ISOMORPHISMS BETWEEN QUANTUM GROUPS $U_q(\mathfrak{sl}_{n+1})$ AND $U_p(\mathfrak{sl}_{n+1})$

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Abstract. Let $\mathbb{K}$ be a field and suppose $p,\ q \in \mathbb{K}^*$ are not roots of unity. We prove that the two quantum groups $U_q(\mathfrak{sl}_{n+1})$ and $U_p(\mathfrak{sl}_{n+1})$ are isomorphic as $\mathbb{K}$-algebras implies that $p = \pm q^{\pm 1}$ when n is even. This new result answers a classical question of Jimbo.

1. **Introduction and the main results**

The Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ over a field $\mathbb{K}$ (see [D1, D2, Ji1, Ja]), associated with a simple finite dimensional Lie algebra $\mathfrak{g}$, plays a crucial role in the study of the quantum Yang-Baxter equations, two dimensional solvable lattice models, the invariants of 3-manifolds, the fusion rules of conformal field theory, and the modular representations (see, for instance, [Ka, Lu1, LZ, RT]). In his fundamental paper [Ji2], Jimbo raised the following

Problem 1.1. *When are the two quantum groups $U_q(\mathfrak{g})$ and $U_p(\mathfrak{g})$ over a field $\mathbb{K}$ isomorphic as $\mathbb{K}$-algebras?*

In [Ji2], Jimbo discovered a close connection between quantum groups and finite dimensional Hecke algebras via R-matrices, then motivated by the connection and the classical result that two finite dimensional Hecke algebras H_q and H_p of same type are always isomorphic by Tits (see Bourbaki [B], see also Lusztig [Lu2] for such an explicit isomorphism), Jimbo conjectured that $U_q(\mathfrak{g})$ and $U_p(\mathfrak{g})$ are always isomorphic as $\mathbb{K}$-algebras, at least 'after appropriate completion'. The above problem is closely related to

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2000 *Mathematics Subject Classification.* 16G10, 16S10, 16W20, 16Z05, 17B10, 20C30.

Key words and phrases. Quantum groups, isomorphisms, automorphisms, center, simple $U_q(\mathfrak{sl}_{n+1})$-modules, Casimir elements, symmetry, PBW type basis, graded algebra structure, multiplicative invariant theory, root lattice, Harish-Chandra isomorphism, invariant subalgebra, Galois theory.

The research of Li-Bin Li was partially supported by NSFC Grant No.10771182.

The research of Jie-Tai Yu was partially supported by an RGC-GRF grant.

Problem 1.2. *Describe the structure of $Aut_{\mathbb{K}}(U_q(\mathfrak{g}))$ for the quantum group $U_q(\mathfrak{g})$ over a field $\mathbb{K}$.*

See, for instance, Alev and Chamarie [AC] and Zha [Z1] for descriptions of $\mathrm{Aut}_{\mathbb{K}}(U_q(\mathfrak{sl}_2))$. See Zha [Z2] for some results about $\mathrm{Aut}_{\mathbb{K}}(U_q(\mathfrak{g}))$. See also Launois [La1, La2], and Launois and Lopes [LL] and references therein for related description of $\mathrm{Aut}_{\mathbb{K}}(U_q^+(\mathfrak{g}))$. In particular, we formulate

Problem 1.3. *When are the two quantum groups $U_q(\mathfrak{sl}_{n+1})$ and $U_p(\mathfrak{sl}_{n+1})$ over a field $\mathbb{K}$ isomorphic as $\mathbb{K}$-algebras?*

The above problems are also motivated by the similar questions regarding the isomorphisms between affine Hecke algebras $\mathbb{H}_q$ and $\mathbb{H}_p$ over a field $\mathbb{K}$ recently considered by Nanhua Xi and Jie-Tai Yu [XY]. See also Rong Yan [Y].
In this paper, we give a necessary condition for the quantum groups $U_q(\mathfrak{sl}_{n+1})$ isomorphic to $U_p(\mathfrak{sl}_{n+1})$ as $\mathbb{K}$-algebras for even n, provided both q and p are not roots of unity in $\mathbb{K}$. This new result answers the classical question of Jimbo.

Theorem 1.4. *Suppose $q \in \mathbb{K}^*$ is not a root of unity in a field $\mathbb{K}$, n is even, then $U_q(\mathfrak{sl}_{n+1})$ and $U_p(\mathfrak{sl}_{n+1})$ are isomorphic as $\mathbb{K}$-algebras impplies that $p = \pm q^{\pm 1}$.*

Based on some more involved methodology, we will prove an 'analogue' of Theorem 1.4 for odd n in a forthcoming paper [LY2], where one sees that for odd n, the situation becomes much more complicated. For the simplest odd case $n = 1$, see L.-B.Li and J.-T.Yu [LY1].

2. **Preliminaries**

In this section, we recall some fundamental facts about the quantum group $U_q(\mathfrak{sl}_{n+1})$ over a field $\mathbb{K}$, where $q \in \mathbb{K}^*$ is not a root of unity in $\mathbb{K}$ (see, for instance, Jantzen [Ja], or Kassel [Ka]). We also prove two technical lemmas, the first classifies the unit elements in $U_q(\mathfrak{sl}_{n+1})$, the second describes a subset of n algebraically independent elements in $\gamma_\rho((U^0_{ev})^W)$ as a minimal generating set of the fractional field of $\gamma_\rho((U^0_{ev})^W)$ for even n by the multiplicative invariant theory. All of these will be used in the proof of the main results in the next section.
Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1}$ be the usual orthogonal unit vectors which form a basis of Euclidean space $\mathbb{R}^{n+1}$ with the usual inner product. It follows that $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n+1\}$ is the root system of $\mathfrak{sl}_{n+1}$ and $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n\}$ is a base of Φ. Note that the reflection s_i corresponding to α_i permutes the subscripts $i,\ i+1$ and

leave all other subscripts fixed. Thus we get that the Weyl group W of $\mathfrak{sl}_{n+1}$ is just the symmetric group $\mathfrak{S}_{n+1}$.

Recall that for given $q \in \mathbb{K}^*$ and $q^2 \neq 1$, the quantum group

$$U := U_q := U_q(\mathfrak{sl}_{n+1})$$

is the associative algebra over $\mathbb{K}$ generated by $K_i,\ K_i^{-1},\ E_i,\ F_i$ for $1 \leq i, j \leq n$ subject to the following defining relations:

$$K_iK_j = K_jK_i,\ K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_j = q^{a_{ij}}E_jK_i, K_iF_j = q^{-a_{ij}}F_jK_i,$$

$$E_iF_j - F_jE_i = \delta_{ij}(K_i - K_i^{-1})/(q - q^{-1}),$$

$$E_iE_j = E_jE_i, F_iF_j = F_jF_i, \quad |i-j| \neq 1,$$

$$E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0, \quad |i-j| = 1,$$

$$F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0, \quad |i-j| = 1,$$

where $A = (a_{ij})$ be the Cartan matrix of type A_{n+1}.
The following lemma describe the unit elements in $U_q(\mathfrak{sl}_{n+1})$. See, for instance, J.-G. Zha [Z1], for a proof..

Lemma 2.1. *An element $u \in U_q(\mathfrak{sl}_{n+1})$ is multiplicative invertible if and only if there exists $\lambda \in \mathbb{K}^*$, $m_i \in \mathbb{Z}$ such that $u = \lambda K_1^{m_1}K_2^{m_2}\dots K_n^{m_n}$.*

Denote by U^0 the subalgebra of $U := U_q(\mathfrak{sl}_{n+1})$ generated by $K_i^{\pm 1}$, and U^+ (U^-) respectively) the subalgebra generated by E_i (F_i respectively) for $1 \leq i \leq n$. It follows that U^0 is the laurent polynomial algebra $\mathbb{K}[K_1^{\pm 1}, \cdots, K_n^{\pm 1}]$. For each λ in the root lattice $\mathbb{Z}\Phi = \mathbb{Z}\Pi = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \cdots \mathbb{Z}\alpha_n$, we define an element K_λ in U^0 by

$$K_\lambda = K_1^{\lambda_1}K_2^{\lambda_2}\cdots K_n^{\lambda_n}, \text{ if } \lambda = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_n\alpha_n \in \mathbb{Z}\Phi.$$

The Weyl group $W = \mathfrak{S}_{n+1}$ acts naturally on U^0 such that

$$w \cdot K_\lambda = K_{\omega(\lambda)}, \text{ for all } w \in W \text{ and } \lambda \in \mathbb{Z}\Phi.$$

Recall that the quantum group $U = U_q(\mathfrak{sl}_{n+1})$ is a $\mathbb{Z}\Phi$-graded $\mathbb{K}$-algebra with the grading on the generators via $\deg(K_i) = \deg(K_i^{-1}) = 0$ and $\deg(E_i) = \alpha_i,\ \deg(F_i) = -\alpha_i$. Suppose that q is not a root of unity, then for $\nu \in \mathbb{Z}\Phi$ we have

$$U_\nu = \{u \in U \mid K_\lambda u = uK_\lambda\}.$$

According to the triangular decomposition $U = U^-U^0U^+$, we have a direct decomposition

$$U = U^0 \oplus \bigoplus_{\nu > 0} U^-_{-\nu}U^0U^+_\nu.$$

Denote $\pi : U_0 \to U^0$ the projection with respect to this decomposition. Then it is easy to check that π is a $\mathbb{K}$-algebra homomorphism. For all $\lambda \in \Lambda$, define the $\mathbb{K}$-algebra automorphism

$$\gamma_\lambda(K_\mu) = q^{(\lambda,\mu)}, \quad forall \mu \in \mathbb{Z}\Phi.$$

We call

$$\gamma_{-\rho} \circ \pi \ : Z(U) \to U^0$$

the Harish-Chandra homomorphism for U, where ρ is half the sum of the positive roots. So we have

$$\rho = \frac{1}{2}\Sigma_{i=1}^{n+1}(n+2-2i)\varepsilon_i$$

and $(\rho, \alpha_i) = 1$ for all $1 \leq i \leq n$.
Set $U_{ev}^0 = \bigoplus_{\lambda \in \mathbb{Z}\Phi \cap 2\Lambda} kK_\lambda$, where Λ is the weight lattice (when we specify the parameter q, we will write $U_{q,ev}^0$). Note that $\mathbb{Z}\Phi \cap 2\Lambda$ is a subgroup of $\mathbb{Z}\Phi \cap \Lambda$ and $\mathbb{Z}\Phi \cap 2\Lambda$ is W-stable. It follows that the action of W maps U_{ev}^0 to itself. The following result is well-known (see [Ja]).

Proposition 2.2. *The Harish-Chandra homomorphism is an isomorphism between between $Z(U_q(\mathfrak{sl}_{n+1}))$ and $(U_{ev}^0)^W$, where*

$$(U_{ev}^0)^W = \{h \in U_{ev}^0 \mid w \cdot h = h, \ \forall \ \ w \in W\}.$$

The following technical lemma is essential for us to obtain the algebraically independent generators of $\gamma_\rho((U_{ev}^0)^W)$. Note that in case the parameter q is to be specified, we will write $\gamma_{\rho,q}((U_{ev}^0)^W)$ instead of $\gamma_\rho((U_{ev}^0)^W)$.

Lemma 2.3. *For the simple Lie algebra $\mathfrak{sl}_{n+1}$ of type A_{n+1}, we have*

$$\mathbb{Z}\Phi \cap 2\Lambda = \begin{cases} \bigoplus_{i=1}^n \mathbb{Z}(2\alpha_i), & for\ evenn, \\ \bigoplus_{i=1}^{n-1} \mathbb{Z}(2\alpha_i) \oplus \mathbb{Z}(\alpha_1 + \alpha_3 + \cdots + \alpha_n), & for\ oddn. \end{cases}$$

Proof. Since $\alpha_i \in \Lambda$ for $1 \leqslant i \leqslant n$, it follows that $2\alpha_i \in \mathbb{Z}\Phi \cap 2\Lambda$ and

$$\bigoplus_{i=1}^{n} \mathbb{Z}(2\alpha_i) \subseteq \mathbb{Z}\Phi \cap 2\Lambda, \quad \bigoplus_{i=1}^{n-1} \mathbb{Z}(2\alpha_i) \subseteq \mathbb{Z}\Phi \cap 2\Lambda.$$

Note that when $n = 2m - 1$ is odd,

$$\lambda_m =$$

$$\frac{1}{2m}[m\alpha_1 + 2m\alpha_2 + \cdots + (m-1)m\alpha_{m-1} + m^2\alpha_m + m(m-1)\alpha_{m+1} + \cdots + m\alpha_m].$$

Thus, we have

$$2\lambda_m = (2\alpha_2 + 2\alpha_3 + 4\alpha_4 + \cdots + 2\alpha_{n-1}) + (\alpha_1 + \alpha_3 + \ldots \alpha_n).$$

It follows that $2\lambda_m \in \mathbb{Z}\Phi \cap 2\Lambda$ and

$$\alpha_1 + \alpha_3 + \ldots \alpha_n \in \mathbb{Z}\Phi \cap 2\Lambda.$$

So we have

$$\bigoplus_{i=1}^{n-1} \mathbb{Z}(2\alpha_i) \bigoplus \mathbb{Z}(\alpha_1 + \alpha_3 + \cdots + \alpha_n) \in \mathbb{Z}\Phi \cap 2\Lambda.$$

On the other hand, since
$\alpha_1 = 2\lambda_1 - \lambda_2, \alpha_2 = -\lambda_1 + 2\lambda_2 - \lambda_3, \cdots, \alpha_n = -\lambda_{n-1} + 2\lambda_n$,
now for any $\alpha = \sum_{i=1}^n k_i\alpha_i \in \mathbb{Z}\Phi \cap 2\Lambda$, where $k_i \in \mathbb{Z}$. Then we have

$$\alpha = (2k_1 - k_2)\lambda_1 + (2k_2 - k_1 - k_3)\lambda_2 + \cdots + (2k_n - k_{n-1})\lambda_n \in 2\Lambda.$$

It follows that

$$k_2 \in 2\mathbb{Z}, k_1 + k_3 \in 2\mathbb{Z}, k_2 + k_4 \in 2\mathbb{Z}, \cdots$$

and

$$k_{n-1} \in 2\mathbb{Z}, k_n + k_{n-2} \in 2\mathbb{Z}, k_{n-1} + k_{n-3} \in 2\mathbb{Z}, \cdots.$$

When n is even, then $k_i \in 2\mathbb{Z}$ for $1 \le i \le n$. So we have

$$\mathbb{Z} \cap 2\Lambda \subseteq \bigoplus_{i=1}^{n} \mathbb{Z}(2\alpha_i).$$

When n is odd, then $k_2, k_4, \cdots, k_{n-1} \in 2\mathbb{Z}$ and $k_1, k_3, \cdots, k_n$ are all odd or all even. Thus, we have

$$\alpha =$$

$$(k_1-k_n)\alpha_1+k_2\alpha_2+(k_3-k_n)\alpha_3+k_4\alpha_4+\cdots+k_{n-1}\alpha_{n-1}+k_n(\alpha_1+\alpha_3+\cdots+\alpha_n),$$

where $k_1 - k_n, k_2, k_3 - k_n, k_4, \cdots, k_{n-1} \in 2\mathbb{Z}$.
So we have

$$\bigoplus_{i=1}^{n-1} \mathbb{Z}(2\alpha_i) \bigoplus \mathbb{Z}(\alpha_1 + \alpha_3 + \cdots + \alpha_n) \supseteq \mathbb{Z}\Phi \cap 2\wedge,$$

when n is odd. □

Remark 2.4. From the above lemma we have already seen for odd n, the situation is more complicated. For details and related multiplicative invariant theory, see L.-B.Li and J.-T.Yu [LY2].

Next we determine a subset of n algebraically independent elements in $\gamma_\rho((U_{ev}^0)^W)$ as a minimal generating set of $\gamma_\rho((U_{ev}^0)^W)$ for even n by the multiplicative invariant theory, see [Lo]. In the sequel we assume that n is even, by Lemma 3.1, we have

$$U_{ev}^0 = \mathbb{K}[K_1^{\pm 2}, K_2^{\pm 2}, \cdots, K_n^{\pm 2}].$$

Put $K_i^2 = y_i = \frac{x_i}{x_{i+1}}$ for $1 \leqslant i \leqslant n$. Define

$$s_i = \sum_{j_1<j_2<\cdots<j_i} x_{j_1}x_{j_2}\cdots x_{j_i},$$

$1 \leqslant i \leqslant n+1$. Then by multiplicative invariant theory (see [Lo]), for $m = 1, 2, \cdots, n-1, n+1$ we obtain the fundamental invariants

$$\frac{s_1^m s_{n-m+1}}{s_{n+1}} \in (U_{ev}^0)^{\mathfrak{S}_{n+1}}.$$

Notice that

$$(x_1 + x_2 + \cdots + x_{n+1})^m = \sum_{\substack{0\leqslant i_1, i_2, \cdots, i_{n+1} \leqslant n+1 \\ i_1+i_2+\cdots+i_{n+1}=m}} \frac{m!}{i_1!i_2!\cdots i_{n+1}!} x_1^{i_1}\cdots x_{n+1}^{i_{n+1}}.$$

By direct calculation, we have

$$\frac{s_1^m s_{n-m+1}}{s_{n+1}}$$

$$= \sum_{\substack{i_1, i_2, \cdots, i_{n+1} \\ i_1+i_2+\cdots+i_{n+1}=m \\ 1\leqslant j_1<j_2<\cdots<j_{n-m+1}}} \frac{m! x_1^{i_1}\cdots x_{j_1}^{i_{j_1}}\cdots x_{j_{n-m+1}}^{i_{n-m+1}+1}\cdots x_{n+1}^{i_{n+1}}}{i_1!i_2!\cdots i_{n+1}! x_1 x_2\cdots x_{n+1}} =$$

$$\sum_{\substack{i_1, i_2, \cdots, i_{n+1} \\ i_1+i_2+\cdots+i_{n+1}=m \\ 1\leqslant j_1<j_2<\cdots<j_{n-m+1}}} \frac{m! y_1^{i_1-1} y_2^{i_1+i_2-1}\cdots y_{j_{n-m+1}}^{i_1+i_2+\cdots i_{j_{n-m+1}}-(n-m+1)}\cdots y_n^{i_{n+1}-1}}{i_1!i_2!\cdots i_{n+1}!}$$

Then substitute $y_i = (qK_i)^2$, we obtain a subset of n algebraically independent elements in $\gamma_\rho((U_{ev}^0)^W)$ as a minimal generating set of the fractional field of $\gamma_\rho((U_{ev}^0)^W)$, which are algebraically independent as follows.

$$\sigma_1 = q^2K_1^2 + q^4K_1^2K_2^2 + \cdots + q^nK_1^2K_2^2\ldots K_n^2$$
$$+ q^{-n}K_1^{-2}K_2^{-2}\ldots K_n^{-2} + \cdots + q^{-4}K_1^{-2}K_2^{-2} + q^{-2}K_1^{-2},$$
$$\sigma_2 = q^6K_1^4K_2^2 + \cdots + q^{-6}K_1^{-2}K_2^{-4},$$
$$\cdots\cdots\cdots$$
$$\sigma_{n-1} =$$
$$q^{n(n-1)}K_1^{2(n-1)}K_2^{2(n-2)}\cdots K_{n-1}^2 + \cdots + q^{-n(n-1)}K_1^{-2}K_2^{-4}\cdots K_n^{-2(n-1)},$$
$$\sigma_{n+1} = q^{n(n+1)}K_1^{2n}K_2^{2(n-1)}\cdots K_n^2 + \cdots + q^{-n(n+1)}K_1^{-2}K_2^{-4}\cdots K_n^{-2n}.$$

Then by the expression of the these elements, it is easy to see that each element admits some special ‘symmetry’, which means that, if we replace qK_i by $(qK_i)^{-1}$ for all i in σ_j as functions for $qK_i \;\; i=1,\dots,n$ for all j, then

$$\sigma_j(qK_1,\dots,qK_n)=\sigma_j((qK_1)^{-1},\dots,(qK_n)^{-1}).$$

Consequently, if we denote by $f_j(qK)$ for the function of qK obtained by substituting $K=K_i$ for all $i=1,2,\cdots,n$ in σ_j, then $f_j(qK)=f_j((qK)^{-1})$ for all j. For example, we write explicitly down these elements for $n=2$ as follows (see [LWP, LWW, Lo] for details) below.

$$\sigma_1=q^2K_1^2+q^4K_1^2K_2^2+q^2K_2^2+q^{-2}K_2^{-2}+q^{-4}K_1^{-2}K_2^{-2}+q^{-2}K_1^{-2}+3,$$
$$\sigma_3=q^6K_1^4K_2^2+K_1^{-2}K_2^2+q^{-6}K_1^{-2}K_2^{-4}$$
$$+3(q^2K_1^2+q^4K_1^2K_2^2+q^2K_2^2+q^{-2}K_2^{-2}+q^{-4}K_1^{-2}K_2^{-2}+q^{-2}K_1^{-2})+6.$$

By the multiplicative invariant theory (see, for instance, [Lo]), we know that the set of these n elements is a minimal (algebraically independent) generating set of the fraction field of $\gamma_\rho((U_{ev}^0)^W)$.

3. Proof of the main result

Suppose $U=:U_q:=U_q(\mathfrak{sl}_{n+1})$ and $U_p:=U_p(\mathfrak{sl}_{n+1})$ are isomorphic as $\mathbb{K}$-algebras. Then we may assume $U=U_q=U_p$. In other words, $U:=U_q(\mathfrak{sl}_{n+1})$ has a $U_p(\mathfrak{sl}_{n+1})$-structure. Namely, $U=U_q$ is also generated by k_i, k_i^{-1}, e_i, f_i for $1\le i,j\le n$ subject to the following defining relations:

$$k_ik_j=k_jk_i,\;\; k_ik_i^{-1}=k_i^{-1}k_i=1,$$
$$k_ie_j=p^{a_{ij}}e_jk_i, k_if_j=p^{-a_{ij}}f_jk_i,$$
$$e_if_j-f_je_i=\delta_{ij}(k_i-k_i^{-1})/(p-p^{-1}),$$
$$e_ie_j=e_je_i, f_if_j=f_jf_i \quad |i-j|\neq 1,$$
$$e_i^2e_j-(p+p^{-1})e_ie_je_i+e_je_i^2=0, \quad |i-j|=1,$$
$$f_i^2f_j-(p+p^{-1})f_if_jf_i+f_jf_i^2=0, \quad for \;\; |i-j|=1,$$

where $A=(a_{ij})$ be the Cartan matrix of type A_{n+1}.

Lemma 3.1. *In $U=U_q(\mathfrak{sl}_{n+1})=U_p(\mathfrak{sl}_{n+1})$,*

$$\mathbb{K}[K_1^{\pm1},\dots,K_n^{\pm1}]=\mathbb{K}[k_1^{\pm1},\dots,k_n^{\pm1}].$$

Proof. Obviously there exists a $\mathbb{K}$-isomorphism from $\mathbb{K}[K_1^{\pm1},\dots,K_n^{\pm1}]$ onto $\mathbb{K}[k_1^{\pm1},\dots,k_n^{\pm1}]$ taking K_i to k_i which induces an isomorphism from the free abelian group generated by K_i ($i=1,\dots,n$) to the free abelian group generated by k_i ($i=1,\dots,n$). But the second isomorphism is indeed a (group) automorphism by Lemma 2.1, so is the first one. Therefore

$$\mathbb{K}[K_1^{\pm1},\dots,K_n^{\pm1}]=\mathbb{K}[k_1^{\pm1},\dots,k_n^{\pm1}]$$

in $U = U_q(\mathfrak{sl}_{n+1}) = U_p(\mathfrak{sl}_{n+1})$. □

Lemma 3.2.

$$\gamma_{\rho,q}((U^0_{q,ev})^W) = \gamma_{\rho,p}((U^0_{p,ev})^W).$$

Proof. We may assume

$$U := U_q = U_q(\mathfrak{sl}_{n+1}) = U_p = U_p(\mathfrak{sl}_{n+1}).$$

Hence $Z(U_q) = Z(U_p)$. By Lemma 3.1,

$$\mathbb{K}[K_1^{\pm 1}, \ldots, K_n^{\pm 1}] = \mathbb{K}[k_1^{\pm 1}, \ldots, k_n^{\pm 1}],$$

hence $\gamma_{\rho,q}(U^0_{q,ev})^W)$, as the image $\pi_q(Z(U_q))$ of $Z(U_q)$ under the projection

$$\pi_q : \ U_{q,0} \to U_q^0 = \mathbb{K}[K_1^{\pm 1}, \ldots, K_n^{\pm 1}],$$

is equal to $\gamma_{\rho,p}(U^0_{p,ev})^W)$, as the image $\pi_p(Z(U_p))$ of $Z(U_p)$ under the projection

$$\pi_p : \ U_{p,0} \to U_p^0 = \mathbb{K}[k_1^{\pm 1}, \ldots, k_n^{\pm 1}]$$

by the definition of the Harish-Chandra isomorphism. □

In Section 2 we already got a subset of n algebraic independent elements $\sigma_j, \ j = 1, \ldots, n$ in $\gamma_{\rho,q}((U^0_{q,ev})^W)$ as a minimal generating set of the fractional field $\gamma_{\rho,q}((U^0_{q,ev})^W)$. Similarly, we can write a subset of n algebraic independent elements $\tau_j, \ j = 1, \ldots, n$ in $\gamma_{\rho,p}((U^0_{p,ev})^W)$ as a minimal generating set of the fraction field of $\gamma_{\rho,p}((U^0_{p,ev})^W)$ as follows.

$$\begin{aligned}
&\tau_1 = p^2 k_1^2 + p^4 k_1^2 k_2^2 + \cdots + p^n k_1^2 k_2^2 \ldots k_n^2 \\
&+ p^{-n} k_1^{-2} k_2^{-2} \ldots k_n^{-2} + \cdots + p^{-4} k_1^{-2} k_2^{-2} + p^{-2} k_1^{-2}, \\
&\tau_2 = p^6 k_1^4 k_2^2 + \cdots + p^{-6} k_1^{-2} k_2^{-4}, \\
&\cdots\cdots\cdots \\
&\tau_{n-1} \\
&= p^{n(n-1)} k_1^{2(n-1)} k_2^{2(n-2)} \cdots k_{n-1}^2 + \cdots + p^{-n(n-1)} k_1^{-2} k_2^{-4} \cdots k_n^{-2(n-1)}, \\
&\tau_{n+1} = p^{n(n+1)} k_1^{2n} k_2^{2(n-1)} \cdots k_n^2 + \cdots + p^{-n(n+1)} k_1^{-2} k_2^{-4} \cdots k_n^{-2n}.
\end{aligned}$$

By Lemma 3.2, there exists a $\mathbb{K}$-birational automorphism of the fraction field

$$\mathbb{K}(\sigma_1, \ldots, \sigma_n) = \mathbb{K}(\tau_1, \ldots, \tau_n)$$

of $\gamma_{\rho,q}((U^0_{ev,q})^W) = \gamma_{\rho,p}((U^0_{ev,p})^W)$ induced by the $\mathbb{K}$-automorphism α of

$$\mathbb{K}[K_1^{\pm 1}, \ldots, K_n^{\pm 1}] = \mathbb{K}[k_1^{\pm 1}, \ldots, k_n^{\pm 1}]$$

taking qK_i to $\alpha(qK_i) = pk_i$ for all i and taking σ_j to $\alpha(\sigma_j) = \tau_j$ for all j. On the other hand, By Galois theory, any $\mathbb{K}$-birational automorphism of

$$\mathbb{K}(\sigma_1, \dots, \sigma_n) = \mathbb{K}(\tau_1, \dots, \tau_n)$$

taking σ_j to τ_j for all j can be lifted a birational $\mathbb{K}$-automorphism of

$$\mathbb{K}(K_1, \dots, K_n) = \mathbb{K}(k_1, \dots, k_n)$$

taking qK_i to pk_i for all i via the finite Galois extension

$$\mathbb{K}(K_1, \dots, K_n)/\mathbb{K}(\sigma_1, \dots, \sigma_n) = \mathbb{K}(k_1, \dots, k_n)/\mathbb{K}(\tau_1, \dots, \tau_n).$$

Put $K := K_1 = \cdots = K_n$, consequently, $k := k_1 = \cdots = k_n$, then α induces an automorphism of $\mathbb{K}(qK + q^{-1}K^{-1})$ taking $qK + q^{-1}K^{-1}$ to $pk + p^{-1}k^{-1}$, and we obtain

$$\mathbb{K}(qK + q^{-1}K^{-1}) = \mathbb{K}(pk + p^{-1}k^{-1}),$$

which forces $qK = (\pm pk)^{\pm 1}$. Therefore $q^2K_i^2 = p^2k_{\theta(i)}^{\pm 2}$ for some $\theta \in \mathfrak{S}_n$ and all i. By definition of σ_1 and τ_1, we see that

$$\sigma_1 = \tau_1(k_{\theta(1)}, \dots, k_{\theta(n)}),$$

if we view $\tau_1 := \tau_1(k_1, \dots, k_n)$ as a polynomial function of $k_1, \dots, k_n$. Consider the scalar action of the central element in $Z(U_q) = Z(U_p)$ corresponding to

$$\sigma_1 = \tau_1(k_{\theta(1)}, \dots, k_{\theta(n)}) \in \gamma_{\rho,q}((U^0_{ev,q})^W) = \gamma_{\rho,p}((U^0_{ev,p})^W)$$

on the ($U_q = U_p =$) U-simple module $L(\lambda)$, where $\lambda = \sum_{i=1}^n m_i\lambda_i \in \Lambda$ for nonnegative integers $m_1, \dots, m_n$, we get

$$\begin{aligned} &q^{2+2m_1} + q^{4+2m_1+2m_2} + \cdots + q^{2n(n+1)+2m_1+2m_2+\cdots+2m_n} \\ &+q^{-2n(n+1)-2m_1-2m_2-\cdots-2m_n} + q^{-4-2m_1-2m_2} + q^{-2-2m_1} \\ &= p^{2+2m_1} + p^{4+2m_1+2m_2} + \cdots + p^{2n(n+1)+2m_1+2m_2+\cdots+2m_n} \\ &+p^{-2n(n+1)-2m_1-2m_2-\cdots-2m_n} + p^{-4-2m_1-2m_2} + p^{-2-2m_1} \end{aligned}$$

for all nonnegative integers $m_1, m_2, \dots, m_n$. That forces $q^2 + q^{-2} = p^2 + p^{-2}$, hence $p^2 = q^{\pm 2}$, therefore $p = \pm q^{\pm 1}$. $\square$

4. Acknowledgements

Jie-Tai Yu is grateful to Yangzhou University (Yangzhou), Yunnan Normal University (Kunming), Shanghai University (Shanghai), Osaka University and Kwansei Gakuin University (Osaka), Saitama University (Saitama), Chern Institute of Mathematical Research (Tianjin), Beijing International Center for Mathematical Research and Chinese Academy of Sciences (Beijing), and Academia Sinica (Taipei) for warm hospitality and stimulating environment during his visits in 2009 when this work was carry out. The authors thank Stephane Launois for valuable comments, in particular for pointing out the references [AC, La1, La2, LL]. The authors also thank I-Chiau Huang and Shigeru Kuroda for providing the reference [Ji1].

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