ON INTERSECTIONS OF CONJUGACY CLASSES AND BRUHAT CELLS

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Abstract. For a connected complex semi-simple Lie group G and a fixed pair (B, B^-) of opposite Borel subgroups of G, we determine when the intersection of a conjugacy class C in G and a double coset BwB^- is non-empty, where w is in the Weyl group W of G. The question comes from Poisson geometry, and our answer is in terms of the Bruhat order on W and an involution $m_C \in W$ associated to C. We study properties of the elements m_C. For $G = SL(n+1, \mathbb{C})$, we describe m_C explicitly for every conjugacy class C, and for the case when $w \in W$ is an involution, we also give an explicit answer to when $C \cap (BwB)$ is non-empty.

1. Introduction

1.1. **The set up and the results.** Let G be a connected complex semi-simple Lie group, and let B and B^- be a pair of opposite Borel subgroups of G. Then $H = B \cap B^-$ is a Cartan subgroup of G. Let $W = N_G(H)/H$ be the Weyl group, where $N_G(H)$ is the normalizer of H in G. One then has the well-known Bruhat decompositions

$$G = \bigsqcup_{w\in W} BwB = \bigsqcup_{w\in W} BwB^- \quad \text{(disjoint unions)}.$$

Subsets of G of the form BwB or $Bw'B^-$, where $w, w' \in W$, will be called Bruhat cells in G. The Bruhat order on W is the partial order on W defined by

$$w_1 \leq w_2 \quad \Longleftrightarrow \quad Bw_1B \subset \overline{Bw_2B}, \quad w_1, w_2 \in W.$$

Given a conjugacy class C of G, let

$$W_C = \{w \in W : \ C \cap (BwB) \neq \emptyset\}, \tag{1.1}$$

$$W_C^- = \{w \in W : \ C \cap (BwB^-) \neq \emptyset\}. \tag{1.2}$$

The sets W_C have been studied by several authors (see, for example, [8, 9] by Ellers and Gordeev and [4] by G. Carnovale) and are not easy to determine even for the case of $G = SL(n, \mathbb{C})$ (see [9]). On the other hand, let m_C be the unique element in W such that $C \cap (Bm_CB)$ is dense in C. It is easy to show (see Lemma 2.4) that m_C is a unique maximal element in W_C with respect to the Bruhat order on W.

Our first result, Theorem 2.5, states that, for every conjugacy class C in G,

$$W_C^- = \{w \in W : w \leq m_C\}.$$

Thus the set W_C^- is completely determined by the element m_C and the Bruhat order on W.

Theorem 2.5 is motivated by Poisson geometry. It is shown in [10] that the connected complex semi-simple Lie group G carries a holomorphic Poisson structure π_0, invariant under conjugation by elements in H, such that the non-empty intersections $C \cap (BwB^-)$ are exactly the H-orbits of symplectic leaves of π_0, where C is a conjugacy class in G and $w \in W$. To describe precisely the symplectic leaves of π_0, one thus first needs to know when an intersection $C \cap (BwB^-)$ is non-empty. By [18, Theorem 1.4], the non-empty intersections $C \cap (BwB^-)$ are always smooth and irreducible. The geometry of such intersections and applications to Poisson geometry will be carried out elsewhere.

The elements m_C play an important role in the study of spherical conjugacy classes, i.e., conjugacy classes in G on which the B-action by conjugation has a dense orbit. In connection with their proof of the de Concini-Kac-Procesi conjecture on representations of the quantized universal enveloping algebra $\mathcal{U}_\epsilon(\mathfrak{g})$ at roots of unity over spherical conjugacy classes, N. Cantarini, G. Carnovale, and M. Costantini proved [2, Theorem 25] that a conjugacy class C in G is spherical if and only if $\dim C = l(m_C) + \mathrm{rank}(1 - m_C)$, where l is the length function on W, and $\mathrm{rank}(1 - m_C)$ is the rank of the operator $1 - m_C$ in the geometric representation of W. It is also shown by M. Costantini [5], again for a spherical conjugacy class C, that the decomposition of the coordinate ring of C as a G-module (for G simply connected) is almost entirely determined by the element m_C (see [5, Theorem 3.22] for the precise statement). When G is simple, a complete list of the m_C's, for C spherical, is given by G. Carnovale in [3, Corollary 4.2].

In this paper, we study some properties of m_C for every conjugacy class C of G. After examining some properties of W_C, we show, in Corollary 2.11, that for each conjugacy class C in G, $m_C \in W$ is one and the only one maximal length element in its conjugacy class in W. In particular, m_C is an involution. When C is spherical, the fact that m_C is an involution is also proved in [2, Remark 4] and [3, Theorem 2.7]. For $m \in W$, denote by $\mathcal{O}_m$ the conjugacy class of m in W. Let

$$\mathcal{M} = \{m \in W : m \text{ is the unique maximal length element in } \mathcal{O}_m\}. \tag{1.3}$$

Then $m_C \in \mathcal{M}$ for every conjugacy class C in G. It is thus desirable to study the set $\mathcal{M}$.

When G is simple, using arguments from [3], it is not hard to give a complete list of elements in $\mathcal{M}$. It turns out that when G is simple, the list of elements in

$\mathcal{M}$ coincides with the list in [3, Corollary 4.2]. See §3 and in particular Theorem 3.10. Consequently, when G is simple, one has

$$\begin{aligned}\mathcal{M} &= \{m_C \in W:\ C \text{ is a conjugacy class in } G\} \\ &= \{m_C \in W:\ C \text{ is a spherical conjugacy class in } G\}.\end{aligned} \tag{1.4}$$

If $G = G_1 \times G_2 \times \cdots \times G_k$ is semi-simple with simple factors G_j and Weyl groups W_j for $1 \leq j \leq k$, then

$$\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_k,$$

where for $1 \leq j \leq k$, $\mathcal{M}_j \subset W_j$ is defined as in (1.3). Hence (1.4) also holds for G semi-simple. We have thus completely described the set $\mathcal{M}$ for any connected semi-simple complex Lie group G.

We consider the case of $G = SL(n+1, \mathbb{C})$ in §4. For any conjugacy class C in $SL(n+1, \mathbb{C})$ and any involution $w \in W \cong S_{n+1}$, we show in Theorem 4.2 that

$$C \cap (BwB) \neq \emptyset \quad \text{iff} \quad l_2(w) \leq r(C),$$

where $l_2(w)$ is the number of distinct 2-cycles in the cycle decomposition of w, and

$$r(C) = \min\{\text{rank}(g - cI):\ c \in \mathbb{C}\}$$

for any $g \in C$. Theorem 4.2 is proved in §4.3 using (a special case of) a criterion by Ellers-Gordeev [9]. Since the proof of the Ellers-Gordeev criterion in [9] involves rather complicated combinatorics, we also give a direct proof of Theorem 4.2 in §4.4. Our direct proof also shows how to explicitly find an element in $C \cap BwB$ when $l_2(w) \leq l(C)$.

Combining Theorem 4.2 and a result of G. Carnovale [3, Theorem 2.7], one has, for a spherical conjugacy class C in $SL(n+1, \mathbb{C})$,

$$W_C = \{w \in S_{n+1}:\ w^2 = 1,\ \ l_2(w) \leq r(C)\}.$$

As another consequence of Theorem 4.2, we show in Corollary 4.4 that for any conjugacy class C in $SL(n+1, \mathbb{C})$, if W_C contains an involution $w \in S_{n+1}$, then W_C contains the whole conjugacy class of w in S_{n+1}.

Finally, let $m_0 = 1$, and for an integer $1 \leq l \leq \left[\frac{n+1}{2}\right]$, let $m_l \in S_{n+1}$ be the involution with the cycle decomposition

$$m_l = (1, n+1)(2, n) \cdots (l,\ n+2-l).$$

Corollary 4.8 says that for any conjugacy class C in $SL(n+1, \mathbb{C})$,

$$m_C = \begin{cases} w_0 & \text{if } r(C) \geq \left[\frac{n+1}{2}\right], \\ m_{r(C)} & \text{if } r(C) < \left[\frac{n+1}{2}\right] \end{cases}.$$

The explicit description of m_C for an arbitrary conjugacy class in other classical groups will be given in [6].

In the study of the symplectic leaves of certain Poisson structures on G as well as on the de Concini-Procesi compactification of G when G is of adjoint type, one needs to consider intersections $C_\delta \cap (BwB^-)$, where δ is an automorphism of G preserving both H and B and C_δ is a δ-twisted conjugacy class in G. See [14]. For such a conjugacy class C_δ in G, we have the element $m_{C_\delta} \in W$ which is the unique maximal length element in its δ-twisted conjugacy class in W. See §2.3.

1.2. **Notation.** Let Δ be the set of all roots of G with respect to H, let $\Delta^+ \subset \Delta$ be the set of positive roots determined by B, and let Γ be the set of simple roots in Δ^+. We also write $\alpha > 0$ (resp. $\alpha < 0$) if $\alpha \in \Delta^+$ (resp. $\alpha \in -\Delta^+$). Define

$$\delta_0 : \ \ \Delta \longrightarrow \Delta : \ \ \delta_0(\alpha) = -w_0(\alpha), \qquad \alpha \in \Delta.$$

Then δ_0 permutes Δ^+ and Γ, and it induces an automorphism, still denoted by δ_0, on W:

$$\delta_0 : \quad W \longrightarrow W : \quad \delta_0(w) = w_0 w w_0, \qquad w \in W.$$

For $\alpha \in \Gamma$, let $s_\alpha \in W$ be the reflection determined by α. For a subset J of Γ, let W_J be the subgroup of W generated by $\{s_\alpha : \alpha \in J\}$, and let $w_{0,J}$ be the maximal length element in W_J. Let $W^J \subset W$ be the set of minimal length representatives of W/W_J. Set $w_0 = W_{0,\Gamma}$, so w_0 is the maximal length element in W. The length function on W is denoted by l.

1.3. **Acknowledgments.** We would like to thank Xuhua He for very helpful discussions. Our research was partially supported by HKRGC grants 703405 and 703407.

2. The sets W_C and W_C^- and the elements m_C

2.1. W_C^- **in terms of** m_C**.** We keep the notation as in §1.1. In particular, for each conjugacy class C in G, we have the subsets W_C and W_C^- of W as in (1.1) and (1.2).

Lemma 2.1. *One has $W_C \subset W_C^-$ for every conjugacy class C in G.*

Proof. Let $w \in W$. If $C \cap (BwB) \neq \emptyset$, then $C \cap (Bw) \neq \emptyset$, so $C \cap (BwB^-) \neq \emptyset$.

Q.E.D.

Lemma 2.2. *For any $w \in W$,*

$$BwB^-B = \bigsqcup_{w' \in W, w \leq w'} Bw'B.$$

Proof. Clearly BwB^-B is the union of some (B,B)-double cosets. Let $w' \in W$. Then

$$Bw'B \subset BwB^-B \iff (Bw'B)\cap(BwB^-B) \neq \emptyset \iff (Bw'B)\cap(BwB^-) \neq \emptyset,$$

which, by [7], is equivalent to $w \leq w'$.

Q.E.D.

Lemma 2.3. *Let C be a conjugacy class in G and let $w \in W$. Then $w \in W_C^-$ if and only if $w \leq w'$ for some $w' \in W_C$.*

Proof. Since C is conjugation invariant,

$$C \cap (BwB^-) \neq \emptyset \iff C \cap (BwB^-B) \neq \emptyset,$$

which, by Lemma 2.2, is equivalent to $w \leq w'$ for some $w' \in W_C$.

Q.E.D.

For a subset X of G, let $\overline{X}$ be the Zariski closure of X in G. The following Lemma 2.4 can also be found in [2, §1].

Lemma 2.4. *Let C be a conjugacy class in G. Then*

1) there is a unique $m_C \in W$ such that $C \cap (Bm_CB)$ is dense in C;

2) $w \leq m_C$ for every $w \in W_C$.

Proof. The decomposition $C = \bigsqcup_{w \in W_C} C \cap (BwB)$ gives

$$\overline{C} = \bigsqcup_{w \in W_C} \overline{C \cap (BwB)}.$$

As C is irreducible, there exists a unique $m_C \in W_C$ such that $\overline{C} = \overline{C \cap (Bm_CB)}$. If $w \in W_C$, then

$$\emptyset \neq C \cap (BwB) \subset \overline{C} = \overline{C \cap (Bm_CB)} \subset \overline{Bm_CB},$$

so $w \leq m_C$.

Q.E.D.

Theorem 2.5. *For every conjugacy class C in G, $W_C^- = \{w \in W : w \leq m_C\}$.*

Proof. Let $w \in W$. If $w \leq m_C$, then $w \in W_C^-$ by Lemma 2.3. Conversely, if $w \in W_C^-$, then again by Lemma 2.3, $w \leq w'$ for some $w' \in W_C$. Since $w' \leq m_C$ by Lemma 2.4, one has $w \leq m_C$.

Q.E.D.

Lemma 2.6. *If C and C' are two conjugacy classes in G such that $C' \subset \overline{C}$, then $m_{C'} \leq m_C$.*

Proof. By definition,

$$\emptyset \neq C' \cap (Bm_{C'}B) \subset \overline{C} = \overline{C \cap (Bm_C B)} \subset \overline{Bm_C B}.$$

Thus $m_{C'} \leq m_C$.

Q.E.D.

2.2. **Some properties of W_C and m_C.** We recall some definitions and results from [8, 11, 12].

Definition 2.7. 1) [8, Definition 3.1] Let $w, w' \in W$. An ascent from w to w' is a sequence $\{\alpha_j\}_{1\leq j\leq k}$ in Γ such that

$$w' = s_{\alpha_k} \cdots s_{\alpha_2} s_{\alpha_1} w \, s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$$

and $l(s_{\alpha_j} \cdots s_{\alpha_2} s_{\alpha_1} w \, s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_j}) \geq l(s_{\alpha_{j-1}} \cdots s_{\alpha_2} s_{\alpha_1} w \, s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{j-1}})$ for every $1 \leq j \leq k$. Write $w' \longleftarrow w$ if there is an ascent from w to w' or if $w' = w$.

2) [12, §2.9] For $w, w', x \in W$, write $w \overset{x}{\sim} w'$ if $l(w) = l(w')$, $w' = xwx^{-1}$, and either $l(w') = l(xw) + l(x)$ or $l(w') = l(x) + l(wx^{-1})$. Write $w \sim w'$ if there exist sequences of $\{x_j\}_{1\leq j\leq k}$ and $\{w_j\}_{1\leq j\leq k}$ in W such that

$$w \overset{x_1}{\sim} w_1 \overset{x_2}{\sim} \cdots \overset{x_k}{\sim} w_k = w'.$$

3) Let $\mathcal{O}$ be a conjugacy class in W. An element $w \in \mathcal{O}$ is called a *maximal length element* in $\mathcal{O}$ if $l(w_1) \leq l(w)$ for all $w_1 \in \mathcal{O}$.

Proposition 2.8. [12, §2.9] *Let $\mathcal{O}$ be any conjugacy class in W.*

1) For any $w \in \mathcal{O}$, there exists a maximal length element $w' \in \mathcal{O}$ such that $w' \longleftarrow w$;

2) If w' and w'' are two maximal length elements in $\mathcal{O}$, then $w' \sim w''$.

Proposition 2.9. *Let C be a conjugacy class in G, and let $w, w' \in W$.*

1) If $w' \longleftarrow w$ and $w \in W_C$, then $w' \in W_C$.

2) If $w \sim w'$ and $w \in W_C$, then $w' \in W_C$.

Proof. 1) is just [8, Proposition 3.4]. To see 2), assume that $w \overset{x}{\sim} w'$ for some $x \in W$, so $w' = xwx^{-1}$, and either $l(w') = l(xw) + l(x)$ or $l(w') = l(x) + l(wx^{-1})$. Assume first that $l(w') = l(xw) + l(x)$. Then

$$C \cap (Bw'B) = C \cap (BxwBx^{-1}B) \supset C \cap (xwBx^{-1}) \neq \emptyset.$$

Thus $C \cap (Bw'B) \neq \emptyset$ and $w' \in W_C$. The case of $l(w') = l(x) + l(wx^{-1})$ is proved similarly.

Q.E.D.

Remark 2.10. We refer to [8, 9] for a more detailed study of the set W_C and in particular for the case of $G = SL(n, \mathbb{C})$. On the other hand, it is proved in [4] by G. Carnovale that a conjugacy class C in G is spherical if and only if W_C consists only of involutions. See also Corollary 4.6 in §4.2.

For $w \in W$, let $\mathcal{O}_w$ be the conjugacy class of w in W.

Corollary 2.11. *For any conjugacy class C in G, m_C is the unique maximal length element in $\mathcal{O}_{m_C}$.*

Proof. By Proposition 2.8, there exists a maximal length element $w' \in \mathcal{O}_{m_C}$ such that $w' \longleftarrow m_C$. By Proposition 2.9, $w' \in W_C$, so $w' \leq m_C$ by Lemma 2.4. Since $l(w') \geq l(m_C)$, one has $w' = m_C$. Thus m_C is a maximal length element in $\mathcal{O}_{m_C}$. If w_1 is any maximal length element in $\mathcal{O}_{m_C}$, then $w_1 \sim m_C$ by Proposition 2.8, so $w_1 \in W_C$ by Proposition 2.9, and thus $w_1 \leq m_C$ by Lemma 2.4. Since $l(w_1) = l(m_C)$, one has $w_1 = m_C$. Thus m_C is the only maximal length element in $\mathcal{O}_{m_C}$.

Q.E.D.

Consider now the bijection

$$\phi: \quad W \longrightarrow W: \quad w \longmapsto w_0 w, \qquad w \in W. \tag{2.1}$$

Then under ϕ, the conjugation action of W on itself becomes the following δ_0-*twisted conjugation action* of W on itself:

$$u \cdot w = \delta_0(u) w u^{-1}, \qquad u, w \in W.$$

For $w \in W$, let $\mathcal{O}_w^{\delta_0}$ be the δ_0-twisted conjugacy class of w, and say an element $w' \in \mathcal{O}_w^{\delta_0}$ has minimal length if $l(w') \leq l(w_1)$ for all $w_1 \in \mathcal{O}_w^{\delta_0}$. Using the fact that $l(w_0 u) = l(w_0) - l(u)$ for any $u \in W$, it is easy to see that for any $w \in W$, ϕ maps maximal length elements in $\mathcal{O}_w$ to minimal length elements in $\mathcal{O}_{w_0 w}^{\delta_0}$.

Corollary 2.12. *For any conjugacy class C in G, $w_0 m_C$ is the unique minimal length element in $\mathcal{O}_{w_0 m_C}^{\delta_0}$.*

Remark 2.13. Let $\tilde{G}$ be the connected and simply connected cover of G, let $\pi: \tilde{G} \to G$ be the covering map, and let $Z = \pi^{-1}(e)$, where e is the identity element of G. Let $\tilde{A} = \pi^{-1}(A)$, where $A \in \{H, B, B^-\}$. Identify the Weyl group for $\tilde{G}$ with W. For any conjugacy class C in G, $\pi^{-1}(C)$ is a union of conjugacy classes in $\tilde{G}$. Since $Z \subset \tilde{H} = \tilde{B} \cap \tilde{B}^-$, it is easy to see that for any conjugacy classes $\tilde{C}$ in $\pi^{-1}(C)$, $W_{\tilde{C}} = W_C$ and $W_{\tilde{C}}^- = W_C^-$, and in particular, $m_C = m_{\tilde{C}}$. Thus the subset $\{m_C : \ C \ \text{ a conjugacy class in } \ G\}$ of W depends only on the isogeneous class of G.

2.3. **δ-twisted conjugacy classes.** Let δ be any automorphism of G such that $\delta(B) = B$ and $\delta(H) = H$. Then G acts on itself by δ-twisted conjugation given by

$$g \cdot_\delta h = \delta(g)hg^{-1}, \quad g, h \in G.$$

A δ-twisted conjugacy class in G is defined to be a G-orbit of the δ-twisted conjugation. Given a δ-twisted conjugacy class C_δ of G, let

$$W_{C_\delta} = \{w \in W : \; C_\delta \cap (BwB) \neq \emptyset\}, \tag{2.2}$$

$$W^-_{C_\delta} = \{w \in W : \; C_\delta \cap (BwB^-) \neq \emptyset\}. \tag{2.3}$$

Then all the arguments in §2.1 carry through when C is replaced by C_δ. In particular, let m_{C_δ} be the unique element in W such that $C_\delta \cap (Bm_{C_\delta}B)$ is dense in C_δ. Then $m_{C_\delta} \in W_{C_\delta}$ and

$$W^-_{C_\delta} = \{w \in W : \; w \leq m_{C_\delta}\}.$$

Recall that Γ is the set of simple roots determined by (B, H). Since $\delta(H) = H$ and $\delta(B) = B$, δ acts on Γ and thus also on W. For any automorphism σ of Γ, define the σ-twisted conjugation of W on itself by

$$u \cdot_\sigma v = \sigma(u)vu^{-1}, \quad u, v \in W,$$

and for $w \in W$, denote by $\mathcal{O}^\sigma_w$ the σ-twisted conjugacy class of w in W. Minimal length elements in σ-twisted conjugacy classes in W have been studied by X. He in [13]. The map ϕ in (2.1) induces a bijection between δ-twisted conjugacy classes and $\delta_0\delta$-twisted conjugacy in W. In particular, for any $w \in W$, ϕ maps maximal length elements in $\mathcal{O}^\delta_w$ to minimal length elements in $\mathcal{O}^{\delta_0\delta}_{w_0w}$. Using the map ϕ, one can translate the notions in [13, Section 3] and [13, Theorem 3.2] on minimal length elements in $\delta_0\delta$-twisted conjugacy classes to the analog of Proposition 2.8 on maximal length elements in δ-twisted conjugacy classes. It is also straightforward to generalize Proposition 2.9 to the case of δ-twisted conjugacy classes in G. We thus have the following conclusion.

Proposition 2.14. *For any δ-twisted conjugacy class C_δ in G, m_{C_δ} is the unique maximal length element in its δ-twisted conjugacy class in W.*

3. Conjugacy classes of W with unique maximal length elements

3.1. **The set $\mathcal{M}$.** Introduce

$$\mathcal{M} = \{m \in W : \; m \text{ is the unique maximal length element in } \mathcal{O}_m\}. \tag{3.1}$$

By Corollary 2.11, $m_C \in \mathcal{M}$ for every conjugacy class C in G. It is thus desirable to have a precise description of elements in $\mathcal{M}$. Clearly $\mathcal{M}$ is in one-to-one correspondence with conjugacy classes in W that have unique maximal length elements.

It is easy to see that if $G = G_1 \times G_2 \times \cdots \times G_k$ is semi-simple with simple factors G_j and Weyl groups W_j for $1 \leq j \leq k$, then

$$\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_k,$$

where for $1 \leq j \leq k$, $\mathcal{M}_j \subset W_j$ is defined as in (3.1). Therefore we only need to determine $\mathcal{M}$ for G simple. This will be done in §3.3.

Remark 3.1. By [13, Corollary 4.5], in any δ_0-twisted conjugacy class in W, a minimal element in the Bruhat order is also a minimal length element. Thus, for $m \in W$, $m \in \mathcal{M}$ if and only if m is the unique maximal element in $\mathcal{O}_m$.

Lemma 3.2. *If $m \in \mathcal{M}$, then $m^2 = 1$.*

Proof. By [11, Corollary 3.2.14], $m^{-1} \in \mathcal{O}_m$. Since $l(m) = l(m^{-1})$, one has $m = m^{-1}$.

Q.E.D.

3.2. **The correspondence between $\mathcal{M}'$ and $\mathcal{J}'$.** Introduce

$$\mathcal{M}' = \{m \in W : \ m^2 = 1 \text{ and } m \text{ is a maximal length element in } \mathcal{O}_m\}.$$

By Lemma 3.2, $\mathcal{M} \subset \mathcal{M}'$. We first determine $\mathcal{M}'$.

It is well-known that elements in $\mathcal{M}'$ correspond to special subsets of the set Γ of simple roots. Indeed, minimal or maximal length elements in conjugacy classes of involutions in W have been studied (see, for example, [11, 13, 16, 17] and especially [11, Remark 3.2.13] for minimal length elements, [16, Theorem 1.1] for maximal length elements, and [13, Lemma 3.6] for minimal length elements in twisted conjugacy classes). We summarize the results on $\mathcal{M}'$ in the following Proposition 3.6, and we give a proof of Proposition 3.6 for completeness.

Lemma 3.3. *Let $m \in W$ be an involution. If $\alpha \in \Gamma$ is such that $l(s_\alpha m s_\alpha) = l(m)$, then $s_\alpha m s_\alpha = m$.*

Proof. This is [11, Exercise 3.18]. If $m(\alpha) > 0$, then $ms_\alpha > m$, and $l(s_\alpha m s_\alpha) = l(m)$ implies that $s_\alpha m s_\alpha < m s_\alpha$. Thus $s_\alpha m(\alpha) < 0$, so $m(\alpha) = \alpha$. Similarly, if $m(\alpha) < 0$, then $m(\alpha) = -\alpha$. In either case, $s_\alpha m s_\alpha = m$.

Q.E.D.

Lemma 3.4. *If $m \in \mathcal{M}'$, then $m = w_0 w_{0,J}$, where $J = \{\alpha \in \Gamma : m(\alpha) = \alpha\}$, and J is δ_0-invariant.*

Proof. Let $m \in \mathcal{M}'$, and let $x = w_0 m$. Then x is a unique minimal length element in its δ_0-twisted conjugacy class $\mathcal{O}_x^{\delta_0}$ in W. Let $x = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$ be a reduced word for x, where $\alpha_j \in \Gamma$ for each $1 \leq j \leq k$. Let $J' = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. Then $x \in W_{J'}$. We first show that $x = w_{0,J'}$. To this end, it is enough to show that $x(\alpha_j) < 0$ for every $1 \leq j \leq k$. Since $xs_{\alpha_k} < x$, we already know that $x(\alpha_k) < 0$. If $k = 1$, we are done. Suppose that $k \geq 2$ and let

$$x_1 = \delta_0(s_{\alpha_k})xs_{\alpha_k} = s_{\delta_0(\alpha_k)}s_{\alpha_1} \cdots s_{\alpha_{k-1}} \in \mathcal{O}_x^{\delta_0}. \tag{3.2}$$

Since x is a minimal length element in $\mathcal{O}_x^{\delta_0}$ and $l(x) = k$, we have $l(x_1) \geq k$. It follows from (3.2) that $l(x_1) \leq k$, so $l(x_1) = k$. Let $m_1 = w_0x_1 = s_{\alpha_k}ms_{\alpha_k}$. Then $l(m_1) = l(m)$. By Lemma 3.3, $m_1 = m$, so $x = x_1$. In particular, $x = s_{\delta_0(\alpha_k)}s_{\alpha_1} \cdots s_{\alpha_{k-1}}$ is a reduced word for x, so $x(\alpha_{k-1}) < 0$. Similar arguments show that $x(\alpha_j) < 0$ for every $1 \leq j \leq k$. Thus $x = w_{0,J'}$, and $m = w_0w_{0,J'}$. It follows from $m^2 = 1$ that J' is δ_0-invariant.

It remains to show that $J' = J$. For any $\alpha \in J'$, since $m(\alpha) > 0$, $l(s_\alpha ms_\alpha) \geq l(m)$. Since $m \in \mathcal{M}'$, one has $l(s_\alpha ms_\alpha) = l(m)$, so by Lemma 3.3, $s_\alpha ms_\alpha = m$ and thus $m(\alpha) = \alpha$. This shows that $J' \subset J$. Since $m(\beta) < 0$ for every $\beta \in \Gamma\backslash J'$, one has $J \subset J'$. Thus $J = J'$.

Q.E.D.

Definition 3.5. A subset J of Γ is said to have Property (1) if J is δ_0-invariant and $-w_0(\alpha) = -w_{0,J}(\alpha)$ for all $\alpha \in J$.

Let $\mathcal{J}'$ be the collection of all subsets J of Γ that have Property (1). For $J \in \mathcal{J}'$, let $m_J = w_0w_{0,J}$. For $m \in \mathcal{M}'$, let

$$J_m = \{\alpha \in \Gamma : \ m(\alpha) = \alpha\} \subset \Gamma.$$

It follows from Lemma 3.4 that $J_m \in \mathcal{J}'$ for every $m \in \mathcal{M}'$.

Proposition 3.6. *1) The map $\psi : \mathcal{M}' \to \mathcal{J}' : m \mapsto J_m$ is bijective with inverse given by $J \mapsto m_J$ for $J \in \mathcal{J}'$.*

2) For $J, K \in \mathcal{J}'$, m_J and m_K are in the same conjugacy class in W if and only if there exists $w \in W$ with $\delta_0(w) = w$ such that $w(J) = K$.

Proof. 1) Since $m = w_0w_{0,J_m}$ for every $m \in \mathcal{M}'$, ψ is injective. To show that ψ is surjective, let $J \in \mathcal{J}'$ and we will prove that $m_J \in \mathcal{M}'$. Since J is δ_0-invariant, m_J is an involution. Property (1) implies that $s_\alpha m_J s_\alpha = m_J$ for every $\alpha \in J$, so $wm_Jw^{-1} = m_J$ for every $w \in W_J$. Thus, if $u = wm_Jw^{-1}$ is an element in $\mathcal{O}_{m_J}$, we can assume that $w \in W^J$ (see notation in §1.2). Then

$$\begin{aligned} l(u) &\leq l(w) + l(m_Jw^{-1}) = l(w) + l(w_0) - l(w_{0,J}w^{-1}) \\ &= l(w) + l(w_0) - l(w_{0,J}) - l(w^{-1}) \\ &= l(m_J). \end{aligned}$$

This shows that m_J is of maximal length in $\mathcal{O}_{m_J}$, so $m_J \in \mathcal{M}'$. To show that $\psi(m_J) = J$, note that $J \subset J_{m_J} = \{\alpha \in \Gamma : m_J(\alpha) = \alpha\}$. Since $m_J(\alpha) < 0$ for every $\alpha \in \Gamma \backslash J$, $J_{m_J} \subset J$. Thus $J_{m_J} = J$, and $\psi(m_J) = J$. This shows that ψ is surjective and that its inverse is given by $\psi^{-1}(J) = m_J$.

2) Assume that $J, K \in \mathcal{J}'$ are such that m_J and m_K are conjugate in W. Since $w m_J w^{-1} = m_J$ for any $w \in W_J$, we may assume that $m_K = w m_J w^{-1}$ for some $w \in W^J$. Then it follows from $m_K w = w m_J$ that for every $\alpha \in J$,

$$m_K w(\alpha) = w m_J(\alpha) = w(\alpha) > 0.$$

Thus $w(\alpha) \in [K]^+$, where $[K]^+$ denotes the set positive roots that are in the linear span of K. Denote similarly by $[J]^+$ the set of positive roots in the linear span of J. Then $w([J]^+) \subset [K]^+$. Since both m_J and m_K are maximal length elements in the same conjugacy class in W, $l(m_J) = l(m_K)$. Since

$$l(m_J) = l(w_0) - |[J]^+| \quad \text{and} \quad l(m_K) = l(w_0) - |[K]^+|,$$

one has $|[J]^+| = |[K]^+|$. Here for a set A, $|A|$ denotes the cardinality of A. Thus $w([J]^+) = [K]^+$. It follows that $w(J) = K$. Now $m_K = w m_J w^{-1}$ implies that $w_{0,K} = \delta_0(w) w_{0,J} w^{-1}$, so $\delta_0(w) = w_{0,K} w w_{0,J} = w$.

Conversely, if $J, K \in \mathcal{J}'$ are such that $w(J) = K$ for some $w \in W$ with $\delta_0(w) = w$, then $w_{0,K} = w w_{0,J} w^{-1} = \delta_0(w) w_{0,J} w^{-1}$, so $m_K = w m_J w^{-1}$.

Q.E.D.

3.3. **The correspondence between $\mathcal{M}$ and $\mathcal{J}$.** We now turn to the set $\mathcal{M}$. Let $\langle\, , \rangle$ be the bilinear form on Γ induced from the Killing form of the Lie algebra of G. For a subset J of Γ, an $\alpha \in J$ is said to be *isolated* if $\langle \alpha, \alpha' \rangle = 0$ for every $\alpha' \in J \backslash \{\alpha\}$. The following Definition 3.7 is inspired by [3, Lemma 4.1].

Definition 3.7. A subset J of Γ is said to have Property (2) if for every isolated $\alpha \in J$, there is no $\beta \in \Gamma \backslash \{\alpha\}$ with the following properties:

a) $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$ and $\langle \beta, \alpha \rangle \neq 0$;

b) $\langle \beta, \alpha' \rangle = 0$ for all $\alpha' \in J \backslash \{\alpha\}$;

c) $-w_0(\beta) = \beta$.

Lemma 3.8. *If $m \in \mathcal{M}$, then J_m has Properties (1) and (2).*

Proof. Let $m \in \mathcal{M}$. By Lemma 3.4, J_m has Property (1). Suppose that $\alpha \in J_m$ is an isolated point and that there exists $\beta \in \Gamma \backslash \{\alpha\}$ with properties a), b) and c) in Definition 3.7. Let $J'_m = J_m \backslash \{\alpha\}$. Since $\alpha \in J_m$ is isolated, on has $w_0(\alpha) = -\alpha$, so,

$$m = w_0 s_\alpha w_{0,J'_m} = s_\alpha w_0 w_{0,J'_m},$$

and by b) and c), $m(\beta) = s_\alpha w_0 w_{0,J'_m}(\beta) = s_\alpha w_0(\beta) = -s_\alpha(\beta) < 0$, and thus

$$s_\beta m s_\beta = m s_{m(\beta)} s_\beta = m s_\alpha s_\beta s_\alpha s_\beta.$$

By a), $s_\alpha s_\beta s_\alpha s_\beta = s_\beta s_\alpha$, so $s_\beta m s_\beta = m s_\beta s_\alpha$, and thus

$$s_\alpha s_\beta m s_\beta s_\alpha = s_\alpha m s_\beta.$$

Since $l(s_\alpha m s_\beta) \geq l(s_\alpha m) - 1 = l(m)$, and since m is the unique maximal length element in $\mathcal{O}_m$, $s_\alpha m s_\beta = m$. It follows from $m s_\alpha m = s_\alpha$ that $s_\alpha s_\beta = 1$ which is a contradiction.

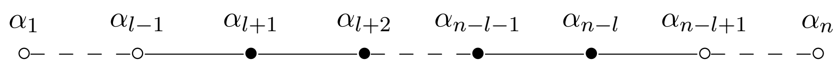
Q.E.D.

Let $\mathcal{J}$ be the collection of all subsets J of Γ with Properties (1) and (2). A $J \in \mathcal{J}$ is said to be non-trivial if Γ is neither empty nor the whole of Γ.

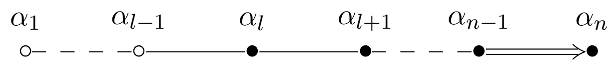
Identify Γ with the Dynkin diagram of G and a subset J of Γ as a sub-diagram of the Dynkin diagram. The following description of $\mathcal{J}$ for G simple is obtained in [3, Corollary 4.2]. We include the list here for the convenience of the reader and for completeness.

Lemma 3.9. *Assume that G is simple and that the rank n of G is at least 2. The following is a complete list of non-trivial $J \in \mathcal{J}$ with points in J painted black:*

(1) *A_n: $J_l = \{\alpha_i : l+1 \leq i \leq n-l\}$ for $1 \leq l \leq \left[\frac{n+1}{2}\right] - 1$:*

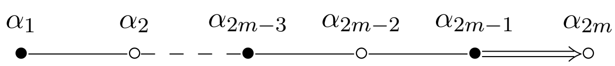


(2) *B_n: $J_{1,l} = \{\alpha_i : l \leq i \leq n\}$ for $2 \leq l \leq n$:*

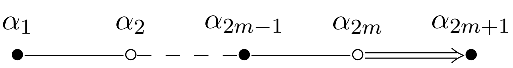


$J_{2,l} = \{\alpha_1, \alpha_3, \ldots, \alpha_{2l-1}\} \cup \{\alpha_i : 2l+1 \leq i \leq n\}$, for $1 \leq l \leq \frac{n}{2} - 1$:

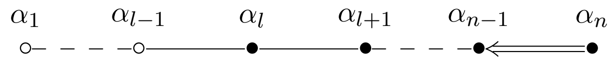
If $n = 2m$, $J_3 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-3}, \alpha_{2m-1}\}$:



If $n = 2m+1$, $J_4 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-1}, \alpha_{2m+1}\}$:

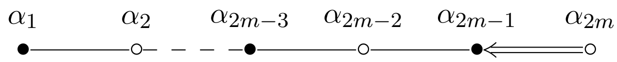


(3) C_n: $J_{1,l} = \{\alpha_i : l \leq i \leq n\}$ *for* $2 \leq l \leq n$:

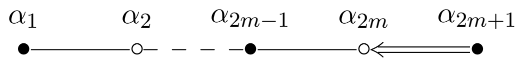


$J_{2,l} = \{\alpha_1, \alpha_3, \ldots, \alpha_{2l-1}\} \cup \{\alpha_i : 2l+1 \leq i \leq n\}$ *for* $1 \leq l \leq \frac{n}{2} - 1$:

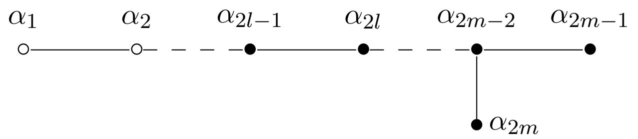
If $n = 2m$, $J_3 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-3}, \alpha_{2m-1}\}$:



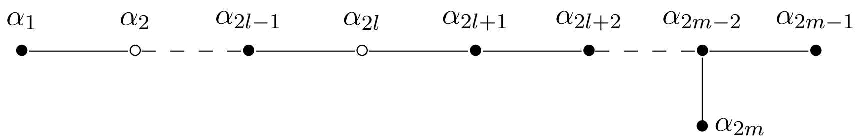
If $n = 2m+1$, $J_4 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-1}, \alpha_{2m+1}\}$:



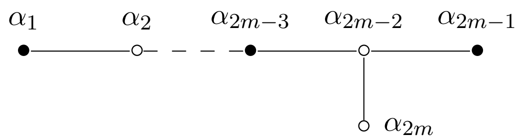
(4) D_{2m}: $J_{1,l} = \{\alpha_i : 2l-1 \leq i \leq 2m\}$ *for* $2 \leq l \leq m$:



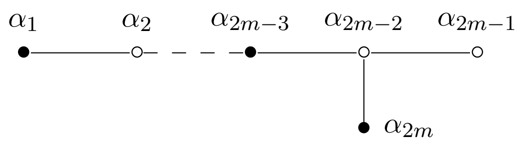
$J_{2,l} = \{\alpha_1, \alpha_3, \ldots, \alpha_{2l-1}\} \cup \{\alpha_i : 2l+1 \leq i \leq 2m\}$ *for* $1 \leq l \leq m-1$:



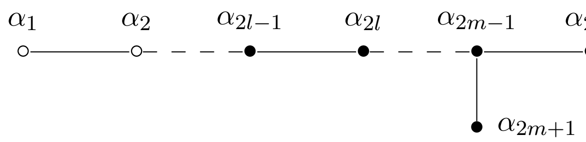
$J_3 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-3}, \alpha_{2m-1}\}$:



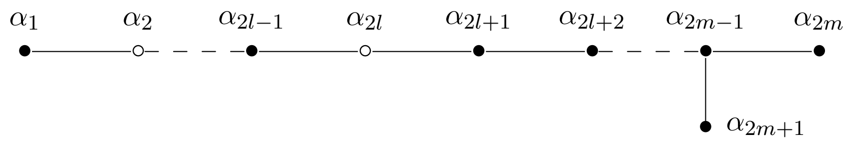
$J_4 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-3}, \alpha_{2m}\}$:



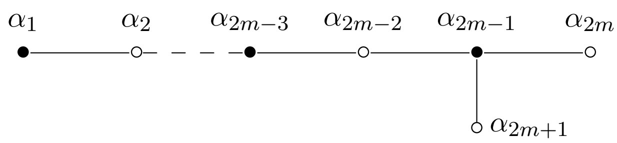
(5) D_{2m+1}: $J_{1,l} = \{\alpha_i : 2l-1 \leq i \leq 2m+1\}$ *for* $2 \leq l \leq m$:



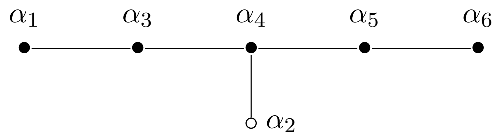
$J_{2,l} = \{\alpha_1, \alpha_3, \ldots, \alpha_{2l-1}\} \cup \{\alpha_i : 2l+1 \leq l \leq 2m+1\}$ *for* $1 \leq l \leq m-1$:

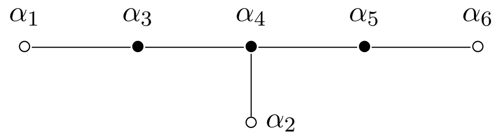


$J_3 = \{\alpha_1, \alpha_3, \ldots, \alpha_{2m-1}\}$:

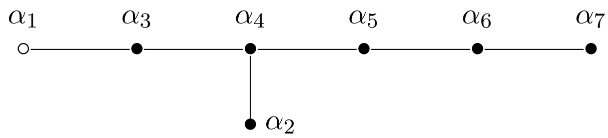


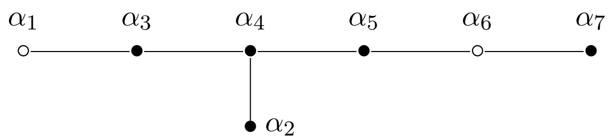
(6) E_6:

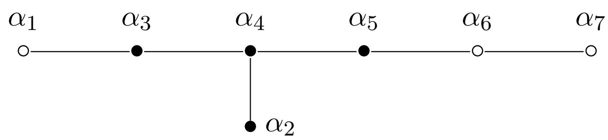
J_1 :


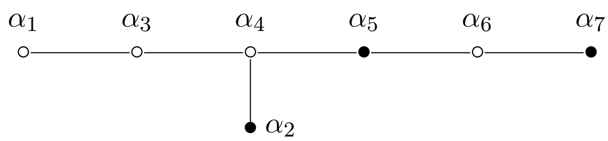
J_2 :


(7) E_7:

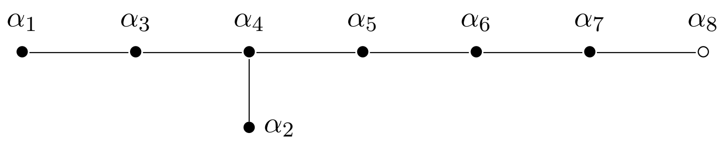
J_1 :


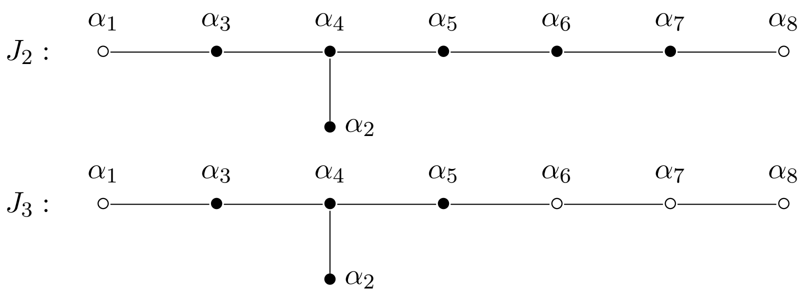
J_2 :


J_3 :


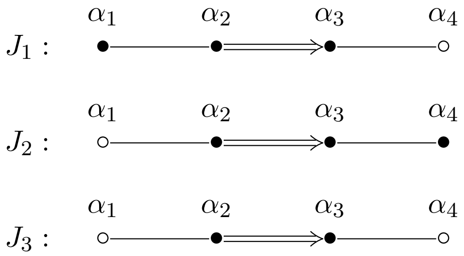
J_4 :


(8) E_8:

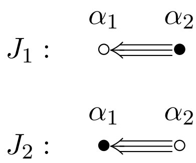
J_1 :




(9) F_4:



(10) G_2:



Theorem 3.10. *When G is simple, the map $\psi : \mathcal{M} \to \mathcal{J} : \ m \mapsto J_m$ is a bijection.*

Proof. It is clear that ψ is injective. To show that ψ is surjective, let $J \in \mathcal{J}$. We need to show that $m_J \in \mathcal{M}$, i.e., m_J is the unique maximal length element in its conjugacy class in $\mathcal{O}_{m_J}$. Let m be any maximal length element in $\mathcal{O}_{m_J}$. By Proposition 3.6, $m = m_K$, where $K \in \mathcal{J}'$ and there exists $w \in W^J$ such that $w(J) = K$ and $\delta_0(w) = w$.

By examining the list of all J's in $\mathcal{J}$ in Lemma 3.9, every $J \in \mathcal{J}$, when regarded as a Dynkin diagram, uniquely embeds in Γ with Property (1) except in the cases of $J_{1,m}, J_3, J_4$ for D_{2m} and J_4 for E_7. In these cases, one can use results in [16] to check directly that $m_J \in \mathcal{M}$.

Q.E.D.

Remark 3.11. By [3, Remark 4.3], for every J in the list in Lemma 3.9, $m_J = m_C$ for some spherical conjugacy class in G, so in particular, $m_J \in \mathcal{M}$. This gives another (indirect) proof of the surjectivity of the map ψ in Theorem 3.10.

4. The case of $G = SL(n+1,\mathbb{C})$

In this section, for an arbitrary conjugacy class C in $SL(n+1,\mathbb{C})$, we give an explicit condition for $C \cap (BwB) \neq \emptyset$ when $w \in W \cong S_{n+1}$ is an involution. In particular, we describe $m_C \in S_{n+1}$ explicitly for every C.

4.1. **Notation.** As is standard, take the Borel subgroup B (resp. B^-) to consist of all upper-triangular (resp. lower triangular) matrices in $SL(n+1,\mathbb{C})$, so that $H = B \cap B^-$ consists of all diagonal matrices in $SL(n+1,\mathbb{C})$. For an integer $p \geq 0$, denote by I_p the identity matrix of size p and by $[p/2]$ the largest integet that is less than or equal to $p/2$.

Identify the Weyl group W of $SL(n+1,\mathbb{C})$ with the group S_{n+1} of permutations on the set of integers between 1 and $n+1$. For $1 \leq i < j \leq n+1$, let (i,j) be the 2-cycle in S_{n+1} exchanging i and j and leaving every other $k \in [1, n+1]$ fixed. If $w \in S_{n+1}$ is an involution, denote by $l_2(w)$ the number of 2-cycles in the cycle decomposition of w.

Every conjugacy class C in $SL(n+1,\mathbb{C})$ contains some g of (upper-triangular) Jordan form. We define the eigenvalues for C to be the eigenvalues of such a $g \in C$ and similarly define the number and sizes of the Jordan blocks of C corresponding to an eigenvalue. For $g \in GL(n+1,\mathbb{C})$, define

$$\begin{aligned}
d(g) &= \max\{\dim\ker(g - cI_{n+1}) : \ c \in \mathbb{C}\} \\
r(g) &= n+1-d(g) = \min\{\mathrm{rank}(g - cI_{n+1}) : c \in \mathbb{C}\} \\
l(g) &= \min\left\{r(g), \ \left[\frac{n+1}{2}\right]\right\}.
\end{aligned}$$

For a conjugacy class C in $SL(n+1,\mathbb{C})$, define

$$d(C) = d(g), \quad r(C) = r(g) \quad \text{and} \quad l(C) = l(g), \quad \text{for any } \ g \in C.$$

Two elements in $SL(n+1,\mathbb{C})$ are in the same conjugacy class in $SL(n+1,\mathbb{C})$ if and only if they are in the same conjugacy class in $GL(n+1,\mathbb{C})$. This fact will be used throughout the rest of this section.

4.2. The main theorem and its consequences.

Lemma 4.1. *Let C be a conjugacy class in $SL(n+1,\mathbb{C})$ and let $w \in S_{n+1}$ be an involution. If $C \cap (BwB) \neq \emptyset$, then $l_2(w) \leq l(C)$.*

Proof. Assume that $C \cap (BwB) \neq \emptyset$. Let $g \in C \cap (BwB)$, and write $g = b_1 \dot{w} b_2$, where $b_1, b_2 \in B$ and $\dot{w}$ is any representative of w in the normalizer of H in G. Then for any non-zero $c \in \mathbb{C}$,

$$\mathrm{rank}(g - cI_{n+1}) = \mathrm{rank}(b_1\dot{w}b_2 - cI_{n+1}) = \mathrm{rank}(\dot{w} - cb_1^{-1}b_2^{-1}).$$

Let $w = (i_1, j_1) \cdots (i_{l_2(w)}, j_{l_2(w)})$ be the decomposition of w into distinct 2-cycles, where $i_1 < \cdots < i_{l_2(w)}$ and $i_k < j_k$ for every $1 \leq k \leq l_2(w)$. It is easy to see that for any $b \in B$, the columns of the matrix $\dot{w} - b$ corresponding to $i_1, \ldots, i_{l_2(w)}$ are linearly independent, so $\mathrm{rank}(\dot{w} - b) \geq l_2(w)$. Thus $\mathrm{rank}(g - cI_{n+1}) \geq l_2(w)$ for every non-zero $c \in \mathbb{C}$. Hence $r(C) = r(g) \geq l_2(w)$. Since $l_2(w) \leq \left[\frac{n+1}{2}\right]$, one has $l_2(w) \leq l(C)$.

Q.E.D.

Theorem 4.2. *Let C be a conjugacy class in $SL(n+1, \mathbb{C})$ and let $w \in S_{n+1}$ be an involution. Then $C \cap (BwB) \neq \emptyset$ if and only if $l_2(w) \leq l(C)$.*

A proof of Theorem 4.2 using a result of Ellers-Gordeev [9] is given in §4.3, and a direct proof of Theorem 4.2 is given in §4.4. We now give some corollaries of Theorem 4.2.

Corollary 4.3. *Let C and C' be two conjugacy classes in $SL(n+1, \mathbb{C})$ such that C' is contained in the closure of C. Let $w \in S_{n+1}$ be an involution. If $w \in W_{C'}$, then $w \in W_C$.*

Proof. It follows from the definition that $r(C') \leq r(C)$, so $l(C') \leq l(C)$. Corollary 4.3 now follows directly from Theorem 4.2.

Q.E.D.

Recall that for $w \in S_{n+1}$, $\mathcal{O}_w$ denotes the conjugacy class of w in S_{n+1}.

Corollary 4.4. *Let $w \in S_{n+1}$ be an involution and let C be a conjugacy class in $SL(n+1, \mathbb{C})$. If $w \in W_C$, then $\mathcal{O}_w \subset W_C$.*

Proof. Since $l_2(w') = l_2(w)$ for every $w' \in \mathcal{O}_w$, Corollary 4.4 follows directly from Theorem 4.2.

Q.E.D.

We now consider spherical conjugacy classes in $SL(n+1, \mathbb{C})$.

Lemma 4.5. [1, 2] *A spherical conjugacy class in $SL(n+1, \mathbb{C})$ is either unipotent or semi-simple.*

1) A unipotent conjugacy class in $SL(n+1, \mathbb{C})$ is spherical if and only if all of its Jordan blocks are of size at most 2.

2) A semi-simple conjugacy class C in $SL(n+1, \mathbb{C})$ is spherical if and only if it has exactly two distinct eigenvalues.

Note that for a unipotent spherical conjugacy class C in $SL(n+1,\mathbb{C})$, $r(C)$ is precisely the number of size 2 blocks in the Jordan form of C, and for a semi-simple spherical conjugacy class, $r(C)$ is equal to the smaller multiplicity of the two eigenvalues. In particular, $l(C) = r(C)$ for every spherical conjugacy class in $SL(n+1,\mathbb{C})$.

Corollary 4.6. *For a spherical conjugacy class C in $SL(n+1,\mathbb{C})$,*

$$W_C = \{w \in S_{n+1} : \ w^2 = 1 \text{ and } l_2(w) \le r(C)\}.$$

Proof. Let C be a spherical conjugacy class in $SL(n+1,\mathbb{C})$. By [3, Theorem 2.7], if $w \in W_C$, then w is an involution, and by Theorem 4.2, $l_2(w) \le r(C)$. Conversely, if $w \in S_{n+1}$ is an involution with $l_2(w) \le r(C)$, then $w \in W_C$ by Theorem 4.2.

Q.E.D.

Remark 4.7. Fix $\xi \in \mathbb{C}$ such that $\xi^{n+1} = -1$. For an integer $0 \le r \le \left[\frac{n+1}{2}\right]$, let

$$h_r = \begin{cases} \mathrm{diag}(I_{n+1-r},\ -I_r) & \text{if } \ r \text{ is even} \\ \mathrm{diag}(\xi I_{n+1-r},\ -\xi I_r) & \text{if } \ r \text{ is odd,} \end{cases}$$

and let C_{h_r} be the conjugacy class of h_r in $SL(n+1,\mathbb{C})$. Every semi-simple spherical conjugacy class in $SL(n+1,\mathbb{C})$ is $SL(n+1,\mathbb{C})$-equivariantly isomorphic to C_{h_r} for some $0 \le r \le \left[\frac{n+1}{2}\right]$, which is also $SL(n+1,\mathbb{C})$-equivariantly isomorphic to the symmetric space

$$X = SL(n+1,\mathbb{C})/S(GL(n+1-r,\mathbb{C}) \times GL(r,\mathbb{C})).$$

Let V be the set of B-orbits on X and let $\phi : V \to \mathcal{I}$ be the map defined in [19, Section 1.6] by Richardson and Springer, where $\mathcal{I}$ is the set of all involutions in S_{n+1}. It is easy to see from the definitions that W_C for $C = C_{h_r}$ is the same as $\mathrm{Im}(\phi)$, the image of ϕ. The fact that $\mathrm{Im}(\phi)$ consists of all $w \in \mathcal{I}$ with $l_2(w) \le r$ is well-known (see, for example, [20]).

We now determine the element m_C for every conjugacy class C in $SL(n+1,\mathbb{C})$. List the simple roots as $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in the standard way. Recall that w_0 is the longest element in S_{n+1} and that for a subset J of Γ, $w_{0,J}$ is the longest element in the subgroup of S_{n+1} generated by simple roots in J. For an integer $0 \le l \le \left[\frac{n+1}{2}\right]$, let

$$J_l = \begin{cases} \{\alpha_{l+1}, \ldots, \alpha_{n-l}\}, & \text{if } \ 0 \le l \le \left[\frac{n+1}{2}\right] - 1 \\ \emptyset, & \text{if } \ l = \left[\frac{n+1}{2}\right] \end{cases},$$

and let $m_l = w_0 w_{0,J_l}$. Thus, $m_0 = 1$, and

$$m_l = (1, n+1)(2, n) \cdots (l,\ n+2-l), \qquad \text{if } \ 1 \le l \le \left[\frac{n+1}{2}\right].$$

In particular, $m_l = w_0$ for $l = \left[\frac{n+1}{2}\right]$. Note that for $0 \leq l_1, l_2 \leq \left[\frac{n+1}{2}\right]$,

$$m_{l_1} \leq m_{l_2} \quad \text{iff} \quad l_1 \leq l_2.$$

Corollary 4.8. *For any conjugacy class C in $SL(n+1, \mathbb{C})$, $m_C = m_{l(C)}$, i.e.,*

$$m_C = \begin{cases} w_0 & \text{if } r(C) \geq \left[\frac{n+1}{2}\right], \\ m_{r(C)} & \text{if } r(C) < \left[\frac{n+1}{2}\right] \end{cases}.$$

Proof. Let C be any conjugacy class in $SL(n+1, \mathbb{C})$. By Corollary 2.11, Lemma 3.8 and Lemma 3.9, $m_C = m_l$ for some $0 \leq l \leq \left[\frac{n+1}{2}\right]$. Since $C \cap (Bm_lB) \neq \emptyset$, $l \leq l(C)$ by Theorem 4.2. Since $C \cap (Bm_{l(C)}B) \neq \emptyset$ by Theorem 4.2, one also has $l(C) \leq l$. Thus $l = l(C)$.

Q.E.D.

4.3. A proof of Theorem 4.2 using the Ellers-Gordeev criterion.

Notation 4.9. First recall (see for example [9, Page 705]) that for an integer $p > 0$, a partition of p is a non-increasing sequence $\lambda = (\lambda_1, \ldots, \lambda_s)$ of positive integers such that $\lambda_1 + \cdots + \lambda_s = p$, and s is called the length of λ. The shape of a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ of p consists of s rows of empty boxes left-aligned with λ_j boxes on the j-th row for each $1 \leq j \leq s$. The partition λ^* of p whose shape is obtained from switching the rows and columns of the shape of λ is called the dual of λ. Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\mu = (\mu_1, \ldots, \mu_t)$ be two partitions of p. Define $\lambda \leq \mu$ if $\sum_{j=1}^k \lambda_j \leq \sum_{j=1}^k \mu_j$ for every $1 \leq k \leq t$. One has (see [15, Section I.1.11]) $\lambda \leq \mu$ if and only if $\mu^* \leq \lambda^*$, where μ^* and λ^* are the partitions of p that are dual to μ and λ respectively.

For integers $p > 0$ and $0 \leq l \leq [p/2]$, let $\lambda(l,p) = (2, \ldots, 2, 1, \ldots, 1)$ be the partition of p with 2 appearing exactly l times.

Lemma 4.10. *Let $p > 0$ be an integer and let $0 \leq l \leq [p/2]$. Then for any partition $\mu = (\mu_1, \ldots, \mu_s)$ of p, $\lambda(l,p) \leq \mu$ if and only if $p - l \geq s$.*

Proof. Let $\lambda(l,p)^*$ and μ^* be the partitions of p that are dual to $\lambda(l,p)$ and μ respectively. Then $\lambda(l,p) \leq \mu$ if and only if $\lambda(l,p)^* \geq \mu^*$, and the latter is equivalent to $p - l \geq s$.

Q.E.D.

We now use [9, Theorem 3.20] to prove Theorem 4.2.

Let C be a conjugacy class in $SL(n+1, \mathbb{C})$ and assume that $w \in S_{n+1}$ is an involution with $l_2(w) \leq l(C)$, or, equivalently, $l_2(w) \leq r(C)$. We need to show that $C \cap (BwB) \neq \emptyset$. By [11, Theorem 3.2.9(a)], there exist w' which is a minimal length element in the conjugacy class of w in W and an ascent from w' to w. Thus, in the notation of [9], there is a tree $\Gamma(w)$ with $w' \in T(\Gamma(w))$. By [9, Theorem 3.20], it is enough to show that $\lambda(w') \leq \tilde{\nu}^*$, where $\lambda(w') = \lambda(l_2(w), n+1)$ is the partition $(2, \ldots, 2, 1, \ldots, 1)$ of $n+1$ with 2 appearing $l_2(w') = l_2(w)$ times, and $\tilde{\nu}^*$ is the partition of $n+1$ associated to C as described at the beginning of [9, Section 3.4]. One checks from the definitions that the partition $\tilde{\nu}^*$ has length $d(C)$. By Lemma 4.10, $\lambda(w') \leq \tilde{\nu}^*$ if and only if $n+1-l_2(w) \geq d(C)$ which is equivalent to $l_2(w) \leq r(C)$. This proves Theorem 4.2.

4.4. **A direct proof of Theorem 4.2.** The proof of Theorem 4.2 in §4.3 uses only a special case of of the Ellers-Gordeev criterion in [9, Theorem 3.20], and the proof of [9, Theorem 3.20] for the general case involves rather complicated combinatorics. We thus think that it is worthwhile to give a direct proof Theorem 4.2. Our direct proof also has the merit that it shows how to explicitly find an element in $C \cap BwB$ when $l_2(w) \leq l(C)$. We will use two lemmas from [9], namely [9, Lemma 3.3] and [9, Lemma 3.24] whose proofs as given in [9] are elementary.

For $g = (g_{i,j}) \in SL(n+1, \mathbb{C})$ and $1 \leq i \leq \left[\frac{n+1}{2}\right]$, let $g^{(i)}$ be the 2×2 matrix

$$g^{(i)} = \begin{pmatrix} g_{2i-1,2i-1} & g_{2i-1,2i} \\ g_{2i,2i-1} & g_{2i,2i} \end{pmatrix}.$$

Recall that a square matrix is said to be regular if its characteristic polynomial is the same as its minimal polynomial. An upper-triangular matrix $A = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in GL(2, \mathbb{C})$ is regular if and only if either $x \neq z$, or $x = z$ and $y \neq 0$, and in this case A is conjugate to some $A_1 = \begin{pmatrix} x_1 & y_1 \\ u & z_1 \end{pmatrix}$ with $u \neq 0$.

Proposition 4.11. *Let C be any conjugacy class C in $SL(n+1, \mathbb{C})$ with $l(C) > 0$. Then there exists $g \in C \cap B$ such that $g^{(i)}$ is regular for every $1 \leq i \leq l(C)$.*

Assuming Proposition 4.11, we now prove Theorem 4.2. Let C be a conjugacy class in $SL(n+1, \mathbb{C})$ and $w \in S_{n+1}$ an involution such that $l_2(w) \leq l(C)$. We will show that $C \cap (wB) \neq \emptyset$.

If $l(C) = 0$, then C consists of only one central element in $SL(n+1, \mathbb{C})$, and C only intersects with B and Theorem 4.2 holds in this case. Thus we will assume that $l(C) > 0$. Since $C \cap B \neq \emptyset$, we will also assume that $l_2(w) > 0$.

Let $g \in C \cap B$ be as in Proposition 4.11 and let $1 \leq i \leq l_2(w)$. Since $g^{(i)} \in GL(2, \mathbb{C})$ is regular, there exists $A_i \in GL(2, \mathbb{C})$ such that $A_i g^{(i)} A_i^{-1} = \begin{pmatrix} x_i & y_i \\ u_i & z_i \end{pmatrix}$ with $u_i \neq 0$. Let $A = \mathrm{diag}(A_1, \cdots, A_{l_2(w)}, I_{n+1-2l_2(w)})$ be the block diagonal matrix in $GL(n+1, \mathbb{C})$. Then $AgA^{-1} \in C \cap BuB$, where

$$u = (1,2)(3,4) \cdots (2l_2(w) - 1,\ 2l_2(w)).$$

Since there is an ascent from some minimal length element in $\mathcal{O}_w$ to w, we can assume that w has minimal length in $\mathcal{O}_w$, so w is the following product of disjoint 2-cycles:

$$w = (i_1,\, i_1+1)\cdots(i_{l_2(w)},\, i_{l_2(w)}+1)$$

where $1 \le i_1 < \cdots < i_{l_2(w)} \le n$. We will now use [9, Lemma 3.3]. Let

$$X = \{\alpha_1,\, \alpha_3,\, \ldots, \alpha_{2l_2(w)-1}\}, \qquad Y = \{\alpha_{i_1},\, \cdots\, \alpha_{i_{l_2(w)}}\},$$

and let ω be any element in S_{n+1} such that $\omega(2s-1) = i_s$ and $\omega(2s) = i_s + 1$ for $1 \le s \le l_2(w)$. Then $\omega^{-1} w \omega = u$, and $Y = \omega(X)$. Applying [9, Lemma 3.3] to the above X, Y, ω and $g_x = AgA^{-1}$, one sees, in the notation of [9, Lemma 3.3], that there exists $g_y \in C \cap BwB$. Thus $C \cap BwB \neq \emptyset$, and Theorem 4.2 is proved.

It remains to prove Proposition 4.11.

Proof of Proposition 4.11 when C has only one eigenvalue. We will use induction on n. It is easy to see that Proposition 4.11 holds for $n = 1$ or $n = 2$. Assume now that $n \ge 3$ and that Proposition 4.11 holds for conjugacy classes C in $SL(p, \mathbb{C})$ for any $p < n+1$ and any C with only one eigenvalue. Assume that C is a conjugacy class in $SL(n+1, \mathbb{C})$ with one eigenvalue c. Since we are assuming that $l(C) > 0$, there exists a Jordan block of C of size at least 2. Since Proposition 4.11 clearly holds when C is regular, we also assume that C has more than one Jordan block.

Case 1. There is a Jordan block of C of size 1. In this case, choose $g \in C$ of the form $g = \begin{pmatrix} g' & 0 \\ 0 & c \end{pmatrix}$, where $g' \in GL(n, \mathbb{C})$ is of Jordan form with c as the only eigenvalue. Then $d(g') = d(g) - 1$ and $r(g') = n - d(g') = r(g)$. Suppose that $r(g) \le \left[\frac{n+1}{2}\right] - 1$. Since $\left[\frac{n+1}{2}\right] - 1 = \left[\frac{n-1}{2}\right] \le \left[\frac{n}{2}\right]$, one has $l(g) = l(g')$, so by induction, Proposition 4.11 holds for C. Suppose that $r(g) \ge \left[\frac{n+1}{2}\right]$. Then $l(g) = \left[\frac{n+1}{2}\right]$ and $l(g') = \left[\frac{n}{2}\right]$. If $n+1$ is odd, then $\left[\frac{n+1}{2}\right] = \left[\frac{n}{2}\right]$, so $l(g) = l(g')$ and by induction, Proposition 4.11 holds for C. If $n+1$ is even, then $l(g') = l(g) - 1$ and one could not use induction. However, since we are assuming that C has a Jordan block of size at least 2, there is an element in C of the form

$$\begin{pmatrix} J_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & J_k & 0 \\ 0 & \cdots & 0 & c \end{pmatrix} \tag{4.1}$$

where $J_1, \cdots, J_k$ are of Jordan form and J_k has size at least 2. By [9, Lemma 3.24], the matrix in (4.1) is conjugate to

$$g_1 = \begin{pmatrix} g_1' & 0 & v \\ 0 & c & 1 \\ 0 & 0 & c \end{pmatrix},$$

where $g_1' \in GL(n-1, \mathbb{C})$ is of Jordan form, and v is the column vector in $\mathbb{C}^{n-1}$ that has 1 for the last coordinate and 0 for all the other coordinates. Now $d(g_1') = d(g) - 1$, so $r(g_1') = n - 1 - d(g_1') = r(g) - 1$. Recall that we are assuming that $n+1$ is even and $r(g) \geq \left[\frac{n+1}{2}\right]$. Let $n+1 = 2m$. Since $r(g_1') = r(g) - 1 \geq m - 1 = \left[\frac{n-1}{2}\right]$, $l(g_1') = m - 1 = l(g) - 1$. Applying induction to g_1', one sees that Proposition 4.11 holds for C.

Case 2. All the Jordan blocks of C have sizes at least 2 and at least one of them has size 2. In this case, choose $g \in C$ of the form

$$g = \begin{pmatrix} c & 1 & 0 \\ 0 & c & 0 \\ 0 & 0 & g' \end{pmatrix},$$

where $g' \in GL(n-1, \mathbb{C})$ is of Jordan form. Then $d(g') = d(g) - 1$, so $r(g') = n - 1 - d(g') = r(g) - 1$. Since all the Jordan blocks have sizes at least 2, one has $2d(g) \leq n+1$, so $r(g) \geq \left[\frac{n+1}{2}\right]$, and $r(g') \geq \left[\frac{n+1}{2}\right] - 1 = \left[\frac{n-1}{2}\right]$. Thus $l(g') = \left[\frac{n-1}{2}\right] = l(g) - 1$. Applying the induction assumption to g', one sees that Proposition 4.11 holds for C.

Case 3. All the Jordan blocks of C have sizes at least 3. Then we can find $g \in C$ of the form

$$g = \begin{pmatrix} c & 1 & 0 \\ 0 & c & v \\ 0 & 0 & g' \end{pmatrix}$$

where v is the row vector in $\mathbb{C}^{n-1}$ which has 1 for the first coordinate and 0 for all the other coordinates, and $g' \in GL(n-1, \mathbb{C})$ is of Jordan form with $d(g') = d(g)$, and thus $r(g') = r(g) - 2$. By assumption $n + 1 \geq 3d(g) \geq 6$, so $r(g) \geq \frac{2(n+1)}{3}$ and $r(g') = r(g) - 2 \geq \left[\frac{n-1}{2}\right]$. Thus $l(g) = \left[\frac{n+1}{2}\right]$ and $l(g') = \left[\frac{n-1}{2}\right] = \left[\frac{n+1}{2}\right] - 1$. Applying the induction assumption to g', one sees that Proposition 4.11 holds for C.

This finishes the proof of Proposition 4.11 in the case when C has only one eigenvalue.

Proof of Proposition 4.11 when C has more that one eigenvalue. We again use induction on n. Proposition 4.11 clearly holds for $n = 0$ or $n = 1$. Assume that Proposition 4.11 holds for $GL(p, \mathbb{C})$ for any $p < n+1$ and any conjugacy class in $GL(p, \mathbb{C})$ with more than one eigenvalue. Let C be a conjugacy class in $SL(n+1, \mathbb{C})$ with distinct eigenvalues $c_1, c_2, \cdots, c_k$, where $k \geq 2$, and for $1 \leq j \leq k$, let d_j be the number of Jordan blocks of C with eigenvalue c_j. We will assume that $d_1 \geq \cdots \geq d_k$. Then $r(C) = n + 1 - d_1$.

Case 1. $r(C) > \left[\frac{n+1}{2}\right]$. Let $g \in C$ be of the form

$$g = \begin{pmatrix} c_1 & 0 & v_1 \\ 0 & c_2 & v_2 \\ 0 & 0 & g' \end{pmatrix}$$

where v_1 and v_2 are row vectors of size $n-1$ and $g' \in GL(n-1, \mathbb{C})$. Then

$$\operatorname{rank}(g' - c_j I_{n-1}) \geq \operatorname{rank}(g - c_j I_{n+1}) - 2, \quad 1 \leq j \leq k.$$

Thus $r(g') \geq r(g) - 2 \geq \left[\frac{n-1}{2}\right]$ and $l(g') = l(g) - 1$. If g' has only one eigenvalue, we have proved that Proposition 4.11 holds for the conjugacy class of g' and thus also holds for C. If g' has more than one eigenvalue, one applies the induction assumption to g' to see that Proposition 4.11 holds for C.

Case 2. $r(C) \leq \left[\frac{n+1}{2}\right]$. Then $d_1 \geq \frac{n+1}{2}$. If all the Jordan blocks of C with eigenvalue c_1 have sizes at least 2, then $n+1 \geq 2d_1 + 1$, and $d_1 \leq \frac{n}{2}$, which is a contradiction. Thus C has at least one Jordan block of size 1. Pick $g \in C$ of the form

$$g = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & v \\ 0 & 0 & g' \end{pmatrix} \tag{4.2}$$

where v is a row vector of size $n-1$ and $g' \in GL(n-1, \mathbb{C})$. Then

$$\begin{aligned} &\operatorname{rank}(g' - c_1 I_{n-1}) = \operatorname{rank}(g - c_1 I_{n+1}) - 1 = n - d_1, \\ &\operatorname{rank}(g' - c_2 I_{n-1}) = \operatorname{rank}(g - c_2 I_{n+1}) - 1 = n - d_2 \geq n - d_1, \quad \text{or} \\ &\operatorname{rank}(g' - c_2 I_{n-1}) = \operatorname{rank}(g - c_2 I_{n+1}) - 2 = n - d_2 - 1. \end{aligned}$$

Moreover, if $k \geq 3$, then for every $3 \leq j \leq k$,

$$\operatorname{rank}(g' - c_j I_{n-1}) = \operatorname{rank}(g - c_j I_{n+1}) - 2 = n - d_j - 1.$$

If $k \geq 3$, then $n - d_j - 1 \geq n - d_1$ for every $3 \leq j \leq k$. Indeed, if $n - d_j - 1 < n - d_1$ for some $j \geq 3$, then $n - d_2 - 1 \leq n - d_j - 1 < n - d_1$, so $d_1 = d_2$. Since $d_1 \geq \frac{n+1}{2}$ and $d_1 + d_2 + d_j \leq n + 1$, one has a contradiction. Thus

$$r(g') = \min\{\operatorname{rank}(g' - c_1 I_{n-1}),\ \operatorname{rank}(g' - c_2 I_{n-1})\}.$$

Consequently, $r(g') = n - d_1 = r(g) - 1$ unless $\operatorname{rank}(g' - c_2 I_{n-1}) = n - d_2 - 1$ and $n - d_2 - 1 < n - d_1$. But in the latter case, $d_1 = d_2 = \frac{n+1}{2}$ so $n+1$ must be even and g is semi-simple. In particular, $\operatorname{rank}(g' - c_2 I_{n-1}) = n - d_2$ which is a contradiction. Thus one always has $r(g') = r(g) - 1$ and $l(g') = l(g) - 1$. Induction on g' again yields Proposition 4.11.

This finishes the proof of Proposition 4.11.

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