pi_j^2}(e)\} \le \lceil d_{\Pi_j}(e)/2 \rceil = \lceil d_\Lambda(e)/2 \rceil$. Combining the above three observations, we conclude that (Λ^1, Λ^2) satisfies (i), (ii), and (iii) of the ESP property, so (21) is justified.

Since Λ dominates Ω, by Lemma 2.3, (Λ^1, Λ^2) is also an equitable subpartition of Ω in $\mathcal{P}_{M,\ell}$. This completes the proof of our theorem. ■

3.4 Vertex covers

Let $G = (V, E)$ be a graph with a nonnegative integral weight function $\boldsymbol{w}$ defined on V. A vertex subset U of G is called a *vertex cover* if $G - U$ contains no edges. The *vertex cover problem* is to find a vertex cover with minimum total weight. As is well known, this NP-hard problem can be approximated within a factor of two. Despite tremendous research effort, no $(2-\epsilon)$-approximation algorithm has been found to date, no matter how small the positive constant ϵ is. Actually it is a widespread belief that 2 is the best approximation ratio we can achieve. One 2-approximation algorithm for the vertex cover problem is based on the following Balinski theorem [2] (see Theorem 64.11 in [19]): Let A be the $E-V$ incidence matrix. Then every vertex of the polytope $P = \{\boldsymbol{x} : A\boldsymbol{x} \geq \mathbf{1},\ \mathbf{1} \geq \boldsymbol{x} \geq \mathbf{0}\}$ is half-integral. The algorithm proceeds by finding a half-integral optimal solution $\boldsymbol{x}^*$ to the LP problem $\min\{\boldsymbol{w}^T\boldsymbol{x} : A\boldsymbol{x} \geq \mathbf{1},\ \mathbf{1} \geq \boldsymbol{x} \geq \mathbf{0}\}$. Set $U = \{v : x^*(v) \geq 1/2\}$. Then U is a vertex cover as desired. In the literature P is called the *fractional vertex cover polytope.* The purpose of this subsection is to present a strengthening of Balinski theorem; that is, the system $A\boldsymbol{x} \geq \mathbf{1}$, $\boldsymbol{x} \geq \mathbf{0}$ is box-TDI/2; what we shall actually prove is the following even stronger statement (recall Corollary 1.5).

Theorem 3.12 *Every graph is PESP.*

Corollary 3.13 *Let A be a $0-1$ matrix with precisely two 1's in each row. Then the linear system $A\boldsymbol{x} \geq \mathbf{1}$, $\boldsymbol{x} \geq \mathbf{0}$ is box-TDI/2*

Proof of Theorem 3.12. Let $G = (V, E)$ be a graph and let Λ be an edge collection of G. We aim to show that Λ admits a pseudo-equitable subpartition. For this purpose, let U be the set of all vertices of G that are incident with some edges in Λ and let $H = (U, \Lambda)$ (possibly H contains multiple edges). Without loss of generality, we may assume that H is connected, otherwise we turn to considering the components of H. Let $H^* = H$ if H is Eulerian and let H^* be obtained from H by adding a new vertex v^* and then making it adjacent to all vertices of odd degree in H otherwise. Then the degree of each vertex of H^* is even, so H^* admits an Eulerian tour T. Let a be the starting vertex of T. Clearly we may assume that

(1) a is precisely v^*, if any, and $d_\Lambda(a) \equiv 2 \pmod 4$ if H is Eulerian and has an odd number of edges.

Let E_1 consist of all odd-numbered edges in T and let E_2 consist of all even-numbered edges in T. It is easy to see that $d_{E_1}(v) = d_{E_2}(v)$ for all vertices v of H^*, except possibly vertex a when H^* has an odd number of edges; in this case $d_{E_1}(a) - d_{E_2}(a) = 2$. Set $\Lambda_i = \Lambda \cap E_i$ for $i = 1, 2$. Then

- $|\Lambda_1| + |\Lambda_2| = |\Lambda|$;
- $d_{\Lambda_1 \cup \Lambda_2}(v) = d_\Lambda(v)$ for all vertices v of H;
- $\max\{d_{\Lambda_1}(v), d_{\Lambda_2}(v)\} \leq \lceil d_\Lambda(v)/2 \rceil$ for all vertices v in H, except possibly a when H is Eulerian and has an odd number of edges; in this case $\max\{d_{\Lambda_1}(a), d_{\Lambda_2}(a)\} = 1 + d_\Lambda(a)/2 = 2\lceil d_\Lambda(a)/4 \rceil$ by (1); and
- $|d_{\Lambda_1}(v) - d_{\Lambda_2}(v)| \leq 2$ for all vertices v of H.

So (i), (ii), (iii'), and (iv) of the PESP property are all satisfied by Λ_1 and Λ_2, and hence (Λ_1, Λ_2) is a pseudo-equitable subpartition of Λ. ∎

3.5 Edge covers

Let $G = (V, E)$ be a graph. An edge subset M of G is called an *edge cover* if each vertex of G is incident with at least one edge in M. Clearly, G has an edge cover if and only if G contains no isolated vertex.

Let B be the $V-E$ incidence matrix and let $Q = \{\boldsymbol{x} : B\boldsymbol{x} \geq \mathbf{1},\ \boldsymbol{x} \geq \mathbf{0}\}$. In the literature Q is called the *fractional edge cover polyhedron* of G. As shown by Balinski [2] (see Theorem 30.10 in [19]), each vertex of Q is half-integral. A closely related theorem of Schrijver asserts that $B\boldsymbol{x} \geq \mathbf{2}$, $\boldsymbol{x} \geq \mathbf{0}$ is TDI/2 (see Corollary 30.11a in [19]), where $\mathbf{2}$ is the all-two vector. The purpose of this subsection is to strengthen these two results as follows: $B\boldsymbol{x} \geq \mathbf{1}$, $\boldsymbol{x} \geq \mathbf{0}$ is box-TDI/2; what we shall actually establish is an even stronger assertion (recall Corollary 1.5). Let us define two terms before presenting the theorem. A *star* centered at a vertex v consists of all edges incident with v in G. Let $\mathcal{S}$ be the set of all stars in G. We call $\mathcal{H} = (E, \mathcal{S})$ the *star hypergraph* of G. Observe that the $\mathcal{S} - E$ incidence matrix is precisely B.

Theorem 3.14 *Every star hypergraph is PESP.*

Corollary 3.15 *Let B be a $0-1$ matrix with precisely two 1's in each column. Then the linear system $B\boldsymbol{x} \geq \mathbf{1}$, $\boldsymbol{x} \geq \mathbf{0}$ is box-TDI/2*

Proof of Theorem 3.14. Let $\mathcal{H} = (E, \mathcal{S})$ be the star hypergraph of a graph $G = (V, E)$ and let Λ be an edge collection of $\mathcal{H}$. We aim to show that Λ admits a pseudo-equitable subpartition. For convenience, we view Λ as a star collection of G. Let $S(v)$ denote the star of G centered at v, let $m_\Lambda(v)$ denote the multiplicity of $S(v)$ in Λ, let X be the set of all vertices v of G with $m_\Lambda(v) \equiv 3 \pmod 4$, and let $Y = V - X$. Now let Λ_1 be the star collection such that $S(v)$ appears $\lceil m_\Lambda(v)/2 \rceil$ times for any $v \in X$ and $\lfloor m_\Lambda(v)/2 \rfloor$ times for any $v \in Y$, and let Λ_2 be the star collection such that $S(v)$ appears $\lfloor m_\Lambda(v)/2 \rfloor$ times for any $v \in X$ and $\lceil m_\Lambda(v)/2 \rceil$ times for any $v \in Y$. Clearly,

(1) $|\Lambda_1| + |\Lambda_2| = |\Lambda|$;

(2) $d_{\Lambda_1 \cup \Lambda_2}(e) = d_\Lambda(e)$ for all edges e of G; and

(3) $|d_{\Lambda_1}(e) - d_{\Lambda_2}(e)| \leq 2$ for all edges e of G.

Moreover, for any edge $e = uv$ of G and $i = 1, 2$, we have $d_{\Lambda_i}(e) = m_{\Lambda_i}(u) + m_{\Lambda_i}(v)$; let us now verify that

(4) $d_{\Lambda_i}(e) \leq 2\lceil d_\Lambda(e)/4 \rceil$.

Assume the contrary: $d_{\Lambda_i}(e) \geq 2\lceil d_\Lambda(e)/4 \rceil + 1$ for some edge $e = uv$ of G and $i = 1$ or 2. Set $j = 3 - i$. By (2), we have $d_{\Lambda_j}(e) \leq 2\lceil d_\Lambda(e)/4 \rceil - 1$. Since $d_{\Lambda_i}(e) - d_{\Lambda_j}(e) \geq 2\lceil d_\Lambda(e)/4 \rceil + 1 - (2\lceil d_\Lambda(e)/4 \rceil - 1) = 2$, from (3) it follows that $d_{\Lambda_i}(e) = 2\lceil d_\Lambda(e)/4 \rceil + 1$ and $d_{\Lambda_j}(e) = 2\lceil d_\Lambda(e)/4 \rceil - 1$. Hence both $d_{\Lambda_i}(e)$ and $d_{\Lambda_j}(e)$ are odd numbers.

On the other hand, $d_{\Lambda_i}(e) - d_{\Lambda_j}(e) = 2$ if and only if $m_{\Lambda_i}(u) - m_{\Lambda_j}(u) = 1$ and $m_{\Lambda_i}(v) - m_{\Lambda_j}(v) = 1$ if and only if $m_{\Lambda_i}(u) = \lceil m_\Lambda(u)/2 \rceil = \lfloor m_\Lambda(u)/2 \rfloor + 1 = m_{\Lambda_j}(u) + 1$ and $m_{\Lambda_i}(v) = \lceil m_\Lambda(v)/2 \rceil = \lfloor m_\Lambda(v)/2 \rfloor + 1 = m_{\Lambda_j}(v) + 1$. So both $m_\Lambda(u)$ and $m_\Lambda(v)$ are odd. From the definition of Λ_1 and Λ_2, we deduce that either $\{u, v\} \subseteq X$ or $\{u, v\} \subseteq Y$. Thus $m_\Lambda(u) \equiv m_\Lambda(v) \equiv 3$ or $1 \pmod 4$ and therefore $d_{\Lambda_i}(e) = m_{\Lambda_i}(u) + m_{\Lambda_i}(v)$ is even. This contradiction justifies (4).

Combining (1)-(4), we conclude that (i), (ii), (iii'), and (iv) of the PESP property are all satisfied by Λ_1 and Λ_2, and hence (Λ_1, Λ_2) is a pseudo-equitable subpartition of Λ. ∎

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