On singularities of generically immersive holomorphic maps between complex hyperbolic space forms

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Abstract

In 1965, Feder proved using a cohomological identity that any holomorphic immersion $\tau : \mathbb{P}^n \to \mathbb{P}^m$ between complex projective spaces is necessarily a linear embedding whenever $m < 2n$. In 1991, Cao-Mok adapted Feder's identity to study the dual situation of holomorphic immersions between compact complex hyperbolic space forms, proving that any holomorphic immersion $f : X \to Y$ from an n-dimensional compact complex hyperbolic space form X into any m-dimensional complex hyperbolic space form Y must necessarily be totally geodesic provided that $m < 2n$. We study in this article singularity loci of generically injective holomorphic immersions between complex hyperbolic space forms. Under dimension restrictions, we show that the open subset U over which the map is a holomorphic immersion cannot possibly contain compact complex-analytic subvarieties of large dimensions which are in some sense sufficiently deformable. While in the finite-volume case it is enough to apply the arguments of Cao-Mok, the main input of the current article is to introduce a geometric argument that is completely local. Such a method applies to $f : X \to Y$ in which the complex hyperbolic space form X is possibly of infinite volume. To start with we make use of the Ahlfors-Schwarz Lemma, as motivated by recent work of Koziarz-Mok, and reduce the problem to the local study of contracting leafwise holomorphic maps between open subsets of complex unit balls. Rigidity results are then derived from a commutation formula on the complex Hessian of the holomorphic map.

Key words

complex hyperbolic space form, holomorphic immersion, total geodesy, holomorphic isometry, leafwise contracting holomorphic map, complex Hessian, commutation formula

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In 1965, Feder [Fe65] proved that any holomorphic immersion $\tau : \mathbb{P}^n \to \mathbb{P}^m$ between complex projective spaces is necessarily a linear embedding whenever $m < 2n$. He did this by using Whitney's formula on Chern classes associated to the tangent sequence of the holomorphic map, thereby proving that the degree of $\tau_* : H_2(\mathbb{P}^n, \mathbb{Z}) \to H_2(\mathbb{P}^m, \mathbb{Z})$ must be 1 under the dimension restriction, noting that the restriction $m < 2n$ forces the vanishing of the n-th Chern class of the normal bundle of the holomorphic immersion. An adaptation of Feder's identity was used by Cao-Mok [CM91] to study the Immersion Problem for the dual situation of holomorphic immersions between compact complex hyperbolic space forms. By an n-dimensional complex hyperbolic space form we mean the quotient of the n-dimensional complex unit ball B^n by a torsion-free discrete group of automorphisms equipped with the complete Kähler metric induced by the canonical complete Kähler-Einstein metric on B^n. By [CM91] any holomorphic immersion $f : X \to Y$ from an n-dimensional compact complex hyperbolic space form X into any m-dimensional complex hyperbolic space form Y must necessarily be totally geodesic provided that $m < 2n$. A generalization of the latter result to the case of complex hyperbolic space forms of finite volume was obtained by To [To93].

By the duality between the complex unit ball $(B^n, ds^2_{B^n})$ equipped with the unique complete Kähler-Einstein metric of constant holomorphic sectional curvature $-K$, $K > 0$, and the projective space $(\mathbb{P}^n, ds^2_{FS})$ equipped with the Fubini-Study metric of constant holomorphic sectional curvature equal to K, the total Chern class of a complex hyperbolic space form is determined by its first Chern class. Given a holomorphic immersion between complex hyperbolic space forms, the first Chern class can be represented by the first Chern form induced on the domain manifold from the canonical Kähler-Einstein metric of the target manifold via the immersion. The main entity in the first Chern form is a nonnegative closed $(1,1)$-form ρ which is derived from the second fundamental form σ on $(1,0)$-vectors of the holomorphic immersion and which enjoys the property that the vanishing of ρ means equivalently the vanishing of σ, i.e., the total geodesy of the immersion. The adaptation by Cao-Mok [CM91] of Feder's identity to the holomorphic immersion $f : X \to Y$ between complex hyperbolic space forms, applied to the tangent sequence $0 \to T_X \to f^*T_Y \to N \to 0$, where N stands for the normal bundle of the holomorphic immersion f, gives the vanishing $\rho^n \equiv 0$ when X is compact, $n := \dim(X)$ and $\dim(Y) := m < 2n$, and the same holds true when X is noncompact and of finite volume by To [To93]. At a general point x of X the kernels of ρ were shown to define a holomorphic foliation $\mathscr{E}$ on a neighborhood U of x the leaves of which are totally geodesic complex submanifolds. This was shown to lead to a contradiction to the fact that X is of finite volume unless the holomorphic foliation $\mathscr{E}$ is trivial.

Recently, Feder's identity has been applied by Koziarz-Mok [KM10] to the Submersion Problem concerning holomorphic submersions between compact complex hyperbolic space forms and more generally between complex hyperbolic space forms of finite-volume. There, given a holomorphic submersion $\pi : X \to Y$ between complex hyperbolic space forms, applying Feder's identity instead to the cotangent sequence $0 \to \pi^* T_Y^* \to T_X^* \to T_\pi^* \to 0$, where $T_\pi = \mathrm{Ker}\,(d\pi)$ stands for the relative tangent bundle $\pi : X \to Y$, yields the vanishing $\mu^{n-m+1} \equiv 0$ for the closed nonnegative $(1,1)$-form $\mu := \omega_X - \pi^*\omega_Y$, where ω_X resp. ω_Y stands for the Kähler form on X resp. on Y of the canonical Kähler-Einstein metric of constant holomorphic sectional curvature $-K$, and the nonnegativity of μ follows from the Ahlfors-Schwarz Lemma. Using the identity $\mu^{n-m+1} \equiv 0$ it was proven in [KM10] that there does not exist any holomorphic submersion between compact complex hyperbolic space forms, and the same was proven in the noncompact finite-volume case provided that the base manifold is of complex dimension ≥ 2.

Motivated by the use of the Ahlfors-Schwarz Lemma in [KM10], in the current article we re-visit the topic of holomorphic immersions $f : X \to Y$ between complex hyperbolic space forms. In [CM91] the closed nonnegative (1,1)-form ρ represents up to a positive constant the cohomology class $\frac{-c_1(X)}{n+1} + \frac{f^*c_1(Y)}{m+1}$. The possibility of representing the latter class by $\rho \geq 0$ results from the constancy of holomorphic sectional curvatures and from the monotonicity of holomorphic bisectional curvatures. The holomorphicity of the foliation defined by $\mathrm{Ker}(\rho)$ then follows from the holomorphicity of the second fundamental form σ on $(1,0)$-vectors. On the other hand, the cohomology class $\frac{-c_1(X)}{n+1} + \frac{f^*c_1(Y)}{m+1}$ is up to a positive constant represented by $\mu := \omega_X - f^*\omega_Y \geq 0$. We will make use simultaneously of the closed nonnegative (1,1)-forms ρ and μ. Motivated by results of [KM10] in the case of compact complex hyperbolic space forms concerning critical values of surjective holomorphic maps, we will study in this article singularity loci of generically injective holomorphic immersions between complex hyperbolic space forms. One of the main results is applicable also to complex hyperbolic space forms of infinite volume. Under dimension restrictions, we will show that the open subset U over which the map is a holomorphic immersion cannot possibly contain compact complex-analytic subvarieties of large dimensions which are in some sense sufficiently deformable.

For results in the finite-volume case it is enough to apply the arguments of Cao-Mok [CM91]. First of all, when X is compact, we observe that the arguments of Cao-Mok [CM91] already imply the estimate that $\dim(\mathrm{Sing}(f)) \geq 2n-m-1$ unless f is totally geodesic. For the proof it suffices to restrict the tangent sequence to linear sections of X (with respect to a projective embedding) which avoid $\mathrm{Sing}(f)$ to deduce total geodesy of f whenever $\dim(\mathrm{Sing}(f)) < 2n-m-1$. In the case of a noncompact complex hyperbolic space form of finite volume the abundant supply of linear sections avoiding $\mathrm{Sing}(f)$ is guaranteed in the arithmetic case by the existence of Satake-Borel-Baily compactifications [Sa60] and [BB66] obtained by adding a finite number of normal isolated singularities, and in the non-arithmetic case by the projective-algebraicity proven in [Mk10] of the complex-analytic compactification obtained in Siu-Yau [SY82] by adding a finite number of points corresponding to the finite number of ends.

As indicated in the above the dimension estimate on $\mathrm{Sing}(f)$ breaks up into two parts. The first half is cohomological. In more precise terms, assuming dim(Sing(f)) $< 2n-m-1$ there exists a q-dimensional compact complex submanifold S of $X' := X - \mathrm{Sing}(f)$ with $q = n-(2n-m-1) = m-n+1$ so that, denoting by N the normal bundle of the holomorphic immersion $f|_{X'} : X' \to Y$ we must have $c_{m-n+1}(N)|_S = 0$ since $\mathrm{rank}(N) = m-n$. Feder's identity and the compactness of S then forces the nonnegative (1,1)-form $\rho|_S$ to have a zero eigenvalue everywhere. By varying S obtained from taking linear sections with respect to a projective embedding one concludes that the closed nonnegative (1,1)-form ρ is degenerate everywhere on X. The second half of the argument is the same as in Cao-Mok [CM91] and [To93] where one derives from the degeneracy of X a holomorphic foliation on some nonempty connected open set by totally geodesic complex submanifolds consisting of maximal integral submanifolds of Re(Ker(ρ)), and where in the proof of the identical vanishing of ρ one requires the fact that the fundamental group of X is a lattice in Aut(B^n). For the sake of brevity we will call the second half the geometric argument.

The main input of the current article is to introduce a geometric argument that is completely local. Such a method applies to $f : X \to Y$ where the complex hyperbolic space form X is possibly of infinite volume, with a conclusion that X' cannot contain a sufficiently deformable $(m-n+1)$-dimensional compact complex-analytic subvariety, where by saying that a q-dimensional compact complex-analytic subvariety $S \subset X - \mathrm{Sing}(f)$ is sufficiently deformable we mean that points corresponding to tangent q-planes of deformations of S fill up a nonempty open subset of the Grassmann bundle of q-planes on X.

Making use of the fact that $\frac{-c_1(X)\big|_{X'}}{n+1} + \frac{f^*c_1(Y)\big|_{X'}}{m+1}$ can be represented by a closed nonnegative (1,1)-form ρ arising from the second fundamental form and another closed nonnegative (1,1)-form μ encoding the failure of f to be an isometry, reinforcing the cohomological argument we obtain a holomorphic foliation on a nonempty open subset U by totally geodesic submanifolds where f restricts to a totally geodesic isometric embedding on each of the totally geodesic leaves. Unless $\rho \equiv 0$ or equivalently $\mu \equiv 0$ we have obtained a nonempty open subset U of B^n, a holomorphic foliation $\mathscr{E}$ on U by totally geodesic complex submanifolds and a holomorphic embedding f of U into some B^m such that f is contracting (distance-decreasing) and it is a totally geodesic isometric embedding when restricted to any leaf of $\mathscr{E}$, and such that $\mathscr{E}_x = \mathrm{Re}\,(\mathrm{Ker}(\rho(x)))$ for any $x \in U$. We call such a map $f : U \to B^m$ a contracting leafwise totally geodesic holomorphic isometric embedding, where implicitly the leaves are assumed to be defined by $\mathrm{Re}(\mathrm{Ker}(\rho))$. Under the dimension restriction $m \leq 2n-4$ we prove that no contracting leafwise totally geodesic holomorphic isometric embedding exists unless $m = n$, in which case f is nothing other than a totally geodesic embedding. This is slightly short of giving a completely local proof for the geometric argument in the dimension estimate for $\mathrm{Sing}(f)$ even in the case where X is compact, where we need the local argument of $m \leq 2n-1$.

Crucial to our geometric argument is a commutation formula concerning the Hessian of the holomorphic map f, more precisely concerning $\nabla\partial f$, the vanishing of which is equivalent to the total geodesy of the map f. The commutation formula applies to any contracting leafwise totally geodesic holomorphic isometric embedding $f : U \to B^m$. However, in the application of the commutation formula, dimension counts are involved, which is the reason why the dimension restriction $m \leq 2n-4$ is imposed. We expect that there is no nontrivial holomorphic embedding $f : U \to B^m$ which is a contracting leafwise totally geodesic isometric embedding, but the latter remains unproved for $m \geq 2n-3$.

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1 Background

By a complex hyperbolic space form we mean the quotient of the n-dimensional complex unit ball B^n for some positive integer n by a torsion-free discrete group of automorphisms equipped with the complete Kähler metric induced by the canonical complete Kähler-Einstein metric on B^n. The total geodesy of holomorphic immersions between complex hyperbolic space forms under dimension restrictions was established in Cao-Mok [CM91] in the compact case and in [To93] in the noncompact finite-volume case. Here the requirement of compactness or of the finiteness of the volume is imposed only on the domain manifold.

Theorem (Cao-Mok [CM91], To [To93]). *Let n,m be positive integers such that $n \geq 2$ and $m < 2n$. Let $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice of biholomorphic automorphisms, $X := B^n/\Gamma$. Let Y be an n-dimensional complex hyperbolic space form. Let $f : X \to Y$ be a holomorphic immersion. Then, f is totally geodesic.*

The proofs of Cao-Mok [CM91] and To [To93] rely on a cohomological argument and a geometric argument. The starting point of the cohomological argument is the vanishing of the n-th Chern class of the normal bundle of the holomorphic immersion $f : X \to Y$, given that the normal bundle is of rank $m-n < n$. The crux of the cohomological argument is the following algebraic identity adapted from Feder [Fe65], in which it was proven that any holomorphic immersion $f : \mathbb{P}^n \to \mathbb{P}^m$ is linear whenever $m < 2n$.

Lemma 1.1. *For the compact complex hyperbolic space form $X = B^n/\Gamma$ let $\alpha, \beta \in H^2(X,\mathbb{R})$. Suppose for $1 \leq k \leq n-m$ there exists $\gamma_k \in H^{2k}(X,\mathbb{R})$ such that $(1+\alpha)^{n+1} = (1+\gamma_1+\cdots\gamma_{n-m})(1+\beta)^{m+1}$. Then, $(\alpha-\beta)^{n-m+1} = 0$.*

In the cohomological argument, the main entity is a closed nonnegative (1,1)-form ρ obtained from the second fundamental form of the holomorphic immersion and enjoying the property that the second fundamental form vanishes identically if and only if $\rho \equiv 0$. By the cohomological argument basing on Lemma 1 one concludes that $\rho^n \equiv 0$ on X. We lift ρ to $\widetilde{\rho}$ defined on some connected open subset $U \subset B^n$ holomorphically foliated by d-dimensional totally geodesic complex submanifolds for some d, $1 \leq d \leq n$. Completing these leaves to totally geodesic complex submanifolds (which are d-dimensional affine-linear sections of $B^n \subset \mathbb{C}^n$) we obtain a subset $S \subset B^n$ swept out by such submanifolds, where S contains $W \cap B^n$ for some neighborhood W of a boundary point $b \in \partial B^n$. The closed nonnegative (1,1)-form $\widetilde{\rho}$ can be extended to $W \cap B^n$. The proofs of the results of Cao-Mok [CM91] and To [To93] are completed by an argument by contradiction. This involves a geometric argument concerning the boundary behavior of $\widetilde{\rho}$ on $W \cap B^n$, where, assuming that $d < n$, $\widetilde{\rho} \geq 0$ is degenerate but not identically 0. From asymptotic properties of the canonical Kähler-Einstein metric on complex unit balls the latter is shown to be asymptotically of zero length as one approaches $W \cap \partial B^n$. Given that $\pi_1(X) = \Gamma$ is a lattice, the asymptotic vanishing of $\widetilde{\rho}$ implies $\rho \equiv 0$, yielding a proof of the theorem by contradiction.

In addition to holomorphic immersions there is naturally the problem of studying holomorphic submersions between complex hyperbolic space forms. In this regard Koziarz-Mok [KM10] has obtained recently the following result.

Theorem (part of Koziarz-Mok [KM10, Theorem 2]). *Let $n > m \geq 1$. Let Z be an m-dimensional compact complex hyperbolic space form. Let $f : X \to Z$ be a surjective holomorphic map and denote by $E \subset Z$ the smallest subvariety such that f is a regular holomorphic fibration over $Z - E$. Then, $E \subset Z$ is of complex codimension 1.*

Normalizing the canonical Kähler-Einstein metrics on X resp. Z with Kähler forms ω_X resp. ω_Z to be of constant holomorphic sectional curvature $-K$ for the same constant $K > 0$, the method of Koziarz-Mok [KM10] relies on Feder's identity as given in Lemma 1 and the use of a closed nonnegative (1,1)-form $\mu := \omega_X - f^*\omega_Z$, where the nonnegativity of μ follows from the Ahlfors-Schwarz Lemma.

Motivated by the above result of Koziarz-Mok [KM10] and the use of a different type of closed nonnegative (1,1)-form μ basing on the Ahlfors-Schwaz Lemma, we re-visit the study of holomorphic immersions between complex hyperbolic space forms, generalizing the context to the study of generically immersive holomorphic maps $f : X \to Y$ between complex hyperbolic space forms where neither X nor Y is required to be of finite volume with respect to the canonical Kähler-Einstein metric. Denoting by $\mathrm{Sing}(f)$ the singular locus of such a map, we are led to consider holomorphic immersions from $X - \mathrm{Sing}(f)$ into Y. In this article we present two main results. The first concerns a lower bound for the complex dimension of $\mathrm{Sing}(f)$ in the case where X is compact or noncompact but of finite volume. We will obtain such a result using essentially the arguments of Cao-Mok [CM91] and of To [To93] by considering furthermore the restriction of the tangent sequence to compact complex-analytic subvarieties of $X - \mathrm{Sing}(f)$. For the noncompact case of finite-volume, to obtain compact complex-analytic subvarieties of $X - \mathrm{Sing}(f)$ we make use of the following result on compactifying not necessarily arithmetic noncompact complex hyperbolic space forms of finite volume.

Theorem (Siu-Yau [SY82], Mok [Mk10]). *Let n be a positive integer, and let $\Gamma \subset \mathrm{Aut}(B^n)$ be a non-uniform torsion-free lattice; $X := B^n/\Gamma$. Then, X can be compactified to a normal projective-algebraic variety $\overline{X}_{\min}$ by adjoining a finite number of isolated normal singularities.*

Thus, in the case of a complex hyperbolic space form $X := B^n/\Gamma$, where Γ is a lattice, for our lower estimate on $\dim(\mathrm{Sing}(f))$ to be given in [§2, Theorem 1] we still rely on the use of the closed (1,1)-form $\rho \geq 0$ arising from the second fundamental form of the immersion on $X - \mathrm{Sing}(f)$. The second main result, to be given in [§5, Theorem 2] concerns the more general case where X may be of infinite volume, and we prove, under certain dimension restrictions, that the open set

$X - \mathrm{Sing}(f)$ does not contain any irreducible compact complex-analytic subvariety of dimension $m - n + 1$ which is in some sense sufficiently deformable. In this result we make use of both the closed (1,1)-forms $\rho \geq 0$, which arises from the second fundamental form, and $\mu \geq 0$, where nonnegativity results from the Ahlfors-Schwarz Lemma, yielding on some holomorphically foliated connected open subset a contracting leafwise totally geodesic holomorphic isometric embedding. On the methodological plane we introduce a method which in principle replaces the geometric argument in Cao-Mok [Ca91] and To [To93] concerning $\rho \geq 0$, which relies on the fact that $\pi_1(X)$ is a lattice, by a local argument resulting from a commutation formula concerning the complex Hessian $\nabla\partial f$. For technical reasons we impose the slightly stronger dimension restriction $m \leq 2n - 4$ for the local argument.

In the formulation of the second main result on complex hyperbolic space forms not necessarily of finite volume, we define the notion of sufficiently deformable compact complex-analytic subvarieties, as follows.

Definition 1 (sufficiently deformable subvariety). *Let N be a complex manifold of dimension n, $0 < q < n$. Let $S \subset N$ be a pure q-dimensional compact complex-analytic subvariety. We say that $S \subset N$ is sufficiently deformable if there exists an irreducible complex space B, $0 \in B$, a complex-analytic subvariety $\mathscr{S} \subset N \times B$ for which the canonical projection $\pi : \mathscr{S} \to B$ is proper with fibers being pure q-dimensional compact complex-analytic subvarieties $S_t := \pi^{-1}(t) \subset N$ for $t \in B$, $S_0 = S$, such that the following holds true. Denoting by $\tau : \mathscr{S} \to Gr(q, T(N))$ the canonical meromorphic map into the Grassmann bundle of q-dimensional vector subspaces of tangent spaces of N, where $\tau(x) = [T_x(S_{\pi(x)})] \in Gr(q, T_x(N))$ whenever x is a smooth point of $S_{\pi(x)}$, there is a point $y \in \mathscr{S}$ such that y is a smooth point of $\mathscr{S}$, $\pi(y)$ is a smooth point of B, π is a holomorphic submersion at y, and $\tau\big|_{U_y} : U_y \to Gr(q, T_x(N))$ is a holomorphic submersion on some open neighborhood U_y of y in $\mathscr{S}$.*

For an n-dimensional projective submanifold N by it is clear that whenever $0 < q < n$, any q-dimensional linear section cut out by $n - q$ hyperplanes is sufficiently deformable in N. The same is true for N being an n-dimensional quasi-projective manifold $N \subset \mathbb{P}^a$, and for any q-dimensional linear section $S \subset \overline{N}$ cut out by $n - q$ hyperplanes such that $S \subset N$, where $\overline{N} \subset \mathbb{P}^a$ denotes the topological closure of N in $\mathbb{P}^a$, $\overline{N} \subset \mathbb{P}^a$ being a projective-algebraic subvariety. Such q-dimensional linear sections S always exist whenever $q < n - d$, where $d = \dim(\overline{N} - N)$.

The first main result concerning singularities of generically immersive maps in the finite-volume case will be explained in §2. In §3-§5 we consider the more general situation in which the domain manifold $X := B^n/\Gamma$ may be of infinite volume. In §3, assuming the existence of a sufficiently deformable compact complex-analytic subvariety of $X - \mathrm{Sing}(f)$ of a certain dimension, we derive the existence of a contracting leafwise totally geodesic holomorphic isometric embedding from some open subset $U \subset B^n$ into B^m. In §4 we establish a commutation formula for the study of the complex Hessian $\nabla\partial f$ adapted to such maps, and in §5 we deduce consequences of the commutation formula, especially proving the second main result concerning compact complex-analytic subvarieties of $X - \mathrm{Sing}(f)$.

2 Singular loci in the finite-volume case

The first main result of the current article is given by the following theorem on the singular loci of generically immersive holomorphic maps between complex hyperbolic space forms in the case where the domain manifold is of finite volume.

Theorem 1. *Let n,m be positive integers such that $n \geq 2$ and $m < 2n$. Let $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free lattice of automorphisms; $X := B^n/\Gamma$; and let Y be any m-dimensional complex hyperbolic space form. Suppose $f : X \to Y$ is a holomorphic map such that df is of rank n at a general point. Assume that the singular locus $\mathrm{Sing}(f)$ of f is of dimension strictly less than $2n-m-1$, then in fact $\mathrm{Sing}(f) = \emptyset$ and f is a totally geodesic map.*

As will be clear from the proof of Theorem 1, there is an obvious analogue of Theorem 1 for the dual case of nonconstant holomorphic maps $f : \mathbb{P}^n \to \mathbb{P}^m$. Such a holomorphic map is automatically an immersion at a general point since no algebraic curve on $\mathbb{P}^n$ can be collapsed to a point, $\mathbb{P}^n$ being of Picard number 1. The dual analogue of Theorem 1 says

Theorem 1'. *Let n,m be positive integers such that $n \geq 2$ and $m < 2n$. Let $f : \mathbb{P}^n \to \mathbb{P}^m$ be a nonconstant holomorphic map. Then, $\mathrm{rank}(df(x))$ is equal to n at a general point $x \in \mathbb{P}^n$, and the singular locus $\mathrm{Sing}(f)$ must be of complex dimension $\geq 2n-m-1$ unless $f : \mathbb{P}^n \to \mathbb{P}^m$ is a projective-linear embedding.*

The inequality $\dim(\mathrm{Sing}(f)) \geq 2n-m-1$ is equivalent to the inequality $\mathrm{codim}(\mathrm{Sing}(f)) \leq n-(2n-m-1) = m-n+1$. For Theorem 1' it says in particular that a nonconstant holomorphic map $f : \mathbb{P}^n \to \mathbb{P}^{n+1}$ is either a projective-linear embedding, or its singular locus is of codimension at most equal to 2. We have the following example which shows in this case that the codimension may be exactly equal to 2.

EXAMPLE Let $f : \mathbb{P}^2 \to \mathbb{P}^3$ be defined by $f([z_0,z_1,z_2]) = [z_0^3,z_1^3,z_2^3,z_0z_1z_2]$ in terms of homogeneous coordinates. Then, f is holomorphic. By a straightforward computation, f is a holomorphic immersion excepting at the three points $[1,0,0],[0,1,0]$ and $[0,0,1]$. For any integer $n \geq 2$ the holomorphic map $f : \mathbb{P}^n \to \mathbb{P}^{n+1}$ defined by $f([z_0,z_1,\cdots,z_n]) = [z_0^{n+1},z_1^{n+1},\cdots z_n^{n+1},z_0z_1\cdots z_n]$ gives an example where $\mathrm{Sing}(f)$ is of codimension 2. In the latter case, $\mathrm{Sing}(f)$ is the union of the $\frac{n(n+1)}{2}$ projective-linear subspaces defined by $z_p = z_q = 0$, $0 \leq p < q \leq n$.

Proof of Theorem 1. Recall that n and m are positive integers, $n < m < 2n$, $\Gamma \subset \mathrm{Aut}(B^n)$ a torsion-free discrete group of automorphisms, $X = B^n/\Gamma$, and $f : X \to Y$ is a generically immersive holomorphic map. Write $X' := X - \mathrm{Sing}(f)$. Consider the tangent sequence $0 \to T_{X'} \to f^*T_Y \to N \to 0$ of X', where $N = f^*T_Y/T_{X'}$ denotes the normal bundle for the holomorphic immersion $f|_{X'} : X' \to Y$. Suppose $S \subset X'$ is an $(m-n+1)$-dimensional compact complex submanifold. Then $N|_S$ is a holomorphic vector bundle of rank $m-n$ and we have $c_{m-n+1}(N)|_S = c_{m-n+1}(N|_S) = 0$. By Feder's identity as given in Lemma 1 it follows that $[\nu^m] = 0$, where $[\cdots]$ denotes the de Rham cohomology class, for any closed smooth $(1,1)$-form ν representing the cohomology class $\frac{-c_1(X)}{n+1} + \frac{f^*c_1(Y)}{m+1}$. Since we have normalized the choice of the canonical Kähler-Einstein metric to be of constant holomorphic sectional curvature -4π, from Cao-Mok [CM91] we can take ν to be ρ, where, denoting by $g = \left(g_{\alpha\overline{\beta}}\right)$ resp. $h = \left(h_{i\overline{j}}\right)$ the canonical Kähler-Einstein metric on B^n resp. B^m of constant holomorphic sectional curvature -4π, we have

$$\rho_{\alpha\overline{\beta}} = \sum_{\alpha,\beta,\gamma,\delta,k,\ell} g^{\alpha\overline{\gamma}} g^{\beta\overline{\delta}} \widehat{h}_{k\overline{\ell}} \sigma^k_{\alpha\beta} \overline{\sigma^\ell_{\gamma\delta}}, \tag{1}$$

where $\left(g^{\alpha\overline{\beta}}\right)$ denotes the conjugate inverse of $\left(g_{\alpha\overline{\beta}}\right)$, $\left(\widehat{h}_{k\ell}\right)$ denotes the Hermitian metric on N induced from h, $\sigma^k_{\alpha\beta}$ denotes the (holomorphic) second fundamental form on $(1,0)$-vectors for the holomorphic immersion $f|_{X'}$, and the summation is performed over the ranges $1 \leq \alpha,\beta,\gamma,\delta \leq n$ and $1 \leq k,\ell \leq n-m$. ¿From the curvature formula for Kähler submanifolds given by the Gauss equation we have in fact

$$\rho = \frac{-c_1(X,f^*h)}{n+1} + \frac{f^*c_1(Y,h)}{m+1}. \tag{2}$$

For the proof of Theorem 1 consider first of all the case where X is compact. Suppose the generically immersive holomorphic map $f : X \to Y$ is not totally geodesic and $\dim(\mathrm{Sing}(f)) < 2n-m-1$, i.e., $\mathrm{codim}(\mathrm{Sing}(f)) > n-(2n-m-1) = m-n+1$. Embedding X as a projective manifold and taking intersections of hyperplane sections for each $x \in X'$ there exists a smooth linear section $S \subset X$ passing through x and of dimension $m-n+1$ such that $S \cap \mathrm{Sing}(f) = \emptyset$. Then, by Feder's identity (Lemma

1) we have $[(\rho|_S)^{m-n+1}] = 0$. From $\rho \geq 0$ it follows that $\rho^{m-n+1}|_S = 0$, so that the smooth (1,1)-form $\rho|_S$ must have a positive-dimensional kernel at each point $s \in S$. Since $x \in X'$ is arbitrary, it follows that $\rho(x)$ must have a positive-dimensional kernel at each $x \in X'$. There is a real-analytic subvariety of $V \subsetneq X'$ such that dim(Ker($\rho(x)$)) is the same integer d, $1 \leq d < n$. In particular there exists a non-empty connected open subset $U \subset X'$ such dim(Ker($\rho(x)$)) $= d$. Since Re(Ker(ρ)) agrees with the kernel of the second fundamental form σ, the assignment of Re(Ker(ρ_x)) to $x \in X$ defines a holomorphic foliation $\mathscr{E}_x$ on U with d-dimensional leaves consisting of totally geodesic locally closed complex submanifolds. Taking U to be simply connected we can lift U in a univalent way to a connected open subset $\widetilde{U} \subset B^n$, ρ to a closed nonnegative (1,1)-form $\widetilde{\rho}$ on $\widetilde{U}$, and $\mathscr{E}$ to a holomorphic foliation $\widetilde{\mathscr{E}}$ on $\widetilde{U}$ consisting of totally geodesic complex submanifolds. By extending each leaf of $\widetilde{\mathscr{E}}$ to a complete totally geodesic complex submanifold of B^n, we sweep out $W \cap B^n$ for some neighborhood W of some boundary point $b \in \partial B^n$. We derive a contradiction exactly as in the argument of Cao-Mok [CM91] from the asymptotic behavior of an extension of $\widetilde{\rho}$ to $W \cap B^n$, which is based on the estimate that the extended closed (1,1)-form $\widetilde{\rho}$ is asymptotically of zero length as one approaches $W \cap \partial B^n$ and hence of zero length everywhere by the compactness of the fundamental domain of B^n modulo the action of Γ.

In the case where $X = B^n/\Gamma$ is noncompact and of finite volume, we adopt the arguments of To [To93], and the only thing that remains to be verified is that, under the assumption that $\dim(\mathrm{Sing}(f)) < 2n-m-1$ there still exists a compact complex submanifold $S \subset X - \mathrm{Sing}(f)$ of complex dimension exactly equal to $m-n+1$ obtained by taking the intersection of $2n-m-1$ hyperplane sections with respect to some projective embedding of X. That this is so follows readily from the existence of a projective-algebraic compactification X obtained by adding a finite number of normal isolated singularities, as follows from Siu-Yau [SY82] and Mok [Mk10] and stated in §1. ■

§3. Contracting leafwise totally geodesic isometric embeddings

Motivated by the use of the Ahlfors-Schwarz Lemma in conjunction with Feder's identity (Lemma 1) in Koziarz-Mok [KM10], we examine further consequences that can be drawn from cohomological arguments by making use of both the Gauss equation (via the second fundamental form σ and hence ρ) and of the Ahlfors-Schwarz Lemma. Thus, in the notation of the proof of Theorem 1, the closed (1,1)-

form ν can be taken to be $\mu = \omega_X - f^*\omega_Y$, where ω_X denotes the Kähler form of g on X, and ω_Y that of h on Y, so that $\mu = \frac{-c_1(X,g)}{n+1} + \frac{f^*c_1(Y,h)}{m+1}$. We have $\rho \geq 0$ from the definition of ρ in terms of σ, and $\mu \geq 0$ by the Ahlfors-Schwarz Lemma. When $S \subset X - \mathrm{Sing}(f)$ is smooth we have by Lemma 1

$$\rho^{m-n+1} = \mu^{m-n+1} = (\rho+\mu)^{m-n+1} = 0$$

identically on S, noting that Lemma 1 can be applied also to $\nu = \frac{\rho+\mu}{2} \geq 0$. When S is singular we can consider a desingularization $\zeta : \widetilde{S} \to S$ and the smooth closed (1,1)-form $\zeta^*\rho$ on $\widetilde{S}$, etc. in place of considering the restriction of ρ to the singular variety S. For the sake of brevity in place of specifying a desingularization we will speak of the restriction of ρ, etc., to the smooth part $\mathrm{Reg}(S)$ of S, written $\rho|_{\mathrm{Reg}(S)}$.

There is a smallest integer $r \geq 1$ such that $[\nu^r] = 0$ for $[\nu] = \frac{-c_1(X)}{n+1} + \frac{f^*c_1(Y)}{m+1} \geq 0$. The positive integer r is determined by the fact that the real-analytic semipositive closed $(1,1)-$form $\rho|_{\mathrm{Reg}(S)}$ has exactly $r-1$ non-zero eigenvalues on a dense open subset of S. Since ρ, μ and $\frac{\rho+\mu}{2}$ are cohomologous when pulled back to a desingularized model $\widetilde{S}$ we have

$$\rho^r = \mu^r = (\rho+\mu)^r = 0$$

on S. Thus, on a dense subset W of S, both ρ and μ have exactly $r-1$ non-zero eigenvalues over W, and they must have the same kernel over W. Note here that for $y \in W$, the vector subspaces $\mathrm{Ker}(\rho(y))$ and $\mathrm{Ker}(\mu(y))$ of $T_y(S)$ must agree with each other. Otherwise, $\dim(\mathrm{Ker}(\rho(y)) \cap \mathrm{Ker}(\mu(y))) < n-r+1$ and $(\rho(y)+\mu(y))^r \neq 0$, while $\rho^r \equiv 0$ over $\mathrm{Reg}(S)$, violating the fact that $\frac{\rho+\mu}{2}$ and ρ are cohomologous to each other when pulled back to a desingularized model $\widetilde{S}$ of S.

Suppose there exists a sufficiently deformable irreducible compact complex-analytic $(m-n+1)$-dimensional subvariety $S \subset X$. In the notations of the definition of such subvarieties as given in §1, Definition 1, without loss of generality we may assume that there exists a holomorphic family $\pi : \mathscr{S} \to B$ of irreducible compact complex-analytic subvarieties $S_t \subset X = \pi^{-1}(t), t \in B$, parametrized by the complex unit ball B of a complex Eulcidean space, such that $S_0 = S$ and such that there exists a point $x \in W \subset S$ so that the holomorphic tangent spaces $T_x(S')$ of those $S' = S_t, t \in B$, passing through x wipes out an open neighborhood of $[T_x(S_0)]$ on $Gr(p, T_x(S))$. Thus $\mathrm{Ker}(\rho(x)|_{T_x(S')})$ and $\mathrm{Ker}((\rho(x)+\mu(x))|_{T_x(S')})$ are of codimension $r-1$ in $T_x(S')$. We conclude from the cohomological argument of the last paragraph that $\mathrm{Ker}(\rho(x)|_{T_x(S')}) = \mathrm{Ker}(\mu(x)|_{T_x(S')})$. For any q-plane $E \subset T_x(X)$ sufficiently close to $T_x(S)$ by assumption there exists some $t \in B$ such that $E = T_x(S')$ for $S' = S_t$. It follows that $\mathrm{Ker}(\rho(x)) = \mathrm{Ker}(\mu(x)) \subset T_x(X)$ is of codimension $r-1$, i.e., of dimension $n-r+1$. For a sufficiently small open neighborhood U of x in the ambient manifold X, the preceding discussion applies with x replaced by $y \in U$ and S replaced by some irreducible compact complex-analytic $(m-n+1)$ dimensional subvariety belonging to $(S_t)_{t\in B}$ and passing through y. Write $d = n-r+1$. Noting that $r \leq m-n+1$, we have $d \geq 2n-m$. Then, U is foliated by a holomorphic family of totally geodesic complex-analytic submanifolds Λ such that $\Lambda \subset X$ is totally geodesic, and such that $f|_\Lambda$ is a totally geodesic holomorphic isometric embedding. Lifting U to B^n and lifting Y locally to B^m, we have a holomorphic map $f : U \to B^m$ which is a leafwise totally geodesic holomorphic isometric embedding. It remains now to investigate whether such holomorphic maps can exist at all. In the next sections we will show that such maps do not exist under certain dimension restrictions, viz., we will show that leafwise totally geodesic holomorphic isometric embeddings are already totally geodesic. In other words, we will derive a contradiction unless $d = n$.

For the sake of convenience we introduce the notion of a contracting leafwise totally geodesic isometric embeddings, as follows.

Definition 2 (Contracting leafwise totally geodesic isometric embedding). *Let n,m be positive integers, $n < m$, $U \subset B^n$ be a connected open subset, and $f : U \to B^m$ be a holomorphic map. We say that f is contracting if and only if it is distance-decreasing when B^n resp. B^m are equipped with the canonical Kähler-Einstein metric $ds^2_{B^n}$ resp. $ds^2_{B^m}$ of constant holomorphic sectional curvature $-K$ for the same constant $K > 0$. Suppose f is an immersion and, denoting by σ the (holomorphic) second fundamental form on $(1,0)$-vectors for the immersion $f : U \to B^m$ with respect to $ds^2_{B^m}$, $\mathrm{Ker}(\sigma(x)) = \mathrm{Ker}(\rho(x)))$ is of the same rank d at every point $x \in U$. Denoting by $\mathscr{E} = \mathrm{Re}(\mathrm{Ker}(\rho))$ the associated integrable holomorphic foliation, assume that for each leaf Λ of $\mathscr{E}$, the restriction $f|_\Lambda : \Lambda \to B^m$ is a totally geodesic isometric embedding. Then, we say that $f : U \to B^m$ is a contracting leafwise totally geodesic isometric embedding (of leaf dimension d).*

REMARKS

1. (a) In place of the complex unit ball B^n resp. B^m we can consider the quotient manifold $X := B^n/\Gamma$ resp. $Y = B^m/\Psi$ with respect to a torsion-free discrete group of automorphisms Γ resp. Ψ, a connected open subset $U \subset X$, and a holomorphic immersion $f : U \to Y$. In this general situation we have analogously the notion of a contracting leafwise totally geodesic immersion, where the restriction of f to each totally geodesic leaf Λ of the analogously defined holomorphic foliation $\mathscr{E}$ is only assumed to be an isometric immersion.

2. (b) By Umehara [Um87] any isometric holomorphic immersion of an open subset of a complex hyperbolic space form into B^m is necessarily totally geodesic. In the terminology of a 'contracting leafwise totally geodesic isometric embedding (immersion)', it is implicit that the mapping is totally geodesic.

Summarizing in terms of the newly introduced terminology we have proven in this section

Proposition 1. *Let n,m be positive integers, $n < m$, $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free discrete group of automorphisms, $X := B^n/\Gamma$, and $\Psi \subset \mathrm{Aut}(B^m)$ be a torsion-free discrete group of automorphisms, $Y := B^n/\Psi$. Let $f : X \to Y$ be a generically immersive holomorphic map. Suppose there exists on $X - \mathrm{Sing}(f)$ a sufficiently deformable compact complex-analytic subvariety S of dimension $m-n+1$. Then, for a general point $x \in X - \mathrm{Sing}(f)$, there exists a connected open neighborhood W of x on $X - \mathrm{Sing}(f)$ and a positive integer d, $2n-m \le d \le n$, such that $f|_W : W \to Y$ is a contracting leafwise totally geodesic isometric immersion of leaf dimension d.*

§4. A commutation formula

In this section we will derive a commutation formula for the Hessian of a contracting leafwise totally geodesic holomorphic isometric embedding $f : U \to B^m$ defined on a connected open subset $U \subset B^m$, where the underlying holomorphic foliation $\mathscr{E}$ is defined by $\mathrm{Re}(\mathrm{Ker}(\rho)) = \mathrm{Re}(\mathrm{Ker}(\sigma))$. We write $E \subset T_U$ for the holomorphic vector subbundle given by $E_x = \mathrm{Ker}(\rho(x))$. Following the conventions in the proof of [§2, Theorem 1] we will normalize holomorphic sectional curvatures to be -4π. Denote by g resp. h the canonical Kähler-Einstein metric on B^n resp. B^m of constant holomorphic sectional curvature -4π. We will be performing covariant differentiation on tensors fields on U. By ∇ we will denote covariant differentiation with respect to the canonical connections associated to g and h. Thus, ∂f is a holomorphic section of $\Omega_U \otimes f^*T_{B^m}$ over U, where Ω_U stands for the holomorphic cotangent bundle T_U^*, and $\nabla\partial f$ is a smooth section of $\Omega_U \otimes \Omega_U \otimes f^*T_{B^m}$ defined in terms of the affine connection on $T_U^* \otimes f^*T_{B^m}$ induced by the Riemannian connection of (B^n, g) and the pull-back of the Riemannian connection on (B^m, h). From the torsion-freeness of Riemannian connections it follows that the tensor field $\nabla\partial f$ takes values in $S^2\Omega_U \otimes f^*T_{B^m}$, i.e., $\nabla_\alpha\partial_\beta f$ is symmetric in α and β. We have the following commutation formula on $\nabla\partial f$.

Proposition 2. *Let $x \in U$, $\xi \in E_x$ and $\nu \in T_x(U)$. Denote by $\widetilde{\xi}$ an extension of $\xi \in E_x$ to a smooth E-valued vector field on some neighborhood of x on U. Then,*

$$\frac{1}{2\pi}\|\nabla_\xi \partial_\nu f\|^2 = \|\xi\| \left(\|\nu\|^2 - \|\partial f(\nu)\|^2\right) - \left(\|\nabla_\nu \widetilde{\xi}\|^2 - \|\partial f(\nabla_\nu \widetilde{\xi})\|^2\right).$$

To simplify notations in what follows we will often write ξ for $\widetilde{\xi}$, etc. whenever there is no risk of confusion. Thus ξ will denote both a vector in E_x and a germ of smooth E-valued vector field at x extending that vector. Additional conditions may be imposed on the choice of smooth extensions for the computations. For the proof of Proposition 2 we start with a lemma regarding values of $\nabla\partial f$, as follows.

Lemma 2.2. *Let $x \in U$ and denote by σ, τ arbitrary smooth $(1,0)$-vector fields on a neighborhood of x, and ξ any smooth E-valued vector field on a neighborhood of x. Then, $h\left(\nabla_\sigma \partial_\tau f(x), \overline{\partial_\xi f}\right) = 0$.*

Proof. For the proof of Lemma 2 without loss of generality we may assume that $\widetilde{\xi}$ is a holomorphic E-valued holomorphic vector field on a neighborhood of s in U. Since f is isometric on each leaf Λ of $\mathscr{E}$, $E \subset \mathrm{Ker}\,(\mu)$ for $\mu := \omega_g - f^*\omega_h \geq 0$, where ω_g is the Kähler form (B^n, g) and ω_h is the Kähler form of (B^m, h). In particular, we have

$$h(\partial_\tau f, \overline{\partial_\xi f}) = g\left(\tau, \overline{\xi}\right). \tag{1}$$

Differentiating against the vector field σ we have

$$\begin{aligned} &h\left(\nabla_\sigma \partial_\tau f, \overline{\partial_\xi f}\right) + h\left(\partial_{\nabla_\sigma \tau} f, \overline{\partial_\xi f}\right) + h\left(\partial_\tau f, \overline{\nabla_{\overline{\sigma}} \partial_\xi f}\right) + h\left(\partial_\tau f, \overline{\partial_{\nabla_{\overline{\sigma}}\xi} f}\right) \\ &\qquad = g\left(\nabla_\sigma \tau, \overline{\xi}\right) + g\left(\tau, \overline{\nabla_{\overline{\sigma}} \xi}\right). \end{aligned} \tag{2}$$

For the last term on the left-hand side of (2), by assumption ξ is a holomorphic vector field, hence $\nabla_{\overline{\sigma}}\xi = 0$ and we have

$$h\left(\partial_\tau f, \overline{\partial_{\nabla_{\overline{\sigma}}\xi} f}\right) = g\left(\tau, \overline{\nabla_{\overline{\sigma}}\xi}\right) = 0. \tag{3}$$

For the third term we have by symmetry

$$\nabla_{\overline{\sigma}} \partial_\xi f = \nabla_\xi \partial_{\overline{\sigma}} f = 0 \tag{4}$$

since f is holomorphic. For the second term, since $\xi(y) \in \mathrm{Ker}\,(\mu(y))$ where defined, we have

$$h\left(\partial_{\nabla_\sigma \tau} f, \overline{\partial_\xi f}\right) = g\left(\nabla_\sigma \tau, \overline{\xi}\right). \tag{5}$$

We conclude therefore from (2) that

$$h\left(\nabla_\sigma \partial_\tau f, \overline{\partial_\xi f}\right) = 0\,. \tag{6}$$

as desired. ■

Next, from standard commutation formulas for covariant differentiation on Hermitian holomorphic vector bundles on Kähler manifolds we have

Lemma 2.3. *Denote by R the curvature tensor on (U,g) and by S the curvature tensor of $(f^*T_{B^m}, f^*h)$ on U. Let σ,τ,ζ be smooth vector fields on U. Then, we have*

$$\nabla_{\overline{\zeta}} \nabla_\sigma \partial_\tau f^k = \sum_\mu R_{\sigma\overline{\zeta}\tau}{}^{\mu}\, \partial_\mu f^k - \sum_\ell S_{\sigma\overline{\zeta}\ell}{}^{k}\, \partial_\tau f^\ell\,,$$

where at $x \in U$, the symbol $\{\mu\}$ runs over the set of indexes of a basis $\{e_\mu\}$ of $T_{U,x}$, and the symbol ℓ runs over the set of indexes of a basis $\{\varepsilon_\ell\}$ of $f^*(T_{B^m,f(x)})$.

Proof. Since f is holomorphic we have $\nabla_{\overline{\zeta}}\partial_\tau f = \nabla_\tau \partial_{\overline{\zeta}} f = 0$. Lemma then follows from standard commutation formulas for Hermitian holomorphic vector bundles for the computation of $\nabla_\sigma \nabla_{\overline{\zeta}} \partial_\tau f^k - \nabla_{\overline{\zeta}} \nabla_\sigma \partial_\tau f^k = -\nabla_{\overline{\zeta}} \nabla_\sigma \partial_\tau f^k$.

We are now ready to derive Proposition 2.

Proof of Proposition 2. Let $x \in U$. We apply Lemma 2 to a special choice of vectors σ,τ,ζ at x, extended to smooth vector fields on U. Let ζ and σ be the same E-valued holomorphic vector field ξ on U such that $\xi(x)$ is of unit length, shrinking the neighborhood U of x if necessary. Let τ be an $E^\perp$-valued smooth vector field ν such that $\nu(x)$ is of unit length. Again shrinking U if necessary let $\{e_\mu\}$ be a smooth basis of T_U which is orthonormal at the point x and which includes at x the orthogonal unit vectors $\xi(x)$ and $\nu(x)$. Then, we have

$$\begin{aligned} R_{\sigma\overline{\zeta}\tau}{}^{\mu}(x) &= R_{\sigma\overline{\zeta}\tau\overline{\mu}}(x) = R_{\xi\overline{\xi}\nu\overline{\mu}}(x) \\ &= \begin{cases} -2\pi & \text{if } e_\mu(x) = \nu(x) \\ 0 & \text{otherwise} \end{cases}. \end{aligned} \tag{1}$$

Denote by R' the curvature tensor of (B^m,h). For (1,0)-vectors α,β,γ at $x \in U$ we have

$$S_{\alpha\overline{\beta}\gamma} = f^*R'\left(\partial f(\alpha), \overline{\partial f(\beta)}; \partial f(\gamma)\right) \in f^*T_{B^m,f(x)}\,. \tag{2}$$

At the point $x \in U$, for the subset $\{e_\lambda(x)\}$ of unit vectors in $\{e_\mu(x)\}$ belonging to E_x, define $\varepsilon_\lambda(x) := \partial f(e_\lambda(x))$. Since $\partial f(x) : T_x(U) \to T_{f(x)}(B^m)$ restricts to a linear isometry on $E_x \subset T_x(U)$, the set $\{\varepsilon_\lambda(x)\}$ constitutes an orthonormal basis of $\partial f(E_x) \subset T_{B^m,f(x)}$. In the sequel for simplicity we will sometimes identify $T_{B^m,f(x)}$

with $f^*T_{B^m,f(x)}$ tautologically in the notation. Complete now $\{\varepsilon_\lambda\}$ to a smooth basis $\{\varepsilon_\ell\}$ of $f^*T_{B^m}$ on a neighborhood of x in such a way that $\{\varepsilon_\ell(x)\}$ is an orthonormal basis of $f^*T_{B^m,f(x)}$ with a further specification, as follows. For $\zeta \in T_{U,x}$ we will also write ζ' for $\partial f(\zeta)$. Since $E_x \subset T_{U,x}$ lies on $\mathrm{Ker}(\mu)$, $\nu'(x) = \partial f(\nu(x))$ is orthogonal to $\partial f(\xi)$ for any $\xi \in E_x$. Thus, the orthonormal basis $\{\varepsilon_\lambda(x)\}$ of $\partial f(E_x)$ can be completed to an orthonormal basis $\{\varepsilon_\ell(x)\}$ such that one of the basis vectors is the unit vector $\nu'' := \frac{\nu'(x)}{\|\nu'(x)\|}$, which is proportional to $\nu'(x)$. We will choose a smooth basis $\{\varepsilon_\ell\}$ of $f^*T_{B^m}$ on some neighborhood of x such that $\{\varepsilon_\ell(x)\}$ is an orthonormal basis of $f^*T_{B^m,f(x)}$ with the latter property. Furthermore such a basis will be chosen such that $\{\varepsilon_\ell\}$ corresponds on a neighborhood of $x \in U$ to $f^*\frac{\partial}{\partial w_\ell}$ for some holomorphic coordinates $(w_1, \cdots, w_m)$ on a neighborhood of $f(x)$ in $\mathbb{C}^m$. We have

$$\begin{aligned} & S_{\sigma\overline{\zeta}\ell} = S_{\nu\overline{\xi}\ell} \\ = R'_{\nu'\overline{\xi'}\ell}(x) = & \begin{cases} -2\pi\|\nu'(x)\|\nu'' & \text{if } \varepsilon_\ell(x) = \nu'' \\ 0 & \text{otherwise} \end{cases} . \end{aligned} \tag{3}$$

Let now ξ be an E-valued holomorphic vector field on U. By Lemma 2 we have

$$h\left(\nabla_\xi \partial_\nu f(x), \overline{\partial_\xi f}\right) = 0\,. \tag{4}$$

Differentiating with respect to $\overline{\nu}$ we have

$$\begin{aligned} & h\left(\nabla_{\overline{\nu}}\nabla_\xi \partial_\nu f, \overline{\partial_\xi f}\right) + h\left(\nabla_{\partial_{\overline{\nu}}\xi}\partial_\nu f, \overline{\partial_\xi f}\right) + h\left(\nabla_\xi \partial_{\nabla_{\overline{\nu}}\nu} f, \overline{\partial_\xi f}\right) \\ & \quad + h\left(\nabla_\xi \partial_\nu f, \overline{\nabla_\nu \partial_\xi f}\right) + h\left(\nabla_\xi \partial_\nu f, \overline{\partial_{\nabla_\nu \xi} f}\right) = 0\,. \end{aligned} \tag{5}$$

By Lemma the second and the third terms on the left-hand side of (5) vanish. By the symmetry of the Hessian we have $\nabla_\nu \partial_\xi f = \nabla_\xi \partial_\nu f$ and hence

$$h\left(\nabla_{\overline{\nu}}\nabla_\xi \partial_\nu f, \overline{\partial_\xi f}\right) + \|\nabla_\xi \partial_\nu f\|^2 + h\left(\nabla_\xi \partial_\nu f, \overline{\partial_{\nabla_\nu \xi} f}\right) = 0\,. \tag{6}$$

We proceed to compute the first and the third terms of the left-hand side of (6). For the first term by Lemma 3 and by the symmetry of the Hessian we have

$$\begin{aligned} \nabla_{\overline{\nu}}\nabla_\xi \partial_\nu f^k(x) = \nabla_{\overline{\nu}}\nabla_\nu \partial_\xi f^k(x) &= \sum_\mu R_{\nu\overline{\nu}\xi}{}^{\mu}\, \partial_\mu f^k(x) - \sum_\ell S_{\nu\overline{\nu}\ell}{}^{k}\, \partial_\xi f^\ell(x) \\ &= \sum_\mu R_{\nu\overline{\nu}\xi}{}^{\mu}\, \partial_\mu f^k(x) - \sum_\ell R'_{\nu'\overline{\nu'}\ell}{}^{k}\, \partial_\xi f^\ell(x)\,. \end{aligned} \tag{7}$$

For the proof of Proposition 2 without loss of generality we may assume that $\xi(x)$ and $\nu(x)$ are (orthogonal) unit vectors. On a neighborhood of x, we use the same choice of a smooth basis $\{e_\mu\}$ and a smooth basis $\{\varepsilon_\ell\}$ of $f^*T_{B^m}$

as in the above, so that in particular $\{e_\mu(x)\}$ is an orthonormal basis of $T_{U,x}$ at x and $\{\varepsilon_\ell(x)\}$ is an orthonormal basis of $f^*T_{B^m,f(x)}$ at x. Write $\xi(x)=e_a$, $\nu(x)=e_b$. Recall the notation $\zeta' := \partial f(\zeta)$ for $(1,0)$-vectors ζ on U. We write also $\xi'(x)=\varepsilon_a$. Recall also $\nu'' := \frac{\nu'(x)}{\|\nu'(x)\|}=\varepsilon_b$. For the first summation on the last line of (7) the only possibly non-zero summand arises when $\mu=a$, giving $R_{\nu\overline{\nu}\xi\overline{\xi}}\partial_\xi f^k(x) = -2\pi\partial_\xi f^a$ when $k=a$ and 0 otherwise. For the second summation the only possibly non-zero summand arises when $\ell=a$, giving $R'_{\nu'\overline{\nu'}\ell}{}^k\ \partial_\xi f^\ell(x) = \|\nu'\|^2 R'_{\nu''\overline{\nu''}\xi'\overline{\xi'}}\partial_\xi f^a(x) = -2\pi\partial_\xi f^a(x)$ when $k=a$ and 0 otherwise. It follows from (7) that

$$\nabla_{\overline{\nu}}\nabla_\nu\partial_\xi f(x) = \sum_k\left(\sum_\mu R_{\nu\overline{\nu}\xi}{}^\mu\ \partial_\mu f^k(x) - \sum_\ell R'_{\nu'\overline{\nu'}\ell}{}^k\ \partial_\xi f^\ell(x)\right)\otimes\varepsilon_k(x)$$
$$= -2\pi\big(\partial_\xi f(x) - \|\partial_\nu f\|^2\partial_\xi f(x)\big)\,. \qquad (8)$$

Plugging into (5) and without assuming that $\xi(x)$ and $\nu(x)$ are of unit length the first term there on the left-hand side is given by

$$h\left(\nabla_{\overline{\nu}}\nabla_\xi\partial_\nu f,\overline{\partial_\xi f}\right) = -2\pi\|\xi\|^2\left(\|\nu\|^2 - \|\partial f(\nu)\|^2\right). \qquad (9)$$

For the proof of Proposition 2 it remains to deal with the last term $h\left(\nabla_\xi\partial_\nu f,\overline{\partial_{\nabla_\nu\xi}f}\right)$ on the left-hand side of (5). Recall that

$$h\left(\partial_\xi f,\overline{\partial_{\nabla_\nu\xi}f}\right) = g\left(\xi,\overline{\nabla_\nu\xi}\right). \qquad (10)$$

Differentiating against ν we have

$$h\left(\nabla_\nu\partial_\xi f,\overline{\partial_\xi f}\right) + h\left(\partial_{\nabla_\nu\xi}f,\overline{\partial_{\nabla_\nu\xi}f}\right) + h\left(\partial_\xi f,\overline{\nabla_{\overline{\nu}}\partial_{\nabla_\nu\xi}f}\right) + h\left(\partial_\xi f,\overline{\partial_{\nabla_{\overline{\nu}}(\nabla_\nu\xi)}f}\right)$$
$$= g\left(\nabla_\nu\xi,\overline{\nabla_\nu\xi}\right) + g\left(\xi,\overline{\nabla_{\overline{\nu}}(\nabla_\nu\xi)}\right). \qquad (11)$$

By the symmetry of the Hessian, the pluriharmonicity of f, i.e., $\overline{\nabla}\partial f=0$, and the identity $h\left(\partial_\xi f,\overline{\partial_\tau f}\right) = g(\xi,\tau)$ for any tangent vector field τ, the equation (11) gives

$$h\left(\nabla_\xi\partial_\nu f,\overline{\partial_\xi f}\right) + h\left(\partial_{\nabla_\nu\xi}f,\overline{\partial_{\nabla_\nu\xi}f}\right) = g\left(\nabla_\nu\xi,\overline{\nabla_\nu\xi}\right). \qquad (12)$$

In other words, we have

$$\begin{aligned} h\left(\nabla_\xi\partial_\nu f,\overline{\partial_\xi f}\right) &= g\left(\nabla_\nu\xi,\overline{\nabla_\nu\xi}\right) - h\left(\partial_{\nabla_\nu\xi}f,\overline{\partial_{\nabla_\nu\xi}f}\right) \\ &= \|\nabla_\nu\xi\|^2 - \|\partial f(\nabla_\nu\xi)\|^2. \end{aligned} \qquad (13)$$

Substituting (9) and (13) into (6) we deduce

$$\frac{1}{2\pi}\|\nabla_\xi \partial_v f\|^2 = \|\xi\|\left(\|v\|^2 - \|\partial f(v)\|^2\right) - \left(\|\nabla_v \xi\|^2 - \|\partial f(\nabla_v \xi)\|^2\right) , \qquad (14)$$

proving Proposition 2, as desired. ■

REMARKS In the proof of Proposition 2, the expression $\nabla_v \xi(x) = \nabla_v \widetilde{\xi}(x)$ depends on the choice of extension of the vector $\xi \in E_x$ to a germ of E-valued holomorphic section $\widetilde{\xi}$ at x, although the notation $\widetilde{\xi}$ is suppressed in the formulas. In the final outcome as given in the identity (14) there, if $\widetilde{\xi}$ is replaced by another smooth extension $\xi^\sharp$, then

$$\nabla_v \xi^\sharp(x) - \nabla_v \widetilde{\xi}(x) := \eta(x) \in E_x \; ; \quad \text{and} \quad \nabla_v \xi^\sharp(x) = \nabla_v \widetilde{\xi}(x) + \eta(x)$$

is an orthogonal decomposition such that $\partial f\big(\nabla_v \xi^\sharp(x)\big) = \partial f\big(\nabla_v \widetilde{\xi}(x)\big) + \partial f\big(\eta(x)\big)$ is again an orthogonal decomposition, while $\|\eta(x)\| = \|\partial f(\eta(x))\|$ since $\eta(x) \in E_x$ and $E_x \subset \mathrm{Ker}\,(\mu(x))$.

§5. Consequences of the commutation formula

We start with the following general result on contracting leafwise totally geodesic holomorphic isometric embeddings from connected open subsets of the complex unit ball into complex unit balls.

Theorem 2. *Let n, m, d be positive integers, $m \le 2n-4$, $3 \le d \le n$. Let $U \subset B^n$ be a nonempty connected open subset, and $\mathscr{E}$ be a holomorphic foliation on U by d-dimensional holomorphic totally geodesic complex submanifolds Λ. Let $f : U \to B^m$ be a contracting (distance-decreasing) holomorphic mapping such that $f|_\Lambda$ is a totally geodesic isometric embedding for each leaf Λ. Assume that the foliation $\mathscr{E}$ is defined by* $\mathrm{Re}(\mathrm{Ker}(\rho))$ *for the closed $(1,1)$-form given by $\rho = \frac{-c_1(X, f^*h)}{n+1} + \frac{f^*c_1(Y,h)}{m+1} \ge 0$, where g (resp. h) stands for the canonical Kähler-Einstein metric on B^n (resp. B^m) of constant holomorphic sectional curvature $-K$ for any fixed $K > 0$. Assume furthermore that* $\mathrm{Ker}(\rho) = \mathrm{Ker}(\mu)$ *for $\mu = \frac{-c_1(X,g)}{n+1} + \frac{f^*c_1(Y,h)}{m+1} \ge 0$. Then, $\rho \equiv 0$, $\mu \equiv 0$, $\mathscr{E}$ is trivial, and $f : U \to B^m$ is a totally geodesic isometric embedding.*

Proof. For the formulation and proof of Theorem 2 the choice of the constant $K > 0$ is unimportant. For the sake of uniformity we will choose K to be 4π as in the statement of [§4, Proposition 2]. In this case μ agrees with the formula $\mu = \omega_g - f^*\omega_h$ given in the proof of [§4, Lemma 2]. Recall that the holomorphic foliation $\mathscr{E}$ on

U corresponds to a d-dimensional holomorphic distribution which we denote by $E \subset T_U$. Let now $x \in U$, $\xi \in E_x$ and $\nu \in T_{U,x}$. By Proposition 2, we have

$$\frac{1}{2\pi}\|\nabla_\xi \partial_\nu f\|^2 = \|\xi\|\left(\|\nu\|^2 - \|\partial f(\nu)\|^2\right) - \left(\|\nabla_\nu \xi\|^2 - \|\partial f(\nabla_\nu \xi)\|^2\right). \qquad (1)$$

Recall from the Remarks after the proof of [§4,Proposition 2] that in the commutation formula (1) it is understood that ξ is extended to a smooth vector field $\widetilde{\xi}$ on a neighborhood of x in U. The expression $\nabla_\nu \widetilde{\xi}(x)$ is then uniquely defined only modulo E_x, but the expression $\left(\|\nabla_\nu \widetilde{\xi}\|^2 - \|\partial f(\nabla_\nu \widetilde{\xi})\|^2\right)(x)$ is independent of the extension $\widetilde{\xi}$ since $\partial f(x)$ is an isometry on E_x, and since $\partial f(E_x)$ is orthogonal to $\partial f(E_x^\perp)$. Define now $T : E^\perp \otimes E \to E^\perp$ by $T(\nu \otimes \xi) = \mathrm{pr}_{E^\perp}(\nabla_\nu(\widetilde{\xi}))$ for $\nu \in E_x^\perp$, $\xi \in E_x$, where $\mathrm{pr}_{E^\perp} : T \to E^\perp$ denotes the orthogonal projection. Then, $\left(\|\nabla_\nu \widetilde{\xi}\|^2 - \|\partial f(\nabla_\nu \widetilde{\xi})\|^2\right)(x) = \|T(\nu,\xi)(x)\|^2$. Here we write $T(\nu,\xi)$ for $T(\nu \otimes \xi)$ and the same notational convention will be adopted for linear maps on tensor products will be adopted elsewhere.

Consider now the linear map $Q : E \otimes E^\perp \to f^*T_{B^m}$ given at $x \in U$ by $Q(\xi \otimes \nu) = Q(\xi,\nu) = \nabla_\xi \partial_\nu f$ for $\xi \in E_x$, $\nu \in E_x^\perp$. The identity (1) translates into an identity of the form

$$\frac{1}{2\pi}\|Q(\xi \otimes \nu)\|^2 = P(\xi \otimes \nu, \overline{\xi \otimes \nu}) - \|T(\xi \otimes \nu)\|^2, \qquad (2)$$

where $P(\cdot,\bar{\cdot})$ is the unique Hermitian bilinear form on $E \otimes E^\perp$ which satisfies $P\left(\xi \otimes \nu, \overline{\xi \otimes \nu}\right) = \|\xi\|^2\left(\|\nu\|^2 - \|\partial f(\nu)\|^2\right)$. Denote by $\pi(\cdot,\bar{\cdot})$ the Hermitian bilinear form on E given by $\pi(\nu,\overline{\mu}) = g(\nu,\overline{\mu}) - h\big(\partial f(\nu), \overline{\partial f(\mu)}\big)$ for $\nu,\mu \in E_x, x \in U$. Then $\pi(\nu,\overline{\nu}) > 0$ whenever $\nu \in E^\perp$ is non-zero since ∂f is strictly distance-decreasing on $E^\perp$. For $x \in U$. Let now $\{\xi_1,\cdots\xi_d\}$ be an orthonormal basis of E_x and $\{\nu_1,\cdots\nu_{n-d}\}$ be an orthonormal basis of $E_x^\perp$ consisting of eigenvectors of the Hermitian form π_x. Thus, $\pi(\nu_j,\overline{\nu_\ell}) = 0$ for $j \neq \ell$, $1 \leq j,\ell \leq n-d$ and $\pi(\nu_j,\overline{\nu_j}) = \lambda_j > 0$. Then, for

$$\tau = \sum_{i=1}^{d}\sum_{j=1}^{n-d} a_{ij}\xi_i \otimes \nu_j. \qquad (3)$$

we have

$$P(\tau,\overline{\tau}) = \sum_{i=1}^{d}\sum_{j=1}^{n-d} \lambda_j^2 |a_{ij}|^2. \qquad (4)$$

In particular $P(\cdot,\bar{\cdot})$ is positive definite. From (2), for $\tau \in E \otimes E^\perp$ we have

$$\frac{1}{2\pi}\|Q(\tau)\|^2 = P(\tau,\overline{\tau}) - \|T(\tau)\|^2, \qquad (5)$$

We examine further the identity (5). Now $\mathrm{rank}(E^{\perp} \otimes E) = (n-d)d$ and $\mathrm{rank}(E^{\perp}) = n-d$. By assumption $d \geq 3$ so that $\dim(\mathrm{Ker}(T_x)) \geq (n-d)(d-1) \geq 2(n-d) > 0$ whenever $3 \leq d < n$. (The case $d = n$ means precisely that f is totally geodesic.) By Lemma 2.2 we have $h\left(\nabla_{\xi}\partial_{\nu} f, \overline{\partial_{\eta} f}\right) = 0$ whenever $\eta \in E_x$, so that $\mathrm{Im}(Q_x)$ lies in the orthogonal complement H_x of $\partial f(E_x)$ in $T_{B^m, f(x)}$, where $\dim(H_x) = m - d$. By the preceding paragraph $\dim(\mathrm{Ker}(T_x)) = (n-d)(d-1)$, hence $\dim(\mathrm{Ker}(T_x) \cap \mathrm{Ker}(Q_x)) \geq (n-d)(d-1) - (m-d) \geq (n-d+1)(d-1) - (2n-5)$. Suppose $(3 \leq) \, d \neq n$. Then, $(n-d+1)(d-1) \geq 2(n-2)$, where equality is attained precisely when $d = 3$ and $d = n-1$, and we have $\dim(\mathrm{Ker}(T_x) \cap \mathrm{Ker}(Q_x)) \geq 1$. Let $\tau \in \mathrm{Ker}(T_x) \cap \mathrm{Ker}(Q_x)$ be a non-zero element. Then from the identity (5) we have

$$P(\tau, \overline{\tau}) = \frac{1}{2\pi} \|Q(\tau)\|^2 + \|T(\tau)\|^2 = 0, \tag{6}$$

violating the positivity of P. Thus, a contradiction arises if $3 \leq d < n$. Since by assumption $d \geq 3$ it follows that the only possibility is that $d = n$. In other words, $f : U \to B^m$ is a totally geodesic embedding, as desired. ■

Dimension restrictions have been placed on n, m and the leaf dimension d of the holomorphic foliation ξ. It is tempting to believe that such dimension restrictions are unnecessary. In the notations used in Theorem 2 we formulate a conjecture as follows.

Conjecture 1. *Let n, m be positive integers. Let $U \subset B^n$ be a nonempty connected open subset, $f : U \to B^m$ be a holomorphic immersion. Suppose there exists a nonzero integrable holomorphic distribution $E \subset T_U$ of rank $d > 0$ such that f is a contracting leafwise totally geodesic isometric embedding with respect to the holomorphic foliation $\mathscr{E}$ defined by* $\mathrm{Re}(E)$. *Assume furthermore that $E = \mathrm{Ker}(\rho) = \mathrm{Ker}(\mu)$ on U. Then, $E = T_U$ and f is totally geodesic.*

As a consequence of Theorem 2 we have the following general result about holomorphic mappings between complex unit balls equivariant with respect to a torsion-free discrete subgroup which is not necessarily a lattice. In particular, they are valid on domain complex hyperbolic space forms X of possibly infinite volume with respect to the canonical Kähler-Einstein metric.

Theorem 3. *Let n, m be positive integers, $\Gamma \subset \mathrm{Aut}(B^n)$ be a torsion-free discrete group of biholomorphic automorphisms, $X := B^n/\Gamma$. Let $\Phi : \Gamma \to \mathrm{Aut}(B^m)$ be a group homomorphism, and $f : B^n \to B^m$ be a holomorphic mapping which is equivariant with respect to Φ, i.e., $f(\gamma x) = \Phi(\gamma)(f(x))$ for any $x \in B^n$ and any $\gamma \in \Gamma$.*

Suppose $m \leq 2n-4$*,* f *is an immersion at a general point* $x \in B^n$*, and* f *is not totally geodesic. Then, writing* $\mathrm{Sing}(f)$ *for the singular locus of* f *(which is necessarily invariant under the action of* Γ*),* E *for the subvariety* $\mathrm{Sing}(f)/\Gamma \subset X$*, and* $Z := X - E$*, there does not exist any sufficiently deformable compact complex-analytic subvariety* $S \subset Z$ *of complex dimension* $m-n+1$*.*

Proof. Suppose there exists a sufficiently deformable compact complex-analytic subvariety $S \subset X$. By Proposition 1 there exists a non-empty connected open subset $U \subset B^n$ such that the restriction $f|_U : U \to B^m$ is a leafwise totally geodesic isometric embedding. Here the leaves of the underlying holomorphic foliation $\mathscr{E}$ by totally geodesic complex submanifolds are of complex dimension d, where d is the rank of $\mathrm{Ker}(\rho)$ and of $\mathrm{Ker}(\mu)$ at each point of U, and $d \geq 1+(n-(m-n+1)) = 2n-m \geq 4$. In particular Theorem 2 applies and f is totally geodesic. ■

Because of the dimension restriction $m \leq 2n-4$, Theorem 3 does not cover [§2, Theorem 2] for the cases where X is compact or noncompact and of finite volume. An affirmative solution of Conjecture 1 in the above would yield a geometric argument (after the global cohomological argument) for the proof of Theorem 2 which is completely local. For the latter purpose it would also suffice to establish a strengthened version of Theorem 2 in which the dimension restrictions in the hypothesis are relaxed to the conditions $m \leq 2n-1$ and $1 \leq d \leq n$.

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