# Optimal selling time in stock market over a finite time horizon*

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**Abstract**

In this paper, we examine the best time to sell a stock as close as possible to its highest price over a finite time horizon $[0, T]$, where the stock price is modelled by a geometric Brownian motion and the 'closeness' is measured by the relative error of the stock price to its highest price over $[0, T]$. More precisely, we want to optimize the expression:

$$V^* = \sup_{0 \leq \tau \leq T} \mathbb{E}[\frac{V_\tau}{M_T}],$$

where $(V_t)_{t\geq 0}$ is a geometric Brownian motion with constant drift $\alpha$ and constant volatility $\sigma > 0$, $M_t = \max_{0\leq s\leq t} V_s$ is the running maximum of the stock price, and the supremum is taken over all possible stopping times $0 \leq \tau \leq T$ adapted to the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of the stock price. The above problem has been considered by Shiryaev, Xu and Zhou (2008) and Du Toit and Peskir (2009). And in this paper we provide a independent proof that when $\alpha = \frac{1}{2}\sigma^2$, a selling strategy is optimal *if and only if* it sells the stock either at the terminal time $T$ or at the moment when the stock price hits its maximum price so far. Besides, when $\alpha > \frac{1}{2}\sigma^2$, selling the stock at the terminal time $T$ is the unique optimal selling strategy. Our approach to the problem is purely probabilistic and has been inspired by relating the notion of dominant stopping $\rho_\tau$ of a hitting time $\tau$ to the optimal stopping strategy arisen in the classical "Secretary Problem".

## 1 Introduction

Denote stock price process by $(V_t)_{t\geq 0}$ which is modelled by a geometric Brownian motion with constant drift $\alpha$ and constant volatility $\sigma > 0$, i.e.,

$$\frac{dV_t}{V_t} = \alpha dt + \sigma d\mathbb{B}_t, \quad V_0 = 1.$$

where $\mathbb{B}_t$ is a standard Brownian Motion. As an application of Ito's lemma

$$V_t = e^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma \mathbb{B}_t}. \tag{1.1}$$

*Our research problem was motivated by a conference talk delivered by Zhou [11] in 2008 International Conference on Mathematics of Finance and Related Applications in Hong Kong. Our approach to the problem was inspired by the unique optimal strategy in the classical "Secretary Problem". The solution to the problem has first been announced by the first author in the AMSS-PolyU Joint Research Workshop 2008 at late June in Hong Kong, and then later in 12th International Congress on Insurance: Mathematics and Economics in July in Dalian and in the 4-th Sino-Japanese Optimization Meeting 2008. We thank the participants of those conferences and seminars for their comments and suggestions.

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We consider the problem:

$$V^* = \sup_{0\le\tau\le T} \mathbb{E}[\frac{V_\tau}{M_T}], \tag{1.2}$$

where $M_t = \max_{0\le s\le t} V_s$ is the running maximum of the stock price. In financial terms, we look for the best time to sell the stock as close as possible to its highest price over $[0, T]$; in mathematical terms, we look for an optimal stopping time to stop a stochastic process as close as possible to its ultimate maximum. For the purpose of presentation, in the rest of this paper we shall use the term 'selling time' (or 'selling strategy') and 'stopping time' interchangeably. To our best knowledge, the above problem was first formulated in the present form by Shiryaev on the first Bachelier Congress held in Paris, 2000 (see Shiryaev (2002)). Problem (1.2) and its variants, known as optimal prediction problems, were initiated by Graversen, Peskir and Shiryaev (2001) in which the 'closeness' was measured by mean square difference:

$$V^* = \inf_{0\le\tau\le T} E\left[(\mathbb{B}_\tau - S_T)^2\right], \tag{1.3}$$

where $\mathbb{B}_t$ is a standard Brownian Motion, and $S_t := \max_{0\le s\le t} \mathbb{B}_s$ is the maximum process of $(\mathbb{B}_t)_{t\ge0}$. In their paper, they first converted the expectation in (1.3) into an expectation of a functional of a certain process adapted to $(\mathcal{F}_t)_{t\ge0}$, that is, they converted the problem into a standard optimal stopping problem (see Peskir and Shiryaev (2006) Chap. I for general theory of optimal stopping), and then used a change of time technique to reduce their problem into one of solving an one-dimensional free-boundary value problem, which leads to an explicit solution of the optimal stopping boundary. Using similar approach, Pedersen (2003) also solved the extended problem

$$V^* = \inf_{0\le\tau\le T} E\left[(S_T - \mathbb{B}_\tau)^q\right],\ 0 < q < \infty. \tag{1.4}$$

However, when one considered problem (1.3) with standard Brownian motion replaced by Brownian motion with drift, the method of time change does not work anymore, that is, the problem

$$V^* = \inf_{0\le\tau\le T} E\left[(\mathbb{B}^\mu_\tau - S^\mu_T)^2\right] \tag{1.5}$$

can-not be solved using old techniques, where $\mathbb{B}^\mu_t = \mu t + \mathbb{B}_t$ is a drifted Brownian Motion and $S^\mu_t = \max_{0\le s\le t} \mathbb{B}^\mu_s$. To attack this problem, Du Toit and Peskir (2007) adopted a new approach: they first convert the problem to a standard optimal stopping problem, an extended version of Ito-Tanaka's formula with local time on curves (See Peskir (2005*a*)) was then applied to characterize the optimal stopping boundary as a unique solution to a coupled system of nonlinear Volterra integral equations of the second kind, which in turn leads to a characterization of the optimal value. However to apply the extended version of Ito-Tanaka's formula, one has to deduce the general shape of the stopping region in advance, which in general is itself a difficult problem.

For our present problem (1.2), it seems that all previous techniques did not work. To solve the problem we developed a conceptual and intuitive approach. With no doubt, the optimal random time (not only stopping time) for Problem (1.2) is the last time the stock price reaches its running maximum; this motivates us that, for any selling time $\tau$, it might be more optimal (than $\tau$) to postpone the time of selling until the first time after (or at) $\tau$ that the stock price reaches its running maximum no later than $T$. In mathematical language, for any stopping time $\tau$, we shall consider a new stopping time after $\tau$:

$$\rho_\tau \triangleq \inf\{\tau \le t : V_t = M_\tau\} \wedge T. \tag{1.6}$$

For if after $\tau$, the stock price can reach its running maximum before $T$, i.e. $V_{\rho_\tau} = M_\tau$, selling the stock at $\rho_\tau$ is at least not worse than selling it at $\tau$; however, there is a possibility

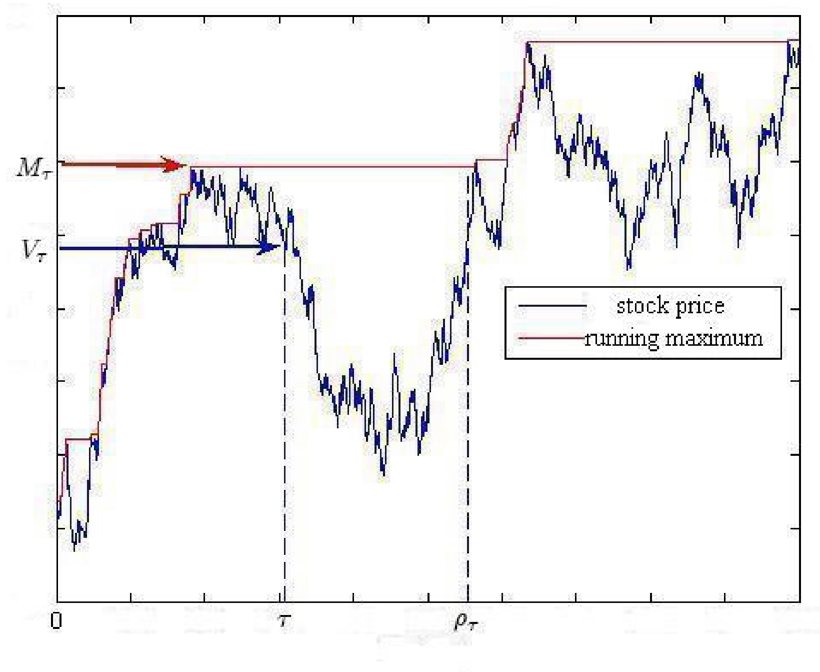


Figure 1: An graphical illustration of the new stopping time $\rho_\tau$.

that the stock price never reaches its running maximum again before $T$ and in this case $V_{\rho_\tau} = V_T$ may be smaller than $V_\tau$. Therefore, it is not clear in general whether $\rho_\tau$ is a better selling strategy than $\tau$. However, it will be proven later that for all $\alpha \geq \frac{1}{2}\sigma^2$, $\rho_\tau$ always dominates $\tau$, i.e.,

$$\mathbb{E}[\frac{V_\tau}{M_T}] \leq \mathbb{E}[\frac{V_{\rho_\tau}}{M_T}]. \tag{1.7}$$

with the equality in (1.7) holds *if and only if*

$$\tau = \rho_\tau \ a.s. \tag{1.8}$$

Hence one may think that the optimal stopping time should be $\rho_\tau$ for some special $\tau$. In fact we proved that when $\alpha = \frac{1}{2}\sigma^2$, $\tau^*$ is an optimal selling strategy *if and only if* $\tau^*$ satisfies (1.7), that is, for any $\tau$, $\rho_\tau$ is optimal stopping time and they are the only ones. Hence in this case, an selling strategy is optimal *if and only if* one sells either at terminal time $T$ or at the moment when the stock price hit its running maximum. Therefore we successfully identify all optimal selling strategies for this critical case. Note that our result in particular implies 0 and $T$ are optimal (since by definition of $\rho_\tau$, $\rho_0 = 0$ and $\rho_T = T$). And suppose one adopted the following strategy: waiting for a period of $\alpha T$ with $0 < \alpha < 1$ and selling the stock at the first time after $\alpha T$ that the stock price reaches the running maximum, then our result implies this strategy is also optimal for all $0 < \alpha < 1$. This kind of strategy had appeared before in the *Secretary Problem* in which the optimal strategy is not to consider the first $r^* - 1$ candidates but to accept the first candidate thereafter who is better than all previous ones. And for the case $\alpha > \frac{1}{2}\sigma^2$, we shall prove $\tau^* = T$ is the unique optimal selling strategy, that is, the only optimal stopping time is $\rho_\tau$ with $\tau = T$.

The problem (1.2) for arbitrary value of $\alpha$ has been arisen within both circles of researchers and practitioners several years ago. We first heard of the formulation of the problem in a conference talk delivered by Zhou (2008*a*) at the 2008 International Conference on Mathematics of Finance and Related Applications in Hong Kong. In the talk, he announced that for $\alpha \geq \sigma^2$, $\tau^* = T$ is the unique optimal stopping time while $\alpha \leq 0$, $\tau^* = 0$ is the unique optimal stopping time. But there was no clue on the solution in the case when $0 < \alpha < \sigma^2$, and from that time on, we proceeded to attack the whole problem altogether from a new perspective as stated above. After we finished our work for the case $\alpha \geq \frac{1}{2}\sigma^2$ in early April, we later found that, after communication with Shiryaev, Xu and Zhou (2008), Du Toit and Peskir (2009) has solved this problem for general $\alpha$ and Shiryaev, Xu and Zhou (2008) has solved it for $\alpha \leq 0$ and $\alpha \geq \frac{\sigma^2}{2}$. Du Toit and Peskir (2009) has proved that (see Theorem 2 there) when $\alpha \geq \frac{\sigma^2}{2}$, the optimal stopping time $\tau^* \equiv T$, when $\alpha \leq \frac{\sigma^2}{2}$, the optimal stopping time $\tau^* \equiv 0$, then when $\alpha = \frac{\sigma^2}{2}$ both $\tau^* \equiv 0$ and $\tau^* \equiv T$ are optimal. Our work provided a independent proof for the case $\alpha \geq \frac{\sigma^2}{2}$ and revealed that the solution to the problem (1.2) for all cases should be given by

$$\tau^* = \begin{cases} T, & if\ \alpha \geq \frac{1}{2}\sigma^2 \\ any\ stopping\ time\ \tau\ satisfies\ (1.8), & if\ \alpha = \frac{1}{2}\sigma^2 \\ 0, & if\ \alpha \leq \frac{1}{2}\sigma^2 \end{cases}$$

Hence correspondingly, a stock with drift $\alpha$ and volatility $\sigma$ is called

1. *Superior* $\quad if\ \alpha > \frac{1}{2}\sigma^2$
2. *Neutral* $\quad if\ \alpha = \frac{1}{2}\sigma^2$
3. *Inferior* $\quad if\ \alpha < \frac{1}{2}\sigma^2$

To resolve the problem (1.2), let us first reduce it to a standard form. In view of (1.1), problem (1.2) is equivalent to

$$V^* = \sup_{0\leq\tau\leq T} \mathbb{E}[\exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right)\tau + \sigma\mathbb{B}_\tau - \max_{0\leq s\leq T}\left(\left(\alpha - \frac{1}{2}\sigma^2\right)s + \sigma\mathbb{B}_s\right)\right)],$$

which by the scaling property of Brownian motion, is equivalent to

$$V^* = \sup_{0\leq\tau'\leq T'} \mathbb{E}[\exp\left(\frac{(\alpha - \frac{1}{2}\sigma^2)}{\sigma^2}\tau' + \mathbb{B}_{\tau'} - \max_{0\leq s'\leq T'}\left(\frac{(\alpha - \frac{1}{2}\sigma^2)}{\sigma^2}s' + \mathbb{B}_{s'}\right)\right)],$$

where $\tau' = \sigma^2\tau$, $T' = \sigma^2 T$ and $s' = \sigma^2 s$.Therefore to solve problem (1.2), it suffices to solve the following problem

$$V^* = \sup_{0\leq\tau\leq T} \mathbb{E}[\exp(\mathbb{B}^\mu_\tau - S^\mu_T)], \tag{1.9}$$

with $\mu = \frac{(\alpha - \frac{1}{2}\sigma^2)}{\sigma^2}$, $\mathbb{B}^\mu_t = \mu t + \mathbb{B}_t$, and $S^\mu_t = \max_{0\leq s\leq t}\mathbb{B}^\mu_s$. That is, it suffices to solve the problem (1.2) with $\sigma = 1$. Clearly the three cases $\alpha > \frac{1}{2}\sigma^2$, $\alpha = \frac{1}{2}\sigma^2$ and $\alpha < \frac{1}{2}\sigma^2$ correspond to $\mu > 0$, $\mu = 0$ and $\mu < 0$ respectively.

In the rest of this paper, we shall first associate to each stopping time $\tau$ a new stopping time $\rho_\tau$ (see (2.1)), and prove that $\rho_\tau$ always dominates $\tau$ in accordance with Theorem 3.1. This suggests us to seek for the optimal stopping time in the form $\rho_\tau$ for some stopping time $\tau$. We shall further prove that if $\mu = 0$, every $\tau$ such that $\tau = \rho_\tau$ *a.s.* is an optimal stopping time, and henceforth we identify all optimal stopping times for this particular case; if $\mu > 0$, the final time $T$ is the only optimal stopping time, which coincides with the conventional wisdom that one should hold superior stocks. In the next section we first establish a few lemmas and then conclude with our main result in Section 3.

## 2 Some preliminary results

Given $\mu \geq 0$,for each stopping time $\tau$, define

$$\rho_\tau \triangleq \inf_t \{\tau \leq t : \mathbb{B}^\mu_t = S^\mu_\tau\} \wedge T \tag{2.1}$$

then $\rho_\tau$ is also a stopping time. We will show in the sequel that whenever $\mu \geq 0$, $\rho_\tau$ always dominate $\tau$ in the sense that the expectation (2.3) will never be smaller than the expectation (2.2), and find out the optimal stopping time with the help of $\rho_\tau$. The next lemma states that our optimal prediction problem can be converted into an standard optimal stopping problem and allows us to compare the following expectations (2.2) and (2.3) evaluated at times $\tau$ and $\rho_\tau$ respectively.

**Lemma 2.1** *For any stopping time $\tau$, we have*

$$\mathbb{E}[\exp\{\mathbb{B}^\mu_\tau - S^\mu_T\}] = \mathbb{E}[G(\tau, X^\mu_\tau)] \tag{2.2}$$

*and*

$$\mathbb{E}[\exp\{\mathbb{B}^\mu_{\rho_\tau} - S^\mu_T\}] = \mathbb{E}[F(\tau, X^\mu_\tau)], \tag{2.3}$$

*where*

$$X^\mu_t \triangleq S^\mu_t - \mathbb{B}^\mu_t, \tag{2.4}$$

$$G(t,x) = \mathbb{E}\left[\exp\left(-\left(x \vee S^\mu_{T-t}\right)\right)\right] \tag{2.5}$$

$$F(t,x) = \mathbb{E}[e^{x - S^\mu_{T-t}} \ \mathbf{1}_{\{S^\mu_{T-t} \geq x\}} + e^{\mathbb{B}^\mu_{T-t} - x} \ \mathbf{1}_{\{S^\mu_{T-t} < x\}}] \tag{2.6}$$

**Proof.** We denote $\mathbb{E}[Z; A]$ to be $\mathbb{E}[Z \ \mathbf{I}_A]$ where $\mathbf{I}$ is the indicator function. Then using the fact that for each stopping time $\tau \leq T$, the post-$\tau$ process of standard Brownian motion is still a standard Brownian motion independent of the $\sigma$-algebra $\mathcal{F}_\tau$, we have

$$\begin{aligned}
& \mathbb{E}\left[\exp\left(\mathbb{B}^\mu_\tau - S^\mu_T\right)|\mathcal{F}_\tau\right] \\
= \ & \mathbb{E}\left[\exp\left(\mathbb{B}^\mu_\tau - \left(S^\mu_\tau \vee \max_{\tau \leq t \leq T} \mathbb{B}^\mu_t\right)\right)|\mathcal{F}_\tau\right] \\
= \ & \mathbb{E}\left[\exp\left(X^\mu_\tau \vee \max_{\tau \leq t \leq T}\left(\mathbb{B}^\mu_t - \mathbb{B}^\mu_\tau\right)\right)|\mathcal{F}_\tau\right] \\
= \ & \mathbb{E}\left[\exp\left(-\left(x \vee S^\mu_{T-t}\right)\right)\right]\Big|_{x = X^\mu_\tau, \ t=\tau} \\
= \ & G\left(\tau, X^\mu_\tau\right)
\end{aligned}$$

where $X^\mu_t = S^\mu_t - \mathbb{B}^\mu_t$. Hence

$$\mathbb{E}\left[\exp\left(\mathbb{B}^\mu_\tau - S^\mu_T\right)\right] = \mathbb{E}\left[G(\tau, X^\mu_\tau)\right].$$

Similarly, we also have

$$\begin{aligned}
& \mathbb{E}\left[\exp\left(\mathbb{B}^\mu_{\rho_\tau} - S^\mu_T\right)|\mathcal{F}_\tau\right] \\
= \ & \mathbb{E}\left[\exp\left(\mathbb{B}^\mu_{\rho_\tau} - \left(S^\mu_\tau \vee \max_{\tau \leq t \leq T} \mathbb{B}^\mu_t\right)\right); \max_{\tau \leq t \leq T} \mathbb{B}^\mu_t \geq S^\mu_\tau | \mathcal{F}_\tau\right] \\
& + \mathbb{E}\left[\exp\left(\mathbb{B}^\mu_{\rho_\tau} - \left(S^\mu_\tau \vee \max_{\tau \leq t \leq T} \mathbb{B}^\mu_t\right)\right); \max_{\tau \leq t \leq T} \mathbb{B}^\mu_t < S^\mu_\tau | \mathcal{F}_\tau\right] \\
= \ & \mathbb{E}\left[\exp\left(S^\mu_\tau - \max_{\tau \leq t \leq T} \mathbb{B}^\mu_t\right); \max_{\tau \leq t \leq T} \mathbb{B}^\mu_t \geq S^\mu_\tau | \mathcal{F}_\tau\right] \\
& + \mathbb{E}\left[\exp\left(\mathbb{B}^\mu_T - S^\mu_\tau\right); \max_{\tau \leq t \leq T} \mathbb{B}^\mu_t < S^\mu_\tau | \mathcal{F}_\tau\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\exp\left((S^\mu_\tau - \mathbb{B}^\mu_\tau) - (\max_{\tau\le t\le T}\mathbb{B}^\mu_t - \mathbb{B}^\mu_\tau)\right); \max_{\tau\le t\le T}\mathbb{B}^\mu_t - \mathbb{B}^\mu_\tau \ge S^\mu_\tau - \mathbb{B}^\mu_\tau|\mathcal{F}_\tau\right] \\
&\quad +\mathbb{E}\left[\exp\left((\mathbb{B}^\mu_T - \mathbb{B}^\mu_\tau) - (S^\mu_\tau - \mathbb{B}^\mu_\tau)\right); \max_{\tau\le t\le T}\mathbb{B}^\mu_t - \mathbb{B}^\mu_\tau < S^\mu_\tau - \mathbb{B}^\mu_\tau|\mathcal{F}_\tau\right] \\
&= \mathbb{E}\left[\exp\left(x - \widetilde{S}^\mu_t\right)\}; \widetilde{S}^\mu_t \ge x\right]\Big|_{x=S^\mu_\tau - \mathbb{B}^\mu_\tau,\ t=T-\tau} \\
&\quad + \mathbb{E}\left[\exp\left(\widetilde{\mathbb{B}}^\mu_t - x\right); \widetilde{S}^\mu_{T-\tau} < x\right]\Big|_{x=S^\mu_\tau - \mathbb{B}^\mu_\tau,\ t=T-\tau} \\
&= F\left(\tau, X^\mu_\tau\right),
\end{aligned}
$$

where $\widetilde{\mathbb{B}}^\mu_t$ is a drifted Brownian motion independent of the $\sigma-$algebra $\mathcal{F}_\tau$, $\widetilde{S}^\mu_t = \max_{0\le s\le t}\widetilde{\mathbb{B}}^\mu_s$, and $X^\mu_t = S^\mu_t - \mathbb{B}^\mu_t$. Therefore

$$
\mathbb{E}\left[\exp\{\mathbb{B}^\mu_{(\rho_\tau\wedge T)} - S^\mu_T\}\right] = \mathbb{E}\left[F\left(\tau, X^\mu_\tau\right)\right].
$$

■

**Remark 1** *In accordance with the above lemma, we deduce that*

$$
V^* = \sup_{0\le\tau\le T}\mathbb{E}\left[G\left(\tau, X^\mu_\tau\right)\right] \tag{2.7}
$$

*It has been shown that that (for example, see Graversen and Shiryaev* (2000)*) the process* $(X^\mu_t)_{0\le t\le T}$ *is equal in law as a process to a reflecting Brownian Motion with drift* $(-\mu)$ *which we now denote by X.. Therefore, the random variables* $X^\mu_\tau$ *and* $X_\tau$ *are identically distributed for any hitting time* $\tau$. *Since an optimal stopping time in* (2.7) *is the first entry time of the process to a closed set, therefore in Problem* (2.7) *we can replace the process* $X^\mu$ *by a reflecting Brownian Motion with drift* $(-\mu)$*, i.e.,*

$$
V^* = \sup_{0\le\tau\le T}\mathbb{E}[G\left(\tau, X_\tau\right)]. \tag{2.8}
$$

*Therefore, the stock selling problem* (1.2) *can be converted into a standard optimal stopping problem* (2.8) *with the underlying process being a reflecting Brownian Motion with drift* $(-\mu)$. *One way of solving* $V^*$ *is to study the extended problem*

$$
V(t,x) = \sup_{0\le\tau\le T-t}\mathbb{E}^{t,x}\left[G\left(\tau, X_\tau\right)\right] = \sup_{0\le\tau\le T-t}\mathbb{E}\left[G\left(t+\tau, X^x_\tau\right)\right]
$$

*where* $(X^x_t)_{t\ge 0}$ *is a reflecting Brownian Motion with drift* $(-\mu)$ *starting at* $x$ *and it can be explicitly constructed as* $(S^\mu_t \vee x - \mathbb{B}^\mu_t)_{t\ge 0}$ *(see Peskir* (2005*b*)*, Theorem* 2.1*). Although solving for the value function* $V(t,x)$ *is desirable, it is not necessary to tackle the problem* (2.8) *where the process* $(t,\ X_t)$ *only starts at* $(0,\ 0)$. *With the aid of the stopping time* $\rho_\tau$ *and the function* $F$*, we shall find out the optimal value* $V^*$ *and the optimal stopping time* $\tau^*$ *directly. See Remark* 2 *for further relevant discussions.*

To investigate the order of magnitude of the functions $F$ and $G$, we first establish some properties for the function $F$ that will be used in the proof of our Theorem 3.1.

**Lemma 2.2** *For any* $\mu$ *(not necessarily non-negative), the function* $F$ *satisfies the partial differential equation ,*

$$
\left(\mathcal{L}F\right)(t,x) = 0,
$$

*for any* $0\le t < T,$, $x > 0$, *with* $\mathcal{L}$ *being the parabolic partial differential operator*

$$
\mathcal{L} = \frac{\partial}{\partial t} - \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}.
$$

*And for any* $\mu > 0$, $F_x(t,0) > 0$ *for all* $0\le t < T$ *where* $F_x$ *denotes* $\frac{\partial F}{\partial x}$; *and for* $\mu = 0$, $F_x(t,0) = 0$ *for any* $0\le t < T$.

**Proof.** Recall that the joint density of

$$\left(\mathbb{B}_{T-t}^{\mu}, S_{T-t}^{\mu}\right) \triangleq \left(\mu\left(T-t\right)+\mathbb{B}_{T-t}, \max_{0\leq s\leq T-t}\left(\mu s+\mathbb{B}_{s}\right)\right) \tag{2.9}$$

is given by (see, e.g., Karatzas and Shreve (1998), page 368)

$$f^{\mu}(t,b,s)=\sqrt{\frac{2}{\pi}}\left(T-t\right)^{-\frac{3}{2}}\left(2s-b\right)\exp\left(-\frac{\left(2s-b\right)^{2}}{2\left(T-t\right)}+\mu\left(b-\frac{\mu\left(T-t\right)}{2}\right)\right), \tag{2.10}$$

and consequently the density of $S_{T-t}^{\mu}$ is

$$\begin{aligned} f_{S}^{\mu}(t,s) &= \int_{-\infty}^{s} f^{\mu}(t,b,s)db \\ &= 2\exp\left(2s\mu\right)\cdot \\ &\quad \left(\frac{1}{\sqrt{2\pi\left(T-t\right)}}\exp\left(-\frac{(s+\mu t)^{2}}{2\left(T-t\right)}\right)-\mu\Phi\left(-\frac{s+\mu\left(T-t\right)}{\sqrt{T-t}}\right)\right) \end{aligned} \tag{2.11}$$

with $\Phi\left(x\right)$ denotes the distribution function for standard normal variable.
Our proof hinges on the the following three identities (which can be verified by direct calculation):

$$e^{b}f^{\mu}(t,b,s)=e^{\left(\mu+\frac{1}{2}\right)(T-t)}f^{\mu+1}(t,b,s) \tag{2.12}$$

$$\left(\frac{\partial}{\partial t}-\mu\frac{\partial}{\partial s}+\frac{1}{2}\frac{\partial^{2}}{\partial s^{2}}\right)f_{S}^{\mu}(t,s)=0 \tag{2.13}$$

$$\left(\frac{\partial}{\partial t}-\mu\frac{\partial}{\partial s}+\frac{1}{2}\frac{\partial^{2}}{\partial s^{2}}\right)\Pr\left(S_{T-t}^{\mu}\leq s\right)=0. \tag{2.14}$$

If we write

$$\begin{aligned} F_{1}(t,x) &= \mathbb{E}\left[e^{x-S_{T-t}^{\mu}}\ \mathbf{1}_{\left\{S_{T-t}^{\mu}\geq x\right\}}\right] \\ F_{2}(t,x) &= \mathbb{E}\left[e^{\mathbb{B}_{T-t}^{\mu}-x}\ \mathbf{1}_{\left\{S_{T-t}^{\mu}<x\right\}}\right], \end{aligned}$$

then

$$F(t,x)=F_{1}(t,x)+F_{2}(t,x)$$

and using (2.12) they can be simplified into

$$\begin{aligned} F_{1}(t,x) &= e^{x}\int_{x}^{\infty}e^{-s}f_{S}^{\mu}(t,s)ds \\ F_{2}(t,x) &= e^{-x}\int_{0}^{x}\int_{-\infty}^{s}e^{b}f^{\mu}(t,b,s)dbds \\ &= e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\int_{0}^{x}\int_{-\infty}^{s}f^{\mu+1}(t,b,s)dbds \\ &= e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\int_{0}^{x}f_{S}^{\mu+1}(t,s)ds \\ &= e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\Pr\left(S_{T-t}^{\mu+1}\leq x\right). \end{aligned}$$

If $(t,x)$ is an interior point, then using (2.13) and (2.14), we have:

$$\begin{aligned}
\frac{\partial}{\partial t}F_1(t,x) &= e^x\int_x^\infty e^{-s}\frac{\partial}{\partial t}f_S^\mu(t,s)ds\\
&= -e^x\int_x^\infty e^{-s}(-\mu\frac{\partial}{\partial s}+\frac{1}{2}\frac{\partial^2}{\partial s^2})f_S^\mu(t,s)\,ds\\
&= \mu e^x\left[e^{-s}f_S^\mu(t,s)\big|_x^\infty+\int_x^\infty e^{-s}f_S^\mu(t,s)ds\right]\\
&\quad -\frac{1}{2}e^x\left[e^{-s}\frac{\partial}{\partial s}f_S^\mu(t,s)\bigg|_x^\infty+\int_x^\infty e^{-s}\frac{\partial}{\partial s}f_S^\mu(t,s)ds\right]\\
&= \mu e^x\left[-e^{-x}f_S^\mu(t,x)+\int_x^\infty e^{-s}f_S^\mu(t,s)ds\right]\\
&\quad -\frac{1}{2}e^x\left[-e^{-x}\frac{\partial}{\partial s}f_S^\mu(t,s)\bigg|_{s=x}+e^{-s}f_S^\mu(t,s)\big|_x^\infty+\int_x^\infty e^{-s}f_S^\mu(t,s)ds\right]\\
&= \mu\left[F_1(t,x)-f_S^\mu(t,x)\right]-\frac{1}{2}[F_1(t,x)-f_S^\mu(t,x)-\frac{\partial}{\partial x}f_S^\mu(t,x)],
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial x}F_1(t,x) &= F_1(t,x)-f_S^\mu(t,x),\\
\frac{\partial^2}{\partial x^2}F_1(t,x) &= \frac{\partial}{\partial x}F_1(t,x)-\frac{\partial}{\partial x}f_S^\mu(t,x),\\
&= F_1(t,x)-f_S^\mu(t,x)-\frac{\partial}{\partial x}f_S^\mu(t,x).
\end{aligned}$$

Therefore

$$\frac{\partial}{\partial t}F_1(t,x)-\mu\frac{\partial}{\partial x}F_1(t,x)+\frac{1}{2}\frac{\partial^2}{\partial x^2}F_1(t,x)=0,$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}F_2(t,x) &= -(\mu+\frac{1}{2})F_2(t,x)+e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\frac{\partial}{\partial t}\Pr\left(S_{T-t}^{\mu+1}\le x\right)\\
&= -(\mu+\frac{1}{2})F_2(t,x)-e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\left(-(\mu+1)\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\Pr\left(S_t^{\mu+1}\le x\right)\\
&= \mu\left[-F_2(t,x)+e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\frac{\partial}{\partial x}\Pr\left(S_t^{\mu+1}\le x\right)\right]\\
&\quad -\frac{1}{2}\left[F_2(t,x)-2e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\frac{\partial}{\partial x}\Pr\left(S_t^{\mu+1}\le x\right)+e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\frac{\partial^2}{\partial x^2}\Pr\left(S_t^{\mu+1}\le x\right)\right]\\
\frac{\partial}{\partial x}F_2(t,x) &= -F_2(t,x)+e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\frac{\partial}{\partial x}\Pr\left(S_t^{\mu+1}\le x\right)\\
\frac{\partial^2}{\partial x^2}F_2(t,x) &= -\frac{\partial}{\partial x}F_2(t,x)-e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\frac{\partial}{\partial x}\Pr\left(S_t^{\mu+1}\le x\right)+e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\frac{\partial^2}{\partial x^2}\Pr\left(S_t^{\mu+1}\le x\right)\\
&= F_2(t,x)-2e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\frac{\partial}{\partial x}\Pr\left(S_t^{\mu+1}\le x\right)+e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\frac{\partial^2}{\partial x^2}\Pr\left(S_t^{\mu+1}\le x\right).
\end{aligned}$$

Hence,

$$\frac{\partial}{\partial t}F_2(t,x)-\mu\frac{\partial}{\partial x}F_2(t,x)+\frac{1}{2}\frac{\partial^2}{\partial x^2}F_2(t,x)=0,$$

and thus, as a whole, we have

$$\frac{\partial}{\partial t}F(t,x)-\mu\frac{\partial}{\partial x}F(t,x)+\frac{1}{2}\frac{\partial^2}{\partial x^2}F(t,x)=0.$$

For our second assertion, direct calculation yields

$$\begin{aligned} F_x(t,0) &= \mathbb{E}\left[e^{-S^{\mu}_{T-t}}\right] - \int_{-\infty}^{0} f^{\mu}(t,b,0)db + \int_{-\infty}^{0} e^{b} f^{\mu}(t,b,0)db \\ &= \mathbb{E}\left[e^{-S^{\mu}_{T-t}}\right] - \left(\int_{-\infty}^{0} \left(1-e^{b}\right) f^{\mu}(t,b,0)db\right) \end{aligned}$$

Let's write $K_1(t,\mu) = \mathbb{E}\left[e^{-S^{\mu}_{T-t}}\right]$ and $K_2(t,\mu) = \int_{-\infty}^{0}\left(1-e^{b}\right) f^{\mu}(t,b,0)db$, then we have

$$K_1(t,\mu) = \int_0^{\infty} e^{-s}\left(\int_{-\infty}^{s} f^{\mu}(t,b,s)db\right)ds, \tag{2.15}$$

and

$$\begin{aligned} K_2(t,\mu) &= \int_{-\infty}^{0}\left(1-e^{b}\right) f^{\mu}(t,b,0)db \\ &= \int_{-\infty}^{0}\left(\int_0^{-b} e^{-s}ds\right) f^{\mu}(t,b,0)db \\ &= \int_0^{\infty} e^{-s}\left(\int_{-\infty}^{-s} f^{\mu}(t,b,0)db\right)ds \\ &= \int_0^{\infty} e^{-s}\left(\int_{-\infty}^{s} f^{\mu}(t,b'-2s,0)db'\right)ds, \end{aligned}$$

where we've used the change of variable $b' = b+2s$. Since $b'$ is just a dummy variable, we can write

$$K_2(t,\mu) = \int_0^{\infty} e^{-s}\left(\int_{-\infty}^{s} f^{\mu}(t,b-2s,0)db\right)ds. \tag{2.16}$$

Comparing (2.15) and (2.16), we can see in order to establish $K_1(t,\mu) \geq K_2(t,\mu)$, it suffices to show $f^{\mu}(t,b,s) \geq f^{\mu}(t,b-2s,0)$. By (2.10),

$$f^{\mu}(t,b,s) = \sqrt{\frac{2}{\pi}}(T-t)^{-\frac{3}{2}}(2s-b)\exp\left(-\frac{(2s-b)^2}{2(T-t)} + \mu b - \frac{\mu^2(T-t)}{2}\right),$$

$$f^{\mu}(t,b-2s,0) = \sqrt{\frac{2}{\pi}}(T-t)^{-\frac{3}{2}}(2s-b)\exp\left(-\frac{(2s-b)^2}{2(T-t)} + \mu(b-2s) - \frac{\mu^2(T-t)}{2}\right).$$

Therefore whenever $\mu > 0$, $f^{\mu}(t,b,s) > f^{\mu}(t,b-2s,0)$ except at the single point $s=0$, hence $K_1(t,\mu) > K_2(t,\mu)$, which means $F_x(t,0) > 0$ for any $t > 0$. And when $\mu = 0$, $f^{\mu}(t,b,s) = f^{\mu}(t,b-2s,0)$, hence $K_1(t,\mu) = K_2(t,\mu)$, which means $F_x(t,0) = 0$ for any $t > 0$. ■

Our next lemma reveals that for $\mu \geq 0$, we always have $F \geq G$ with the equality holds only on the $x-$ and $t-$axes. It further leads us to our main result that any stopping time $\tau$ is dominated by $\rho_\tau$ in the sense in accordance with Theorem 3.1.

**Lemma 2.3** *$F(t,x) > G(t,x)$ for any $t > 0, x > 0$, while $F(t,x) = G(t,x)$ if either $t = 0$ or $x = 0$.*

**Proof.** We are going to prove our claim by contradiction. Suppose that there is some $(t,x)$ such that

$$F(t,x) < G(t,x).$$

We first note that for $x = 0$, we have

$$F(t,x) = G(t,x) = \mathbb{E}\left[e^{-S^{\mu}_{T-t}}\right],$$

while for $t = 0$, we also have

$$F(t,x) = G(t,x) = e^{-x}.$$

Henceforth, we have $F - G = 0$ on the boundaries $t = 0$ and $x = 0$. Since for each $0 \le t \le T$,

$$\lim_{x\to\infty} (F - G)(t,x) = 0,$$

so there is a finite point $(t_0, x_0)$ at which $F - G$ attains its minimum value that must be strictly less than 0. As $(t_0, x_0)$ is a local minimum point, we must have

$$F_x(t_0, x_0) = G_x(t_0, x_0).$$

On the other hand, from (2.5) we have

$$G_x(t,x) = -\mathbb{E}[e^{-x}\ \mathbf{1}_{\{S^{\mu}_{T-t}<x\}}]$$

In Lemma 2.1 we already define

$$\begin{aligned} F_1(t,x) &= \mathbb{E}\left[e^{x-S^{\mu}_{T-t}}\ \mathbf{1}_{\left\{S^{\mu}_{T-t}\ge x\right\}}\right] \\ F_2(t,x) &= \mathbb{E}\left[e^{\mathbb{B}^{\mu}_{T-t}-x}\ \mathbf{1}_{\left\{S^{\mu}_{T-t}<x\right\}}\right], \end{aligned}$$

Then in terms of $F_1, F_2$ and $G_x$, (2.5) and (2.6) becomes:

$$F(t,x) = F_1(t,x) + F_2(t,x) \tag{2.17}$$

$$G(t,x) = F_1(t,x)e^{-x} - G_x(t,x). \tag{2.18}$$

we next claim that $\varphi(t,x) = F_2(t,x) + F_x(t,x) \ge 0$ for all $t > 0$ and $x > 0$. Lemma 2.2 suggests that

$$(\mathcal{L}\varphi)(t,x) = 0, \tag{2.19}$$

with $\mathcal{L} = \frac{\partial}{\partial t} - \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}$. Since on the boundary,

$$\begin{aligned} \lim_{x\to\infty} \varphi(t,x) &= \lim_{x\to\infty} (F_2(t,x) + F_x(t,x)) \\ &= \lim_{x\to\infty} \left(F_1(t,x) - f^{\mu}_S(t,x) + e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\frac{\partial}{\partial x}\Pr\left(S^{\mu+1}_{T-t} \le x\right)\right) \\ &= 0 \qquad (2.20) \\ \varphi(T,x) &= \lim_{t\to T} (F_2(t,x) + F_x(t,x)) \\ &= \lim_{t\to 0} \left(F_1(t,x) - f^{\mu}_S(t,x) + e^{-x}e^{(\mu+\frac{1}{2})(T-t)}\frac{\partial}{\partial x}\Pr\left(S^{\mu+1}_{T-t} \le x\right)\right) \\ &= 0 \qquad (2.21) \\ \varphi(t,0) &= F_x(t,0) \ge 0, \qquad (2.22) \end{aligned}$$

where (2.22) follows from Lemma 2.2. Thus minimum principle in PDE theory implies that $\varphi(t,x) \ge 0$ for all $t > 0$ and $x > 0$, or equivalently

$$F_2(t,x) \ge -F_x(t,x)\ \textit{for all}\ t > 0\ \textit{and}\ x > 0. \tag{2.23}$$

Henceforth, at the local minimum point $(t_0, x_0)$, combining (2.17), (2.18) and (2.23) we have,

$$\begin{aligned} F(t_0,x_0) < G(t_0,x_0) &\Rightarrow F_1(t_0,x_0) + F_2(t_0,x_0) < F_1(t_0,x_0)e^{-x_0} - G_x(t_0,x_0) \\ &\Rightarrow F_2(t_0,x_0) < -G_x(t_0,x_0) \\ &\Rightarrow -F_x(t_0,x_0) \le F_2(t_0,x_0) < -G_x(t_0,x_0) \\ &\Rightarrow F_x(t_0,x_0) > G_x(t_0,x_0), \end{aligned}$$

which is a contradiction! This shows that $F - G \ge 0$ at the minimum point $(t_0, x_0)$, and hence $(F - G)(t, x) \ge 0$ for all $t > 0$ and $x > 0$.
Furthermore, using the same argument as above, we can also establish that for any $x > 0$ such that $F(t, x) = G(t, x)$, we then have

$$F_x(t,x) > G_x(t,x).$$

In other words, if there is an interior point $(t, x)$ such that $(F - G)(t, x) = 0$, then $(F - G)$ can assume a negative value in a small neighborhood of $(t, x)$ which also contradicts to what we have just proven. Therefore $F(t, x) > G(t, x)$ for $t > 0, x > 0$. ■

# 3 Main theorem

We now conclude with our main theorem:

**Theorem 3.1** *Consider Problem* (1.2) *with* $\mu \ge 0$.
(1) *when* $\mu = 0$, *for any stopping time* $\tau$,

$$2e^{\frac{T}{2}}\Phi(-\sqrt{T}) = \mathbb{E}\left[\exp\left(\mathbb{B}_{\rho_\tau} - S_T\right)\right] \ge \mathbb{E}\left[\exp\left(\mathbb{B}_\tau - S_T\right)\right], \tag{3.1}$$

*where the inequality becomes equality if and only if* $\tau^* = \rho_{\tau^*}$ *a.s.. Henceforth for any stopping time* $\tau$, $\rho_\tau$ *is an optimal stopping time for our optimal prediction problem, i.e.,*

$$\mathbb{E}\left[\exp\left(\mathbb{B}_{\rho_\tau} - S_T\right)\right] = \sup_{0\le\sigma\le T} \mathbb{E}\left[\exp\left(\mathbb{B}_\sigma - S_T\right)\right], \tag{3.2}$$

*where the supremum ranges over all possible stopping time* $\sigma \le T$. *And* $\tau^*$ *is an optimal stopping time if and only if*

$$\tau^* = \rho_{\tau^*} \quad a.s. \tag{3.3}$$

(2) *when* $\mu > 0$, *for any stopping time* $\tau$,

$$\mathbb{E}\left[\exp\left(\mathbb{B}^\mu_T - S^\mu_T\right)\right] \ge \mathbb{E}\left[\exp\left(\mathbb{B}^\mu_{\rho_\tau} - S^\mu_T\right)\right] \ge \mathbb{E}\left[\exp\left(\mathbb{B}^\mu_\tau - S^\mu_T\right)\right], \tag{3.4}$$

*where the first inequality becomes equality if and only if* $T = \rho_\tau$ *a.s., and the second one becomes equality if and only if* $\tau = \rho_\tau$ *a.s.. Henceforth* $T$ *is the unique optimal stopping time for our optimal prediction problem when* $\mu > 0$, *i.e.,*

$$\mathbb{E}\left[\exp\left(\mathbb{B}^\mu_T - S^\mu_T\right)\right] = \sup_{0\le\sigma\le T} \mathbb{E}\left[\exp\left(\mathbb{B}^\mu_\sigma - S^\mu_T\right)\right], \tag{3.5}$$

*where the supremum ranges over all possible stopping time* $\sigma \le T$.

**Proof.** From Lemma 2.2,

$$\left(\frac{\partial}{\partial t} - \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) F(t,x) = 0 \tag{3.6}$$

for all $0 \leq t < T$ and $x > 0$. For any $0 \leq t < T$ , Ito formula implies that

$$\begin{aligned} F(t, X_t) &= F(0,0) + \int_0^t F_t(s, X_s) ds \\ &\quad + \int_0^t F_x(s, X_s) dX_s \\ &\quad + \frac{1}{2} \int_0^t F_{xx}(s, X_s) d < X, X >_s . \end{aligned} \tag{3.7}$$

1. When $\mu = 0$,

$$X_t = |\mathbb{B}_t| = \int_0^t \text{sign}\,(\mathbb{B}_s)\,\mathbf{I}_{\{\mathbb{B}_s \neq 0\}} d\mathbb{B}_s + l_t^0(\mathbb{B}.),$$

with the first equality holds in distribution sense, i.e., $X_t$ equal $|\mathbb{B}_t|$ in law as process, and the second equality is Tanaka's formula. Simplifying the expression (3.7) upon using $d < X, X >_t = d$ and $\Pr(\mathbb{B}_t = 0) = 0$, we deduce that

$$\begin{aligned} F(t, X_t) &= F(0,0) + \int_0^t \left( \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) F(s, |\mathbb{B}_s|) ds \\ &\quad + \int_0^t F_x(s, |\mathbb{B}_s|) \text{sign}\,(\mathbb{B}_s)\,\mathbf{I}_{\{\mathbb{B}_s \neq 0\}} d\mathbb{B}_s \\ &\quad + \int_0^t F_x(s, |\mathbb{B}_s|) dl_s^0\,(\mathbb{B}.) \\ &= F\,(0,0) + M_t, \end{aligned}$$

where

$$M_t = \int_0^t F_x(s, |\mathbb{B}_s|) \text{sign}\,(\mathbb{B}_s)\,\mathbf{I}_{\{\mathbb{B}_s \neq 0\}} d\mathbb{B}_s \tag{3.8}$$

is a martingale. Note that the second last equality holds because $F_x(t, 0) = 0$ (see Lemma 2.2) and the local time $l_\cdot^0(\mathbb{B}.)$ has full measure on the set $\{s \leq T : \mathbb{B}_s = 0\}$. Therefore

$$\begin{aligned} & \mathbb{E}[F(\tau, X_\tau)] \\ = \; & F(0,0) + \mathbb{E}[M_\tau] \\ = \; & F(0,0) \end{aligned}$$

upon using Optional Sampling Theorem. Hence

$$\begin{aligned} \mathbb{E}[F(\tau, X_\tau)] &= F(0,0) \\ &= 2e^{\frac{T}{2}} \Phi(-\sqrt{T}) \end{aligned} \tag{3.9}$$

which is a constant for any $0 \leq \tau \leq T$.

On the other hand, using Lemma 2.3, we also have

$$\mathbb{E}[F(\tau, X_\tau)] \geq \mathbb{E}\left[G(\tau, X_\tau)\right]$$

where the equality holds if and only if

$$\tau = \rho_\tau \;\; a.s.$$

The only if part holds since the process $X.$ is a continuous process, $F - G$ equals to zero only on the boundary and $(T - \tau, X_\tau) \in \{0\} \times [0, \infty] \cup [0, T] \times \{0\}$ exactly means $\tau = \rho_\tau$. Therefore, from Lemma 2.1, we have

$$\mathbb{E}[\exp\left(\mathbb{B}_{\rho_\tau} - S_T\right)] \geq \mathbb{E}[\exp\left(\mathbb{B}_\tau - S_T\right)], \tag{3.10}$$

with the equality holds if and only if

$$\tau = \rho_\tau \ \ a.s.$$

Taking supremum over all stopping time $\sigma$ on the right hand side of (3.10) and using (3.9), we get

$$2e^{\frac{T}{2}}\Phi(-\sqrt{T}) = \mathbb{E}[\exp\left(\mathbb{B}_{\rho_\tau} - S_T\right)] \geq \sup_{0\leq\sigma\leq T} \mathbb{E}[\exp\left(\mathbb{B}_\sigma - S_T\right)],$$

while $\rho_\tau$ is itself a stopping time, henceforth, we actually have the identity:

$$2e^{\frac{T}{2}}\Phi(-\sqrt{T}) = \mathbb{E}[\exp\left(\mathbb{B}_{\rho_\tau} - S_T\right)] = \sup_{0\leq\sigma\leq T} \mathbb{E}[\exp\left(\mathbb{B}_\sigma - S_T\right)],$$

for any $0 \leq \sigma \leq T$. As a consequence, $\tau^*$ is an optimal stopping time if and only if $\tau^* = \rho_{\tau^*}$ $a.s.$, i.e., $\tau^*$ is optimal if either $\mathbb{B}_{\tau^*} = S_{\tau^*}$ or $\tau^* = T$.

2. When $\mu > 0$, it is known that (see Graversen and Shiryaev (2000))

$$X_t = |Y_t|,$$

where the equality holds in distribution sense, i.e., $X$ and $Y$ equal in law as process, and $Y$ is the unique strong solution to the stochastic differential equation:

$$dY_t = -\mu \mathrm{sign}\left(Y_t\right) dt + d\mathbb{B}_t$$

with $Y_0 = 0$. Thus Tanaka's formula implies

$$dX_t = -\mu \mathbf{I}_{\{Y_s \neq 0\}} dt + \mathrm{sign}\left(Y_t\right) \mathbf{I}_{\{Y_s \neq 0\}} d\mathbb{B}_t + dl_t^0\left(Y_\cdot\right). \tag{3.11}$$

Inserting (3.11) into (3.7), and using that $d < X, X >_t = dt$ and $\Pr\left(Y_t = 0\right) = 0$, we get

$$\begin{aligned} F(t, X_t) &= F(0,0) + \int_0^t \left(\frac{\partial}{\partial t} - \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) F(s, |Y_s|) ds \\ &\quad + \int_0^t F_x(s, |Y_s|) \mathrm{sign}\left(Y_s\right) d\mathbb{B}_s \\ &\quad + \int_0^t F_x(s, |Y_s|) dl_s^0\left(Y_\cdot\right) \\ &= F\left(0,0\right) + M_t + \int_0^t F_x(s, |Y_s|) dl_s^0\left(Y_\cdot\right) \end{aligned} \tag{3.12}$$

where $M_t = \int_0^t F_x(s, X_s)\mathrm{sign}(Y_s)\, d\mathbb{B}_s$ is a martingale. Note that the last equality holds because the local time $l_\cdot^0(Y_\cdot)$ has full measure on the set $\{s \leq T : Y_s = 0\}$.
From (3.12), we have

$$\begin{aligned} \mathbb{E}[F(\tau, X_\tau)] &= F(0,0) + \mathbb{E}[M_\tau] + \mathbb{E}\left[\int_0^\tau F_x(s,0) dl_s^0\left(Y_\cdot\right)\right] \\ &= F(0,0) + \mathbb{E}\left[\int_0^\tau F_x(s,0) dl_s^0\left(Y_\cdot\right)\right] \end{aligned}$$

upon using Optional Sampling Theorem. By Lemma 2.2, when $\mu > 0$, $F_x(t,0) > 0$ for all $t > 0$, hence $F_x(t,0) > 0$ for all $T \geq t > 0$.Therefore

$$\begin{aligned}
\mathbb{E}[F(\tau, X_\tau)] &= F(0,0) + E\left[\int_0^\tau F_x(s,0)dl_s^0(Y.)\right] \\
&\leq F(0,0) + E\left[\int_0^T F_x(s,0)dl_s^0(Y.)\right] \\
&= \mathbb{E}[F(T, X_T)] \\
&= \mathbb{E}[\exp\{\mathbb{B}^\mu_{\rho_T} - S^\mu_T\}] \\
&= \mathbb{E}[\exp\{\mathbb{B}^\mu_T - S^\mu_T\}], \qquad (3.13)
\end{aligned}$$

where the last two equalities hold by Lemma 2.1 and the fact that $\rho_T = T$ $a.s.$.It's clearly that the inequality above becomes equality if and only if $\tau = T$ $a.s.$.By Lemma 2.3, we also have

$$\mathbb{E}[F(\tau, X_\tau)] \geq \mathbb{E}\left[G(\tau, X_\tau)\right] \qquad (3.14)$$

where the equality holds if and only if

$$\tau = \rho_\tau \ \ a.s.$$

The only if part holds since the process $X.$ is a continuous process, $F - G$ equals to zero only on the boundary and $(\tau, X_\tau) \in \{T\} \times [0,\infty] \cup [0,T] \times \{0\}$ exactly means $\tau = \rho_\tau$. Therefore, combining (3.13) and (3.14), and by Lemma 2.1, we get

$$\mathbb{E}[\exp\{\mathbb{B}^\mu_T - S^\mu_T\}] \geq \mathbb{E}[\exp\left(\mathbb{B}^\mu_{\rho_\tau} - S^\mu_T\right)] \geq \mathbb{E}[\exp(\mathbb{B}^\mu_\tau - S^\mu_T)],$$

and hence

$$\mathbb{E}[\exp\{\mathbb{B}^\mu_T - S^\mu_T\}] \geq \mathbb{E}[\exp(\mathbb{B}^\mu_\tau - S^\mu_T)], \qquad (3.15)$$

with the equality holds if and only if

$$\tau = T \ \ a.s.$$

Taking supremum over all stopping time $\sigma$ on the right hand side of (3.15), we get

$$\mathbb{E}[\exp(\mathbb{B}^\mu_T - S^\mu_T)] \geq \sup_{0\leq\sigma\leq T} \mathbb{E}[\exp(\mathbb{B}^\mu_\sigma - S^\mu_T)],$$

while $T$ is itself a stopping time, henceforth, we actually have the identity:

$$\mathbb{E}[\exp(\mathbb{B}^\mu_T - S^\mu_T)] = \sup_{0\leq\sigma\leq T} \mathbb{E}[\exp(\mathbb{B}^\mu_\sigma - S^\mu_T)],$$

for any $0 \leq \sigma \leq T$. Therefore when $\mu > 0$, $\tau^* = T$ is the unique optimal stopping time for our problem (1.2) .

■

**Remark 2** *In the case $\mu = 0$, the same argument as in above proof shows that $F$ is the smallest superharmonic majorant of $G$. Therefore, in accordance with general theory of optimal stopping, $F$ coincides with the value function $V$ which is defined in Remark* 1. *And in standard terminology of optimal stopping theory, our claims* (3.1) *and* (3.3) *is equivalent to saying that the optimal stopping boundary $b(t)$ for this problem equals to constant* 0 *when* $\mu = 0$.

# 4 Conclusion

We conclude this paper by pointing out further extensions of Problem (1.9) in recent years. For example, Yam, Yung and Zhou (2009) considered Problem (1.9) by replacing Brownian motion by Bernoulli random walks; in the same work, they also considered maximizing the probability that the stock price is precisely sold at the ultimate maximum, and it was shown that the solution to both problems is also in bang-bang type. Besides, Allaart (2009*a*) also extended Problem (1.9) to the case when the exponential function is replaced by an arbitrary nonincreasing convex function while the underlying process can be either a Brownian motion or a Bernoulli random walk. Almost the same time, Allaart (2009*b*) further extended his work to a large class of general random walks and/or Levy processes. From a PDE point of views, Dai and Zhong (2009) solved Problem (1.9) with the ultimate maximum replaced by the ultimate average. Recently, Yam, Yung and Zhou (2010) generalized Problem (1.9) to the case when the exponential function is replaced by a monotone convex function, while the ultimate maximum is replaced by a general class of nonanticipative benchmark. This is the most general result known so far that include all aforementioned formulations as special cases. A limitation of the present formulation of Problem (1.9) is that it assumes both the drift and volatility of the stock price process to be constant. Therefore it would be interesting to see what one can obtain by relaxing this constant parameter assumption.

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# References

[1] ALLAART, P. C. (2009a). A general "bang-bang" principle for predicting the maximum of a random walk. arXiv:0910.0545.

[2] ALLAART, P. C. (2009b). A "bang-bang" principle for predicting the supremum of a random walk or Levy process. arXiv:0912.0615

[3] DAI, M., ZHONG, Y.F. (2009). Optimal stock selling/buying strategy with reference to the ultimate average. to appear in *Math. Finan.*

[4] DU. TOIT, J. and PESKIR. G. (2007) The trap of complacency in predicting the maximum. *Ann. Probab.* **35** (340-365).

[5] DU. TOIT, J. and PESKIR. G. (2009) Selling a Stock at the Ultimate Maximum. *Ann. Appl. Probab.* **19** (983-1014).

[6] GRAVERSEN, S.E., PESKIR, G. and SHIRYAEV, A.N. (2001). Stopping Brownian motion without anticipation as close as possible to its ultimate maximum. *Theory Probab. Appl.* **45** (125-136).

[7] GRAVERSEN, S. E. and SHIRYAEV, A. N. (2000). An extension of P. Levy's distributional properties to the case of a Brownian motion with drift. *Bernoulli* **6** (615-620).

[8] KARATZAS, I. and SHREVE, S. E. (1998). *Methods of Mathematical Finance*. Springer.

[9] PEDERSEN, J.L. (2003). Optimal prediction of the ultimate maximum of Brownian motion. Stoch. Stoch. Rep. **75** (205-219).

[10] PESKIR, G.(2005a). A change-of-variable formula with local time on curves. *J. Theoret. Probab.* **18** (499-535).

[11] PESKIR, G. (2005b). On Reflecting Brownian Motion with Drift. *Proc. Symp. Stoch. Syst.* (Osaka, 2005), ISCIE Kyoto, 2006, (1-5).

[12] PESKIR, G. and SHIRYAEV, A. N. (2006). *Optimal Stopping and Free-Boundary Problems*. Lectures in Mathematics, ETH Zürich, Birkhäuser.

[13] SHIRYAEV, A.N. (2002). Quickest detection problems in the technical analysis of the fancial data. Proc. Math. Finance Bachelier Congress (Paris, 2000), Springer (487-521).

[14] SHIRYAEV, A.N. XU, Z. and ZHOU, X.Y. (2008). Thou shalt buy and hold. *Quantitative Finance*, **8** (765-776).

[15] Yam, S. C. P., Yung, S. P. and Zhou, W. (2009). Two rationales behind 'buy-and-hold or sell-at-once'. J. Appl.Probab. **46** 651-668.

[16] Yam, S. C. P., Yung, S. P. and Zhou, W. (2010). A unified 'bang-bang' principle with respect to a class of nonanticipative benchmarks. Submitted.

[17] Zhou X. Y. (2008). Thou Shalt Buy and Hold. A conference talk in 2008 International Conference on Mathematics of Finance and Related Applications in Hong Kong.