ON MODULAR SIGNS

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Abstract. We consider some questions related to the signs of Hecke eigenvalues or Fourier coefficients of classical modular forms. One problem is to determine to what extent those signs, for suitable sets of primes, determine uniquely the modular form, and we give both individual and statistical results. The second problem, which has been considered by a number of authors, is to determine the size, in terms of the conductor and weight, of the first sign-change of Hecke eigenvalues. Here we improve the recent estimate of Iwaniec, Kohnen and Sengupta.

1. Introduction

There are many results in the arithmetic of modular forms which are concerned with various ways of characterizing a given primitive cusp form f from its siblings, starting from the fact that Fourier coefficients, hence the L-function, determine uniquely a cusp form f relative to a congruence subgroup Γ of $SL(2,\mathbb{Z})$. Among such results are stronger forms of the multiplicity one theorem for automorphic forms or representations, various explicit forms of these statements, where only finitely many coefficients are required (say at primes $p \leqslant X$, for some explicit X depending on the parameters defining f), and a number of interesting "statistic" versions of the last problem, where X can be reduced drastically, provided one accepts some possible exceptions. Among other papers, we can cite [4], [16], [3] or [8].

Some of these statements were strongly suggested by the analogy with the problem of the least quadratic non-residue, which is a problem of great historic importance in analytic number theory, and there are many parallels between the results which have been obtained. However, this parallel breaks down sometimes. For instance, in [13], Lau and Wu note that one result of [12] for the least quadratic non-residue is highly unlikely to have a good analogue for modular forms. This result (see [12, Th. 3]) is a precise estimate for the number of primitive real Dirichlet characters of modulus $q \leqslant D$ for which the least n with $\chi(n) = -1$ is $\gg \log D$, and the difficulty is that this estimate can be understood by assuming that the values $\chi(p)$, for p of moderate size compared with D, behave like independent random variables taking values ± 1 equally often. However, Hecke eigenvalues may take many more than two values, and thus assuming that they coincide should definitely be a much more stringent condition.

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In this paper, we consider a way to potentially recover a closer analogy: namely (narrowing our attention to forms with real eigenvalues) instead of looking at the values of the Hecke eigenvalues, we consider only their *signs* (where we view 0 as being of both signs simultaneously, to increase the possibility of having same sign). Then classical questions for Dirichlet characters and modular forms have the following analogues for signs of Hecke eigenvalues $\lambda_f(p)$ of a classical modular form f:

- What is the first sign-change, i.e., the smallest $n \geqslant 1$ (or prime p) for which $\lambda_f(n) < 0$ (or $\lambda_f(p) < 0$)? (Analogues of the least quadratic non-residue). Note a small difference with quadratic characters: it is not true here that the smallest integer with negative Hecke eigenvalue is necessarily prime; finding one or the other are two different questions.[1]
- Given arbitrary signs $\varepsilon_p \in \{\pm 1\}$ for all primes, what is the number of f (in a suitable family) for which $\lambda_f(p)$ has sign ε_p for all $p \leqslant X$, for various values of X? (Analogue of the question in [12]).
- In particular, is there a finite limit X such that coincidences of signs of $\lambda_f(p)$ and ε_p for all $p \leqslant X$ implies that f is uniquely determined? (Analogue of the multiplicity one theorem).

Of these three problems, only the first one seems to have be considered earlier, with the best current result due to Iwaniec, Kohnen and Sengupta [5]. We will improve it, and obtain some first results concerning the other two problems. We will also suggest further questions that may be of interest.

Before stating our main theorems, here are the basic notation about modular forms (see, e.g., [6, Ch. 14] for a survey of these facts). We denote by $\mathrm{H}_k^*(N)$ the finite set of all primitive forms of weight k for $\Gamma_0(N)$, where $k \geqslant 2$ is an even integer and $N \geqslant 1$ is an integer. The restriction to trivial Nebentypus ensures that all Fourier coefficients are real, and for any $f \in \mathrm{H}_k^*(N)$, we denote

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e(nz), \qquad e(z) = e^{2i\pi z}, \qquad\qquad (\Im m z > 0),$$

its Fourier expansion at infinity. Since f is primitive, the $\lambda_f(n)$ are the normalized eigenvalues of the Hecke operators T_n, and satisfy the well-known Hecke relations

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d \mid (m,n) \\ (d,N)=1}} \lambda_f\Big(\frac{mn}{d^2}\Big), \tag{1.1}$$

for all integers $m \geqslant 1$ and $n \geqslant 1$. In particular, λ_f is a multiplicative function of n (so $\lambda_f(1) = 1$) and moreover the following important special case

$$\lambda_f(p)^2 = 1 + \lambda_f(p^2) \tag{1.2}$$

holds for all primes $p \nmid N$.

Furthermore, it is also known that $\lambda_f(n)$ satisfies the deep inequality

$$|\lambda_f(n)| \leqslant \tau(n) \tag{1.3}$$

[1] E.g., as a random example, for the cusp form of weight 2 associated to the elliptic curve $y^2 = x^3 + x$, the first negative coefficient is $\lambda(9) = -3$, and the first negative prime coefficient is $\lambda(13) = -6$.

for all $n \geqslant 1$, where $\tau(n)$ is the divisor function (this is the Ramanujan-Petersson conjecture, proved by Deligne). In particular, we have $\lambda_f(p) \in [-2, 2]$ for $p \nmid N$, and hence there exists a unique angle $\theta_f(p) \in [0, \pi]$ such that

$$\lambda_f(p) = 2\cos\theta_f(p). \tag{1.4}$$

Our other notation is standard in analytic number theory: for instance, $\pi(x)$ denotes the number of primes $\leqslant x$ and $P^+(n)$ (resp. $P^-(n)$) denotes the largest (resp. smallest) prime factor of n, with the convention $P^+(1) = 1$ (resp. $P^-(1) = \infty$).

We now describe our results.

1.1. **The first negative Hecke eigenvalue.** For $f \in \mathrm{H}_k^*(N)$, $k \geqslant 2$ and $N \geqslant 1$, it is well-known that the coefficients $\lambda_f(n)$ change sign infinitely often. We denote by n_f the smallest integer $n \geqslant 1$ such that $(n, N) = 1$ and

$$\lambda_f(n) < 0. \tag{1.5}$$

The analogue (or one analogue) of the least-quadratic non-residue problem is to estimate n_f in terms of the analytic conductor $Q := k^2N$. Iwaniec, Kohnen & Sengupta [5] have shown recently that

$$n_f \ll Q^{29/60} = (k^2N)^{29/60}$$

(here, standard methods lead to $n_f \ll_\varepsilon Q^{1/2+\varepsilon}$, so the significance is that the exponent is $< 1/2$).

Our first result is a sharpening of this estimate:

Theorem 1. *Let $k \geqslant 2$ be an even integer and $N \geqslant 1$. Then for all $f \in \mathrm{H}_k^*(N)$, we have*

$$n_f \ll Q^{10/21} = (k^2N)^{10/21}, \tag{1.6}$$

where the implied constant is absolute.

Theorem 1 will be proved by refining the method of Iwaniec, Kohnen and Sengupta, which is based on sieve methods and a clever use of the Hecke relation (1.2); see the introduction to Section 2 for a further discussion of the basic ideas. Also, notice that, as observed by Iwaniec, Kohnen and Sengupta, an immediate improvement of their bound follows from any subconvexity bound for L-functions on the critical line. Like them, we do not need any such result to prove Theorem 1, but we will state below the precise relation.

Note that we do not know if the estimate of Theorem 1 holds for the first negative Hecke eigenvalue at a prime argument.

1.2. **Statistic study of the first sign-change.** The upper bound (1.6) is probably far from optimal. Indeed, one can show that under the Grand Riemann Hypothesis we have

$$n_f \ll (\log(kN))^2$$

where the implied constant is absolute. Our next result confirms this unconditionally for almost all f. It closely parallels the case of Dirichlet characters (see [12]). Precisely, we first recall that

$$|\mathrm{H}_k^*(N)| \asymp k\varphi(N),$$

where $\varphi(N)$ is the Euler function, as k, $N \to +\infty$, and we prove:

Theorem 2. *Let $\nu \geqslant 1$ be a fixed integer and $\mathscr{P}$ be a set of prime numbers of positive density in the following sense:*

$$\sum_{\substack{z<p\leqslant 2z\\ p\in\mathscr{P}}} \frac{1}{p} \geqslant \frac{\delta}{\log z} \qquad (z \geqslant z_0)$$

for some constants $\delta > 0$ and $z_0 > 0$. Let $\{\varepsilon_p\}_{p\in\mathscr{P}}$ be a sequence of real numbers such that $|\varepsilon_p| = 1$ for all p. Let $k \geqslant 2$ be an even integer and $N \geqslant 1$ be squarefree. Then there are two positive constants C and c such that the number of primitive cusp forms $f \in \mathrm{H}_k^*(N)$ satisfying

$$\varepsilon_p \lambda_f(p^\nu) > 0 \quad \textit{for} \quad p \in \mathscr{P}, \quad p \nmid N \quad \textit{and} \quad C\log(kN) < p \leqslant 2C\log(kN)$$

is bounded by

$$\ll_{\nu,\mathscr{P}} kN \exp\Big(-c\frac{\log kN}{\log\log kN}\Big).$$

Here C, c and the implied constant depend on ν and $\mathscr{P}$ only.

Taking $\mathscr{P}$ the set of all primes, $\varepsilon_p = 1$ and $\nu = 1$ in Theorem 2, we immediately get:[2]

Corollary 1. *Let $k \geqslant 2$ be an even integer and $N \geqslant 1$ be squarefree. There is an absolute positive constant c such that we have*

$$n_f \ll \log(kN),$$

for all $f \in \mathrm{H}_k^*(N)$, except for f in an exceptional set with

$$\ll kN \exp\Big(-c\frac{\log kN}{\log\log kN}\Big)$$

elements, where the implied constants are absolute.

It is very natural to ask whether this result is optimal (as the analogue is known to be for real Dirichlet characters). In this direction, we can prove the following:

Theorem 3. *Let N be a squarefree number and $k \geqslant 2$ an even integer, and let (ε_p) be a sequence of signs indexed by prime numbers. For any $\varepsilon > 0$, $\varepsilon < 1/2$, there exists $c > 0$ such that*

$$\frac{1}{|\mathrm{H}_k^*(N)|}|\{f \in \mathrm{H}_k^*(N) \mid \lambda_f(p) \textit{ has sign } \varepsilon_p \textit{ for } p \leqslant z,\ p \nmid N\}| \geqslant \Big(\frac{1}{2} - \varepsilon\Big)^{\pi(z)}$$

for $z = c\sqrt{(\log kN)(\log\log kN)}$, provided kN is large enough.

One may expect that the same result would be true for $z \leqslant c\log kN$ (note that

$$\Big(\frac{1}{2}\Big)^{\pi(c\log kN)} \geqslant \exp\Big(-c_1\frac{\log kN}{\log\log kN}\Big)$$

so this result would be quite close to the statistic upper-bound of Corollary 1, and would essentially be best possible, confirming that the signs of $\lambda_f(p)$ behave almost like independent (and unbiased) random variables in that range of p).

[2] The cases where $\nu \geqslant 2$ can be interpreted as similar statements for the ν-th symmetric powers.

Theorems 2 and 3 will be proved in Section 3, using the method in [13] and quantitative equidistribution statements for Hecke eigenvalues, respectively.

1.3. **Recognition of modular forms by signs of Hecke eigenvalues.** Here we consider whether it is true that a primitive form f is determined uniquely by the sequence of signs of its Fourier coefficients $\lambda_f(p)$, where we recall that we interpret the sign of 0 in a relaxed way, so that 0 has the same sign as both positive and negative numbers.

The answer to this question is, indeed, yes, and in fact (in the non-CM case) an analogue of the strong multiplicity one theorem holds: not only can we exclude finitely many primes, or a set of primes of density zero, but even a set of sufficiently small positive density. Here, the density we use is the analytic density defined as follows: a set E of primes has density $\kappa > 0$ if and only if

$$\sum_{p\in E} \frac{1}{p^\sigma} \sim \kappa \sum_{p} \frac{1}{p^\sigma} \sim -\kappa \log(\sigma - 1) \qquad (\sigma \to 1+). \tag{1.7}$$

We will prove:

Theorem 4. *Let $k_1, k_2 \geqslant 2$ be even integers, let $N_1, N_2 \geqslant 1$ be integers and $f_1 \in \mathrm{H}^*_{k_1}(N_1)$, $f_2 \in \mathrm{H}^*_{k_2}(N_2)$.*

(1) *If the signs of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ are the same for all p except those in a set of analytic density 0, then $f_1 = f_2$.*

(2) *Assume that neither of f_1 and f_2 is of CM type, for instance assume that N_1 and N_2 are squarefree. Then, if $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ have same sign for every prime p, except those in a set E of analytic density κ, with $\kappa \leqslant 0.00109$, it follows that $f_1 = f_2$.*

Recall that a form $f \in \mathrm{H}^*_k(N)$ is of CM type if there exists a non-trivial primitive real Dirichlet character χ such that $\lambda_f(p) = \chi(p)\lambda_f(p)$ for all but finitely many primes p. In that case, $\lambda_f(p) = 0$ for all p such that $\chi(p) = -1$, and hence its signs coincide (in our relaxed sense) with those of any other modular form for a set of primes of density at least 1/2.

Of course, Theorem 4 is also valid for the natural density, since the existence of the latter implies that of the analytic density, and that they are equal. As a corollary, we get of course:

Corollary 2. *For any sequence of signs (ε_p) indexed by primes, there is at most one pair (k, N) and one $f \in \mathrm{H}^*_k(N)$ such that $\lambda_f(p)$ has sign ε_p for all primes.*

Theorem 4 is proved in Section 4; although the argument is quite short and simple, it is of interest to note that it depends crucially on the deep results of Kim and Shahidi proving the holomorphy of the 6-th symmetric power L-function of (non-CM) holomorphic modular forms.

1.4. **Further remarks and questions.** The main remark is that, underlying most of the problems we consider is the Sato-Tate conjecture, which we recall (see Mazur's survey [14]): provided f is not of CM type (for instance, if N is squarefree), one should have

$$\lim_{x\to+\infty} \frac{1}{\pi(x)} |\{p \leqslant x \mid \theta_f(p) \in [\alpha,\beta]\}| = \int_\alpha^\beta \mathrm{d}\mu_{ST}$$

for any $\alpha < \beta$, where μ_{ST} is the Sato-Tate measure

$$\mu_{ST} = \frac{2}{\pi} \sin^2 \theta \, \mathrm{d}\theta,$$

on $[0, \pi]$. Since $\mu_{ST}([0, \pi/2]) = \mu_{ST}([\pi/2, \pi])$, this indicates in particular that the signs of $\lambda_f(p)$ should be equitably shared between $+1$ and -1. This suggests and motivates many of our results and techniques of proof.

We also remark that there is much ongoing progress on the Sato-Tate conjecture; for $f \in \mathrm{H}_k^*(N)$, non-CM, a proof of the conjecture has been announced by Barnet-Lamb, Geraghty, Harris and Taylor [1, Th. B]. However, knowing its truth does not immediately simplify our arguments much (except parts of Section 4).

Finally, here are a number of further questions concerning our results:

– What is the optimal density κ one can obtain in Theorem 4? If one assumes that f_1 and f_2 obey the pair-Sato-Tate conjecture, namely that for $a_1 < b_1$, $a_2 < b_2$ the set of primes

$$\big\{p \mid \lambda_{f_1}(p) \in [a_1, b_1] \text{ and } \lambda_{f_2}(p) \in [a_2, b_2]\big\}$$

has density equal to $\mu_{ST}([a_1, b_1])\mu_{ST}([a_2, b_2])$ (in other words, the Fourier coefficients at primes are *independently* Sato-Tate distributed), one may easily get the result for any $\kappa < \frac{1}{2}$ (corresponding to the probability for $\mu_{ST} \otimes \mu_{ST}$ of having the same sign). But this can only hold if f_1 and f_2 are not related by quadratic twists, of course (and in that case, for elliptic curves, Mazur [14, Footnote 12] mentions progress made by Harris). If, on the other hand, $f_2 = f_1 \otimes \chi$ for a real character χ, the coefficients are of the same sign for a set of primes of density exactly $\frac{1}{2}$.

– Another natural problem suggested by Theorem 4 is to estimate the size, as a function of the weight and conductor, of the smallest integer n_{f_1,f_2} for which the sign of $\lambda_{f_1}(n)$ and $\lambda_{f_2}(n)$ are different. If we enlarge slightly our setting to allow f_2 to be an Eisenstein series such as the Eisenstein series of weight 4:

$$E_4(z) = 1 + 240 \sum_{n \geqslant 1} \Big(\sum_{d \mid n} d^3\Big) e(nz),$$

where all Hecke eigenvalues are positive, then the question becomes (once more) that of finding the first negative Hecke eigenvalue for f_1, i.e., the problem considered in Theorem 1. Hence, we know that

$$n_{f_1,E_4} \ll (k_1^2 N_1)^{10/21},$$

where the implied constant is absolute, but it would be interesting to obtain a more general version, in particular a uniform one with respect to both f_1 and f_2.

At least our statistic result (Theorem 2) generalizes immediately if one of the forms is fixed: taking $\mathscr{P}$ to be the set of all primes, $\nu = 1$ and $\varepsilon_p = \operatorname{sign} \lambda_{f_2}(p)$ if $\lambda_{f_2}(p) \neq 0$, and 1 otherwise, we get immediately the following corollary:

Corollary 3. *Let $k_1, k_2 \geqslant 2$ be even integers and $N_1, N_2 \geqslant 1$ squarefree. For any fixed $f_2 \in \mathrm{H}_{k_2}^*(N_2)$, there is an absolute positive constant c such that*

$$n_{f_1,f_2} \ll_{f_2} \log(k_1 N_1),$$

for all $f \in \mathrm{H}_{k_1}^*(N_1)$ except for those in an exceptional set with

$$\ll k_1 N_1 \exp\Big(-c \frac{\log k_1 N_1}{\log\log k_1 N_1}\Big)$$

elements, where the implied constants depend only on f_2.

2. Proof of Theorem 1

The basic ideas are similar to those of Iwaniec, Kohnen and Sengupta. For $x \geqslant y \geqslant z > 1$, we define (as usual)

$$\Theta(x, y, z) := \sum_{\substack{n \leqslant x \\ z < P^-(n),\, P^+(n) \leqslant y}} 1, \qquad \Phi(x, y) := \Theta(x, x, y).$$

Denote by $\omega(t)$ the Buchstab function, defined as the unique continuous solution of the difference-differential equation

$$\begin{cases} t\omega(t) = 1 & \text{if } 1 \leqslant t \leqslant 2, \\ (t\omega(t))' = \omega(t-1) & \text{if } t \geqslant 2. \end{cases}$$

It is very well-known that

$$\Phi(x, z) = \omega\left(\frac{\log x}{\log z}\right)\frac{x}{\log z} + O\left(\frac{z}{\log z} + \frac{x}{(\log z)^2}\right) \tag{2.1}$$

and $\omega(t) \to e^{-\gamma}$ as $t \to \infty$, where γ is the Euler constant (see, e.g., [23, Theorem III.6.3, p. 400]).

We need a lower bound for $\Theta(x, y, z)$, which is an improvement of [5, Lemma 3].

Lemma 2.1. *Let* $\delta \in (0, \frac{1}{2})$. *Then we have*

$$\Theta(x, y, z) \geqslant \frac{x}{\log z}\left\{J(\delta) + O_\delta\left(\frac{\log\log z}{\log z}\right)\right\}$$

uniformly for

$$y \geqslant 10, \qquad z = y^\delta, \qquad (y/\log y)^2 \leqslant x \leqslant (yz)^2,$$

where

$$J(\delta) = \inf_{t \geqslant 4} \omega(t) - (\log 2 + \delta) \sup_{t \geqslant 2} \omega(t) - \frac{\delta}{2}\log\left(\frac{2-\delta}{1-\delta}\right) - \frac{\delta\log(2+2\delta)\log(1+2\delta)}{2+\delta}.$$

The implied constant depends only on δ.

We defer the proof until the end of this section, and we now explain how to prove Theorem 1. Thus, assume $y \geqslant 1$ is such that $\lambda_f(n) \geqslant 0$ for all integers n with $1 \leqslant n \leqslant y$ and $(n, N) = 1$. Write $x := yz$ with $z := y^\delta$ and $0 < \delta < \frac{1}{2}$.

We consider the sum

$$S_f(x) = \sum_{\substack{n \leqslant x \\ (n,N)=1}} \lambda_f(n)\log\left(\frac{x}{n}\right) = S_f^+(x) + S_f^-(x), \tag{2.2}$$

where

$$S_f^+(x) := \sum_{\substack{n \leqslant x,\, (n,N)=1 \\ \lambda_f(n) > 0}} \lambda_f(n)\log\left(\frac{x}{n}\right),$$

and similarly for $S_f^-(x)$ with $\lambda_f(n) < 0$. The goal is to obtain upper and lower bounds for $S_f(x)$, using the positivity assumption, which are incompatible if y is too large.

For the upper bound, we simply use [5, (2.2)], which gives

$$S_f(x) \ll \sqrt{x}(k^2N)^{1/4}\log(k^2N)\prod_{p\mid N}(1+p^{-1/2}), \tag{2.3}$$

with an absolute implied constant. This does not use the assumed positivity of $\lambda_f(n)$ for $n \leqslant y$, and in fact is easily obtained from the convexity bound for the L-function of f on the critical line [5, (1.6)]. We note that, more generally, if we have

$$L(f, 1/2+it) \ll (k^2N(1+|t|)^2)^\eta,$$

for $t \in \mathbb{R}$, where $\eta > 0$, then we get in the same way

$$S_f(x) \ll \sqrt{x}(k^2N)^\eta\prod_{p\mid N}(1+p^{-1/2}) \tag{2.4}$$

(the recent work of Michel and Venkatesh [15] provides such a uniform result for some $\eta < 1/4$).

To derive a lower bound for $S_f(x)$, we get separately lower bounds for $S_f^\pm(x)$. The key to our improvement of [5] is in getting a better lower bound for $S_f^+(x)$, though there is a trade-off in the lower bound of $S_f^-(x)$.

We start by the lower bound for $S_f^-(x)$. From the multiplicativity of λ_f, our assumption and the choice of x and y, it follows that each integer $n \geqslant 1$ which occurs in the sum $S_f^-(x)$, so that $\lambda_f(n) < 0$, has exactly one prime power factor p^ν with $y < p^\nu < x$ and $(p, N) = 1$. Thus, using (1.1) and (1.3), we deduce

$$\begin{aligned} S_f^-(x) &= \sum_{\substack{dp^\nu \leqslant x,\, p^\nu > y,\, \lambda_f(p^\nu)<0 \\ (d,N)=(p,dN)=1}} \lambda_f(dp^\nu)\log\Big(\frac{x}{dp^\nu}\Big) \\ &\geqslant -\sum_{\substack{d\leqslant z \\ (d,N)=1}} \lambda_f(d) \sum_{p^\nu \leqslant x/d} (\nu+1)\log\Big(\frac{x}{dp^\nu}\Big). \end{aligned}$$

It is clear from the prime number theorem that

$$\begin{aligned} \sum_{p\leqslant x/d} 2\log\Big(\frac{x}{dp}\Big) &= \frac{2(x/d)}{\log(x/d)}\Big\{1+O\Big(\frac{1}{\log(x/d)}\Big)\Big\} \\ &\leqslant \frac{2x}{d\log y}\Big\{1+O\Big(\frac{1}{\log y}\Big)\Big\} \end{aligned}$$

and

$$\sum_{p^\nu \leqslant x/d,\, \nu\geqslant 2} (\nu+1)\log\Big(\frac{x}{dp^\nu}\Big) \ll \Big(\frac{x}{d}\Big)^{1/2}(\log x)^2 \ll \frac{\sqrt{xz}}{d}(\log x)^2 \ll \frac{x}{d(\log x)^2}$$

for $d \leqslant z$. Inserting these into the preceding relation, we obtain

$$S_f^-(x) \geqslant -2\frac{x}{\log y}\left\{1+O\left(\frac{1}{\log y}\right)\right\} \sum_{d\leqslant z,\,(d,N)=1} \frac{\lambda_f(d)}{d}. \tag{2.5}$$

On the other hand, we deduce that

$$\begin{aligned}\sum_{\substack{d\leqslant z\\(d,N)=1}} \frac{\lambda_f(d)}{d} &= \sum_{\substack{d\leqslant z,\,P^+(d)\leqslant\sqrt{z}\\(d,N)=1}} \frac{\lambda_f(d)}{d} + \sum_{\substack{\sqrt{z}<p\leqslant z\\p\nmid N}} \frac{\lambda_f(p)}{p} \sum_{\substack{d\leqslant z/p\\(d,N)=1}} \frac{\lambda_f(d)}{d}\\ &\leqslant \left\{1+2\log 2+O\left(\frac{1}{\log z}\right)\right\} \sum_{\substack{d\leqslant z,\,P^+(d)\leqslant\sqrt{z}\\(d,N)=1}} \frac{\lambda_f(d)}{d}\end{aligned}$$

by Deligne's bound $|\lambda_f(p)| \leqslant 2$, the prime number theorem and positivity. Combining with (2.5), this yields

$$S_f^-(x) \geqslant -\left\{2+4\log 2+O\left(\frac{1}{\log z}\right)\right\}\frac{x}{\log y}\Lambda_f(z), \tag{2.6}$$

where

$$\Lambda_f(z) = \sum_{\substack{d\leqslant z,\,P^+(d)\leqslant\sqrt{z}\\(d,N)=1}} \frac{\lambda_f(d)}{d}. \tag{2.7}$$

Next we derive a lower bound for $S_f^+(x)$. By positivity and (1.2), we have

$$\lambda_f(p)^2 = 1+\lambda_f(p^2) \geqslant 1 \qquad (p\nmid N,\ p\leqslant\sqrt{y})$$

and hence $\lambda_f(\ell) \geqslant 1$ for squarefree ℓ with $P^+(\ell) \leqslant \sqrt{y}$ and $(\ell, N) = 1$. Thus

$$\begin{aligned}S_f^+(x) &\geqslant \sum_{\substack{d\leqslant z,\,P^+(d)\leqslant\sqrt{z}\\(d,N)=1}} \lambda_f(d) \sum_{\substack{\ell\leqslant x/d,\,(\ell,N)=1\\p\mid\ell\Rightarrow\sqrt{z}<p\leqslant\sqrt{y},\,\ell\,\text{squarefree}}} \lambda_f(\ell)\log\left(\frac{x}{d\ell}\right)\\ &\geqslant \sum_{\substack{d\leqslant z,\,P^+(d)\leqslant\sqrt{z}\\(d,N)=1}} \lambda_f(d) \sum_{\substack{\ell\leqslant x/d,\,(\ell,N)=1\\p\mid\ell\Rightarrow\sqrt{z}<p\leqslant\sqrt{y},\,\ell\,\text{squarefree}}} \log\left(\frac{x}{d\ell}\right)\\ &\geqslant \sum_{\substack{d\leqslant z,\,P^+(d)\leqslant\sqrt{z}\\(d,N)=1}} \lambda_f(d) \sum_{\substack{\ell\leqslant x/d\\p\mid\ell\Rightarrow\sqrt{z}<p\leqslant\sqrt{y}}} \log\left(\frac{x}{d\ell}\right) - R,\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}R :=& \sum_{\substack{d\leqslant z,\, P^+(d)\leqslant\sqrt{z}\\(d,N)=1}} \lambda_f(d)\biggl\{\sum_{\substack{p\mid N\\p>\sqrt{z}}}\sum_{\ell\leqslant x/(dp)}\log\biggl(\frac{x}{d\ell p}\biggr)+\sum_{p>\sqrt{z}}\sum_{\ell\leqslant x/(dp^2)}\log\biggl(\frac{x}{d\ell p^2}\biggr)\biggr\}\\ \ll& \, x(\log x)\sum_{\substack{d\leqslant z,\, P^+(d)\leqslant\sqrt{z}\\(d,N)=1}}\frac{\lambda_f(d)}{d}\biggl(\sum_{\substack{p\mid N\\p>\sqrt{z}}}\frac{1}{p}+\sum_{p>\sqrt{z}}\frac{1}{p^2}\biggr)\\ \ll& \,(\log(2N))\frac{x}{\sqrt{z}}\sum_{\substack{d\leqslant z,\, P^+(d)\leqslant\sqrt{z}\\(d,N)=1}}\frac{\lambda_f(d)}{d}\ll\frac{x}{z^{1/3}}\sum_{\substack{d\leqslant z,\, P^+(d)\leqslant\sqrt{z}\\(d,N)=1}}\frac{\lambda_f(d)}{d}=xz^{-1/3}\Lambda_f(z)\end{aligned}$$

(see (2.7)), since we can obviously assume that $z \geqslant (\log(2N))^6$ without loss of generality.

Now we estimate the sum over ℓ on the right-hand side of (2.8), with the help of Lemma 2.1. Note that

$$\sum_{\substack{\ell\leqslant x/d\\p\mid\ell\Rightarrow\sqrt{z}<p\leqslant\sqrt{y}}}\log\biggl(\frac{x}{d\ell}\biggr)=\int_{y/(\log y)^2}^{x/d}\frac{\Theta(s,\sqrt{y},\sqrt{z})}{s}\,\mathrm{d}s+O\biggl(\frac{y}{(\log y)^2}\biggr).$$

Setting $(x, y, z) = (s, \sqrt{y}, \sqrt{z})$ where $y/(\log y)^2 \leqslant s \leqslant x/d$, it is easy to check this lemma is applicable, and we get

$$\sum_{\substack{\ell\leqslant x/d\\p\mid\ell\Rightarrow\sqrt{z}<p\leqslant\sqrt{y}}}\log\biggl(\frac{x}{d\ell}\biggr)\geqslant\frac{2x}{d\log z}\biggl\{J(\delta)+O\biggl(\frac{\log\log z}{\log z}\biggr)\biggr\}.$$

Inserting this into (2.8), it follows

$$S_f^+(x)\geqslant\frac{2x}{\log z}\biggl\{J(\delta)+O\biggl(\frac{\log\log z}{\log z}\biggr)\biggr\}\Lambda_f(z) \tag{2.9}$$

Combining (2.6) and (2.9) with (2.2) and noting that $\Lambda_f(z) \geqslant 1$ by positivity, we conclude that

$$S_f(x)\geqslant\frac{2x}{\log y}\biggl\{\frac{J(\delta)}{\delta}-1-2\log 2+O\biggl(\frac{1}{\log z}\biggr)\biggr\}.$$

According to [25, Lemme 12], we have

$$\inf_{u\geqslant3}\omega(u)\geqslant0.5608\qquad\text{and}\qquad\sup_{u\geqslant2}\omega(u)\leqslant0.5672.$$

Then a simple numerical computation gives

$$20J(\tfrac{1}{20})-1-2\log2>0.$$

Since $J(\delta)$ is a continuous function of δ, we take $\delta = \frac{1}{20} + \delta_0$ for a suitably small positive constant δ_0, so that we obtain finally the desired lower bound

$$S_f(x)\gg\frac{x}{\log y}. \tag{2.10}$$

Now we can quickly conclude: comparing this estimate with (2.3), it follows that

$$x \ll (k^2N)^{1/2}\Big(\log(k^2N)\prod_{p\mid N}(1+p^{-1/2})\Big)^2,$$

and since $x = yz = y^{1+\delta}$, we have

$$y \ll (k^2N)^{1/(2(1+\delta))}\Big(\log(k^2N)\prod_{p\mid N}(1+p^{-1/2})\Big)^{2/(1+\delta)},$$

which implies the desired result (1.6). Note that if we used an estimate of the type (2.4), we would get

$$n_f \ll_\varepsilon (k^2N)^{2\eta/(1+\delta)+\varepsilon} = (k^2N)^{40\eta/21+\varepsilon}.$$

This completes the proof of Theorem 1, except for the proof of Lemma 2.1.

Proof of Lemma 2.1. First, it is clear that

$$\Theta(x,y,z) = \Phi(x,z) - \sum_{\substack{1<q\leqslant x\\ P^-(q)>y}} \Phi\Big(\frac{x}{q}, z\Big).$$

Since $x/z > y$ and $\Phi(x/q, z) = 1$ for $x/z < q \leqslant x$, we can split further as follows,

$$\Theta(x,y,z) = \Phi(x,z) - \sum_{\substack{1<q\leqslant x/z\\ P^-(q)>y}} \Phi\Big(\frac{x}{q}, z\Big) - \Phi(x,y) + \Phi\Big(\frac{x}{z}, y\Big).$$

By the asymptotic formula (2.1), it follows that

$$\begin{aligned}\Theta(x,y,z) = \omega\Big(\frac{\log x}{\log z}\Big)\frac{x}{\log z} - \sum_{\substack{1<q\leqslant x/z\\ P^-(q)>y}} \omega\Big(\frac{\log(x/q)}{\log z}\Big)\frac{x}{q\log z}\\ - \omega\Big(\frac{\log x}{\log y}\Big)\frac{x}{\log y} + O\Big(\frac{x}{(\log z)^2}\Big).\end{aligned} \tag{2.11}$$

Next we divide the sum on the right-hand side of (2.11) into two parts according to $1 < q < x/z^2$ or $x/z^2 < q \leqslant x/z$. In the first case, q must be a prime since $x/z^2 \leqslant y^2$. Thus

$$\sum_{y<p\leqslant x/z^2} \omega\Big(\frac{\log(x/p)}{\log z}\Big)\frac{1}{p} \leqslant \sup_{t\geqslant 2}\omega(t)\sum_{y<p\leqslant x/z^2}\frac{1}{p} \leqslant \sup_{t\geqslant 2}\omega(t)\log 2 + O\Big(\frac{1}{\log y}\Big) \tag{2.12}$$

by using again $x/z^2 \leqslant y^2$.

In the second case, we have $1 \leqslant \log(x/q)/\log z \leqslant 2$ and q has at most two prime factors (because of $x/z \leqslant y^2 z \leqslant y^{5/2}$). In view of our assumption $x \geqslant (y/\log y)^2 \geqslant yz^2$ and the relation $\omega(t) = 1/t$ if $1 \leqslant t \leqslant 2$, we can write

$$\sum_{\substack{x/z^2<q\leqslant x/z\\ P^-(q)>y}} \omega\Big(\frac{\log(x/q)}{\log z}\Big)\frac{1}{q} = \sum_{x/z^2<p\leqslant x/z}\frac{\log z}{p\log(x/p)} + S,$$

where the second term, namely

$$S := \sum_{y<p_1\leqslant x/(yz)} \sum_{y<p_2\leqslant x/(zp_1)} \frac{\log z}{p_1p_2\log(x/p_1p_2)},$$

does not appear if $x \leqslant y^2z$.

By integration by parts, the prime number theorem leads to

$$\begin{aligned}\sum_{x/z^2<p\leqslant x/z} \frac{\log z}{p\log(x/p)} &= \frac{\log z}{\log x}\int_{x/z^2}^{x/z} \frac{\mathrm{d}\pi(t)}{t(1-\log t/\log x)}\\ &= \frac{\log z}{\log x}\bigg\{1+O\bigg(\frac{1}{\log z}\bigg)\bigg\}\int_{x/z^2}^{x/z}\frac{\mathrm{d}t}{t(\log t)(1-\log t/\log x)}\\ &= \frac{\log z}{\log x}\bigg\{1+O\bigg(\frac{1}{\log z}\bigg)\bigg\}\int_{\log(x/z^2)/\log x}^{\log(x/z)/\log x}\frac{\mathrm{d}u}{u(1-u)}\\ &\leqslant \frac{\delta}{2}\log\left(\frac{2-\delta}{1-\delta}\right)+O\bigg(\frac{\log\log z}{\log z}\bigg),\end{aligned}$$

since

$$\frac{\log(x/z)}{\log(x/z^2)} \leqslant \frac{\log(y^2/z(\log y)^2)}{\log(y^2/z^2(\log y)^2)} = \frac{2-\delta}{2(1-\delta)}+O\bigg(\frac{\log\log y}{\log y}\bigg)$$

and

$$\frac{\log z}{\log x} \leqslant \frac{\log z}{\log(y/\log y)^2} = \frac{\delta}{2}+O\bigg(\frac{\log\log y}{\log y}\bigg)$$

for $x \geqslant (y/\log y)^2$ and $z = y^\delta$.

Similarly when $y^2z \leqslant x \leqslant (yz)^2$, we have

$$\begin{aligned}S &= \bigg\{1+O\bigg(\frac{1}{\log z}\bigg)\bigg\}\frac{\log z}{\log x}\int_y^{x/(yz)}\frac{\mathrm{d}t_1}{t_1\log t_1}\int_y^{x/(zt_1)}\frac{\mathrm{d}t_2}{t_2(\log t_2)(1-\log(t_1t_2)/\log x)}\\ &= \bigg\{1+O\bigg(\frac{1}{\log z}\bigg)\bigg\}\frac{\log z}{\log x}\int_{\log y/\log x}^{1-\log(yz)/\log x}\frac{\mathrm{d}u_1}{u_1}\int_{\log y/\log x}^{1-u_1-\log z/\log x}\frac{\mathrm{d}u_2}{u_2(1-u_1-u_2)}.\end{aligned}$$

For $\log y/\log x \leqslant u_1 \leqslant 1-\log(yz)/\log x$, we have

$$\begin{aligned}\int_{\log y/\log x}^{1-u_1-\log z/\log x}\frac{\mathrm{d}u_2}{u_2(1-u_1-u_2)} &= \frac{1}{1-u_1}\log\left(\frac{\log(x^{1-u_1}/z)}{\log z}\frac{\log(x^{1-u_1}/y)}{\log y}\right)\\ &\leqslant \frac{1}{1-u_1}\log\left(\frac{\log(x/yz)}{\log z}\frac{\log(x/y^2)}{\log y}\right)\\ &\leqslant \frac{1}{1-u_1}\log(2+2\delta)\end{aligned}$$

since $x \leqslant (yz)^2$ and $z = y^\delta$. Inserting into the preceding formula, it follows that

$$\begin{aligned} S &\leqslant \left\{1 + O\left(\frac{1}{\log z}\right)\right\} \frac{\log z}{\log x} \log(2+2\delta) \int_{\log y/\log x}^{1-\log(yz)/\log x} \frac{\mathrm{d}u}{u(1-u)} \\ &= \left\{1 + O\left(\frac{1}{\log z}\right)\right\} \frac{\log z}{\log x} \log(2+2\delta) \log\left(\frac{\log(x/yz)}{\log(yz)} \frac{\log(x/y)}{\log y}\right) \\ &\leqslant \left\{1 + O\left(\frac{1}{\log z}\right)\right\} \frac{\delta}{2+\delta} \log(2+2\delta)\log(1+2\delta), \end{aligned}$$

as $y^2 z \leqslant x \leqslant (yz)^2$ and $z = y^\delta$. This inequality is trivial when $x \leqslant y^2 z$ for $S = 0$ in this case. Thus we always have

$$\begin{aligned} \sum_{\substack{x/z^2 < q \leqslant x/z \\ P^-(q) > y}} \omega\left(\frac{\log(x/q)}{\log z}\right) \frac{1}{q} &\leqslant \frac{\delta}{2} \log\left(\frac{2-\delta}{1-\delta}\right) + \frac{\delta \log(2+2\delta)\log(1+2\delta)}{2+\delta} \\ &\qquad + O\left(\frac{\log\log z}{\log z}\right). \end{aligned} \tag{2.13}$$

Inserting (2.12) and (2.13) into (2.11), we obtain the required inequality. This completes the proof. □

3. Statistical results

Our goal is now to prove Theorems 2 and 3. For the first, the main tool is the following type of large sieve inequality.

Lemma 3.1 ([13], Theorem 1). *Let $\nu \geqslant 1$ be a fixed integer and let $\{b_p\}_p$ be a sequence of real numbers indexed by prime numbers such that $|b_p| \leqslant B$ for some constant B and for all primes p. Then we have*

$$\sum_{f \in \mathrm{H}_k^*(N)} \bigg| \sum_{\substack{P < p \leqslant Q \\ p \nmid N}} b_p \frac{\lambda_f(p^\nu)}{p} \bigg|^{2j} \ll_\nu k\varphi(N) \left(\frac{96B^2(\nu+1)^2 j}{P \log P}\right)^j + (kN)^{10/11} \left(\frac{10BQ^{\nu/10}}{\log P}\right)^{2j}$$

uniformly for

$$B > 0, \qquad j \geqslant 1, \qquad 2 \mid k, \qquad 2 \leqslant P < Q \leqslant 2P, \qquad N \geqslant 1 \quad (\textit{squarefree}).$$

The implied constant depends on ν only.

Proof of Theorem 2. The basic idea is that for all forms f with coefficients $\lambda_f(p^\nu)$ of the same sign ε_p, the sums

$$\sum_{\substack{P < p \leqslant 2P \\ p \in \mathscr{P}}} \frac{\varepsilon_p \lambda_f(p^\nu)}{p}$$

exhibit no cancellation due to variation of signs. The large sieve implies this is very unlikely to happen, except if the $\lambda_f(p^\nu)$ are very small in absolute value. The Hecke relations are used to control this other possibility by relating it to $\lambda_f(p^{2\nu})$ being large which can not happen too often either.[3]

[3] Variants of this well-known trick have been used in a number of other contexts, as in [3], but note that the large sieve inequality proved there would not work for this problem, due to the lack of multiplicative stability of the sign conditions (it would also be much less efficient).

For the details, we first denote

$$\mathscr{P}_N := \{p \in \mathscr{P} \mid p \nmid N\},$$

and define

$$\mathscr{E}_k^*(N, P; \mathscr{P}) := \big\{f \in \mathrm{H}_k^*(N) \mid \varepsilon_p \lambda_f(p^\nu) > 0 \ \text{ for } \ p \in \mathscr{P}_N \cap (P, 2P]\big\},$$

$$\mathscr{E}_k^{\nu'}(N, P; \mathscr{P}) := \bigg\{f \in \mathrm{H}_k^*(N) \ \Big| \ \bigg|\sum_{\substack{P<p\leqslant 2P\\ p\in\mathscr{P}_N}} \frac{\lambda_f(p^{2\nu'})}{p}\bigg| \geqslant \frac{\delta}{2\nu\log P}\bigg\} \quad (1 \leqslant \nu' \leqslant \nu).$$

To prove Theorem 2, clearly we only need to show that there are two positive constants $C = C(\nu, \mathscr{P})$ and $c = c(\nu, \mathscr{P})$ such that

$$|\mathscr{E}_k^*(N, P; \mathscr{P})| \ll_{\nu,\mathscr{P}} kN \exp\Big(-c\frac{\log kN}{\log\log kN}\Big) \tag{3.1}$$

uniformly for

$$2 \mid k, \qquad N \quad \text{(squarefree)}, \qquad kN \geqslant X_0, \qquad C\log(kN) \leqslant P \leqslant (\log(kN))^{10}$$

for some sufficiently large number $X_0 = X_0(\nu, \mathscr{P})$.

The definition of $\mathscr{E}_k^*(N, P; \mathscr{P})$ and Deligne's inequality allow us to write

$$\begin{aligned}\sum_{f\in\mathscr{E}_k^*(N,P;\mathscr{P})} \bigg|\sum_{\substack{P<p\leqslant 2P\\ p\in\mathscr{P}_N}} \frac{\lambda_f(p^\nu)^2}{p}\bigg|^{2j} &\leqslant \sum_{f\in\mathscr{E}_k^*(N,P;\mathscr{P})} \bigg|\sum_{\substack{P<p\leqslant 2P\\ p\in\mathscr{P}_N}} (\nu+1)\varepsilon_p \frac{\lambda_f(p^\nu)}{p}\bigg|^{2j}\\ &\leqslant \sum_{f\in\mathrm{H}_k^*(N)} \bigg|\sum_{\substack{P<p\leqslant 2P\\ p\in\mathscr{P}_N}} (\nu+1)\varepsilon_p \frac{\lambda_f(p^\nu)}{p}\bigg|^{2j}.\end{aligned}$$

Choosing

$$b_p = \begin{cases}(\nu+1)\varepsilon_p & \text{if } p \in \mathscr{P},\\ 0 & \text{otherwise}\end{cases}$$

in Lemma 3.1, we find that

$$\begin{aligned}\sum_{f\in\mathscr{E}_k^*(N,P;\mathscr{P})} \bigg|\sum_{\substack{P<p\leqslant 2P\\ p\in\mathscr{P}_N}} \frac{\lambda_f(p^\nu)^2}{p}\bigg|^{2j} &\leqslant \sum_{f\in\mathrm{H}_k^*(N)} \bigg|\sum_{\substack{P<p\leqslant 2P\\ p\nmid N}} b_p \frac{\lambda_f(p^\nu)}{p}\bigg|^{2j}\\ &\ll kN\bigg(\frac{96(\nu+1)^4 j}{P\log P}\bigg)^j + (kN)^{10/11}P^{\nu j/2}.\end{aligned} \tag{3.2}$$

In view of the Hecke relation (1.1), the left-hand side of (3.2) is

$$\begin{aligned}&\geqslant \sum_{f\in\mathscr{E}_k^*(N,P;\mathscr{P})\setminus(\cup_{\nu'=1}^{\nu}\mathscr{E}_k^{\nu'}(N,P;\mathscr{P}))} \bigg(\sum_{\substack{P<p\leqslant 2P\\ p\in\mathscr{P}_N}} \frac{1}{p} - \sum_{1\leqslant\nu'\leqslant\nu} \bigg|\sum_{\substack{P<p\leqslant 2P\\ p\in\mathscr{P}_N}} \frac{\lambda_f(p^{2\nu'})}{p}\bigg|\bigg)^{2j}\\ &\geqslant \sum_{f\in\mathscr{E}_k^*(N,P;\mathscr{P})\setminus(\cup_{\nu'=1}^{\nu}\mathscr{E}_k^{\nu'}(N,P;\mathscr{P}))} \bigg(\sum_{\substack{P<p\leqslant 2P\\ p\in\mathscr{P}_N}} \frac{1}{p} - \frac{\delta}{2\log P}\bigg)^{2j}.\end{aligned}$$

Let $\omega(n)$ be the number of distinct prime factors of n. Using the hypothesis on $\mathscr{P}$ and the classical inequality

$$\omega(n) \leqslant \{1+o(1)\}\frac{\log n}{\log\log n},$$

we infer that

$$\begin{aligned}\sum_{\substack{P<p\leqslant 2P\\ p\in\mathscr{P}_N}}\frac{1}{p}-\frac{\delta}{2\log P} &\geqslant \sum_{\substack{P<p\leqslant 2P\\ p\in\mathscr{P}}}\frac{1}{p}-\sum_{\substack{P<p\leqslant 2P\\ p\,|\,N}}\frac{1}{p}-\frac{\delta}{2\log P}\\ &\geqslant \frac{\delta}{2\log P}-\frac{\omega(N)}{P}\geqslant\frac{\delta/2-2/C}{\log P}\geqslant\frac{\delta}{6\log P},\end{aligned}$$

provided $C\geqslant 6/\delta$. Combining this with (3.2), we infer that

$$|\mathscr{E}_k^*(N,P;\mathscr{P})\setminus(\cup_{\nu'=1}^{\nu}\mathscr{E}_k^{\nu'}(N,P;\mathscr{P}))| \ll kN\bigg(\frac{3456(\nu+1)^4 j\log P}{\delta^2 P}\bigg)^j+(kN)^{10/11}P^j.$$

Now we bound the size of the sets $\mathscr{E}_k^{\nu'}(N,P;\mathscr{P})$ to finish the proof. Taking

$$B=1,\qquad \nu=2\nu',\qquad Q=2P\qquad\text{and}\qquad b_p=\begin{cases}1 & \text{if } p\in\mathscr{P}\\ 0 & \text{otherwise}\end{cases}$$

in Lemma 3.1, we get

$$\begin{aligned}\bigg(\frac{\delta}{2\log P}\bigg)^{2j}|\mathscr{E}_k^{\nu'}(N,P;\mathscr{P})| &\leqslant \sum_{f\in\mathrm{H}_k^*(N)}\bigg|\sum_{\substack{P<p\leqslant 2P\\ p\nmid N}} b_p\frac{\lambda_f(p^{2\nu'})}{p}\bigg|^{2j}\\ &\ll kN\bigg(\frac{96(2\nu'+1)^2 j}{P\log P}\bigg)^j+(kN)^{10/11}\bigg(\frac{10(2P)^{\nu'/5}}{\log P}\bigg)^{2j}.\end{aligned}$$

Hence,

$$|\mathscr{E}_k^{\nu'}(N,P;\mathscr{P})|\ll kN\bigg(\frac{3456\nu^4 j\log P}{\delta^2 P}\bigg)^j+(kN)^{10/11}P^{\nu j}\quad(1\leqslant\nu'\leqslant\nu) \tag{3.3}$$

provided $P\geqslant 2(20\nu/\delta)^{10/(3\nu)}$.

Combining this with (3.3), we finally obtain

$$|\mathscr{E}_k^*(N,P;\mathscr{P})|\ll kN\bigg(\frac{3456(\nu+1)^4 j\log P}{\delta^2 P}\bigg)^j+(kN)^{10/11}P^{\nu j} \tag{3.4}$$

uniformly for

$$2\mid k,\qquad N\quad\text{(squarefree)},\qquad C\log(kN)\leqslant P\leqslant(\log(kN))^{10},\qquad j\geqslant 1.$$

Now, take

$$j=\bigg[\delta^*\frac{\log(kN)}{\log P}\bigg]$$

where $\delta^*=\delta^2/(10(\nu+1))^4$. We can ensure $j>1$ once X_0 is chosen to be suitably large. A simple computation gives that

$$\bigg(\frac{3456(\nu+1)^4 j\log P}{\delta^2 P}\bigg)^j\ll\exp\Big(-c\frac{\log kN}{\log\log kN}\Big)$$

for some positive constant $c = c(\nu, \mathscr{P})$ and $P^{\nu j} \ll (kN)^{1/1000}$, provided X_0 is large enough. Inserting them into (3.4), we get (3.1) and complete the proof. $\square$

We now come to the lower bound of Theorem 3. Our basic tool here is an equidistribution theorem for Hecke eigenvalues which is of some independent interest: it shows (quantitatively) that, after suitable average over $\mathrm{H}_k^*(N)$, the Hecke eigenvalues corresponding to the first primes are independently Sato-Tate distributed (thus, it is related to the earlier work of Sarnak [20] for Maass forms and Serre [21] and Royer [19] for holomorphic forms).

First, we recall the definition (1.4) of the angle $\theta_f(p) \in [0, \pi]$ associated to any $f \in \mathrm{H}_k^*(N)$ and prime $p \nmid N$. We also recall that the Chebychev *functions* X_n, $n \geqslant 0$, defined by

$$X_n(\theta) = \frac{\sin((n+1)\theta)}{\sin\theta} \tag{3.5}$$

for $\theta \in [0, \pi]$, form an orthonormal basis of $L^2([0, \pi], \mu_{ST})$. Hence, for any $\omega \geqslant 1$, the functions of the type

$$(\theta_1, \ldots, \theta_\omega) \mapsto \prod_{1 \leqslant j \leqslant \omega} X_{n_j}(\theta_j)$$

for $n_j \geqslant 0$, form an orthonormal basis of $L^2([0, \pi]^\omega, \mu_{ST}^{\otimes\omega})$.

Proposition 1. *Let N be a squarefree number, $k \geqslant 2$ an even integer, $s \geqslant 1$ an integer and $z \geqslant 2$ a real number. For any prime $p \leqslant z$ coprime with N, let*

$$Y_p(\theta) = \sum_{j=0}^{s} \hat{y}_p(j) X_j(\theta)$$

be a "polynomial" of degree $\leqslant s$ expressed in the basis of Chebychev functions on $[0, \pi]$. Then we have

$$\sum_{f \in \mathrm{H}_k^*(N)} \omega_f \prod_{\substack{p \leqslant z \\ (p,N)=1}} Y_p(\theta_f(p)) = \prod_{\substack{p \leqslant z \\ (p,N)=1}} \hat{y}_p(0) + O(C^{\pi(z)} D^{sz} (\tau(N) \log 2N)^2 (Nk^{5/6})^{-1})$$

where

$$\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}\langle f, f\rangle} \frac{N}{\varphi(N)}, \quad \langle f, f\rangle \text{ the Petersson norm of } f,$$
$$C = \max_{p,j} |\hat{y}_p(j)|,$$

and $D \geqslant 1$ and the implied constant are absolute.

By linearity, clearly, we get an analogue result for

$$\sum_{f \in \mathrm{H}_k^*(N)} \omega_f \varphi((\theta_p)_{p \leqslant z}), \qquad \varphi = \sum_j \varphi_j,$$

where each φ_j is a function which is a product of polynomials as in the statement.

Proof. Using the fact that for any $n_p \geqslant 0$, we have

$$\prod_{\substack{p \leqslant z \\ (p,N)=1}} X_{n_p}(\theta_f(p)) = \lambda_f\Big(\prod_{\substack{p \leqslant z \\ (p,N)=1}} p^{n_p} \Big), \tag{3.6}$$

(which is another form of the Hecke multiplicativity), we expand the product and get

$$\prod_{\substack{p\leqslant z\\(p,N)=1}} Y_p(\theta_f(p)) = \sum_{d\mid P_N(z)^s}\Bigl(\prod_{p\mid P_N(z)} \hat{y}_p(v_p(d))\Bigr)\lambda_f(d)$$

where $v_p(d)$ is the p-adic valuation of an integer and $P_N(z)$ is the product of the primes $p\leqslant z$, $p\nmid N$.

We now sum over f and appeal to the following Petersson formula for primitive forms:

$$\sum_{f\in \mathrm{H}_k^*(N)} \omega_f\lambda_f(m) = \delta(m,1) + O(m^{1/4}\tau(N)^2(\log 2mN)^2(Nk^{5/6})^{-1}),$$

for all $m\geqslant 1$ coprime with N (this is a simplified version of that in [7, Cor. 2.10]; note our slightly different definition of ω_f, which explains the absence of $\varphi(N)/N$ on the right-hand side); the result then follows easily from simple estimates for the sum over d of the remainder terms. □

We now deduce Theorem 3 from this, assuming $\varepsilon_p = 1$ for all p (handling the other choices of signs being merely a matter of complicating the notation).

To simplify notation, we write $P = P_N(z)$ the product of primes $\leqslant z$ coprime with N, and ω the number of such primes.

First, if we wanted only to have $\lambda_f(p)\geqslant 0$ for a fixed (finite) set of primes (i.e., for z fixed), we would be immediately done: Proposition 1 shows[4] that the $(\theta_f(p))_{p\mid P}$ become equidistributed as $kN\to+\infty$ with respect to the product Sato-Tate measure, if we weigh modular forms with ω_f, and hence

$$\sum_{\substack{f\in\mathrm{H}_k^*(N)\\ p\mid P\Rightarrow\lambda_f(p)\geqslant 0}} \omega_f \longrightarrow \mu_{ST}([0,\pi/2])^{\omega} = \Bigl(\frac{1}{2}\Bigr)^{\omega}$$

which is of the desired type, except for the presence of the weight.[5] However, we want to have

$$\lambda_f(p)\geqslant 0 \text{ for } p\leqslant z,\ (p,N)=1,$$

where z grows with kN, and this involves quantitative lower bounds for approximation in large dimension, which requires more care. We use a result of Barton, Montgomery and Vaaler [2] for this purpose; although it is optimized for uniform distribution modulo 1 instead of the Sato-Tate context, but it is not difficult to adapt it here and this gives a quick and clean argument.[6]

Precisely, we consider $[0,\pi]^{\omega}$, with the product Sato-Tate measure, and we will write $\theta = (\theta_p)$ for the elements of this set; we also consider $[0,1]^{\omega}$ and we write $x = (x_p)$ for elements there.

[4] For Maass forms, this is essentially one of the early results of Sarnak [20].

[5] Using the trace formula instead of the Petersson formula (as in [19]), the unweighted analogue of Proposition 1 holds with a product of local Plancherel measures, but each still gives measure 1/2 to the two signs.

[6] This result was also used recently by Y. Lamzouri [11, §7], in a somewhat related context.

For any positive odd integer L, we get from [2, Th. 7] two explicit trigonometric polynomials[7] on $[0,1]^\omega$, denoted $A_L(x)$, $B_L(x)$, such that

$$A_L(\theta/\pi) - B_L(\theta/\pi) \leqslant \prod_{\substack{p\leqslant z\\(p,N)=1}} \chi(\theta_p)$$

for all $\theta = (\theta_p) \in [0,\pi]^\omega$, where $\chi(\theta_p)$ is the characteristic function of $[0,\pi/2] \subset [0,\pi]$ (precisely, we consider the functions denoted $A(x)$, $B(x)$ in [2], with parameters $N = \omega$ and $u_n = 0$, $v_n = 1/2$ for all $n \leqslant \omega$; since $(v_n - u_n)(L+1) = (L+1)/2$ is a positive integer, we are in the situation $\Phi_{u,v} \in \mathcal{B}_N(L)$ of loc. cit.).

Thus we have the lower bound

$$\sum_{\substack{f\in \mathrm{H}_k^*(N)\\ \lambda_f(p)\geqslant 0 \text{ for } p\mid P}} \omega_f \geqslant \sum_{f\in \mathrm{H}_k^*(N)} \omega_f\Big(A_L(\theta_f/\pi) - B_L(\theta_f/\pi)\Big), \tag{3.7}$$

where $\theta_f = (\theta_f(p))_p$.

Moreover, as we will explain below, $A_L(\theta/\pi)$ is a product of polynomials over each variable, and $B_L(\theta/\pi)$ is a sum of ω such products, and we can now apply Proposition 1 (and the remark following it) to the terms on the right-hand side. More precisely, we claim that the following lemma holds:

Lemma 3.2. *With notation as above, we have:*

(1) *For any $\varepsilon \in (0,1/2)$, there exists constants $L_0 \geqslant 1$, and $c > 0$, such that the contribution Δ of the constant terms of the Chebychev expansions of $A_L(\theta/\pi)$ and $B_L(\theta/\pi)$ satisfies*

$$\Delta \geqslant \Big(\frac{1}{2} - \varepsilon\Big)^{\pi(z)},$$

if L is the smallest odd integer $\geqslant c\pi(z)$ and if $L \geqslant L_0$.

(2) *All the coefficients in the expansion in terms of Chebychev functions of the factors in $A_L(\theta/\pi)$ or in the terms of $B_L(\theta/\pi)$ are bounded by* 1.

(3) *The degrees, in terms of Chebychev functions, of the factors of $A_L(\theta/\pi)$ and of the terms of $B_L(\theta/\pi)$, are $\leqslant 2L$.*

Using this lemma, fixing $\varepsilon \in (0,1/2)$ and taking L as in Part (1) (we can obviously assume $L \geqslant L_0$, since otherwise z is bounded) we derive from Proposition 1 that

$$\sum_{f\in \mathrm{H}_k^*(N)} \omega_f\Big(A_L(\theta_f/\pi) - B_L(\theta_f/\pi)\Big) = \Delta + O(D^{z\pi(z)}(\tau(N)\log 2N)^2(Nk^{5/6})^{-1})$$

for some absolute constants D, with $\Delta \geqslant (1/2-\varepsilon)^{\pi(z)}$. This is then $\gg (1/2-\varepsilon)^{\pi(z)}$, provided

$$D^{z\pi(z)}(\tau(N)\log 2N)^2(Nk^{5/6})^{-1} \ll \Big(\frac{1}{2} - \varepsilon\Big)^{\pi(z)}.$$

This condition is satisfied for

$$z \leqslant c\sqrt{(\log kN)(\log\log kN)}$$

[7] Meaning, standard trigonometric polynomials of the type $\sum_\ell \alpha_\ell e(\ell\cdot x)$.

where $c > 0$ is an absolute constant, and this gives Theorem 3 when counting with the weight ω_f. But, using well-known bounds for $\langle f, f\rangle$, we have

$$\omega_f \ll kN(\log kN)(\log\log 6N) \ll kN(\log kN)^2,$$

with an absolute implied constant. Hence, for $z = c\sqrt{(\log kN)(\log\log kN)}$, we get

$$\begin{aligned}\frac{1}{|\mathrm{H}_k^*(N)|}|\{f \in \mathrm{H}_k^*(N) \mid \lambda_f(p) \geqslant 0 \text{ for } p \leqslant z,\ p \nmid N\}| &\gg \frac{1}{(\log kN)^2}\Bigl(\frac{1}{2} - \varepsilon\Bigr)^{\pi(z)} \\ &\gg \Bigl(\frac{1}{2} - 2\varepsilon\Bigr)^{\pi(z)}\end{aligned}$$

if kN is large enough, and so we obtain Theorem 3 as stated.

Proof of Lemma 3.2. We must now refer to the specific construction in [2]. We start with $A_L(x)$: we have the product formula

$$A_L(x) = \prod_{p|P} \alpha_L(x_p),$$

where α_L is a trigonometric polynomial in one variable of degree $\leqslant L$, i.e., of the type

$$\alpha_L(x) = \sum_{|\ell|\leqslant L} \hat{\alpha}_L(\ell) e(\ell x),$$

with $\hat{\alpha}_L(0) = 1/2$ (see [2, (2.2), Lemma 5, (2.17)]). In particular, the constant term (in the Chebychev expansion) for $A_L(\theta/\pi)$ is given by

$$\Bigl(\int_0^\pi \alpha_L(\theta/\pi)\, \mathrm{d}\mu_{ST}\Bigr)^\omega,$$

and we will bound it below. For the moment, we observe further that, from [2, Lemma 5], we know that $0 \leqslant \alpha_L(x) \leqslant 1$ for all $x \in [0, 1]$, and so we can simply bound all the coefficients in the Chebychev expansion, using the Cauchy-Schwarz inequality and orthonormality:

$$\begin{aligned}\Bigl|\int_0^\pi \alpha_L(\theta/\pi) X_n(\theta)\, \mathrm{d}\mu_{ST}\Bigr|^2 &\leqslant \int_0^\pi |\alpha_L(\theta/\pi)|^2\, \mathrm{d}\mu_{ST} \times \int_0^\pi |X_n(\theta)|^2\, \mathrm{d}\mu_{ST} \\ &\leqslant \int_0^\pi \mathrm{d}\mu_{ST} \times \int_0^\pi |X_n(\theta)|^2\, \mathrm{d}\mu_{ST} = 1.\end{aligned}$$

It is also clear using the definition of $X_n(\theta)$ that the n-th coefficient is zero as soon as $n + 2 > 2L$.

We now come to $B_L(x)$, which is a sum of ω product functions, as already indicated: we have

$$B_L(x) = \sum_{p|P} \beta_L(x_p) \prod_{\substack{q|P\\ q\neq p}} \alpha_L(x_q),$$

where $\beta_L(x)$ is another trigonometric polynomial of degree L, given explicitly by

$$\begin{aligned}\beta_L(x) &= \frac{1}{2L+2}\Big(\sum_{|\ell|\leqslant L}\Big(1-\frac{|\ell|}{L+1}\Big)e(\ell x)+\sum_{|\ell|\leqslant L}\Big(1-\frac{|\ell|}{L+1}\Big)e(\ell(x-1/2))\Big)\\ &= \frac{1}{2L+2}\Big(2+2\sum_{1\leqslant\ell\leqslant L}\Big(1-\frac{\ell}{L+1}\Big)(1+(-1)^\ell)\cos(2\pi\ell x)\Big),\end{aligned}$$

(see [2, p. 342, (2.3), p. 339]).

We now see immediately that Part (3) of the lemma is valid, and moreover, we see that $|\beta_L(x)|\leqslant 1$, so the same Cauchy-Schwarz argument already used for α_L implies that Part (2) holds.

To conclude, we look at the constant term in the Chebychev expansion for B_L, which is given by

$$\omega\Big(\int_0^\pi \alpha_L(\theta/\pi)\,\mathrm{d}\mu_{ST}\Big)^{\omega-1}\int_0^\pi \beta_L(\theta/\pi)\,\mathrm{d}\mu_{ST}.$$

Using the expression

$$\beta_L(\theta/\pi)=\frac{1}{2L+2}\Big(2+2\sum_{1\leqslant\ell\leqslant L}\Big(1-\frac{|\ell|}{L+1}\Big)(1+(-1)^\ell)\cos(2\ell\theta)\Big),$$

where the second term doesn't contribute after integrating against $\sin^2\theta=(1-\cos 2\theta)/2$ (the term with $\ell=1$ is zero), we get the formula

$$\Delta=\Big(\int_0^\pi \alpha_L(\theta/\pi)\,\mathrm{d}\mu_{ST}\Big)^{\omega-1}\Big(\int_0^\pi \alpha_L(\theta/\pi)\,\mathrm{d}\mu_{ST}-\frac{\omega}{L+1}\Big)$$

for the contribution of $A_L(x)-B_L(x)$.

Now we come back to a lower bound for the constant term for α_L. The point is that, as $L\to+\infty$, α_L converges in $L^2([0,1])$ to the characteristic function χ of $[0,1/2]$: from [2, (2.6)], and the definition of α_L, we get

$$|\chi(x)-\alpha_L(x)|\leqslant\beta_L(x),\qquad 0\leqslant x\leqslant 1,$$

and from the Fourier expansion of β_L we have

$$\|\beta_L\|_{L^2}^2\leqslant\frac{1}{(2L+2)^2}\times(4L+4)\to 0.$$

Hence, we know that

$$\frac{2}{\pi}\int_0^\pi \alpha_L(\theta/\pi)\sin^2\theta\,\mathrm{d}\theta\to\int_0^\pi \chi(\theta/\pi)\,\mathrm{d}\mu_{ST}=1/2.$$

For given $\varepsilon\in(0,1/2)$, the integral is $\geqslant(1/2-\varepsilon/2)$ if $L\geqslant L_0$, for some constant L_0. Then, if $L+1\geqslant 2\varepsilon^{-1}\omega$, we derive

$$\Delta\geqslant\Big(\frac12-\frac{\varepsilon}{2}\Big)^{\omega-1}\Big(\frac12-\varepsilon\Big)\geqslant\Big(\frac12-\varepsilon\Big)^{\omega},$$

which gives Part (1) of the lemma. □

4. Proof of Theorem 4

The simple idea of the proof of Theorem 4 is that the assumption translates to $\lambda_{f_1}(p)\lambda_{f_2}(p) \geqslant 0$ for all primes p (with few exceptions), but it is well-known from Rankin-Selberg theory that if $f_1 \neq f_2$, we have

$$\sum_p \frac{\lambda_{f_1}(p)\lambda_{f_2}(p)}{p^\sigma} = O(1) \qquad (\sigma \to 1+) \tag{4.1}$$

(see, e.g, [6, §5.12] for a survey and references; the underlying fact about automorphic forms is due to Mœglin and Waldspurger). Thus we only need to find a lower bound for the left-hand side (which is a sum of non-negative terms) which is unbounded as σ tends to $1+$. Since Rankin-Selberg theory also gives

$$\sum_p \frac{\lambda_{f_1}(p)^2}{p^\sigma} \sim -\log(\sigma - 1) \qquad (\sigma \to 1+), \tag{4.2}$$

the only difficulty is that one might fear that the coefficients of f_1 and f_2 are such that whenever $\lambda_{f_1}(p)$ is not small, the value of $\lambda_{f_2}(p)$ *is* very small.[8] In other words, we must show that the smaller order of magnitude of (4.1) compared with (4.2) is not due to the small size of the summands, but to sign compensations.

In the non-CM case, this would be quite easy if we knew (or used) the Sato-Tate conjecture, but we can avoid it by using small symmetric power L-functions. Here is the precise lemma that we will use:

Lemma 4.1. *Let $N \geqslant 1$ be an integer, $k \geqslant 2$ be an even integer and $f \in \mathrm{H}_k^*(N)$ a primitive cusp form of level N and weight k which is not of CM type. Then there exists a constant $\alpha > 0$ and $\delta > \frac{1}{2}$ such that*

$$\sum_{|\lambda_f(p)| > \alpha} \frac{1}{p^\sigma} \geqslant \delta \sum_p \frac{1}{p^\sigma} + O(1),$$

for $\sigma > 1$. In fact, one can take $\alpha = 0.231$ and $\delta = \frac{1}{2} + \frac{1}{24}$.

Lemma 4.1 does not hold for CM forms, because we then have $\lambda_f(p) = 0$ for a set of primes of density $\frac{1}{2}$. But the distribution of $\lambda_f(p)$ is even better known in the CM case, so we will be able to treat the case (1) of Theorem 4 involving CM forms without too much difficulty.

Proof. It is convenient here to work with the Chebychev polynomials U_n instead of the Chebychev functions X_n considered in the previous section: recall that for $n \geqslant 0$, we have

$$X_n(\theta) = U_n(2\cos\theta)$$

where $U_n \in \mathbb{R}[x]$ is a polynomial of degree n. Then (3.6) gives $U_n(\lambda_f(p)) = \lambda_f(p^n)$ for any $f \in \mathrm{H}_k^*(N)$, $p \nmid N$, and $n \geqslant 0$.

We then claim that there exists a polynomial

$$Y = \beta_0 + \beta_2 U_2 + \beta_4 U_4 + \beta_6 U_6 \in \mathbb{R}[x]$$

with the following properties:

(i) $\beta_0 > \frac{1}{2}$;

[8] See the remark after the proof for an example of which potential situations must be excluded.

(ii) for some $\alpha > 0$ and $x \in [-2,2]$, we have

$$Y(x) \leqslant \chi_A(x), \tag{4.3}$$

where $A := \{x \in [-2,2] \mid |x| > \alpha\}$.

Assuming this, we conclude as follows: by (ii), we have

$$\sum_{|\lambda_f(p)|>\alpha} \frac{1}{p^\sigma} \geqslant \sum_{p\nmid N} \frac{Y(\lambda_f(p))}{p^\sigma} = \beta_0 \sum_{p\nmid N} \frac{1}{p^\sigma} + \sum_{1\leqslant i\leqslant 3} \beta_{2i} \sum_{p\nmid N} \frac{U_{2i}(\lambda_f(p))}{p^\sigma}.$$

By the holomorphy and non-vanishing at $s = 1$ of the second, fourth and sixth symmetric power L-functions (see [9, Th. 3.3.7, Prop. 4.3] for the last two, noting that non-CM forms are not dihedral, and [22] for a survey concerning those L-functions), since $U_n(\lambda_f(p))$ is exactly the p-th coefficient of the n-th symmetric power for $p \nmid N$, standard analytic arguments show that

$$\sum_{p\nmid N} \frac{U_{2i}(\lambda_f(p))}{p^\sigma} = O(1)$$

for $\sigma \geqslant 1$ and $i = 1, 2, 3$. Hence the result follows with $\delta = \beta_0 > \frac{1}{2}$.

Now to check the claim, and verify the values of α and δ, we just exhibit a suitable polynomial, namely

$$Y = \tfrac{1}{2} + \tfrac{1}{24} + \tfrac{1}{4}U_2 - \tfrac{1}{4}U_4 + \tfrac{136}{1000}U_6 = \tfrac{17}{125}x^6 - \tfrac{93}{100}x^4 + \tfrac{227}{125}x^2 - \tfrac{283}{3000},$$

since

$$\begin{cases} U_0 = 1, \quad U_1 = x, \quad U_2 = x^2 - 1, \quad U_4 = x^4 - 3x^2 + 1, \\ U_5 = x^5 - 4x^3 + 3x, \quad U_6 = x^6 - 5x^4 + 6x^2 - 1. \end{cases} \tag{4.4}$$

This polynomial is even, and its graph on $[-2,2]$ is in Figure 1 (see Remark 1 below for an explanation of the origin of this polynomial).

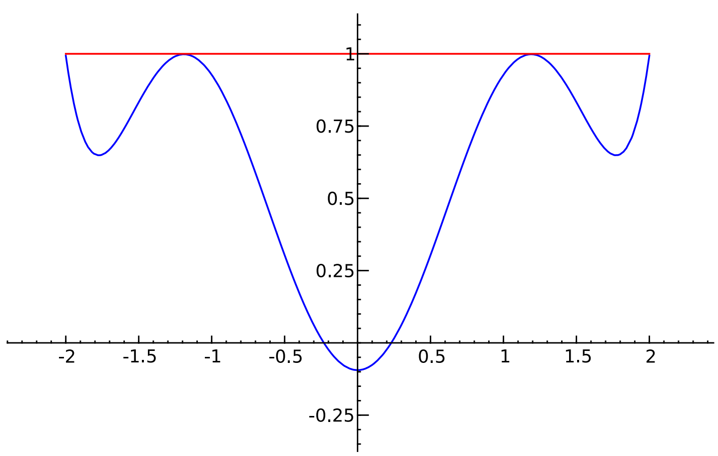

FIGURE 1

The value of α is an approximation (from below) to the real root

$$\alpha_0 = 0.23107202470801418176315245050693402580\ldots$$

of Y in $[0,2]$; the maximum value of Y on $[0,2]$ is very close to 1. $\square$

Proof of Theorem 4. We first consider Part (2), i.e., the case where f_1 and f_2 are not of CM type. Let E be the exceptional set of primes where the signs of $\lambda_{f_1}(p)$ and $\lambda_{f_2}(p)$ do not coincide. As already noticed, the assumption implies that for all primes $p \notin E$, we have $\lambda_{f_1}(p)\lambda_{f_2}(p) \geqslant 0$. Moreover, we have $|\lambda_{f_1}(p)\lambda_{f_2}(p)| \leqslant 4$ for all p, by the Deligne bound. Hence, with $\alpha > 0$ and $\delta > \frac{1}{2}$ as in Lemma 4.1, we have

$$\begin{aligned}
\sum_p \frac{\lambda_{f_1}(p)\lambda_{f_2}(p)}{p^\sigma} &= \sum_{p\notin E} \frac{\lambda_{f_1}(p)\lambda_{f_2}(p)}{p^\sigma} + \sum_{p\in E} \frac{\lambda_{f_1}(p)\lambda_{f_2}(p)}{p^\sigma} \\
&\geqslant \alpha^2 \sum_{\substack{|\lambda_{f_1}(p)|>\alpha,\, |\lambda_{f_2}(p)|>\alpha \\ p\notin E}} \frac{1}{p^\sigma} - 4\kappa|\log(\sigma-1)| + O(1) \\
&\geqslant \alpha^2 \Big(\sum_p - \sum_{p\in E} - \sum_{|\lambda_{f_1}(p)|\leqslant\alpha} - \sum_{|\lambda_{f_2}(p)|\leqslant\alpha} \Big) \frac{1}{p^\sigma} - 4\kappa|\log(\sigma-1)| + O(1) \\
&\geqslant \alpha^2 \Big\{ \big(1-\kappa-2(1-\delta)\big) \sum_p \frac{1}{p^\sigma} \Big\} - 4\kappa|\log(\sigma-1)| + O(1) \\
&= \big(\alpha^2(2\delta-\kappa-1) - 4\kappa\big)|\log(\sigma-1)| + O(1),
\end{aligned}$$

for any $\sigma > 1$. Since $2\delta > 1$, we find that the left-hand side goes to $+\infty$ as $\sigma \to 1+$ under the condition

$$\kappa < \frac{\alpha^2(2\delta-1)}{4+\alpha^2} = \frac{53361}{48640332} = 0.001097052\ldots$$

(with the values of Lemma 4.1). However, as already mentioned, the theory of Rankin-Selberg L-functions shows that if $f_1 \neq f_2$, we have

$$\sum_p \frac{\lambda_{f_1}(p)\lambda_{f_2}(p)}{p^\sigma} = O(1) \qquad (\sigma \to 1+),$$

since there is no pole (or zero) of $L(s, f_1 \times f_2)$ at $s = 1$ (recall that f_1 and f_2 have real coefficients so $f_2 = \bar{f_2}$). So we must indeed have $f_1 = f_2$.

There remains to consider Part (1) of Theorem 4 when one of the forms is of CM type (and the exceptional set E now has density 0). We will be brief since there are less difficulties here. The main point is the following well-known result concerning the distribution of the angles $\theta_f(p)$ for a CM form $f \in \mathrm{H}_k^*(N)$, with $k \geqslant 2$: there exists a real, non-trivial, primitive Dirichlet character χ_f such that $\lambda_f(p) = 0$ when $\chi_f(p) = -1$ (a set of primes I_f of density 1/2), and for $p \notin I_f$, the $\theta_f(p) \in [0, \pi]$ for $p \leqslant x$ become *uniformly distributed* as $x \to +\infty$, i.e., we have

$$\frac{2}{\pi(x)} \sum_{\substack{p\notin I_f \\ p\leqslant x}} e^{2im\theta_f(p)} \to 0,$$

for all non-zero integers $m \in \mathbb{Z}$ (see, e.g., [17, p. 197], where this is explained for elliptic curves, with slightly different notation). In particular, for any $\alpha > 0$, the density of the set of primes where $|\lambda_f(p)| > \alpha$ exists and is equal to

$$\frac{1}{\pi}\arccos(\alpha/2)$$

and this density goes to $1/2$ as $\alpha \to 0$.

Now assume f_1 is a CM form and f_2 is not; according to Lemma 4.1, we find $\alpha > 0$ and a set of primes P_2 of analytic density $\delta > 1/2$ where $|\lambda_{f_2}(p)| > \alpha$, and then the set $P_2 \cap I_{f_1}$ has analytic density > 0, thus for small enough α', it contains a set G with positive analytic density where $|\lambda_{f_1}(p)| > \alpha'$. Hence we have

$$\begin{aligned}\sum_p \frac{\lambda_{f_1}(p)\lambda_{f_2}(p)}{p^\sigma} &\geqslant \sum_{p\in G} \frac{\lambda_{f_1}(p)\lambda_{f_2}(p)}{p^\sigma} + o(\log|\sigma-1|^{-1})\\ &\geqslant \alpha\alpha' \sum_{p\in G}\frac{1}{p^\sigma} + o(\log|\sigma-1|^{-1}), \qquad \text{as } \sigma \to 1+,\end{aligned}$$

which is in fact a contradiction (since f_1 can not be equal to f_2).

Finally, assume f_1 and f_2 are CM forms. Because of independence of primitive real characters, the union $I_{f_1} \cup I_{f_2}$ has density *at most* $3/4$ (the complement contains the set of primes totally split in a Galois extension of $\mathbb{Q}$ of degree at most 4). For small enough $\alpha > 0$, the complement must contain a set of primes of positive analytic density where $|\lambda_{f_1}(p)| > \alpha$, $|\lambda_{f_2}(p)| > \alpha$, and we can conclude as before that the Rankin-Selberg convolution has a pole at $s = 1$, so that $f_1 = f_2$ in that case also. □

It is known that the 8-th symmetric power are holomorphic for non-CM forms (still due to Kim and Shahidi); using this, one could improve the constant slightly. However, it seems more interesting to show that using the sixth symmetric power (and thus the deep results of Kim and Shahidi) is necessary. For this, note that the sequences $\{x_p\}_{p\,\text{primes}}$ and $\{y_p\}_{p\,\text{primes}}$ defined by $x_2 = y_2 = 0$ and for primes $p \geqslant 3$ by

$$x_p = \begin{cases} 0 & \text{if } p \equiv 3 \,(\mathrm{mod}\, 4),\\ (-1)^{(p-1)/4}\sqrt{2} & \text{if } p \equiv 1 \,(\mathrm{mod}\, 4),\end{cases}$$

$$y_p = \begin{cases} (-1)^{(p-3)/4}\sqrt{2} & \text{if } p \equiv 3 \,(\mathrm{mod}\, 4),\\ 0 & \text{if } p \equiv 1 \,(\mathrm{mod}\, 4),\end{cases}$$

have the "right" moments of order 1 to 5 for being Sato-Tate distributed,[9] i.e., we have

$$\sum_p \frac{X_k(x_p)}{p^\sigma} = O(1) \qquad \text{and} \qquad \sum_p \frac{X_k(y_p)}{p^\sigma} = O(1)$$

for $\sigma > 1$ and $1 \leqslant k \leqslant 5$, and yet $x_p y_p \geqslant 0$ for all p, in fact $x_p y_p = 0$, so that we most certainly have

$$\sum_p \frac{x_p y_p}{p^\sigma} = O(1) \qquad (\sigma > 1).$$

Remark 1. We now explain how the polynomial of Lemma 4.1 was found. First – confirming the previous remark – there is no polynomial

$$Y = \beta_0 + \beta_2 X_2 + \beta_4 X_4 \in \mathbb{R}[x], \qquad \text{with } \beta_0 > \tfrac{1}{2},$$

such that $Y \leqslant \chi_I$ for some interval $I = [\alpha, 2] \subset (0, 2]$; indeed, such a polynomial would have to satisfy

[9] The sixth moment fails: it is 4 instead of 5 for the Sato-Tate distribution.

(iv) $Y(0) \leqslant 0$;
(v) $Y \leqslant 1$ on $[0,2]$;
(vi) $\beta_0 > \frac{1}{2}$.

But then, expressing $Y \in \mathbb{R}[x]$ in the basis of powers of x, we check that

$$Y(0) + Y(\sqrt{2}) = (\beta_0 - \beta_2 + \beta_4) + (\beta_0 - \beta_2 + \beta_4 + 2\beta_2 - 6\beta_4 + 4\beta_4) = 2\beta_0$$

so condition (vi) leads to $Y(0) + Y(\sqrt{2}) > 1$, and if $Y(0) \leqslant 0$, this means that $Y(\sqrt{2}) > 1$, showing that (iv) and (v) are then incompatible. (This formula is a very special case of "Gaussian quadrature" using zeros of orthogonal polynomials.)

Looking at this argument, however, reveals that it barely fails: the polynomial

$$Y_0 = \tfrac{1}{2} + \tfrac{1}{4}X_2 - \tfrac{1}{4}X_4 = x^2 - \tfrac{1}{4}x^4 = x^2(1 - \tfrac{1}{2}x)(1 + \tfrac{1}{2}x)$$

satisfies $0 \leqslant Y_0 \leqslant 1$ on $[0,2]$ and $\beta_0 = \frac{1}{2}$; in fact

$$Y_0(0) = 0, \qquad \beta_0 = \tfrac{1}{2}, \qquad \max_{x\in[0,2]} Y_0(x) = Y_0(\sqrt{2}) = 1.$$

So we constructed our polynomial by "deforming" slightly this example, increasing the constant coefficient to make it $> \frac{1}{2}$ and compensating with a small multiple of X_6. We did spend some time trying to adjust the parameters to maximize $\frac{\alpha^2(2\delta-1)}{4+\alpha^2}$, but we do not know what is the best possible result with polynomials of degree 6. Maybe the method of Rankin (cf. [18, 26]) could be helpful to solve this optimization problem.

Remark 2. Lemma 4.1 is somewhat "dual" to well-known investigations of consequences of holomorphy of the first symmetric power L-functions towards the Sato-Tate conjecture, explained in particular in Serre's letter to Shahidi included as an Appendix to [22] (and refined most recently by Kim and Shahidi [9, §4]). In those works, one is interested in finding $c \in [-2,2]$, as large as possible, such that $\lambda_f(p) > c$ for infinitely many p and $\lambda_f(p) < -c$ for infinitely many p. In our lemma, the value of c (i.e., α) is not the most important, but the density of the set of primes has to be quite large.

Remark 3. The limit of the argument we used (for non-CM forms) is fairly easy to determine: if the Sato-Tate conjecture is valid, we have all even symmetric power L-functions at our disposal and hence we can use polynomials Y approximating arbitrarily closely (in L^1 or L^2) to $\chi_A(x)$ for any $\alpha \in (0,2]$. The value $\delta = \beta_0$ is then the probability under μ_{ST} distribution of $A = \{x \,:\, |x| \geqslant \alpha\}$, namely

$$\beta_0 = \frac{2}{\pi}\arccos\Big(\frac{\alpha}{2}\Big) - \frac{1}{\pi}\sin\Big(2\arccos\Big(\frac{\alpha}{2}\Big)\Big)$$

and we are led to maximize over $[0,2]$ the quantity

$$\kappa = \frac{\alpha^2}{4+\alpha^2}\Big(\frac{4}{\pi}\arccos\Big(\frac{\alpha}{2}\Big) - \frac{2}{\pi}\sin\Big(2\arccos\Big(\frac{\alpha}{2}\Big)\Big) - 1\Big)$$

we find numerically that the best value is around $\alpha \simeq 0.522179$, allowing to take $\kappa < 0.021875\ldots$ (i.e., knowing only the individual Sato-Tate conjecture, we can allow a bit more than 2% of the primes to be in the exceptional set).

Remark 4. As a final remark, one can think of other ways (than looking at signs) of reducing Fourier coefficients of modular forms to a fixed finite set: the most obvious, at least if f has integral coefficients $\lambda_f(n)n^{(k-1)/2}$, is to look at the coefficients modulo some fixed prime number ℓ. However, the situation there can be drastically different: for instance, for all (infinitely many) elliptic curves with full rational 2-torsion, given for instance by equations

$$y^2 = (x-a)(x-b)(x-c)$$

with a, b, c distinct integers, the reduction modulo 2 of the odd prime coefficients of the corresponding L-function (or modular form) is the same!

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