arXiv:1008.5333[math.SG]
HKU-IMR preprint IMR2010:#11
August 2010 (revised: October 2010)

Projective flatness in the quantisation of bosons and fermions

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Abstract

We compare the quantisation of linear systems of bosons and fermions. We recall the appearance of projectively flat connection and results on parallel transport in the quantisation of bosons. We then discuss pre-quantisation and quantisation of fermions using the calculus of fermionic variables. We then define a natural connection on the bundle of Hilbert spaces and show that it is projectively flat. This identifies, up to a phase, equivalent spinor representations constructed by various polarisations. We introduce the concept of metaplectic correction for fermions and show that the bundle of corrected Hilbert spaces is naturally flat. We then show that the parallel transport in the bundle of Hilbert spaces along a geodesic is the rescaled projection or the Bogoliubov transformation provided that the geodesic lies within the complement of a cut locus. Finally, we study the bundle of Hilbert spaces when there is a symmetry.

Keywords: geometric quantisation, projectively flat bundles, Hermitian symmetric spaces
MSC(2000): Primary 53D50; Secondary 32M15, 15A66, 58C50

1 Introduction

One of the central questions in geometric quantisation is whether the quantum Hilbert spaces constructed from different choices of polarisations can be naturally identified. Since a quantum state actually corresponds to a ray of vectors, identification is only required for the projectivisation of the Hilbert spaces. This amounts to the existence of a natural projectively flat connection on the bundle of Hilbert spaces over the space of polarisations. When the symplectic manifold is Kähler, it is convenient to consider a subclass of polarisations that come from complex structures compatible with the symplectic form. Given such a complex structure, the quantum Hilbert space is the space of holomorphic sections of the pre-quantum line bundle. However, under quite general conditions (satisfied by, for example, the 2-sphere), there is no naturally projectively flat connection in the bundle of quantum Hilbert spaces [10].

The next best scenario is that projective flatness holds if we limit the polarisations to a smaller subset, for example, to those respecting the symmetry of the system. For a symplectic vector space polarised by linear complex structures, there is indeed a natural projectively flat connection in the bundle of Hilbert spaces [1]. Moreover, the connection is flat if we include metaplectic correction [29, 15]. Parallel transport in the bundle yields the familiar Fourier and Segal-Bargmann transforms that are usually used to identify wave functions in various pictures [15]. (The Segal-Bargmann transform can be generalised to relate polarisations on the cotangent bundles of compact Lie groups [11, 9].) Another example of projective flatness is from quantising the space of flat connections on a compact orientable surface [13, 1]; in this case the complex structures are induced by those on the surface.

Let (V,ω) be a symplectic vector space and J, a compatible linear complex structure on V. The qunatum Hilbert space is a representation of Heisenberg algebra, generated by tensor powers of V subject to the canonical commutation relation. The existence of projective flatness is related to the celebrated Stone-von Neumann theorem [18], which asserts that the irreducible representation of the Heisenberg algebra is unique up to a unitary equivalence. Moreover, by Schur's lemma, any two unitary equivalences between two irreducible representations have to differ only by a phase. So between the fibres over two linear complex

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structures in the bundle of Hilbert spaces, there is a unitary identification which is unique up to a phase. This is the hallmark of projectively flat bundles.

The main purpose of this paper is to establish a similar structure of projective flatness in the quantisation of fermions. The phase space of a linear fermionic system is a Euclidean space (V, g) and quantisation means finding an irreducible representation of the Clifford algebra, which is the fermionic analog of the Heisenberg algebra. (See [16, 7] for formal similarities between the two algebras.) Such a representation is the spinor representation, and just like the bosonic case, it is unique (when $\dim V$ is even) or nearly unique (when $\dim V$ is odd) [5]. In the construction of the spinor representation, one needs to choose a compatible complex structure (see for example [3], §3.2), which is the fermionic counterpart of polarisation. We therefore expect a projectively flat bundle (of spinor representations) over space of such complex structures.

The rest of the paper is organised as follows. In §2, we review the pre-quantisation and quantisation of bosonic systems whose phase spaces are symplectic vector spaces. We then recall the natural connection on the bundle of Hilbert spaces and give a straightforward proof of its projective flatness [1]. The connection becomes flat after metaplectic correction is included [29, 15]. We present, in a coordinate-free way, the results in [15] on the parallel transport in the bundle of Hilbert spaces along geodesics in the base space. Finally, when there is a group acting symplectically on the vector space, we decompose the bundle of Hilbert spaces into a direct sum of projectively flat sub-bundles and identify the invariant part as the bundle from quantising the symplectic quotient. §3 is devoted to the quantisation of fermions when the phase space is an even-dimensional Euclidean space. We discuss the pre-quantisation and quantisation of fermions using calculus of fermionic variables. We then define a natural connection on the bundle of Hilbert spaces and show that it is projectively flat. This identifies, up to a phase, constructions of the spinor representation under various polarisations. We introduce the concept of metaplectic correction for fermions and show that the bundle of corrected Hilbert spaces is naturally flat. We then show that the parallel transport in the bundle of Hilbert spaces along a geodesic is the rescaled projection or the Bogoliubov transformation provided the geodesic lies within the complement of a cut locus. The decomposition of the bundle of Hilbert spaces when there is a symmetry is also studied. In §4, we conclude by highlighting the similarities and differences in the quantisation of bosons and fermions. In Appendix A, we consider the geometry of the spaces of complex structures compatible to a symplectic or Euclidean structure, which are classical Hermitian symmetric spaces [21, 14]. We describe cut locus in the space of polarisations of a fermionic system. Appendix B is on the calculus of fermionic variables. We describe fermionic coherent states and the fermionic analog of Bergman kernel. In Appendix C, we collect some facts on real and quaternionic representations and on complex structures invariant under a representation.

2 Quantisation of bosonic systems

2.1 Pre-quantisation and quantisation

Let (V, ω) be a symplectic vector space of dimension $2n$. A pre-quantum line bundle ℓ over V is a line bundle with a connection whose curvature is $\omega/\sqrt{-1}$. The pre-quantum Hilbert space is $\mathcal{H}_0 = L^2(V, \ell)$, the space of L^2-sections of ℓ with respect to the symplectic volume form $\varepsilon_\omega = \omega^{\wedge n}/n!$ or $\tilde{\varepsilon}_\omega = \varepsilon_\omega/(2\pi)^n$ on V. The covariant derivative ∇_x along a constant vector field on V parallel to $x \in V$ is a skew-self-adjoint operator on $\mathcal{H}_0$ and satisfies the commutation relation $[\nabla_x, \nabla_y] = \omega(x, y)/\sqrt{-1}$ for any $x, y \in V$. As V is contractible, ℓ is topologically trivial and is unique up to an isomorphism. We can choose a trivialisation of ℓ identifying $\mathcal{H}_0$ with $L^2(V, \mathbb{C})$ such that

$$\nabla_x = L_x + \tfrac{1}{2}\sqrt{-1}\,\iota_x\omega, \quad x \in V.$$

Here $\iota_x\omega \in V^*$ is regarded as a linear function on V multiplying on the sections of ℓ or on $L^2(V, \mathbb{C})$. For any $\alpha \in V^*$, the corresponding pre-quantum operator acting on $\mathcal{H}_0$ is

$$\hat{\alpha} = \sqrt{-1}\,\nabla_{\nu^{-1}(\alpha)} + \alpha = \sqrt{-1}\,L_{\nu^{-1}(\alpha)} + \tfrac{1}{2}\,\alpha.$$

These operators are self-adjoint on $\mathcal{H}_0$ and satisfy Heisenberg's canonical commutation relation $[\hat{\alpha}, \hat{\beta}] = \sqrt{-1}\,\omega^{-1}(\alpha, \beta)$, where $\alpha, \beta \in V^*$.

Consider the space $\mathcal{J}_\omega$ of compatible complex structures on (V, ω). (We refer the reader to §A.1-2 for notations and results on complex structures.) For each $J \in \mathcal{J}_\omega$, the complex subspaces $V_J^{1,0}$, $V_J^{0,1}$ of $V^{\mathbb{C}}$ are Lagrangian with respect to ω and they determine a (linear) complex polarisation of (V, ω). The quantum Hilbert space associated to J is

$$\mathcal{H}_J = \{\psi \in \mathcal{H}_0 \,|\, \nabla_x\psi = 0,\ \forall x \in V_J^{0,1}\}.$$

So a vector $\psi \in \mathcal{H}_J$ is a holomorphic L^2-section of ℓ. The compatibility condition on J guarantees that the space $\mathcal{H}_J$ is non-empty; in fact, it is infinite dimensional. Note that for any $J \in \mathcal{J}_\omega$, $\mathcal{H}_J$ is a subspace of $\mathcal{H}_0$. Thus we have a bundle of quantum Hilbert spaces $\mathcal{H} \to \mathcal{J}_\omega$ whose fibre over $J \in \mathcal{J}_\omega$ is $\mathcal{H}_J$.

The following results are well known.

Proposition 2.1 *1. Any $\psi \in \mathcal{H}_J$ is of the form*

$$\psi = e^{-\frac{1}{4}q_J}\phi$$

for a unique J-holomorphic function ϕ on V, where $q_J \in \omega(\cdot, J\cdot) \in \mathrm{Sym}^2(V^*)$ is regarded as a quadratic function on V.
2. If ϕ is a J-holomorphic function on V, then $\psi = e^{-\frac{1}{4}q_J}\phi$ is in $\mathcal{H}_J$ if and only if its norm

$$\left(\int_V e^{-\frac{1}{2}q_J}|\phi|^2\,\tilde{\varepsilon}_\omega\right)^{1/2}$$

weighted by $e^{-\frac{1}{2}q_J}$ is finite, in which case it is equal to the norm of $\psi \in \mathcal{H}_0$.
3. For any $\alpha \in V^*$, $\hat{\alpha}$ preserves $\mathcal{H}_J$ and is self-adjoint on $\mathcal{H}_J$. It acts on ϕ by

$$\hat{\alpha}\colon \phi \mapsto \sqrt{-1}\,L_{\nu^{-1}(\alpha^{0,1})}\phi + \alpha^{1,0}\phi.$$

If x^i $(1 \le i \le n)$ are the complex coordinates on $V_J^{1,0}$ with respect to a basis $\{e_i\}_{i=1,\dots,n}$, then the covariant derivative along $\bar{e}_{\bar{j}}$ is $\nabla_{\bar{j}} = \frac{\partial}{\partial \bar{x}^{\bar{j}}} + \frac{1}{2}\,q_{i\bar{j}}x^i$, where $q_{i\bar{j}} = q_J(e_i, \bar{e}_{\bar{j}})$. A section $\psi \in \mathcal{H}_J$ can be identified as a function of the form

$$\psi(x) = \phi(x)\,\exp[-\tfrac{1}{2}\,q_{i\bar{j}}x^i\bar{x}^{\bar{j}}],$$

where $\phi(x)$ is a holomorphic function in $x = (x^1, \cdots, x^n) \in \mathbb{C}^n$.

2.2 Projectively flat connection and metaplectic correction

The vector bundle $\mathcal{H} \to \mathcal{J}_\omega$ of quantum Hilbert spaces is a sub-bundle of the product bundle $\mathcal{J}_\omega \times \mathcal{H}_0 \to \mathcal{J}_\omega$ of pre-quantum Hilbert spaces. The trivial connection on the latter induces a natural connection on $\mathcal{H}$ by orthogonal projection. In [1], it was shown that the connection on $\mathcal{H}$ is projectively flat. For completeness and for comparison with the fermionic case (§3.2), we give a simple derivation of this result.

We first study the effect of the variation δJ on $\mathcal{H}_J$. We choose a basis $\{e_i\}_{1\le i\le n}$ of $V_J^{1,0}$. Suppose $\psi \in \mathcal{H}_J$, i.e., $\nabla_{\bar{k}}\psi = 0$ $(1 \le k \le n)$. As J changes to $J + \delta J$, the infinitesimal parallel transport $\psi + \delta\psi \in \mathcal{H}_{J+\delta J}$ of $\psi \in \mathcal{H}_J$ is the orthogonal projection of ψ on $\mathcal{H}_{J+\delta J}$. Thus we have two conditions: $\delta\psi \perp \mathcal{H}_J$, i.e., $\delta\psi \perp \ker\nabla_{\bar{i}}$ for $1 \le i \le n$, and $\psi + \delta\psi \in \mathcal{H}_{J+\delta J}$, or

$$\nabla_{\bar{e}_{\bar{i}}+\delta\bar{e}_{\bar{i}}}(\psi + \delta\psi) = 0,$$

where $\delta\bar{e}_{\bar{i}} = (\delta\bar{P})_{\bar{i}}^{\;j}e_j$ (see §A.2). This implies, to the first order, that $\delta\psi$ satisfies the equation

$$\nabla_{\bar{i}}\,(\delta\psi) = -(\delta\bar{P})_{\bar{i}}^{\;j}\nabla_j\psi = (\delta P)_{\bar{i}}^{\;j}\nabla_j\psi.$$

We claim that

$$\delta\psi = \tfrac{1}{2}\sqrt{-1}\,\nabla_i(\delta P)^{ij}\nabla_j\psi = \tfrac{1}{2}\sqrt{-1}\,(\delta P)^{ij}\nabla_i\nabla_j\psi$$

is the (unique) solution satisfying the above conditions. (The second equality holds because $(\delta P)^{ij}$ is a constant tensor on V.) First, this $\delta\psi$ is orthogonal to $\mathcal{H}_J$ as $\nabla_{\bar{i}}$ is the formal adjoint of ∇_i. Second, as $\nabla_{\bar{i}}\psi = 0$, $(\delta P)^{ij} = (\delta P)^{ji}$ and $[\nabla_{\bar{i}}, \nabla_j] = \omega_{\bar{i}j}/\sqrt{-1}$, we get

$$\begin{aligned}\nabla_{\bar{k}}\,(\delta\psi) &= \tfrac{1}{2}\sqrt{-1}\,(\delta P)^{ij}[\nabla_{\bar{k}}, \nabla_i\nabla_j]\psi = \tfrac{1}{2}\sqrt{-1}\,(\delta P)^{ij}([\nabla_{\bar{k}}, \nabla_i]\nabla_j + \nabla_i[\nabla_{\bar{k}}, \nabla_j])\psi\\ &= \omega_{\bar{k}i}(\delta P)^{ij}\nabla_j\psi = (\delta P)_{\bar{k}}^{\;j}\nabla_j\psi.\end{aligned}$$

The uniqueness is clear from the geometric interpretation.

The connection 1-form $\mathsf{A}^{\mathcal{H}}$ on $\mathcal{H}$ satisfies $(\delta + \mathsf{A}^{\mathcal{H}})\psi = 0$. Therefore $\mathsf{A}^{\mathcal{H}} = -\frac{\sqrt{-1}}{2}(\delta P)^{ij}\nabla_i\nabla_j$; it is an operator-valued 1-form on $\mathcal{J}_\omega$. We then calculate

$$\begin{aligned}\delta\,\mathsf{A}^{\mathcal{H}} &= \tfrac{\sqrt{-1}}{2}\,(\delta P)^{ij}\wedge\nabla_{\delta e_i}\nabla_j = \tfrac{\sqrt{-1}}{2}\,(\delta P)^{ij}\wedge(\delta P)_i^{\;\bar{k}}\nabla_{\bar{k}}\nabla_j\\ &= \tfrac{1}{2}\,\omega_{\bar{k}j}(\delta P)^{ji}\wedge(\delta P)_i^{\;\bar{k}} = \tfrac{1}{2}\,\mathrm{tr}(P\,\delta P\wedge\delta P\,P),\end{aligned}$$

ignoring the terms that vanish on $\mathcal{H}_J$, and

$$\mathsf{A}^{\mathcal{H}} \wedge \mathsf{A}^{\mathcal{H}} = -\tfrac{1}{4}\,\nabla_i\nabla_j\nabla_k\nabla_l\,(\delta P)^{ij}\wedge(\delta P)^{kl} = 0.$$

Therefore the curvature of the connection on $\mathcal{H}$ is

$$\mathsf{F}^{\mathcal{H}} = \tfrac{1}{2}\,\mathrm{tr}(P\,\delta P\wedge\delta P\,P)\,\mathrm{id}_{\mathcal{H}} = \upsigma_\omega/2\sqrt{-1}\,\mathrm{id}_{\mathcal{H}}.$$

Since it is a 2-form on $\mathcal{J}$ times the identity operator on the fibre, the connection on $\mathcal{H}$ is indeed projectively flat [1]. (In §2.1 of [26], it was shown directly, without the orthogonal projection from $\mathcal{H}_0$, that the formula for $\mathsf{A}^{\mathcal{H}}$ defines a connection on $\mathcal{H}$ which is projectively flat.)

We now incorporate metaplectic correction. Denote the restriction of $\mathcal{K} = (\det\mathcal{V})^* \to \mathcal{J}$ to $\mathcal{J}_\omega$ (see §A.1) by the same notation $\mathcal{K}$. Since $\mathcal{J}_\omega$ is contractible, there is a unique bundle $\sqrt{\mathcal{K}} \to \mathcal{J}_\omega$ such that $(\sqrt{\mathcal{K}})^{\otimes 2} = \mathcal{K}$. The curvature of the natural connection on $\sqrt{\mathcal{K}}$ is (see [15] or §A.1)

$$\mathsf{F}^{\sqrt{\mathcal{K}}} = \tfrac{1}{2}\,\mathsf{F}^{\mathcal{K}} = -\tfrac{1}{2}\,\mathrm{tr}(P\,\delta P\wedge\delta P\,P) = -\upsigma_\omega/2\sqrt{-1}\,.$$

We consider the bundle $\hat{\mathcal{H}} = \mathcal{H}\otimes\sqrt{\mathcal{K}}$. Its fibre $\hat{\mathcal{H}}_J = \mathcal{H}_J\otimes\sqrt{\mathcal{K}_J}$ over $J\in\mathcal{J}_\omega$ is the metaplectically corrected quantum Hilbert space with the polarisation J. Since the curvatures of $\mathcal{H}$ and $\sqrt{\mathcal{K}}$ cancel, the bundle $\hat{\mathcal{H}} \to \mathcal{J}_\omega$ is canonically flat [29, 15]. The flatness of the bundle indicates that for the symplectic linear space, quantisation is independent of the choice of polarisations. We summarise the results in the following

Theorem 2.2 ([1, 29, 15]) *Consider the quantisation of a bosonic system whose phase space is a finite dimensional symplectic vector space (V,ω).*
1. The bundle of quantum Hilbert spaces $\mathcal{H}\to\mathcal{J}_\omega$ is projectively flat, with curvature $\upsigma_\omega/2\sqrt{-1}$.
2. The bundle of quantum Hilbert spaces with metaplectic correction $\hat{\mathcal{H}}\to\mathcal{J}_\omega$ is flat.

2.3 Parallel transport along geodesics and the Bogoliubov transformations

We recall various results in [15]. Let (V,ω) be a symplectic vector space. The space $(\mathcal{J}_\omega,\upeta_\omega)$ of compatible complex structures is non-positively curved and there is a unique geodesic connecting any two points. Let $J_0, J_1\in\mathcal{J}_\omega$ define two complex polarisations. We want to study the parallel transport $\mathcal{U}^{\mathcal{H}}_{J_1J_0}$ and $\mathcal{U}^{\hat{\mathcal{H}}}_{J_1J_0}$ in the bundles $\mathcal{H}$ and $\hat{\mathcal{H}}$, respectively, along the geodesic from J_0 to J_1. A related notion is the orthogonal projection $\mathcal{P}_{J_1J_0}$ from $\mathcal{H}_{J_0}$ to $\mathcal{H}_{J_1}$ in $\mathcal{H}_0$.

Theorem 2.3 ([15]) *Let $J_0,J_1\in\mathcal{J}_\omega$ and let $\gamma=\{J_t\}_{0\le t\le 1}$ be the (unique) geodesic from J_0 to J_1, t being proportional to the arc-length parameter. Then*
1. the parallel transport in $\mathcal{H}$ along γ is $\mathcal{U}^{\mathcal{H}}_{J_1J_0} = \big(\det\frac{J_0+J_1}{2}\big)^{1/4}\,\mathcal{P}_{J_1J_0}$;
2. the parallel transport in $\sqrt{\mathcal{K}}$ along γ is $\mathcal{U}^{\sqrt{\mathcal{K}}}_{J_1J_0} = \sqrt{{}^T\big(\det J_{1/2}/\sqrt{-1}\,|_{V^{1,0}_{J_0}}\big)^{-1}}$, and

$$\langle\mathcal{U}^{\sqrt{\mathcal{K}}}_{J_1J_0}\sqrt{\mu_0'},\sqrt{\mu_0}\,\rangle = \big(\det\tfrac{J_0+J_1}{2}\big)^{-1/4}\,\langle\sqrt{\mu_0'},\sqrt{\mu_0}\,\rangle$$

for any $\mu_0,\mu_0'\in\bigwedge^n(V^{1,0}_{J_0})^*$;
3. the parallel transport in $\hat{\mathcal{H}}$ along γ, which is $\mathcal{U}^{\hat{\mathcal{H}}}_{J_1J_0} = \mathcal{U}^{\mathcal{H}}_{J_1J_0}\otimes\mathcal{U}^{\sqrt{\mathcal{K}}}_{J_1J_0}$, corresponds to the pairing between $\hat{\mathcal{H}}_{J_0}$ and $\hat{\mathcal{H}}_{J_1}$ given by

$$\langle\psi_1\otimes\sqrt{\mu_1},\psi_0\otimes\sqrt{\mu_0}\,\rangle = \langle\psi_1,\psi_0\rangle\langle\sqrt{\mu_1},\sqrt{\mu_0}\,\rangle, \qquad \psi_l\in\mathcal{H}_{J_l},\ \mu_l\in\textstyle\bigwedge^n(V^{1,0}_{J_l})^*\ \ (l=0,1).$$

Proof: Part 1 is Theorem 3.4 of [15]. Part 2 follows from Theorem 3.3.2 and formula (3.9) of [15], except the parallel transport itself is expressed more intrinsically using Proposition A.1. Part 3 is Corollary 3.7 of [15]. □

We note that $\frac{J_0+J_1}{2}$ is always invertible for $J_0,J_1\in\mathcal{J}_\omega$. The factor $\big(\det\frac{J_0+J_1}{2}\big)^{1/4}$ appeared in [8, 28] and was used to rescale the projection $\mathcal{P}_{J_1J_0}$ to a unitary operator called the Bogoliubov transformation [28, 29]. Therefore Theorem 2.3.1 shows that the parallel transport $\mathcal{U}^{\mathcal{H}}_{J_1J_0}$ along the geodesic coincides with the Bogoliubov transformation. (The induced parallel transport on the creation and annihilation operators gives the more traditional version of Bogoliubov transformations.) The parallel transport can also be expressed using the Bergman kernel (Proposition 3.6 of [15]):

Corollary 2.4 ([15]) *Let $\psi = \phi\, e^{-\frac14 q_{J_0}} \in \mathcal{H}_{J_0}$. Then for $x \in V$,*

$$(\mathcal{U}^{\mathcal{H}}_{J_1 J_0}\psi)(x) = \big(\det \tfrac{J_0+J_1}{2}\big)^{1/4}\, e^{-\frac14 q_{J_1}(x)} \int_V \exp[\sqrt{-1}\,\omega(x^{1,0}_{J_1}, y) - \tfrac14\, q_{J_1}(y) - \tfrac14\, q_{J_0}(y)]\, \phi(y)\, \tilde\varepsilon_\omega(y).$$

Of particular interest is the parallel transport of a coherent state

$$c^\alpha_J(x) = \exp[q_J(\bar\alpha, x) - \tfrac14\, q_J(x)] = \exp[\sqrt{-1}\,\omega(\bar\alpha, x^{1,0}_J) - \tfrac14\, q_J(x)], \quad x \in V,$$

where $J \in \mathcal{J}_\omega$ and $\alpha \in V^{1,0}_J$ is a parameter. We recall some results from [15], but in a coordinate-free way.

Theorem 2.5 ([15]) *Under the assumptions of Theorem 2.3,*
1. the parallel transport along γ of the coherent state $c^\alpha_{J_0} \in \mathcal{H}_{J_0}$ is, for $x \in V$,

$$(\mathcal{U}^{\mathcal{H}}_{J_1 J_0} c^\alpha_{J_0})(x) = \big(\det \tfrac{J_0+J_1}{2}\big)^{-1/4}\, e^{-\frac14 q_{J_1}(x)} \exp\big[\tfrac12\, \omega\big(x^{1,0}_{J_1} - \bar\alpha,\, \big(\tfrac{J_0+J_1}{2}\big)^{-1}(x^{1,0}_{J_1} - \bar\alpha)\big)\big];$$

2. the parallel transport along γ of any state $\psi = e^{-\frac14 q_{J_0}}\phi \in \mathcal{H}_{J_0}$ is, for $x \in V$,

$$(\mathcal{U}^{\mathcal{H}}_{J_1 J_0}\psi)(x) = \big(\det \tfrac{J_0+J_1}{2}\big)^{-1/4}\, e^{-\frac14 q_{J_1}(x)} \int_V \exp\big[\tfrac12\, \omega\big(x^{1,0}_{J_1} - y^{0,1}_{J_0},\, \big(\tfrac{J_0+J_1}{2}\big)^{-1}(x^{1,0}_{J_1} - y^{0,1}_{J_0})\big) - \tfrac12\, q_{J_0}(y)\big]\, \phi(y)\, \tilde\varepsilon_\omega(y).$$

Proof: Part 1 is Theorems 3.3.1 of [15], where it was proved by solving the equation of parallel transport. As remarked in [15] (after Corollary 3.7), the result also follows from the Bergman kernel by Theorem 2.3.1 or Corollary 2.4. Since the latter approach will be adapted in the proof of Theorem 3.5.1 for fermions, we include the details here for comparison. Indeed, for any $x \in V$,

$$\begin{aligned}(\mathcal{P}_{J_1 J_0} c^\alpha_{J_0})(x) &= e^{-\frac14 q_{J_1}(x)} \int_V \exp\big[\sqrt{-1}\,\omega(y, x^{1,0}_{J_1}) - \tfrac12\, \omega\big(y, \tfrac{J_0+J_1}{2}\, y\big)\big]\, e^{\sqrt{-1}\,\omega(\bar\alpha, y)}\, \tilde\varepsilon_\omega(y)\\ &= e^{-\frac14 q_{J_1}(x)} \exp\big[\tfrac12\, \omega\big(x^{1,0}_{J_1} - \bar\alpha,\, \big(\tfrac{J_0+J_1}{2}\big)^{-1}(x^{1,0}_{J_1} - \bar\alpha)\big)\big] \int_V e^{-\frac12\,\omega\left(y', \frac{J_0+J_1}{2}\, y'\right)}\, \tilde\epsilon(y')\\ &= \big(\det \tfrac{J_0+J_1}{2}\big)^{-1/2}\, e^{-\frac14 q_{J_1}(x)} \exp\big[\tfrac12\, \omega\big(x^{1,0}_{J_1} - \bar\alpha,\, \big(\tfrac{J_0+J_1}{2}\big)^{-1}(x^{1,0}_{J_1} - \bar\alpha)\big)\big],\end{aligned}$$

where the change of variable is $y' = y - \sqrt{-1}\,(\frac{J_0+J_1}{2})^{-1}(x^{1,0}_{J_1} - \bar\alpha)$. Here we used the Gaussian integral

$$\int_V e^{-\frac12 \omega(x, Ax)}\, \tilde\varepsilon_\omega(x) = (\det A)^{-1/2}$$

for any $A \in \mathrm{End}(V)$ such that $\omega(\cdot, A\cdot)$ is a symmetric, positive-definite bilinear form. (This implies $\det A > 0$.) Part 2 is Theorems 3.8 of [15]. □

In particular, when $J_1 = J_0$, the above reduces to the identity transformation. When $n = 1$, we use the parametrisation in §A.2. We note that $\alpha \in V^{1,0}_{J_0}$ and $x \in V \cong V^{1,0}_{J_1}$ can be identified with complex numbers. If the geodesic $\{J_t\}_{0 \le t \le 1}$ from J_0 to J_1 is given by $z(t) = \tanh bt$, then Theorem 2.5 gives

$$(\mathcal{U}^{\mathcal{H}}_{J_1 J_0} c^\alpha_{J_0})(x) = \sqrt{\operatorname{sech} b}\, \exp\big[\,\bar\alpha x \tanh b + \tfrac12\, (\bar\alpha^2 - x^2) \operatorname{sech} b - \tfrac14\, |x|^2\,\big].$$

This is an important case ($n = 1$) of Proposition 3.2 in [15].

2.4 Bosonic systems with symmetries

Let (V, ω) be a symplectic vector space of dimension $2n$. The action of $\mathrm{Sp}(V, \omega)$ on $\mathcal{J}_\omega$ can be lifted to $\mathcal{H}$, preserving the projectively flat connection. It can be lifted to an action of the metaplectic group $\mathrm{Mp}(V, \omega)$, which is a double cover of $\mathrm{Sp}(V, \omega)$, on $\sqrt{\mathcal{K}}$ and hence on $\hat{\mathcal{H}}$. The lifted action preserves the flat connection.

Let K be a compact Lie group with Lie algebra $\mathfrak{k}$. Suppose there is a representation of K on V preserving ω. Then it also acts on the bundles $\mathcal{H}$ and $\hat{\mathcal{H}}$ preserving the connections. Over the fixed-point set $(\mathcal{J}_\omega)^K$, the group K acts on the fibres of $\mathcal{H}$. Each fibre splits orthogonally into a direct sum of subspaces of various representation types. Since K preserves the connection, the restriction of the bundle $\mathcal{H}$ to $(\mathcal{J}_\omega)^K$, together with the projectively flat connection, splits into sub-bundles with fibre-wise K-actions. The sub-bundle $\mathcal{H}^K$ on which K acts trivially is related to the quantisation of the symplectic quotient.

The action of K on V is Hamiltonian with a moment map $\mu_K\colon V \to \mathfrak{k}^*$ given by

$$\langle \mu_K(x), A\rangle = \tfrac{1}{2}\,\omega(x, Ax), \quad x \in V,\ A \in \mathfrak{k}.$$

The symplectic quotient $V/\!\!/K = \mu_K^{-1}(0)/K$ is a stratified symplectic space [23]. Choosing $J \in (\mathcal{J}_\omega)^K$, the action of K extends to that of $K^{\mathbb{C}}$. Let $\pi\colon V \to V/K^{\mathbb{C}}$ be the quotient map. With the above moment map, every point in V is semi-stable, i.e., $V^{\rm ss} = V$ (see Example 2.3 of [22]). The quotient $V/K^{\mathbb{C}} = V/\!\!/K$ is also a stratified analytic space; a function f on an open set $U \subset V/K^{\mathbb{C}}$ is analytic if π^*f is so on $\pi^{-1}(U)$. On the singular space $V/\!\!/K$, this analytic structure replaces the notion of polarisation. The sheaf of invariant sections $\pi^K_*\ell$ on $V/K^{\mathbb{C}}$ defined by $\mathcal{O}(\pi^K_*\ell)(U) = \Gamma(\pi^{-1}(U), \mathcal{O}(\ell))^K$ plays the role of a pre-quantum line bundle. We have (cf. Proposition 2.14 and Theorem 2.18 of [22])

$$\Gamma(V/\!\!/K, \mathcal{O}(\pi^K_*\ell)) \cong \Gamma(V, \mathcal{O}(\ell))^K.$$

Restricting to the L^2-subspaces, we can identify $(\mathcal{H}_J)^K$ with the quantum Hilbert space arising from the quantisation of $V/\!\!/K$ with a complex structure induced from J.

We have a projectively flat bundle $\mathcal{H}^K \to (\mathcal{J}_\omega)^K$ whose fibres are quantum Hilbert spaces of $V/\!\!/K$ with complex structures from $(\mathcal{J}_\omega)^K$. The connection is unitary if the inner product in the fibres $(\mathcal{H}_J)^K$ is the restriction of that in $\mathcal{H}_J$. This is the case, for example, in the quantisation of Chern-Simons gauge theory [1]. The inner product on $(\mathcal{H}_J)^K$ from quantisation of $V/\!\!/K$ is usually different. In [12], it was shown that for a compact symplectic manifold with a Hamiltonian group action and when metaplectic correction is included, the two inner products agree in the semi-classical limit.

Unless the moment map μ_K is proper, the symplectic quotient $V/\!\!/K$ is non-compact and the quantum Hilbert space $(\mathcal{H}_J)^K$ is expected to be infinite dimensional. When μ_K is proper however, the base space of the bundle $\mathcal{H}^K \to (\mathcal{J}_\omega)^K$ is a point.

Proposition 2.6 *If $\mu_K\colon V \to \mathfrak{k}^*$ is proper, then $(\mathcal{J}_\omega)^K = \{J_0\}$.*

Proof: If $(\mathcal{J}_\omega)^K \neq \{J_0\}$, then by Proposition C.4.1, there is a non-zero K-invariant complex subspace (V', J_0) of (V, J_0) and a K-invariant real structure R on V' such that $\omega(Rx, Ry) = -\omega(x, y)$ for all $x, y \in V'$. For any $A \in \mathfrak{k}$, $x \in V'$, we have

$$\langle \mu_K(Rx), A\rangle = \tfrac{1}{2}\,\omega(Rx, ARx) = \tfrac{1}{2}\,\omega(Rx, RAx) = -\tfrac{1}{2}\,\omega(x, Ax) = -\langle \mu_K(x), A\rangle.$$

Therefore $\mu_K = 0$ on $V'_0 = (V')^R$ and hence μ_K is not proper. □

3 Quantisation of fermionic systems

3.1 Pre-quantisation and quantisation

We consider pre-quantisation [17] and quantisation [28] of linear fermionic systems. The phase space is given by a finite-dimensional real vector space V equipped with a Euclidean inner product g. More precisely, it is a fermionic copy ΠV of V (see §B.1). The pre-quantum line bundle does not exists in the usual sense, but its "sections" and the operators acting on them do. Motivated by the bosonic case (§2.1), we take the pre-quantum Hilbert space $\mathcal{H}_0$ of fermions as $\bigwedge^\bullet (V^{\mathbb{C}})^*$, the space of "functions" on ΠV. On $\mathcal{H}_0$, there is an Hermitian form given by the Berezin integral (see §B.1 for definition and notations)

$$\langle \psi, \psi'\rangle_0 = \int_{\Pi V} \bar{\psi} \wedge \star_0\, \psi'\ \epsilon_g, \quad \psi, \psi' \in \mathcal{H}_0,$$

where $\bar{\psi}$ is the standard complex conjugation of ψ, $\star_0$ is the Hodge star defined by the metric $\frac{1}{2}\,g$. The covariant derivatives take the form

$$\nabla_x = \iota_x - \tfrac{1}{2}\,\nu(x) \wedge \cdot\,, \quad x \in V$$

and satisfy the relation

$$\{\nabla_x, \nabla_y\} = -g(x, y), \quad x, y \in V.$$

So the "curvature" is a symmetric bilinear form; the minus sign is enforced by the requirement, as in the bosonic case, that the covariant derivatives are skew-self-adjoint operators on $\mathcal{H}_0$.

A linear functional $\alpha \in V^*$ is a "classical observable" that can be pre-quantised, giving rise to a self-adjoint operator

$$\hat{\alpha} = \nabla_{\nu^{-1}(\alpha)} + \alpha \wedge \cdot\ = \iota_{\nu^{-1}(\alpha)} + \tfrac{1}{2}\,\alpha \wedge \cdot$$

on $\mathcal{H}_0$. These operators satisfy the canonical anti-commutation relation or the Clifford algebra relation

$$\{\hat{\alpha}, \hat{\beta}\} = g^{-1}(\alpha, \beta), \quad \alpha, \beta \in V^*,$$

making $\mathcal{H}_0$ a (reducible) Clifford module.

We now assume that V is even dimensional; let $\dim V = 2n$. Recall from §A.2 the space $\mathcal{J}_g$ of complex structures on V compatible with the metric g and the orientation. Each $J \in \mathcal{J}_g$ defines a polarisation, a maximally isotropic complex subspace $V_J^{1,0}$ of $V^{\mathbb{C}}$. The quantum Hilbert space (with the choice of polarisation J) is

$$\mathcal{H}_J = \{\psi \in \mathcal{H}_0 \,|\, \nabla_x \psi = 0,\ \forall x \in V_J^{0,1}\}.$$

We have a bundle of quantum Hilbert spaces $\mathcal{H} \to \mathcal{J}_g$ whose fibre over $J \in \mathcal{J}_g$ is $\mathcal{H}_J$.

On $\mathcal{H}_0$, there is an involution $\psi \mapsto \psi^*$ defined as the unique linear extension of the operation $(\alpha_1 \wedge \cdots \wedge \alpha_k)^* = \bar{\alpha}_k \wedge \cdots \wedge \bar{\alpha}_1$, where $\alpha_1, \ldots, \alpha_k \in (V^{\mathbb{C}})^*$.

Proposition 3.1 *1. Any $\psi \in \mathcal{H}_J$ is of the form*

$$\psi = e^{\frac{\sqrt{-1}}{2}\varpi_J} \wedge \phi$$

for a unique $\phi \in \bigwedge^\bullet (V_J^{1,0})^*$, where $\varpi_J = g(J\cdot, \cdot) \in \bigwedge^2 V^*$. Consequently, $\dim_{\mathbb{C}} \mathcal{H}_J = 2^n$.
2. Suppose $\psi, \psi' \in \mathcal{H}_J$ correspond to $\phi, \phi' \in \bigwedge^\bullet (V_J^{1,0})^*$, respectively, then

$$\langle \psi, \psi' \rangle_0 = \int_{\Pi V} \bar{\phi} \wedge \star \phi'\ \epsilon_g = \int_{\Pi V} \phi^* \wedge \phi' \wedge e^{\sqrt{-1}\,\varpi_J}\, \tilde{\epsilon}_g,$$

where $\star$ is the Hodge star defined by g and $\tilde{\epsilon}_g = \sqrt{-1}^{\,n}\, \epsilon_g$.
3. For any $\alpha \in V^*$, $\hat{\alpha}$ preserves $\mathcal{H}_J$ and remains self-adjoint on $\mathcal{H}_J$. It acts on $\phi \in \bigwedge^\bullet (V_J^{1,0})^*$ by

$$\hat{\alpha} \colon \phi \mapsto \iota_{\nu^{-1}(\alpha^{0,1})} \phi + \alpha^{1,0} \wedge \phi.$$

Proof: 1. Write $\psi = e^{\frac{\sqrt{-1}}{2}\varpi_J} \wedge \phi$ for some (unique) $\phi \in \mathcal{H}_0$. Then for any $x \in V$, we have

$$\nabla_x \psi = e^{\frac{\sqrt{-1}}{2}\varpi_J} \wedge (\iota_x \phi - \nu(x^{1,0}) \wedge \phi).$$

Therefore $\psi \in \mathcal{H}_J$ if and only if $\iota_x \phi = 0$ for all $x \in V_J^{0,1}$. This implies $\phi \in \bigwedge^\bullet (V_J^{1,0})^*$.
2. We choose a basis $\{e_i\}$ of $V_J^{1,0}$ such that $g(e_i, \bar{e}_j) = \delta_{ij}$ and $\epsilon_g = e_1 \wedge \cdots \wedge e_n$. Assume, without loss of generality, that $\phi = \phi' = e_1^* \wedge \cdots \wedge e_k^*$ $(1 \le k \le n)$. Then

$$\psi = \phi \wedge \sum_{r=0}^{n-k} 2^{-r} \sum_{k+1 \le i_1 < \cdots < i_r \le n} \bar{e}_{i_1}^* \wedge e_{i_1}^* \wedge \cdots \wedge \bar{e}_{i_r}^* \wedge e_{i_r}^*.$$

Since $\star_0 = 2^{p-n} \star$ on $\bigwedge^p V^*$, we have

$$\star_0\, \psi = \frac{(-1)^{\frac{k(k-1)}{2}}}{\sqrt{-1}^{\,n}}\, \phi \wedge \sum_{r=0}^{n-k} (-1)^r\, 2^{r+k-n} \sum_{k+1 \le j_1 < \cdots < j_{n-r} \le n} \bar{e}_{j_1}^* \wedge e_{j_1}^* \wedge \cdots \wedge \bar{e}_{j_{n-k-r}}^* \wedge e_{j_{n-k-r}}^*.$$

So

$$\langle \psi, \psi' \rangle_0 = \int_{\Pi V} \bar{\psi} \wedge \star_0\, \psi\ \epsilon_g = \sum_{r=0}^{n-k} 2^{k-n} \binom{n-k}{r} = 1.$$

The two integrals in the equality are clearly 1 (cf. proof of Proposition B.2).
3. This follows from the identity

$$\hat{\alpha}(e^{\frac{\sqrt{-1}}{2}\varpi_J} \wedge \phi) = e^{\frac{\sqrt{-1}}{2}\varpi_J} \wedge (\iota_{\nu^{-1}}(\alpha)\phi + \alpha^{1,0} \wedge \phi),$$

which yields the result when $\phi \in \bigwedge^\bullet (V_J^{1,0})^*$. $\square$

We note that the space $\bigwedge^\bullet (V_J^{1,0})^*$ with the action of $\alpha \in V^*$ in Proposition 3.1.3 is the standard construction of the irreducible Clifford module of spinors. Here it arises naturally in the quantisation of fermionic systems. The factor $e^{\frac{\sqrt{-1}}{2}\varpi_J}$ is the fermionic analogue of the Gaussian in Proposition 2.1.1. It is crucial in achieving projective flatness for the bundle $\mathcal{H} \to \mathcal{J}_g$ (§3.2), as the bundle $\bigwedge^\bullet \mathcal{V}^* \to \mathcal{J}_g$ without the fermionic Gaussian factor is not projectively flat.

The results in this section can be explained using "fermionic coordinates". We refer the reader to §B.2 where this is done.

3.2 Projectively flat connection and metaplectic correction

We study the geometry of the bundle $\mathcal{H} \to \mathcal{J}_g$ of Hilbert spaces of the fermionic system (V, g). Following §2.2, we define a connection on $\mathcal{H}$ by orthogonal projection of the trivial connection on the product bundle $\mathcal{J}_g \times \mathcal{H}_0 \to \mathcal{J}_g$. We now show that this connection is also projectively flat.

Along a variation δJ of $J \in \mathcal{J}_g$, $\psi \in \mathcal{H}_J$ changes to $\psi + \delta\psi \in \mathcal{H}_{J+\delta J}$ by parallel transport. Since $\psi + \delta\psi$ is the (infinitesimal) orthogonal projection of ψ to $\mathcal{H}_{J+\delta J}$, we have, as in §2.2, $\delta\psi \perp \mathcal{H}_J$ and

$$\nabla_{\bar{i}}(\delta\psi) = -(\delta\bar{P})_{\bar{i}}^{\;j}\nabla_j\psi = (\delta P)_{\bar{i}}^{\;j}\nabla_j\psi.$$

The (unique) solution that satisfies the above two conditions is

$$\delta\psi = -\tfrac{1}{2}\,\nabla_i(\delta P)^{ij}\nabla_j\psi = -\tfrac{1}{2}\,(\delta P)^{ij}\nabla_i\nabla_j\psi.$$

First, this $\delta\psi$ is orthogonal to $\mathcal{H}_J$ as $\nabla_{\bar{i}}$ is the formal adjoint of ∇_i. Second, as $\nabla_{\bar{i}}\psi = 0$, $(\delta P)^{ij} = -(\delta P)^{ji}$ and $\{\nabla_{\bar{i}}, \nabla_j\} = -g_{\bar{i}j}$, we get

$$\begin{aligned}\nabla_{\bar{k}}(\delta\psi) &= -\tfrac{1}{2}\,(\delta P)^{ij}[\nabla_{\bar{k}}, \nabla_i\nabla_j]\psi = -\tfrac{1}{2}\,(\delta P)^{ij}(\{\nabla_{\bar{k}}, \nabla_i\}\nabla_j - \nabla_i\{\nabla_{\bar{k}}, \nabla_j\})\psi \\ &= g_{\bar{k}i}(\delta P)^{ij}\nabla_j\psi = (\delta P)_{\bar{k}}^{\;j}\nabla_j\psi.\end{aligned}$$

The connection $\mathsf{A}^{\mathcal{H}}$ on $\mathcal{H}$ is determined by $(\delta + \mathsf{A}^{\mathcal{H}})\psi = 0$ and thus $\mathsf{A}^{\mathcal{H}} = \frac{1}{2}\,(\delta P)^{ij}\nabla_i\nabla_j$. Following the calculations in §2.2, we get

$$\begin{aligned}\delta\,\mathsf{A}^{\mathcal{H}} &= -\tfrac{1}{2}\,(\delta P)^{ij} \wedge \nabla_{\delta e_i}\nabla_j = -\tfrac{1}{2}\,(\delta P)^{ij} \wedge (\delta P)_i^{\;\bar{k}}\nabla_{\bar{k}}\nabla_j \\ &= -\tfrac{1}{2}\,g_{\bar{k}j}(\delta P)^{ji} \wedge (\delta P)_i^{\;\bar{k}} = -\tfrac{1}{2}\,\mathrm{tr}(P\,\delta P \wedge \delta P\,P)\end{aligned}$$

and

$$\mathsf{A}^{\mathcal{H}} \wedge \mathsf{A}^{\mathcal{H}} = \tfrac{1}{4}\,\nabla_i\nabla_j\nabla_k\nabla_l\,(\delta P)^{ij} \wedge (\delta P)^{kl} = 0.$$

Therefore the curvature of the connection $\mathsf{A}^{\mathcal{H}}$ is

$$\mathsf{F}^{\mathcal{H}} = -\tfrac{1}{2}\,\mathrm{tr}(P\,\delta P \wedge \delta P\,P)\,\mathrm{id}_{\mathcal{H}} = \sigma_g/2\sqrt{-1}\,\mathrm{id}_{\mathcal{H}}.$$

Since it is a 2-form on $\mathcal{J}_g$ times the identity operator on the fibre, the connection is projectively flat.

We propose a metaplectic correction for fermions. Recall the line bundle $\mathcal{K}^{-1} = \det\mathcal{V} \to \mathcal{J}_g$ whose fibre over J is $\mathcal{K}_J^{-1} = \bigwedge^n V_J^{1,0}$. We claim that $c_1(\mathcal{K})$ is even. This can be seen from the holonomy of the bundle $\mathcal{H}$ with curvature $\mathsf{F}^{\mathcal{H}} = -\frac{1}{2}\mathsf{F}^{\mathcal{K}}\,\mathrm{id}_{\mathcal{H}}$. Since $\mathcal{J}_g$ is simply connected, there is a unique line bundle $\sqrt{\mathcal{K}^{-1}} \to \mathcal{J}_g$ such that $(\sqrt{\mathcal{K}^{-1}})^{\otimes 2} = \mathcal{K}^{-1}$. The bundle $\sqrt{\mathcal{K}^{-1}}$ has a connection (§A.1) whose curvature is

$$\mathsf{F}^{\sqrt{\mathcal{K}^{-1}}} = -\tfrac{1}{2}\,\mathsf{F}^{\mathcal{K}} = \tfrac{1}{2}\,\mathrm{tr}(P\,\delta P \wedge \delta P\,P) = -\sigma_g/2\sqrt{-1}\,.$$

For any $J, J' \in \mathcal{J}_g$, there is a pairing between $\mathcal{K}_J^{-1} = \bigwedge^n V_J^{1,0}$ and $\mathcal{K}_{J'}^{-1} = \bigwedge^n V_{J'}^{1,0}$. For any $\mu \in \mathcal{K}_J^{-1}$ and $\mu' \in \mathcal{K}_{J'}^{-1}$, $\langle\mu', \mu\rangle$ is the ratio of $\bar{\mu}' \wedge \mu$ and $\tilde{\epsilon}_g$. Since $\langle\mu, \mu\rangle > 0$ if $\mu \neq 0$, there is an inner product on $\sqrt{\mathcal{K}_J}^{-1}$ defined by $\langle\sqrt{\mu}, \sqrt{\mu}\rangle = \sqrt{\langle\mu, \mu\rangle}$.

We consider the bundle $\hat{\mathcal{H}} = \mathcal{H} \otimes \sqrt{\mathcal{K}^{-1}}$. The fibre $\hat{\mathcal{H}}_J = \mathcal{H}_J \otimes \sqrt{\mathcal{K}_J^{-1}}$ over $J \in \mathcal{J}_g$ is called the metaplectically corrected quantum Hilbert space in polarisation J. Since the curvatures of $\mathcal{H}$ and $\sqrt{\mathcal{K}^{-1}}$ cancel, the bundle $\hat{\mathcal{H}} \to \mathcal{J}$ is canonically flat. The flatness of the bundle indicates that for the fermionic system whose phase space is a linear space, quantisation is independent of the choice of polarisations. We note here that in contrast to the bosonic case, the metaplectic correction is obtained by tensoring $\mathcal{H}_J$ with $\sqrt{\mathcal{K}_J^{-1}}$ instead of $\sqrt{\mathcal{K}_J}$. This is clearly related to the opposite way a fermionic measure transforms under coordinate changes. We recall from §A.2 that the pseudo-Kähler form σ restricts to σ_ω on $\mathcal{J}_\omega$ but to $-\sigma_g$ on $\mathcal{J}_g$. Consequently, for both bosonic and fermionic systems, the line bundle of half-forms has a negative first Chern form on the space of polarisations.

We summarise the results in the following

Theorem 3.2 *Consider the quantisation of a fermionic system whose phase space is given by a finite dimensional Euclidean vector space (V, g).*
1. The bundle of quantum Hilbert spaces $\mathcal{H} \to \mathcal{J}_g$ is projectively flat, with curvature $\sigma_g/2\sqrt{-1}$.
2. The bundle of quantum Hilbert spaces with metaplectic correction $\hat{\mathcal{H}} \to \mathcal{J}_g$ is flat.

3.3 Parallel transport along geodesics and the Bogoliubov transformations

We investigate the parallel transport in the bundles $\mathcal{H}$ and $\hat{\mathcal{H}}$ along geodesics in $\mathcal{J}_g$. Unlike $\mathcal{J}_\omega$, which is contractible and non-positively curved, the space $\mathcal{J}_g$ is compact and non-negatively curved. The geodesics through two conjugate points in $\mathcal{J}_g$ are not unique. Nevertheless, we show that if $J_0, J_1 \in \mathcal{J}_g$ are not in the cut loci (see §A.3) of each other, then the parallel transport $\mathcal{U}^{\mathcal{H}}_{J_1 J_0}$ along the (unique) length-minimising geodesic from J_0 to J_1 is the rescaled orthogonal projection $\mathcal{P}_{J_1 J_0}$ from $\mathcal{H}_{J_0}$ to $\mathcal{H}_{J_1}$ in $\mathcal{H}_0$. The latter was called the Bogoliubov transformation of fermionic systems [28]. The inner product in $\sqrt{\mathcal{K}_{J_0}^{-1}}$ extends to a pairing between $\sqrt{\mathcal{K}_{J_0}^{-1}}$ and $\sqrt{\mathcal{K}_{J_2}^{-1}}$ along the geodesic, which is non-degenerate as long as J_1 is not on the cut locus of J_0 (Corollary A.5).

Theorem 3.3 *Let $J_0, J_1 \in \mathcal{J}_g$ and let $\gamma = \{J_t\}_{0 \le t \le 1}$ be a geodesic from J_0 to J_1, t being proportional to the arc-length parameter. Assume that the geodesic lies completely in the complement of the cut locus of J_0. Then*
1. the parallel transport in $\mathcal{H}$ along γ is $\mathcal{U}^{\mathcal{H}}_{J_1 J_0} = \big(\det \frac{J_0 + J_1}{2}\big)^{-1/4} \mathcal{P}_{J_1 J_0}$;
2. the parallel transport in $\sqrt{\mathcal{K}^{-1}}$ along γ is $\mathcal{U}^{\sqrt{\mathcal{K}^{-1}}}_{J_1 J_0} = \sqrt{\det J_{1/2}/\sqrt{-1}\,|_{V^{1,0}_{J_0}}}$, and

$$\langle \mathcal{U}^{\sqrt{\mathcal{K}^{-1}}}_{J_1 J_0} \sqrt{\mu_0'}, \sqrt{\mu_0}\,\rangle = \big(\det \tfrac{J_0 + J_1}{2}\big)^{1/4} \langle \sqrt{\mu_0'}, \sqrt{\mu_0}\,\rangle$$

for any $\mu_0, \mu_0' \in \bigwedge^n V^{1,0}_{J_0}$;
3. the parallel transport in $\hat{\mathcal{H}}$ along γ, which is $\mathcal{U}^{\hat{\mathcal{H}}}_{J_1 J_0} = \mathcal{U}^{\mathcal{H}}_{J_1 J_0} \otimes \mathcal{U}^{\sqrt{\mathcal{K}^{-1}}}_{J_1 J_0}$, corresponds to the pairing between $\hat{\mathcal{H}}_{J_0}$ and $\hat{\mathcal{H}}_{J_1}$ given by

$$\langle \psi_1 \otimes \sqrt{\mu_1}, \psi_0 \otimes \sqrt{\mu_0} \rangle = \langle \psi_1, \psi_0 \rangle \langle \sqrt{\mu_1}, \sqrt{\mu_0} \rangle, \qquad \psi_l \in \mathcal{H}_{J_l},\ \mu_l \in \textstyle\bigwedge^n V^{1,0}_{J_l}\ (l = 0, 1).$$

Proof: 1. Choosing a unitary basis $\{e_i\}$ of $V^{1,0}_{J_0}$, the geodesics in $\mathcal{J}_g \cong \mathrm{SO}(2n)/\mathrm{U}(n)$ are given by Proposition A.3.2. We can assume without loss of generality (cf. the proof of Theorem 3.4 in [15]) that $n = 2$ and $k = 1$, $b = b_1 > 0$; the case $n = 1$ is trivial. Then as in the proof of Proposition A.4, the vectors $e^{(t)}_1 = \cos bt\ e_1 - \sin bt\ \bar{e}_{\bar{2}}$, $e^{(t)}_2 = \cos bt\ e_2 + \sin bt\ \bar{e}_{\bar{1}}$ form a unitary basis of $V^{1,0}_{J_t}$. Since $(\delta P)\, e^{(t)}_1 = -b\, e^{(t)}_2$ and $(\delta P)\, e^{(t)}_2 = b\, e^{(t)}_1$, we have $(\delta P)_1^{\bar{2}} = -b$ and hence $(\delta P)^{12} = b$; here the tensor indices correspond to the basis $\{e^{(t)}_1, e^{(t)}_2\}$. We want to find $\alpha(t)$ such that the quantity $\alpha(t)\langle \psi', \psi_t \rangle$ is independent of t for any $\psi' \in \mathcal{H}_{J_0}$ if $\psi_t \in \mathcal{H}_{J_t}$ is a parallel transport of $\psi_0 \in \mathcal{H}_{J_0}$. This would imply $\mathcal{U}^{\mathcal{H}}_{J_t J_0} = \alpha(t)^{-1} \mathcal{P}_{J_t J_0}$. Since ψ_t satisfies the differential equation

$$\dot{\psi}_t = -\tfrac{1}{2} (\delta P)^{ij} \nabla_i \nabla_j \psi_t = -b \nabla_1 \nabla_2 \psi_t,$$

we have

$$\begin{aligned} \frac{d}{dt}\big(\alpha(t)\langle \psi', \psi_t \rangle\big) &= \dot{\alpha}(t)\langle \psi', \psi_t \rangle - b\, \alpha(t) \langle \psi', \nabla_1 \nabla_2 \psi_t \rangle \\ &= \dot{\alpha}(t)\langle \psi', \psi_t \rangle - b\, \alpha(t) \langle \psi', (\sec bt\, \nabla^{(0)}_1 - \tan bt\, \nabla_{\bar{2}}) \nabla_2 \psi_t \rangle \\ &= \dot{\alpha}(t)\langle \psi', \psi_t \rangle + b\, \alpha(t) \tan bt\, \langle \psi', \{\nabla_{\bar{2}}, \nabla_2\} \psi_t \rangle \\ &= (\dot{\alpha}(t) - b\, \alpha(t) \tan bt)\, \langle \psi', \psi_t \rangle. \end{aligned}$$

Solving $\dot\alpha - b\alpha\tan bt = 0$ with $\alpha(0)=1$, we get $\alpha(t) = (\cos bt)^{-1}$. By Proposition A.4, the assumption on the geodesic means that $\frac{J_0+J_t}{2}$ is invertible for all $t\in[0,1]$. Since $\det\frac{J_0+J_t}{2} = \cos^4 bt$, this means $|b| < \frac{\pi}{2}$ and the result follows.
2. The formula for $\mathcal{U}^{\sqrt{\mathcal{K}^{-1}}}_{J_1J_0}$ follows from Lemma A.1. It suffices to show the rest when $n=2$ using the above parametrisation. If we take $\mu_0 = e_1^{(0)}\wedge e_2^{(0)}$, then $\langle\mu_0,\mu_0\rangle = 1$ and

$$\mathcal{U}^{\mathcal{K}^{-1}}_{J_1J_0}\,\mu_0 = e_1^{(t)}\wedge e_2^{(t)} = \cos^2 bt\,\mu_0 + \cdots,$$

where the omitted part has a factor from $V^{0,1}_{J_0}$. The result then follows from

$$\langle\,\mathcal{U}^{\mathcal{K}^{-1}}_{J_1J_0}\,\mu_0,\mu_0\,\rangle = \cos^2 bt = \big(\det\tfrac{J_0+J_t}{2}\big)^{1/2}.$$

3. This is an immediately consequence of parts 1 and 2. □

Notice that although power of the factor $\big(\det\frac{J_0+J_1}{2}\big)^{-1/4}$ in Theorem 3.3.1 is opposite to that in Theorem 2.3.1, both are greater than 1. Using the fermionic analog of the Bergman kernel projection in Proposition B.2, we have

Corollary 3.4 *Let $\psi = e^{\frac{\sqrt{-1}}{2}\varpi_{J_0}}\wedge\phi\in\mathcal{H}_{J_0}$. Then for fermionic but real $\theta\in\Pi V$,*

$$(\mathcal{U}^{\mathcal{H}}_{J_1J_0}\psi)(\theta) = \big(\det\tfrac{J_0+J_1}{2}\big)^{-1/4}\,e^{\frac{\sqrt{-1}}{4}\varpi_{J_1}(\theta)}\int_{\Pi V}\exp\big[\,g(\theta^{1,0}_{J_1},\chi) + \tfrac{\sqrt{-1}}{4}\,\varpi_{J_1}(\chi) + \tfrac{\sqrt{-1}}{4}\,\varpi_{J_0}(\chi)\,\big]\,\phi(\chi)\,\tilde\epsilon_g(\chi).$$

We recall from §B.2 the notion of fermionic coherent states.

Theorem 3.5 *Under the assumptions of Theorem 3.3,*
1. the parallel transport along γ of the coherent state $c^\alpha_{J_0}$ is, for $\theta\in\Pi V$,

$$(\mathcal{U}^{\mathcal{H}}_{J_1J_0}c^\alpha_{J_0})(\theta) = \big(\det\tfrac{J_0+J_1}{2}\big)^{1/4}\,e^{\frac{\sqrt{-1}}{4}\varpi_{J_1}(\theta)}\,\exp\big[\tfrac{\sqrt{-1}}{2}\,g\big(\theta^{1,0}_{J_1}-\bar\alpha,\,\big(\tfrac{J_0+J_1}{2}\big)^{-1}(\theta^{1,0}_{J_1}-\bar\alpha)\big)\big];$$

2. the parallel transport along γ of any state $\psi = e^{\frac14\varpi_{J_0}}\wedge\phi\in\mathcal{H}_{J_0}$ is, for $\theta\in\Pi V$,

$$(\mathcal{U}^{\mathcal{H}}_{J_1J_0}\psi)(\theta) = \big(\det\tfrac{J_0+J_1}{2}\big)^{1/4}e^{\frac14\varpi_{J_1}(\theta)}\int_{\Pi V}\exp\big[\tfrac{\sqrt{-1}}{2}\,g\big(\theta^{1,0}_{J_1}-\chi^{0,1}_{J_0},\big(\tfrac{J_0+J_1}{2}\big)^{-1}(\theta^{1,0}_{J_1}-\chi^{0,1}_{J_0})\big)+\tfrac{\sqrt{-1}}{4}\,\varpi_{J_0}(\chi)\big]\phi(\chi)\,\tilde\epsilon_g(\chi).$$

Proof: 1. We follow the proof of Theorem 2.5.1. By Lemma B.1 and Proposition B.2, we get, for fermionic but real $\theta\in\Pi V$,

$$\begin{aligned}(\mathcal{P}_{J_1J_0}c^\alpha_{J_0})(\theta) &= e^{\frac{\sqrt{-1}}{4}\varpi_{J_1}(\theta)}\int_{\Pi V}\exp\big[\,g(\theta^{1,0}_{J_1},\chi) + \tfrac{\sqrt{-1}}{2}\,g\big(\tfrac{J_0+J_1}{2}\,\chi,\chi\big)\big]\,e^{g(\chi,\bar\alpha)}\,\tilde\epsilon_g(\chi)\\ &= e^{\frac{\sqrt{-1}}{4}\varpi_{J_1}(\theta)}\exp\big[\tfrac{\sqrt{-1}}{2}\,g\big(\theta^{1,0}_{J_1}-\bar\alpha,\,\big(\tfrac{J_0+J_1}{2}\big)^{-1}(\theta^{1,0}_{J_1}-\bar\alpha)\big)\big]\int_{\Pi V}e^{\frac{\sqrt{-1}}{2}g\left(\frac{J_0+J_1}{2}\chi',\chi'\right)}\,\tilde\epsilon(\chi')\\ &= \big(\det\tfrac{J_0+J_1}{2}\big)^{1/2}\,e^{\frac{\sqrt{-1}}{4}\varpi_{J_1}(\theta)}\,\exp\big[\tfrac{\sqrt{-1}}{2}\,g\big(\theta^{1,0}_{J_1}-\bar\alpha,\,\big(\tfrac{J_0+J_1}{2}\big)^{-1}(\theta^{1,0}_{J_1}-\bar\alpha)\big)\big],\end{aligned}$$

where we made a change of variable $\chi' = \chi - \sqrt{-1}\,(\frac{J_0+J_1}{2})^{-1}(\theta^{1,0}_{J_1}-\bar\alpha)$ and used Lemma B.1. The condition that $\frac{J_0+J_t}{2}$ is invertible for all $0\le t\le 1$ implies that $\frac{J_0+J_1}{2}$ is in the same connected component of invertible skew-symmetric operators as J_0. The result then follows from Theorem 3.3.1.
2. By Proposition B.2, we have

$$\psi(\theta,\bar\theta) = \int_{\Pi V}c^\chi_J(\theta)\,e^{-g(\chi,\bar\chi)}\,\phi(\chi,\bar\chi)\,\tilde\epsilon(\chi).$$

The result then follows from part 1 and the linearity of fermionic integration. □

When $n=2$, if the geodesic $\{J_t\}_{0\le t\le 1}$ from J_0 to J_1 is given by $z(t) = \tan bt$, where $|b| < \frac{\pi}{2}$, then Theorem 3.5 gives

$$(\mathcal{U}_{J_1J_0}c^\alpha_{J_0})(\theta) = \cos b\,\exp\big[(\theta^1\bar\alpha^{\bar1}+\theta^2\bar\alpha^{\bar2})\sec b + \tfrac12\,(\bar\alpha^{\bar1}\bar\alpha^{\bar2}+\theta^1\theta^2)\tan b - \tfrac12\,(\theta^1\bar\theta^{\bar1}+\theta^2\bar\theta^{\bar2})\big],$$

which can also be obtained by solving the equation of parallel transport as in the bosonic case (cf. [15]).

3.4 Fermionic systems with symmetries

Let (V, g) be a Euclidean space of dimension $2n$. The action of $\mathrm{SO}(V, g)$ on $\mathcal{J}_g$ can be lifted to $\mathcal{H}$, preserving the connection. It can be lifted to an action of the spin group $\mathrm{Spin}(V, g)$, which is a double cover of $\mathrm{SO}(V, g)$, on $\sqrt{\mathcal{K}^{-1}}$ and hence on $\hat{\mathcal{H}}$. The lifted action preserves the flat connection.

Suppose K is a compact Lie group with Lie algebra $\mathfrak{k}$ acting on (V, g) by an orthogonal representation. Then it also acts on the bundles $\mathcal{H}$ and $\hat{\mathcal{H}}$ preserving the connections. Over the fixed-point set $(\mathcal{J}_g)^K$, the group K acts on the fibres of $\mathcal{H}$. Each fibre splits orthogonally into a direct sum of subspaces of various representation-types of K. Since K preserves the connection, the restriction of the bundle $\mathcal{H}$ to $(\mathcal{J}_g)^K$ together with the projectively flat connection splits into sub-bundles on which K acts fibre-wisely. In particular, we have a projectively flat sub-bundle $\mathcal{H}^K \to (\mathcal{J}_g)^K$ on which K acts trivially on the fibres.

We now study the fermionic reduced phase space $\Pi V /\!\!/ K$ and its quantisation. The action of the Lie algebra $\mathfrak{k}$ yields Hamiltonian "vector fields" on ΠV [17]. The moment map μ_K is given by, for any $A \in \mathfrak{k}$, $\langle \mu_K, A \rangle = \frac{1}{2} g(A\cdot, \cdot) \in \bigwedge^2 V^*$ or

$$\langle \mu_K(\theta), A \rangle = \tfrac{1}{2}\, g(A\theta, \theta), \quad \theta \in \Pi V,\ A \in \mathfrak{k}$$

using fermionic variables. Following the construction of the usual symplectic quotients, the fermionic analogue $\Pi V /\!\!/ K$ should be the "spec" of the non-commutative ring $(\bigwedge^\bullet (V^*)^{\mathbb{C}} / \langle \mu_K \rangle)^K$, where $\langle \mu_K \rangle$ is the ideal generated by $\langle \mu_K, A \rangle$ for all $A \in \mathfrak{k}$. The "space" $\Pi V /\!\!/ K$ is not usually a graded manifold in the sense of [17]; it would be so if 0 were a regular value of μ_K [2]. So fermionic symplectic quotients are interesting examples of supermanifolds with curved fermionic coordinates. Consider the example $V = \mathbb{R}^{2n}$ with $K = S^1$ acting by weights $\lambda_1, \ldots, \lambda_r \neq 0$, $\lambda_{r+1} = \cdots = \lambda_n = 0$. Then the "coordinate ring" of $\Pi\mathbb{R}^{2n} /\!\!/ S^1$ is generated by $1, \theta^1\theta^2, \ldots, \theta^{2r-1}\theta^{2r}, \theta^{2r+1}, \ldots, \theta^{2n}$ subject to a relation $\lambda_1\theta^1\theta^2 + \cdots + \lambda_r\theta^{2r-1}\theta^{2r} = 0$. Here $\theta^1, \ldots, \theta^{2n}$ are the fermionic coordinates on $\Pi\mathbb{R}^{2n}$. When $r = 1$, the above ring is the exterior algebra on $\theta^3, \ldots, \theta^{2n}$ and thus $\Pi\mathbb{R}^{2n} /\!\!/ S^1 \cong \Pi\mathbb{R}^{2n-2}$.

Proposition 3.6 *If* $\dim(\mathcal{J}_g)^K > 0$*, then there is a non-zero K-invariant complex subspace* (V', J_0) *of* (V, J_0) *with a K-invariant quaternionic structure such that the restriction of μ_K to* $\Pi V'$ *is invariant under the scalar multiplication by quaternions of unit norm.*

Proof: By Proposition C.4.2, there is a non-zero K-invariant complex subspace (V', J_0) of (V, J_0) and a K-invariant quaternionic structure Q on V' such that $g(Q\cdot, Q\cdot) = g(\cdot, \cdot)$ on V'. For any $A \in \mathfrak{k}$, we have

$$\langle \mu_K(Q\theta), A \rangle = \tfrac{1}{2}\, g(Q\theta, AQ\theta) = \tfrac{1}{2}\, g(Q\theta, QA\theta) = \tfrac{1}{2}\, g(\theta, A\theta) = \langle \mu_K(\theta), A \rangle,$$

where $\theta \in \Pi V'$. The result then follows easily from $g(J_0\cdot, J_0\cdot) = g(\cdot, \cdot)$ and $QJ_0 = -J_0Q$. □

4 Concluding remarks

We end with a comparison of the quantisation of bosons and fermions. As explained in §1, the existence of projectively flat connection is due largely to the fact that the irreducible representation of the operator algebra (Heisenberg algebra for bosons and Clifford algebra for fermions) is unique up to unitary equivalence. This enables us to identify, up to a phase, states in Hilbert spaces constructed from various linear polarisations. Moreover, the geometric structure of the bundle of Hilbert spaces and results on parallel transport are rather similar in the bosonic and fermionic cases.

We note however some conceptual differences. For bosons, the positivity condition is on the polarisation whereas for fermions, it is on the Euclidean structure g. Indeed, the unitarity of the representation of the Heisenberg algebra is not sensitive to the sign of ω, whereas for fermions, the positivity condition on g is the requirement for unitarity [25]. On the other hand, the positivity condition on polarisation for bosons guarantees the existence of holomorphic sections rather than elements in higher cohomology groups. For fermions, the cohomology is always at the zeroth degree; this reflects the Dirac sea picture in physics.

Mathematically, the spaces of allowed polarisations are Hermitian symmetric spaces in both cases. For bosons, the space non-compact, non-positively curved. Though it is contractible, the difficulty occurs at the boundary at infinity, where the limit of parallel transport should be carefully taken [15]. For fermions, the space of polarisation is compact, non-negatively curved. Though it has no boundary, interesting phenomena (non-uniqueness of geodesics, degeneracy of the half-form pairing) because of cut locus (§A.3 and §3.3). Furthermore, the half-form bundles in metaplectic correction are of opposite powers of the canonical bundle in the two cases in order to cancel the curvature of the projectively flat connection. We hope that these observations are useful to future research on the quantisation of more general symplectic or graded symplectic manifolds.

Appendix

A Geometry of the space of complex structures

A.1 Complex structures on a vector space

Let V be a real vector space of dimension $2n$. Consider the set $\mathcal{J}$ of complex structures on V compatible with a given orientation on V. For each $J \in \mathcal{J}$, there is a decomposition $V^{\mathbb{C}} = V_J^{1,0} \oplus V_J^{0,1}$, where $V_J^{1,0}$, $V_J^{0,1}$ are the holomorphic and anti-holomorphic subspaces, on which $J = \pm\sqrt{-1}$, respectively. Similarly, there is a decomposition $(V^*)^{\mathbb{C}} = (V_J^{1,0})^* \oplus (V_J^{0,1})^*$. For $x \in V$, $\alpha \in V^*$, we write, accordingly,

$$x = x_J^{1,0} + x_J^{0,1}, \quad \alpha = \alpha_J^{1,0} + \alpha_J^{0,1}.$$

The space $\mathcal{J}$ is a connected smooth manifold of real dimension $2n^2$. At any $J_0 \in \mathcal{J}$, the tangent space of $\mathcal{J}$ is $T_{J_0}\mathcal{J} \cong \mathrm{Hom}_{\mathbb{C}}(V_{J_0}^{1,0}, V_{J_0}^{0,1})$. Moreover, a dense open subset of $\mathcal{J}$ can be parametrised by $Z \in \mathrm{Hom}_{\mathbb{C}}(V_{J_0}^{1,0}, V_{J_0}^{0,1})$: for any such Z, the corresponding subspace $V_J^{1,0}$ is the graph of Z, that is, $V_J^{1,0}$ consists of the vectors of the form $\binom{x}{Zx}$ under the decomposition $V^{\mathbb{C}} = V_{J_0}^{1,0} \oplus V_{J_0}^{0,1}$, where $x \in V_{J_0}^{1,0}$. These open sets (for various J_0) have complex coordinates and form an open cover of $\mathcal{J}$. This defines a complex structure on $\mathcal{J}$. For a fixed $J_0 \in \mathcal{J}$, the complement of the open set consists of $J \in \mathcal{J}$ such that $V_J^{1,0} \cap V_{J_0}^{0,1} \neq \{0\}$, which happens when $J + J_0$ is not invertible. On the other hand, not every $Z \in \mathrm{Hom}_{\mathbb{C}}(V_{J_0}^{1,0}, V_{J_0}^{0,1})$ corresponds to a complex structure. If it does, then the condition $V_J^{1,0} \cap V_J^{0,1} = \{0\}$ implies that $1 - \bar{Z}Z \in \mathrm{End}(V_{J_0}^{1,0})$ is invertible. An element Z that violates this condition is on the "boundary" of $\mathcal{J}$. Finally, the projection onto $V_J^{1,0}$ along $V_J^{0,1}$ is $P_J = \frac{1}{2}(1 - \sqrt{-1}\,J)$. Given J_0, on the dense set where J can be parametrised by Z, the projection is

$$P_J = \begin{pmatrix} 1 \\ Z \end{pmatrix}(1 - \bar{Z}Z)^{-1}(1,\ -\bar{Z}).$$

Suppose δJ is an infinitesimal variation of $J \in \mathcal{J}$. (The symbol δ can be interpreted as the differential on $\mathcal{J}$.) Since the change $\delta P = -\frac{\sqrt{-1}}{2}\delta J$ of $P = P_J$ anti-commutes with J, it is off-diagonal with respect to the decomposition $V^{\mathbb{C}} = V_J^{1,0} \oplus V_J^{0,1}$. The new holomorphic subspace $V_{J+\delta J}^{1,0}$ is the graph of the $\mathrm{Hom}_{\mathbb{C}}(V_J^{1,0}, V_J^{0,1})$-component of δP. Since $P + \bar{P} = \mathrm{id}_V$, we have $\delta\bar{P} = -\delta P$.

We consider the vector bundle $\mathcal{V} \to \mathcal{J}$ whose fibre over $J \in \mathcal{J}$ is $V_J^{1,0}$. This is a sub-bundle of the product bundle $\mathcal{J} \times V^{\mathbb{C}}$ and has a connection defined by the projection P_J. This connection on $\mathcal{V}$ is $\mathsf{A}^{\mathcal{V}} = -(\delta P_J)P_J$ and its curvature is

$$\begin{aligned} \mathsf{F}^{\mathcal{V}} &= P_J\,\delta P_J \wedge \delta P_J\,P_J = -\tfrac{1}{4}\,P_J\,\delta J \wedge \delta J\,P_J \\ &= -\begin{pmatrix} 1 \\ Z \end{pmatrix}(1 - \bar{Z}Z)^{-1}\delta\bar{Z} \wedge (1 - Z\bar{Z})^{-1}\delta Z(1 - \bar{Z}Z)^{-1}(1,\ -\bar{Z}). \end{aligned}$$

The curvature of the line bundle $\det\mathcal{V} = \bigwedge^n \mathcal{V} \to \mathcal{J}$ is the 2-form

$$\mathsf{F}^{\det\mathcal{V}} = \mathrm{tr}(P_J\,\delta P_J \wedge \delta P_J\,P_J) = \mathrm{tr}((1 - Z\bar{Z})^{-1}\delta Z \wedge (1 - \bar{Z}Z)^{-1}\delta\bar{Z}).$$

(All expressions in terms of Z are valid on a dense open set of $\mathcal{J}$ only.) The canonical line bundle over V is the product bundle $V \times \mathcal{K}_J$, where $\mathcal{K}_J = \bigwedge^n (V_J^{1,0})^*$. We have a line bundle $\mathcal{K} \to \mathcal{J}$ whose fibre over $J \in \mathcal{J}$ is $\mathcal{K}_J$. In fact, $\mathcal{K} = (\det\mathcal{V})^*$ and its curvature is $\mathsf{F}^{\mathcal{K}} = -\mathsf{F}^{\det\mathcal{V}}$.

The space $\mathcal{J}$ has a transitive action of $\mathrm{GL}_+(V)$, the identity component of $\mathrm{GL}(V) \cong \mathrm{GL}(2n, \mathbb{R})$ preserving the orientation on V. At $J_0 \in \mathcal{J}$, the isotropy subgroup $\mathrm{GL}(V, J_0) \cong \mathrm{GL}(n, \mathbb{C})$ consists of elements in $\mathrm{GL}(V)$ that commutes with J_0. Therefore $\mathcal{J}$ can be identified as the homogeneous space $\mathrm{GL}_+(V)/\mathrm{GL}(V, J_0)$. The Lie algebra of $\mathrm{GL}_+(V)$ has the decomposition $\mathfrak{gl}(V) = \mathfrak{gl}(V, J_0) \oplus \mathfrak{m}$, where $\mathfrak{gl}(V, J_0)$, $\mathfrak{m}$ are the subspaces in $\mathfrak{gl}(V)$ of elements that commute, anti-commute with J_0, respectively. Since $\mathfrak{m}$ is invariant under the adjoint action of $\mathrm{GL}(V,J_0)$ and since $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{gl}(V,J_0)$, $\mathrm{GL}_+(V)/\mathrm{GL}(V,J_0)$ is a reductive symmetric space. The trace form on $\mathfrak{m}\cong T_{J_0}\mathcal{J}$ is a $\mathrm{GL}(V,J_0)$-invariant non-degenerate symmetric bilinear form and it induces a pseudo-Riemannian metric

$$\eta=2\,\mathrm{tr}(P_J\,\delta P_J\,\delta P_J\,P_J)=2\,\mathrm{tr}((1-Z\bar{Z})^{-1}\delta Z\,(1-\bar{Z}Z)^{-1}\delta\bar{Z})$$

on $\mathcal{J}$. It is pseudo-Kähler with the pseudo-Kähler form

$$\sigma=\mathsf{F}^{\mathcal{K}}/\sqrt{-1}=\sqrt{-1}\;\mathrm{tr}(P_J\,\delta P_J\wedge\delta P_J\,P_J)=\sqrt{-1}\;\mathrm{tr}((1-Z\bar{Z})^{-1}\delta Z\wedge(1-\bar{Z}Z)^{-1}\delta\bar{Z}).$$

The group $\mathrm{GL}_+(V)$ acts on Z by fractional linear transformations preserving σ.

The map $J\mapsto J_0^{-1}JJ_0$ is an isometric reflection on $\mathcal{J}$ that induces minus the identity map on the tangent space $T_{J_0}\mathcal{J}\cong\mathfrak{m}$. As $\mathcal{J}$ is reductive, a geodesic in $\mathcal{J}$ is of the form $t\mapsto[g_t]$ with $g_t=g_0\,e^{tM}$ (see [19]), where $g_0\in\mathrm{GL}_+(V)$, $M\in\mathfrak{m}$ and $[g]$ denotes the coset in $\mathrm{GL}_+(V)/\mathrm{GL}(V,J_0)\cong\mathcal{J}$ represented by $g\in\mathrm{GL}_+(V)$. The parameter $t\in\mathbb{R}$ is proportional to the arc-length on the geodesic.

Proposition A.1 *For any t, $J_{t/2}/\sqrt{-1}$ is the parallel transport in $\mathcal{V}$ along the geodesic from J_0 to J_t.*

Proof: Since the geodesic starts from J_0, we have $g_0=1$, hence $g_tg_{t'}=g_{t+t'}$ and $g_t^{-1}=g_{-t}$. The reflection at J_0 reverses the direction of each geodesic passing through J_0, as $J_0^{-1}g_tJ_0=g_{-t}$. The complex structure at $[g_t]$ is $J_t=g_tJ_0g_t^{-1}=g_{2t}J_0=J_0g_{-2t}$, and hence $J_0J_{-t}=J_tJ_0=-g_{2t}$. $J_0/\sqrt{-1}$ is the identity map on $V^{1,0}_{J_0}$, $J_{t/2}=g_tJ_0$ maps $V^{1,0}_{J_0}$ to $V^{1,0}_{J_t}$, and $\frac{d}{dt}J_{t/2}=g_tMJ_0$ maps $V^{1,0}_{J_0}$ to $V^{0,1}_{J_t}$. So $J_{t/2}/\sqrt{-1}$ satisfies all the requirements that uniquely defines the parallel transport in $\mathcal{V}$. □

Corollary A.2 *If J, J' and J'' are three points on a geodesic such that J' bisects the segment between J and J'', then $J'J=J''J'$ and $J'/\sqrt{-1}$, which maps $V^{1,0}_J$ to $V^{1,0}_{J''}$, is the parallel transport in $\mathcal{V}$ from J to J'' along the geodesic.*

A.2 Complex structures compatible with a symplectic or Euclidean structure

Given a symplectic form ω on V, a complex structure J on V is compatible to ω if $\omega(J\cdot,J\cdot)=\omega(\cdot,\cdot)$ and $\omega(\cdot,J\cdot)>0$. (The second condition implies that the orientation defined by J coincides with that of the volume form $\omega^n/n!$.) Let $\mathcal{J}_\omega$ be the set of such J. The symplectic form defines an isomorphism $\nu=\nu_\omega\colon V\to V^*$ by $x\in V\mapsto\iota_x\omega\in V^*$ and a holomorphic involution $\mathsf{s}=\mathsf{s}_\omega$ on $\mathcal{J}$ by $J\mapsto\nu^{-1}\circ{}^TJ\circ\nu$. The fixed-point set $\mathcal{J}^{\mathsf{s}}$ consists of $J\in\mathcal{J}$ satisfying $\omega(J\cdot,J\cdot)=\omega(\cdot,\cdot)$. The connected components of $\mathcal{J}^{\mathsf{s}}$, one of which is $\mathcal{J}_\omega$, are labelled by the signature of the symmetric bilinear form $\omega(\cdot,J\cdot)$. Since s is an isometry, $\mathcal{J}^{\mathsf{s}}$ is a totally geodesic submanifold. In fact, $\mathcal{J}_\omega$ is a Kähler manifold since the pseudo-Kähler metric η restricts to a positive-definite metric η_ω on $\mathcal{J}_\omega$. So the restriction σ_ω of σ to $\mathcal{J}_\omega$ is a Kähler form. Given $J_0\in\mathcal{J}_\omega$, the whole space $\mathcal{J}_\omega$ can be parametrised by $Z\in\mathrm{Hom}_{\mathbb{C}}(V^{1,0}_{J_0},V^{0,1}_{J_0})$. We denote by the same notation $\nu\colon V^{0,1}_{J_0}\to(V^{1,0}_{J_0})^*$ the restriction of the isomorphism $\nu\colon V^{\mathbb{C}}\to(V^*)^{\mathbb{C}}$. Then $J\in\mathcal{J}^{\mathsf{s}}$ if and only if $\nu\circ Z\in\mathrm{Sym}^2(V^{1,0}_{J_0})^*$, or equivalently, $Z\circ\nu^{-1}\in\mathrm{Sym}^2V^{0,1}_{J_0}$. The tangent space $T_{J_0}\mathcal{J}_\omega$ consists of Z satisfying this condition. The condition $\omega(\cdot,J\cdot)>0$ is equivalent to $1-\bar{Z}Z>0$ with respect to the Hermitian form $h_0(x,y)=\omega(x,J_0\bar{y})=-\sqrt{-1}\,\omega(x,\bar{y})$, $x,y\in V^{1,0}_{J_0}$. The symplectic group $\mathrm{Sp}(V,\omega)$ acts transitively on $\mathcal{J}_\omega$ and the isotropic subgroup at J_0 is the unitary group $\mathrm{U}(V,h_0)$. So $\mathcal{J}_\omega\cong\mathrm{Sp}(V,\omega)/\mathrm{U}(V,h_0)$. It can be identified holomorphically with a bounded Hermitian symmetric domain or with the Siegel upper-half space; the two are related by a Cayley transform.

If instead there is a Euclidean inner product g on V, let $\mathcal{J}_g$ be the set of complex structures J that is compatible with g, i.e., $g(J\cdot,J\cdot)=g(\cdot,\cdot)$, and the orientation on V. There is an isomorphism $\nu=\nu_g\colon V\to V^*$ defined by $x\in V\mapsto\iota_xg\in V^*$ and an involution $\mathsf{s}=\mathsf{s}_g\colon J\mapsto\nu^{-1}\circ{}^TJ\circ\nu$ on $\mathcal{J}$. The fixed-point set is $\mathcal{J}^{\mathsf{s}}=\mathcal{J}_g$. As s is an isometry, $\mathcal{J}_g$ is totally geodesic in $\mathcal{J}$ as in the symplectic case. $\mathcal{J}_g$ is Kähler since the restriction η_g of $-\eta$ to $\mathcal{J}_g$ is positive definite; the restriction σ_g of $-\sigma$ to $\mathcal{J}_g$ is the Kähler form. Given $J_0\in\mathcal{J}_g$, we denote also by $\nu\colon V^{0,1}_{J_0}\to(V^{1,0}_{J_0})^*$ the restriction of $\nu\colon V^{\mathbb{C}}\to(V^*)^{\mathbb{C}}$. On the dense set of $\mathcal{J}$ that can be parametrised by Z, $J\in\mathcal{J}_g$ if and only if the corresponding Z satisfies $\nu\circ Z\in\bigwedge^2(V^{1,0}_{J_0})^*$, or equivalently, $Z\circ\nu^{-1}\in\bigwedge^2V^{0,1}_{J_0}$. The tangent space $T_{J_0}\mathcal{J}_g$ consists of Z satisfying this condition. For any such Z, we always have $1-\bar{Z}Z>0$ with respect to the Hermitian form $h_0(x,y)=g(x,\bar{y})$, $x,y\in V^{1,0}_{J_0}$. The group $\mathrm{SO}(V,g)$ acts transitively on $\mathcal{J}_g$ and the isotropic subgroup at J_0 is $\mathrm{U}(V,h_0)$. The space $\mathcal{J}_g\cong\mathrm{SO}(V,g)/\mathrm{U}(V,h_0)$ is a compact Hermitian symmetric space. Finally, if in addition there is a symplectic form ω on V such that $\omega(\cdot,J_0\cdot)$ is proportional to g, then $\mathcal{J}_\omega$ and $\mathcal{J}_g$ intersects at J_0 orthogonally with respect to the pseudo-Kähler metric η on $\mathcal{J}$.

We describe the results using tensor indices. Let $\{e_i\}_{1\le i\le n}$ be a basis of $V^{1,0}_{J_0}$. Then $\{\bar e_{\bar i}\}$ is a basis of $V^{0,1}_{J_0}$. We represent $Z\in\mathrm{Hom}_{\mathbb C}(V^{1,0}_{J_0},V^{0,1}_{J_0})$ by a matrix $Z_i^{\,\bar j}$ such that $Ze_i=Z_i^{\,\bar j}\bar e_{\bar j}$. If (V,ω) is a symplectic vector space, we have $\omega_{\bar i j}=\omega(\bar e_{\bar i},e_j)=-\omega_{j\bar i}$. Set $Z_{ij}=(\nu\circ Z)_{ij}=Z_i^{\,\bar k}\omega_{\bar k j}$ and $Z^{\bar i\bar j}=(Z\circ\nu^{-1})^{\bar i\bar j}=\omega^{\bar i k}Z_k^{\,\bar j}$. Then Z determines an element in $\mathcal J_\omega$ if and only if $Z_{ij}=Z_{ji}$ (or $Z^{\bar i\bar j}=Z^{\bar j\bar i}$) and the matrix $\delta_i^{\,j}-Z_i^{\,\bar k}\bar Z_{\bar k}^{\,j}$ is positive definite. If (V,g) is a Euclidean space instead, then $g_{\bar i j}=g(\bar e_{\bar i},e_j)=g_{j\bar i}$. Set $Z_{ij}=(\nu\circ Z)_{ij}=Z_i^{\,\bar k}g_{\bar k j}$ and $Z^{\bar i\bar j}=(Z\circ\nu^{-1})^{\bar i\bar j}=g^{\bar i k}Z_k^{\,\bar j}$. Then Z determines an element in $\mathcal J_g$ if and only if $Z_{ij}=-Z_{ji}$ (or $Z^{\bar i\bar j}=-Z^{\bar j\bar i}$). If there is a variation δJ of $J\in\mathcal J$, then we have tensors $(\delta P)_i^{\,\bar j}=-(\delta\bar P)_i^{\,\bar j}$ and $(\delta P)_{\bar i}^{\,j}=-(\delta\bar P)_{\bar i}^{\,j}$. We note that $\{e_i+\delta e_i\}$, where $\delta e_i=(\delta P)_i^{\,\bar j}\bar e_{\bar j}$, is a basis of the new holomorphic subspace $V^{1,0}_{J+\delta J}$, whereas $\{\bar e_{\bar i}+\delta\bar e_{\bar i}\}$, where $\delta\bar e_{\bar i}=(\delta\bar P)_{\bar i}^{\,\bar j}\bar e_{\bar j}=-(\delta P)_{\bar i}^{\,\bar j}\bar e_{\bar j}$, is a basis of $V^{0,1}_{J+\delta J}$. If (V,ω) is symplectic and $J\in\mathcal J_\omega$, then $J+\delta J\in\mathcal J_\omega$ (to the first order) if and only if any of the tensors $(\delta P)_{ij},(\delta P)^{ij},(\delta P)^{\bar i\bar j},(\delta P)_{\bar i\bar j}$ is symmetric. If (V,g) is Euclidean and $J\in\mathcal J_g$, then $J+\delta J\in\mathcal J_g$ (to the first order) if and only if any of the above tensors is anti-symmetric.

Choosing a unitary basis $\{e_i\}_{1\le i\le n}$ of $V^{1,0}_{J_0}$ in both the symplectic and the orthogonal cases, we have $\mathcal J_\omega\cong\mathrm{Sp}(2n,\mathbb R)/\mathrm U(n)$ and $\mathcal J_g\cong\mathrm{SO}(2n)/\mathrm U(n)$, respectively, where J_0 is identified with the coset o of the identity element. Using the basis $\{e_i,\bar e_{\bar i}\}$ of $V^{\mathbb C}$, the Lie groups and/or their Lie algebras that appear in the above identifications are

$$\mathrm U(n)=\left\{\begin{pmatrix}U&0\\0&\bar U\end{pmatrix}\;\middle|\;{}^T\bar UU=I_n\right\},\quad \mathfrak u(n)=\left\{\begin{pmatrix}A&0\\0&\bar A\end{pmatrix}\;\middle|\;{}^T\bar A=-A\right\},$$

$$\mathfrak{sp}(2n,\mathbb R)=\left\{\begin{pmatrix}A&B\\\bar B&\bar A\end{pmatrix}\;\middle|\;\begin{smallmatrix}{}^T\bar A=-A,\\ {}^TB=B\end{smallmatrix}\right\},\quad \mathfrak{so}(2n)=\left\{\begin{pmatrix}A&B\\\bar B&\bar A\end{pmatrix}\;\middle|\;\begin{smallmatrix}{}^T\bar A=-A,\\ {}^TB=-B\end{smallmatrix}\right\},$$

where U,A,B are $n\times n$ complex matrices. The following results on geodesics are well-known:

Proposition A.3 ([21, 14]) *1. A geodesic γ in $\mathrm{Sp}(2n,\mathbb R)/\mathrm U(n)$ from o is of the form*

$$\gamma(t)=\left[k\begin{pmatrix}\cosh Bt&\sinh Bt\\\sinh Bt&\cosh Bt\end{pmatrix}k^{-1}\right],$$

where $k\in\mathrm U(n)$ and $B=\mathrm{diag}\{b_1,\cdots,b_r,0,\cdots,0\}$ for some $b_1,\dots,b_r>0$, $r\le n$.
2. A geodesic γ in $\mathrm{SO}(2n)/\mathrm U(n)$ from o is of the form

$$\gamma(t)=\left[k\begin{pmatrix}\cos\sqrt{-B^2}t&\frac{B}{\sqrt{-B^2}}\sin\sqrt{-B^2}t\\\frac{B}{\sqrt{-B^2}}\sin\sqrt{-B^2}t&\cos\sqrt{-B^2}t\end{pmatrix}k^{-1}\right],$$

where $k\in\mathrm U(n)$ and $B=\mathrm{diag}\left\{\begin{pmatrix}&b_1\\-b_1&\end{pmatrix},\cdots,\begin{pmatrix}&b_r\\-b_r&\end{pmatrix},0,\cdots,0\right\}$ for some $b_1,\dots,b_r>0$, $r\le[n/2]$. (In this case, $\sqrt{-B^2}=\mathrm{diag}\{b_1,b_1,\cdots,b_r,b_r,0,\cdots,0\}$.)

Proof: Writing $\mathfrak{sp}(2n,\mathbb R)=\mathfrak u(n)\oplus\mathfrak m$ and $\mathfrak{so}(2n)=\mathfrak u(n)\oplus\mathfrak m$, respectively, geodesics are of the form $\gamma(t)=[e^{tM}]$ for some $M=\begin{pmatrix}0&B'\\\bar B'&0\end{pmatrix}\in\mathfrak m$. Since ${}^TB'=\pm B'$, by Theorems 5 and 7 in [14], respectively, there exists an $n\times n$ complex matrix U, ${}^T\bar UU=I_n$, such that $B'=UB\,{}^TU$, where B is of the required form. The results then follow from simple calculations with $k=\begin{pmatrix}U&0\\0&\bar U\end{pmatrix}\in\mathrm U(n)$. □

When $n=1$, $\mathcal J_\omega=\mathcal J$ because every complex structure compatible with the orientation is compatible with the symplectic form. Choosing a base vector e_1 of the 1-dimensional vector space $V^{1,0}_{J_0}$, $\mathcal J_\omega$ can be parametrised by $z=Z_1^{\,\bar 1}\in\mathbb C$ such that $|z|<1$, with $z=0$ for J_0. The Kähler form and metric are, respectively,

$$\sigma_\omega=\frac{2\sqrt{-1}\,dz\wedge d\bar z}{(1-|z|^2)^2},\qquad \eta_\omega=\frac{4\,dz\,d\bar z}{(1-|z|^2)^2}.$$

A geodesic through $z=0$ is of the form $z(t)=e^{\sqrt{-1}\,\alpha}\tanh t$ ($0\le\alpha<2\pi$), where $t\in\mathbb R$ is half of the arc-length parameter. For Euclidean space (V,g), the first non-trivial case is, when $n=2$, $\mathcal J_\omega\cong\mathrm{SO}(4)/\mathrm U(2)=S^2$. Choose a basis $\{e_1,e_2\}$ of $V^{1,0}_{J_0}$ such that $g_{1\bar 1}=g_{2\bar 2}$, $g_{1\bar 2}=g_{2\bar 1}=0$. Then the dense subset $\mathcal J_\omega\backslash\{-J_0\}$ can be parametrised by $Z=\begin{pmatrix}&z\\-z&\end{pmatrix}$, where $z\in\mathbb C$. On $\mathcal J_\omega$, the point $-J_0$ (which would be $z=\infty$) is conjugate to J_0 ($z=0$). The Kähler form and metric are, respectively,

$$\sigma_g=\frac{2\sqrt{-1}\,dz\wedge d\bar z}{(1+|z|^2)^2},\qquad \eta_\omega=\frac{4\,dz\,d\bar z}{(1+|z|^2)^2}.$$

A geodesic through $z = 0$ is of the form $z(t) = e^{\sqrt{-1}\,\alpha} \tan t$ $(0 \le \alpha < 2\pi)$, where $t \in \mathbb{R}$ is half of the arc-length parameter. Note that $z(t) = \infty$ at $t = \frac{\pi}{2}$ corresponds to the antipodal point $-J_0$ of J_0.

A.3 Cut and first conjugate loci in $\mathcal{J}_g$

Given a point o in a Riemannian manifold M, the first conjugate point p of o along a geodesic γ from o is a point such that there is a Jacobi field along γ that is zero at o and p but nowhere zero in between. The collection of such points form the first conjugate locus of o. The cut point of o along a geodesic γ from o is the point p such that γ is length-minimising between o and p but fails to be so beyond p. There is an open cell B in T_oM such that the exponential map is a diffeomorphism from B onto a (connected) open subset of M whose compliment is the cut locus of o. The image of the closure $\bar{B}$ under the exponential map is M.

While the structure of cut loci or first conjugate loci for general Riemannian manifolds is quite complicated (see for example [24]), there is a Lie-theoretical description for compact Riemannian symmetric spaces. For simply connected symmetric spaces (such as the space $\mathcal{J}_g$ above), the cut locus and first conjugate locus coincide [6], though this fails to be true in general [20]. For example, the cut and first conjugate loci of Grassmannian manifolds are known explicitly in terms of Schubert varieties [27] (see however Remark 4.3 of [20]). We determine the cut (or the first conjugate) locus of $\mathcal{J}_g$, which is the space of polarisations of fermionic systems.

Proposition A.4 *Let $J_0, J \in \mathcal{J}_g$. The following statements are equivalent:*
(a) J is on the cut locus of J_0;
(b) $\det\left(\frac{J_0+J}{2}\right) = 0$;
(c) the pairing between $\mathcal{K}_{J_0}^{-1}$ and $\mathcal{K}_J^{-1}$ is degenerate.

Proof: Choosing a unitary basis $\{e_i\}_{1\le i\le n}$ of $V_{J_0}^{1,0}$, we have $\mathcal{J}_g \cong \mathrm{SO}(2n)/\mathrm{U}(n)$. The geodesics from o are given by Proposition A.3.2. Since $k \in \mathrm{U}(n)$ acts as an isometry, we can assume $k = 1$ as well as $b_1 \ge \cdots \ge b_r > 0$ without loss of generality. For any t, $V_{J_t}^{1,0}$ has a unitary basis consisting of vectors $e_{2i-1}^{(t)} = \cos b_i t\; e_{2i-1} - \sin b_i t\; \bar{e}_{\overline{2i}}$, $e_{2i}^{(t)} = \cos b_i t\; e_{2i} + \sin b_i t\; \bar{e}_{\overline{2i-1}}$ $(1 \le i \le r)$ and $e_j^{(t)} = e_j$ $(2r+1 \le j \le n)$.

(a)⇒(b): The cut point of o along the above geodesic is at $t = \pi/2b_1$. It is clear that $e_1^{(\pi/2b_1)} = -\bar{e}_{\bar{2}} \in V_{J_{\pi/2b_1}}^{1,0} \cap V_{J_0}^{0,1}$ and hence $\det\left(\frac{J_0+J_{\pi/2b_1}}{2}\right) = 0$.

(b)⇒(a): Consider a geodesic γ from J_0 to J of the above form. Then $\det\left(\frac{J_0+J}{2}\right) = 0$ implies that $t = \pi/2b_i$ for some $i = 1, \ldots, r$. Assume $i = 1$. For $2 \le j \le r$, let b'_j be defined such that $|b'_j| \le b_1$ and $b'_j = b_j$ mod $2b_1$. Let γ' be the geodesic from o corresponding to $B' = \mathrm{diag}\left\{\begin{pmatrix} & b_1 \\ -b_1 & \end{pmatrix}, \begin{pmatrix} & b'_2 \\ -b'_2 & \end{pmatrix}, \cdots, \begin{pmatrix} & b'_r \\ -b'_r & \end{pmatrix}, 0, \cdots, 0\right\}$. Then $J = \gamma'(\pi/2b_1)$ is the cut point of o along γ'.

(b)⇔(c): Along the geodesic, let $\mu_t = e_1^{(t)} \wedge \cdots \wedge e_n^{(t)}$. Then

$$\langle \mu, \mu_0 \rangle = \prod_{i=1}^{r} \cos^2 b_i = \det\left(\tfrac{J_0+J}{2}\right)$$

and hence the result. ☐

Corollary A.5 *If $J \in \mathcal{J}_g$ is not on the cut locus of J_0, then*
1. $\det\left(\frac{J_0+J}{2}\right) > 0$;
2. the inner product on $\sqrt{\mathcal{K}_{J_0}^{-1}}$ extends continuously to a non-degenerate pairing between $\sqrt{\mathcal{K}_{J_0}^{-1}}$ and $\sqrt{\mathcal{K}_J^{-1}}$.

Proof: Consider the geodesic in the proof of Proposition A.4.
1. $\det\left(\frac{J_0+J}{2}\right) = \prod_{i=1}^{r} \cos^2 b_i > 0$ if $t < \pi/2b_i$ for all $i = 1, \ldots, r$.
2. Consider μ_t in the proof of Proposition A.4. The pairing is given by $\langle \sqrt{\mu_t}, \sqrt{\mu_0} \rangle = \prod_{i=1}^{r} \cos b_i$. ☐

B Berezin integral and the fermionic Bergman kernel

B.1 Calculus of fermionic variables

Let V be an n-dimensional real vector space with a (non-zero) volume element $\epsilon = \epsilon_V \in \bigwedge^n V$. The Berezin integral of a form $\alpha \in \bigwedge^\bullet (V^*)^{\mathbb{C}}$ on V is

$$\int_{\Pi V} \alpha\, \epsilon_V = \langle \alpha^{(n)}, \epsilon_V \rangle,$$

where the pairing is between the top-degree component $\alpha^{(n)}$ and ϵ_V. To highlight its formal similarity with the usual integration, the Berezin integral is often expressed, as in the physics literature, as an "integration" over fermionic variables. While the setting is well known, we recall it here to fix the sign convention. For a standard reference, see for example, §1.4-7 of [30]. For a mathematical treatment of graded manifolds or supermanifolds, especially in the context of geometric quantisation, see [17].

We imagine a copy ΠV of the vector space that is identical as V except it has fermionic coordinates, which are "numbers" satisfying the same law of addition but anti-commute when they are multiplied. If we choose a basis $\{e_i\}_{1\le i\le n}$ of V, then a "vector" $\theta \in \Pi V$ has the form $\theta = \theta^i e_i$, where $\theta^1, \dots, \theta^n$ the fermionic coordinates. Although ΠV does not exist as a set of points, the "functions" on ΠV are elements of the exterior algebra $\bigwedge^\bullet (V^*)^{\mathbb{C}}$. In fact, any form

$$\alpha = \sum_{k=0}^{n} \sum_{1\le i_1<\cdots<i_k\le n} \alpha_{i_1\dots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}$$

on V determines a "function"

$$\alpha(\theta) = \sum_{k=0}^{n} \sum_{1\le i_1<\cdots<i_k\le n} \alpha_{i_1\dots i_k} \theta^{i_1} \cdots \theta^{i_k}$$

on ΠV. The "derivative" of such functions corresponds the usual contraction on forms:

$$\frac{\partial}{\partial \theta^i} \alpha(\theta) = (\iota_{e_i} \alpha)(\theta).$$

Suppose the basis $\{e_i\}$ spans a unit volume, i.e., $\epsilon = e_1 \wedge \cdots \wedge e_n$. The volume element provides a "measure" $\epsilon(\theta) = d\theta^1 \cdots d\theta^n$ on ΠV. The fermionic integral is defined by

$$\int_{\Pi V} \alpha(\theta)\, \epsilon(\theta) = \int_{\Pi \mathbb{R}^n} \alpha_{12\dots n} \theta^1 \cdots \theta^n \, d\theta^1 \cdots d\theta^n = (-1)^{\frac{n(n-1)}{2}} \alpha_{12\dots n}.$$

This differs from the Berezin integral by a sign because $d\theta^i$ also anti-commutes with θ^j. As a useful example, we calculate

$$\int_{\Pi\mathbb{R}^2} e^{\sqrt{-1}\, a\theta^1\theta^2} \sqrt{-1}\, d\theta^1 d\theta^2 = \int_{\Pi\mathbb{R}^2} (1 + \sqrt{-1}\, a\theta^1\theta^2) \sqrt{-1}\, d\theta^1 d\theta^2 = a \int_{\Pi\mathbb{R}} \theta^1\, d\theta^1 \int_{\Pi\mathbb{R}} \theta^2\, d\theta^2 = a,$$

where $a \in \mathbb{R}$.

Suppose V is even dimensional, say $\dim V = 2n$. Let g be a Euclidean metric on V and $\epsilon = \epsilon_g$, a unit volume element. Set $\tilde\epsilon_g = \sqrt{-1}^{\,n}\, \epsilon_g$.

Lemma B.1 *If $A \in \mathrm{End}(V)$ is skew-symmetric with respect to g, then*

$$\int_{\Pi V} e^{\frac{\sqrt{-1}}{2} g(A\theta,\theta)}\, \tilde\epsilon_g(\theta) = \mathrm{Pf}(A).$$

Proof: We choose an orthonormal basis of V so that A decomposes as a direct sum of 2×2 skew-symmetric matrices. The result then follows from the example computed above. □

Now assume that J is a complex structure on V compatible with g and the orientation given by ϵ_g. If A is invertible and if A and J are in the same connected component of invertible, skew-symmetric operators on V, then

$$\mathrm{Pf}(A) = (\det A)^{1/2},$$

where the square root is chosen so that $(\det J)^{1/2} = 1$. Compare this with the usual Gaussian integral in the proof of Theorem 2.5.1, in which A is symmetric and the determinant factor $(\det A)^{1/2}$ appears in the denominator.

B.2 The fermionic Bergman kernel and projection

We work with the pre-quantum data of the fermionic system in §3.1: a real Euclidean space (V, g) of dimension $2n$ with a complex structure J compatible with g and a unit volume element ϵ_g which agrees with the orientation of J. Choosing a basis $\{e_i\}_{1\le i\le n}$ of $V_J^{1,0}$, we have complex fermionic coordinates $\theta^1, \ldots, \theta^n$ of $\theta \in \Pi V_J^{1,0}$ or ΠV. An element ψ in the pre-quantum Hilbert space $\mathcal{H}_0$ can be regarded as a "function" $\psi(\theta, \bar\theta)$ of θ^i and $\bar\theta^{\bar i}$ $(1 \le i \le n)$. The covariant derivative along e_i and $\bar e_{\bar j}$ are, respectively,

$$\nabla_i = \tfrac{\partial}{\partial\theta^i} - \tfrac{1}{2}\, g_{i\bar j}\bar\theta^{\bar j}, \qquad \nabla_{\bar j} = \tfrac{\partial}{\partial\bar\theta^{\bar j}} - \tfrac{1}{2}\, g_{i\bar j}\theta^i,$$

where $g(\theta, \bar\theta) = g_{i\bar j}\theta^i\bar\theta^{\bar j}$. Any $\psi = e^{\frac{\sqrt{-1}}{2}\varpi_J} \wedge \phi \in \mathcal{H}_J$ can be written as (cf. Theorem 3.1.1)

$$\psi(\theta, \bar\theta) = \phi(\theta)\, e^{-\frac{1}{2}g(\theta,\bar\theta)},$$

where $\phi(\theta)$ is a "holomorphic function", that is, it depends on θ^i only. By Theorem 3.1.2, the inner product of $\psi = e^{\frac{\sqrt{-1}}{2}\varpi_J} \wedge \phi$ and $\psi' = e^{\frac{\sqrt{-1}}{2}\varpi_J} \wedge \phi'$ in $\mathcal{H}_J$ is

$$\langle\psi, \psi'\rangle = \int_{\Pi V} \phi(\theta)^*\, \phi'(\theta)\, e^{-g(\theta,\bar\theta)}\, \tilde\epsilon_g(\theta).$$

Here $\phi(\theta)^*$ is obtained from $\phi(\theta)$ by complex conjugation and reversing the order in the multiplication, i.e.,

$$(\theta^{i_1}\cdots\theta^{i_k})^* = \bar\theta^{\bar i_k}\cdots\bar\theta^{\bar i_1}, \quad 1 \le i_1 < \cdots < i_k \le n,\ 0 \le k \le n.$$

The formula bears a formal resemblance with that in Proposition 2.1.2 of the bosonic case. Moreover, the projection from $\mathcal{H}_0$ to $\mathcal{H}_J$ is given by the fermionic counterpart of the Bergman kernel.

Proposition B.2 *The orthogonal projection from $\psi \in \mathcal{H}_0$ onto $\mathcal{H}_J$ is*

$$\psi(\theta, \bar\theta) \longmapsto e^{-\frac{1}{2}g(\theta,\bar\theta)} \int_{\Pi V} e^{g(\theta,\bar\chi)-\frac{1}{2}g(\chi,\bar\chi)}\, \psi(\chi, \bar\chi)\, \tilde\epsilon(\chi).$$

Proof: Suppose the basis is unitary, i.e., $g(e_i, \bar e_{\bar j}) = \delta_{ij}$. We write $\theta\bar\chi = \theta^1\bar\chi^{\bar 1} + \cdots + \theta^n\bar\chi^{\bar n}$ for two fermionic vectors ψ, χ in ΠV. The fermionic measure can be written as

$$\tilde\epsilon(\theta) = d\theta^1 d\bar\theta^{\bar 1}\cdots d\theta^n d\bar\theta^{\bar n} = d\theta d\bar\theta.$$

It is easy to check that $\mathcal{H}_J$ has an unitary basis $\{\theta^{i_1}\cdots\theta^{i_k}e^{-\frac{1}{2}\theta\bar\theta} \,|\, 0 \le k \le n, 1 \le i_1 < \cdots < i_k \le n\}$. So the Bergman kernel that produces the orthoganal projection from $\mathcal{H}_0$ to $\mathcal{H}_J$ is

$$K(\theta, \bar\chi) = e^{-\frac{1}{2}\theta\bar\theta-\frac{1}{2}\chi\bar\chi} \sum_{k=0}^{n} \sum_{1\le i_1<\cdots<i_k\le n} \theta^{i_1}\cdots\theta^{i_k}\bar\chi^{\bar i_k}\cdots\bar\chi^{\bar i_1} = e^{\theta\bar\chi-\frac{1}{2}\theta\bar\theta-\frac{1}{2}\chi\bar\chi}.$$

□

A fermionic coherent state c_J^α is of the form

$$c_J^\alpha(\theta) = \exp[g(\theta, \bar\alpha) - \tfrac{1}{2}\, g(\theta, \bar\theta)],$$

where $\alpha \in \Pi V_J^{1,0}$ is a fermionic parameter. The above projection can be written as

$$\psi(\theta, \bar\theta) \longmapsto e^{-\frac{1}{2}g(\theta,\bar\theta)} \int_{\Pi V} c_J^\theta(\chi)^*\, \psi(\chi, \bar\chi)\, \tilde\epsilon(\chi).$$

Finally, we find the relation to real fermionic coordinates. Let $\theta_0 = \theta + \bar\theta \in \Pi V$. We have $2\sqrt{-1}\, g(\theta, \bar\theta) = \varpi_J(\theta_0, \theta_0)$, which we write as $\varpi_J(\theta_0)$ for short. Then the fermionic Gaussian factor becomes $e^{\frac{\sqrt{-1}}{4}\varpi_J(\theta_0)}$.

C Invariant real, complex and quaternionic structures

C.1 Representations of real and quaternionic types

Let W be a finite-dimensional complex vector space. An operator $C\colon W \to W$ is conjugate linear if C is real linear and $C(ax) = \bar{a}(Cx)$ for all $a \in \mathbb{C}$, $x \in W$. Such operators are in $\mathrm{Hom}_{\mathbb{C}}(W, \bar{W})$, where $\bar{W}$ the complex vector space which is equal to W as an Abelian group but whose scalar multiplication is given by $(a, x) \in \mathbb{C} \times W \mapsto \bar{a}x \in W$. A real structure on W is a conjugate-linear operator R on W such that $R^2 = \mathrm{id}_W$. Such an R determines a real vector space $W_0 = W^R \subset W$ fixed by R and $W \cong (W_0)^{\mathbb{C}}$ as complex vector spaces. A quaternionic structure on W is a conjugate-linear operator Q on W such that $Q^2 = -\mathrm{id}_W$. This makes W a quaternionic vector space whith the scalar multiplication $(a + bj, x) \in \mathbb{H} \times W \mapsto ax + bQx \in W$ (where $a, b \in \mathbb{C}$).

Suppose a group K acts on W by a complex representation. The representation is of real type if there is a K-invariant real structure on W. Such a representation is the complexification of a real representation of K on W_0. The representation is of quaternionic type if there is a K-invariant quaternionic structure on W. Such a representation is quaternionic-linear with the above scalar multiplication by $\mathbb{H}$. We refer the reader to [4] for the standard properties of real- and quaternionic-type representations. We collect here some more results that will be used in §C.2.

Unless stated otherwise, we assume from now on that K is a finite group or a compact Lie group. By averaging over K, there is a K-invariant Hermitian form $h\colon W \times \bar{W} \to \mathbb{C}$ on W. Our convention of Hermitian forms on W is that they are complex linear in the first variable but conjugate linear in the second. (However, we took the opposite convention of physicists for pre-quantum or quantum Hilbert spaces.)

Lemma C.1 *Consider a representation of K on a complex vector space W. Then*
1. the representation is of real (quaternionic, respectively) type if and only if there is a non-degenerate K-invariant symmetric (skew-symmetric, respectively) bilinear form on W;
2. there is a non-zero sub-representation $W' \subset W$ of real (quaternionic, respectively) type if and only if there is a non-zero K-invariant symmetric (skew-symmetric, respectively) bilinear form on W.

Proof: Part 1 is well known; see for example Proposition II.6.4 in [4] or the proof of Lemma C.2 below. Part 2 follows immediately by taking W' as the orthogonal complement (with respect to a K-invariant Hermitian form) of the kernel of the bilinear form. □

Lemma C.2 *Under the same conditions as in Lemma C.1, suppose h is a K-invariant Hermitian form on W. If the representation of K on W is of real (quaternionic, respectively) type, then there is a K-invariant real structure R (quaternionic structure Q, respectively) on W such that $h(Rx, Ry) = h(y, x)$ ($h(Qx, Qy) = h(y, x)$, respectively) for all $x, y \in W$.*

Proof: Suppose C_0 is a K-invariant real (quaternionic, respectively) structure on W. Then $C_0^2 = \epsilon\, \mathrm{id}_W$, where $\epsilon = \pm 1$, respectively. Consider the complex bilinear form β on W given by $\beta(x, y) = h(x, C_0 y) + \epsilon\, h(y, C_0 x)$, where $x, y \in W$. Then β is symmetric (skew-symmetric, respectively) when $\epsilon = \pm 1$. Moreover, β is non-degenerate as $\beta(C_0 x, x) = h(x, x) + h(C_0 x, C_0 x)$ for any $x \in W$. Following the proof of Proposition II.6.4 in [4], we define an invertible, conjugate-linear operator C on W by $\beta(x, y) = h(x, Cy)$, $x, y \in W$. Then $h(C^2 x, y) = \epsilon\, h(Cy, Cx) = h(x, C^2 y)$. Consequently, $\epsilon\, C^2$ is self-adjoint and positive definite with respect to h. The space W decomposes as a direct sum of eigenspaces of $\epsilon\, C^2$, each of which is K-invariant since C^2 is so. Without loss of generality, assume that W is the eigenspace of $\epsilon\, C^2$ of a single eigenvalue $\lambda > 0$. Since $C^2 = \epsilon\lambda\, \mathrm{id}_W$ and $\lambda h(x, y) = h(Cy, Cx)$ for all $x, y \in W$, $\lambda^{-1/2} C$ is the desired real (quaternionic, respectively) structure on W when $\epsilon = \pm 1$. □

The results in Lemma C.2 can also be explained in matrix language; we do so when the representation is of real type. Choosing a real basis of the real subspace W_0, the Hermitian form h corresponds to a positive definite Hermitian matrix H. There is a unitary matrix U such that $D = {}^T U H \bar{U}$ is a diagonal matrix of positive entries. With the representation of K, U can be chosen to commute with K. Let $R = U\, {}^T U$. Since $R\bar{R}$ is the identity matrix, R defines a real structure on W. The result follows form the identity ${}^T R H \bar{R} = U D {}^T \bar{U} = \bar{H}$.

C.2 Complex structures invariant under a representation

If V is a real vector space and J is a complex structure on V, we denote by (V, J) the complex vector space whose underlying real vector space is V and on which the scalar multiplication by $\sqrt{-1}$ is the action of J.

Clearly, $(V,J) \cong V_J^{1,0}$ as complex vector spaces. Let K be a finite or a compact Lie group. A presentation of K on V is a complex representation on (V,J) if and only if $J \in \mathfrak{J}$ is invariant under K.

Proposition C.3 *1. Suppose (V,ω) is a symplectic vector space and $J \in \mathfrak{J}_\omega$. If a representation of K on (V,J) is of real type and preserves ω, then there is a K-invariant real structure R on (V,J) such that $\omega(Rx,Ry) = -\omega(x,y)$ for all $x,y \in V$.*
2. Suppose (V,g) is an oriented Euclidean vector space of even dimension and $J \in \mathfrak{J}_g$. If a representation of K on (V,J) is of quaternionic type and preserves g, then there is a K-invariant quaternionic structure Q on (V,J) such that $g(Qx,Qy) = g(x,y)$ for all $x,y \in V$.

Proof: 1. Let $h(x,y) = \omega(x,Jy) - \sqrt{-1}\,\omega(x,y)$, $x,y \in V$. Then h is a K-invariant Hermitian form, as $h(Jx,y) = -h(x,Jy) = \sqrt{-1}\,h(x,y)$. By Lemma C.2, there is a K-invariant real structure R on (V,J) (i.e., $RJ = -JR$, $R^2 = \mathrm{id}_V$) such that $h(Rx,Ry) = h(y,x)$. This is equivalent to $\omega(Rx,Ry) = -\omega(x,y)$.
2. Let $h(x,y) = g(x,y) - \sqrt{-1}\,g(Jx,y)$, $x,y \in V$. Then h is a K-invariant Hermitian form, as $h(Jx,y) = -h(x,Jy) = \sqrt{-1}\,h(x,y)$. By Lemma C.2, there is a K-invariant quaternionic structure Q on (V,J) (i.e., $QJ = -JQ$, $Q^2 = -\mathrm{id}_V$) such that $h(Qx,Qy) = h(y,x)$. This is equivalent to $g(Qx,Qy) = g(x,y)$. □

Since the group K acts on $\mathfrak{J}_\omega$ ($\mathfrak{J}_g$, respectively) by isometry, the fixed-point set $(\mathfrak{J}_\omega)^K$ ($(\mathfrak{J}_g)^K$, respectively) is a totally geodesic submanifold.

Proposition C.4 *1. Suppose $J_0 \in \mathfrak{J}_\omega$ is preserved by a symplectic representation of K on (V,ω). Then $T_{J_0}(\mathfrak{J}_\omega)^K \cong (\mathrm{Sym}^2(V_{J_0}^{1,0}))^K$. Moreover, the following statements are equivalent:*
- *(a) $(\mathfrak{J}_\omega)^K$ contains a point other than J_0;*
- *(b) $(\mathrm{Sym}^2(V_{J_0}^{1,0}))^K \neq \{0\}$;*
- *(c) there is a non-zero complex sub-representation (V',J_0) of (V,J_0) of real type.*

In this case, V' can be chosen as a symplectic subspace and there is a K-invariant real structure R on (V',J_0) such that $\omega(Rx,Ry) = -\omega(x,y)$ for all $x,y \in V'$.
2. Suppose $J_0 \in \mathfrak{J}_g$ is preserved by an orthogonal representation of K on (V,g). Then $T_{J_0}(\mathfrak{J}_g)^K \cong (\bigwedge^2(V_{J_0}^{1,0}))^K$. Moreover, the following statements are equivalent:
- *(a) $(\mathfrak{J}_g)^K$ is not a discrete set;*
- *(b) $(\bigwedge^2(V_{J_0}^{1,0}))^K \neq \{0\}$;*
- *(c) there is a non-zero complex sub-representation (V',J_0) of (V,J_0) of quaternionic type.*

In this case, there is a K-invariant quaternionic structure Q on (V',J_0) such that $g(Qx,Qy) = g(x,y)$ for all $x,y \in V'$.

Proof: 1. The result on the tangent space follows from §A.2. Since $\mathfrak{J}_\omega$ is also K-equivariantly diffeomorphic to $\mathrm{Sym}^2(V_{J_0}^{1,0})$, the equivalence of (a) and (b) is clear. The equivalence with (c) is a consequence of Lemma C.1.2 and the rest follows from Proposition C.3.1.
2. Although $(\mathfrak{J}_g)^K$ is not $(\bigwedge^2(V_{J_0}^{1,0}))^K$ globally, the latter is isomorphic to the tangent space $T_{J_0}(\mathfrak{J}_g)^K$, which is zero if and only if $(\mathfrak{J}_g)^K$ is a discrete set. The rest of the proof is similar to that of part 1. □

By compactness, the set $(\mathfrak{J}_g)^K$ is finite if it is discrete. When K is a compact torus group, the number of elements in $(\mathfrak{J}_g)^K$ is equal to the Euler characteristic $\chi(\mathfrak{J}_g) = 2^{n-1}$. (This is the quotient of the order of the Weyl group of $\mathrm{SO}(2n)$ by that of $\mathrm{U}(n)$.) For example, $\mathfrak{J}_g \cong S^2$ when $n=2$. If $K = S^1$ acts on $\mathbb{C}^2$ with weights 1 and -1, then the fixed-point set in $\mathfrak{J}_g$ is $(\mathfrak{J}_g)^K = \{\pm J_0\}$, whose cardinal is $2 = \chi(S^2)$.

Acknowledgments. Part of the work has been reported at various conferences, in Oberwolfach (2004), Sendai (2005), Tianjin (2005), Hong Kong (2006). The author thanks W. D. Kirwin, N. Mok and R. Sjamaar for helpful discussions. The work is supported in part by CERG HKU705407P, Fondation Sciences Mathématiques de Paris and Université Paris–Diderot (Paris 7).

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