ON INTERSECTIONS OF CERTAIN PARTITIONS OF A GROUP COMPACTIFICATION

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Abstract. Let G be a connected semi-simple algebraic group of adjoint type over an algebraically closed field, and let $\overline{G}$ be the wonderful compactification of G. For a fixed pair (B, B^-) of opposite Borel subgroups of G, we look at intersections of Lusztig's G-stable pieces and the $B^- \times B$-orbits in $\overline{G}$, as well as intersections of $B \times B$-orbits and $B^- \times B^-$-orbits in $\overline{G}$. We give explicit conditions for such intersections to be non-empty, and in each case, we show that every non-empty intersection is smooth and irreducible, that the closure of the intersection is equal to the intersection of the closures, and that the non-empty intersections form a strongly admissible partition of $\overline{G}$.

1. Introduction

1.1. Let Z be an irreducible algebraic variety over an algebraically closed field $\mathbf{k}$. By a *partition* of Z we mean a finite disjoint union $Z = \bigsqcup_{i \in \mathcal{I}} X_i$ such that each X_i is a smooth irreducible locally closed subset of Z and that the closure of each X_i in Z is the union of some $X_{i'}$'s for $i' \in \mathcal{I}$.

Throughout the paper, if a group M acts on a set S, we use $m.s$ to denote the action of $m \in M$ on $s \in S$, and for a subset M' of M and $s \in S$, we set $M'.s = \{m.s : m \in M'\}$.

1.2. Let G be a connected semi-simple algebraic group of adjoint type over an algebraically closed field $\mathbf{k}$. Regard G as a $G \times G$ homogeneous space by the action

$$(g_1, g_2).g = g_1 g g_2^{-1}, \quad g_1, g_2, g \in G.$$

The De Concini-Procesi wonderful compactification $\overline{G}$ of G is a smooth $(G \times G)$-equivariant compactification of G (see [2, 3]).

Let B and B^- be a pair of opposite Borel subgroups of G. The partition of $\overline{G}$ into the $B \times B$-orbits was studied in [1, 25]. In his study of parabolic character sheaves on $\overline{G}$ in [19, 20], G. Lusztig introduced a decomposition of $\overline{G}$ into finitely many G-stable pieces, where G is identified with the diagonal G_{diag} of $G \times G$. It was later proved in [10] that Lusztig's G-stable pieces form a partition of $\overline{G}$.

This paper concerns with

1) intersections of $B \times B$-orbits and $B^- \times B^-$-orbits in $\overline{G}$,

2) intersections of the G-stable pieces with $B^- \times B$-orbits in $\overline{G}$.

Our motivation partially comes from Poisson geometry. Let $H = B \cap B^-$. When $\mathbf{k} = \mathbb{C}$, there is [5, 18] a natural $H \times H$-invariant Poisson structure Π_1 on $\overline{G}$ whose $H \times H$-orbits of symplectic leaves are the non-empty intersections of $B \times B$ and $B^- \times B^-$-orbits. Similarly, there is natural H_{diag}-invariant Poisson structure Π_2 on $\overline{G}$ whose H_{diag}-orbits of symplectic leaves are the non-empty intersections of G_{diag}-orbits and $B^- \times B$-orbits. The restrictions of Π_1 and Π_2 to $G \subset \overline{G}$ are closely related to the quantized universal enveloping algebra of the Lie algebra of G and its dual (as a Hopf algebra). See [6, 16].

The closures of such intersections also appear in the study of algebro-geometric properties of $\overline{G}$. In the joint work [13] of He and Thomsen, it was proved that in positive characteristics, there exists a Frobenius splitting on $\overline{G}$ which compatibly splits all the nonempty intersections of the closures of $B \times B$-orbits and $B^- \times B^-$-orbits in $\overline{G}$. In particular, all such closures are weakly normal and reduced. Moreover, the closure of a $B \times B$-orbit is globally F-regular in positive characteristic and is normal and Cohen-Macaulay for arbitrary characteristic.

Later, in the joint work [14] of He and Thomsen, it was proved that in positive characteristics, there exists a Frobenius splitting on $\overline{G}$ which compatibly splits all the nonempty intersections of the closures of G-stable pieces and $B^- \times B$-orbits in $\overline{G}$. In particular, all such closures are weakly normal and reduced. However, the closure of a G-stable piece is not normal in general [14, No. 11.2].

We would like to point out that analogs of Lusztig's G-stable pieces in reductive monoids and in some $G \times G$-compactifications of G have been studied by M. Putcha [21] using the work of L. Renner [22] on $B \times B$-orbits in such settings. The resulting posets of the G-stable pieces can sometimes be worked out explicitly. See [21] for detail.

1.3. To state our results more precisely, we introduce some notation. Let $N_G(H)$ be the normalizer of H in G, and let $W = N_G(H)/H$ be the Weyl group. Let Γ be the set of simple roots determined by the pair (H, B). For $J \subset \Gamma$, let W_J be the subgroup of W generated by J, and let $W^J \subset W$ the set of minimal length representatives of W/W_J in W. If J' is a another subset of Γ, and $x \in W$, let $\min(W_{J'}xW_J)$ and $\max(W_{J'}xW_J)$ be respectively the unique minimal and maximal length elements in the double coset $W_{J'}xW_J$.

For $x, y \in W$, let $x * y \in W$ be such that $B(x * y)B$ is the unique dense (B, B)-double coset in $BxByB$. The operation $*$ makes W into a monoid which will be denoted by $(W, *)$. See [24].

Let $\delta \in \text{Aut}(G)$ be such that $\delta(H) = H$ and $\delta(B) = B$. Let $G_\delta = \{(g, \delta(g)) : g \in G\} \subset G \times G$ be the graph of δ. We will in fact work with G_δ-stable pieces in $\overline{G}$ (see definition below).

Recall that the $G \times G$-orbits in $\overline{G}$ are in one to one correspondence with subsets of Γ. For $J \subset \Gamma$, let Z_J be the corresponding $G \times G$-orbit in $\overline{G}$. One has $\overline{Z_J} = \bigsqcup_{K \subset J} Z_K$, and $\overline{Z_J}$ is smooth [2, 3]. Let h_J be a distinguished point in Z_J (see §3.1). For $w \in W^J$ and $(x, y) \in W^J \times W$, let

$$\begin{aligned} Z_{J,\delta,w} &= G_\delta (B \times B)(w, 1).h_J, \\ [J, x, y] &= (B \times B)(x, y).h_J, \\ [J, x, y]^{-,+} &= (B^- \times B)(x, y).h_J, \\ [J, x, y]^{-,-} &= (B^- \times B^-)(x, y).h_J. \end{aligned}$$

The $Z_{J,\delta,w}$'s are called the G_δ-stable pieces in $\overline{G}$. By [10, 25], one has the following partitions of $\overline{G}$:

$$\begin{aligned} \overline{G} &= \bigsqcup_{J \subset \Gamma, (x,y) \in W^J \times W} [J, x, y] = \bigsqcup_{J \subset \Gamma, (x,y) \in W^J \times W} [J, x, y]^{-,-} \\ &= \bigsqcup_{J \subset \Gamma, (x,y) \in W^J \times W} [J, x, y]^{-,+} = \bigsqcup_{J \subset \Gamma, w \in W^J} Z_{J,\delta,w}. \end{aligned} \tag{1}$$

For a subset X of $\overline{G}$, let $\overline{X}$ be the Zariski closure of X in $\overline{G}$.

We prove (see Proposition 3.1, Theorem 3.2, and Theorem 3.3) that for any $J \subset \Gamma$, $w \in W^J$, and $(x, y), (u, v) \in W^J \times W$,

1) $[J, x, y] \cap [J, u, v]^{-,-} \neq \emptyset$ if and only if $x \leqslant u$ and $v \leqslant \max(yW_J)$, and in this case, $[J, x, y] \cap [J, u, v]^{-,-}$ is smooth and irreducible, and

$$\overline{[J, x, y] \cap [J, u, v]^{-,-}} = \overline{[J, x, y]} \cap \overline{[J, u, v]^{-,-}}.$$

2) $Z_{J,\delta,w} \cap [J, x, y]^{-,+} \neq \emptyset$ if and only if $\min(W_J\, \delta(w)) \leqslant y^{-1} * \delta(x)$, and in this case, $Z_{J,\delta,w} \cap [J, x, y]^{-,+}$ is smooth and irreducible, and

$$\overline{Z_{J,\delta,w} \cap [J, x, y]^{-,+}} = \overline{Z_{J,\delta,w}} \cap \overline{[J, x, y]^{-,+}}.$$

Let

$$\begin{aligned} \mathcal{J} = \{(J, x, y, u, v) : \ & J \subset \Gamma,\ (x, y), (u, v) \in W^J \times W, \\ & x \leqslant u,\ v \leqslant \max(yW_J)\}, \\ \mathcal{K} = \{(J, w, x, y) : \ & J \subset \Gamma,\ (w, x, y) \in W^J \times W^J \times W, \\ & \min(W_J\, \delta(w)) \leqslant y^{-1} * \delta(x)\}. \end{aligned}$$

One then has two more partitions of $\overline{G}$:

$$\begin{aligned} \overline{G} &= \bigsqcup_{(J,x,y,u,v) \in \mathcal{J}} [J, x, y] \cap [J, u, v]^{-,-} \\ &= \bigsqcup_{(J,w,x,y) \in \mathcal{K}} Z_{J,\delta,w} \cap [J, x, y]^{-,+} \end{aligned} \tag{2}$$

We introduce the notion of *admissible partitions* and *strongly admissible* partitions of $\overline{G}$ (see Definition 3.1) and show that the six partitions in (1) and (2) are all strongly admissible (Proposition 3.2, Theorem 3.2, and Theorem 3.3). Moreover, the first two partitions in (1), as well as the last two in (1), are shown to be *compatible*. As consequences, we prove

1) if $J \subset \Gamma$ and if X is a subvariety of Z_J appearing in any of the six partitions in (1) and (2), then for any $K \subset J$, $\overline{X} \cap Z_K \neq \emptyset$, and $\overline{X}$ and $\overline{Z_K}$ intersect properly in $\overline{Z_J}$. Moreover, we describe the irreducible components of $\overline{X} \cap \overline{Z_K}$ in each case (Corollaries 3.3 and 3.4). This result for $\overline{G} = \bigsqcup_{J \subset \Gamma, (x,y) \in W^J \times W} [J, x, y]^{-,+}$ was also obtained by M. Brion [1];

2) if $X = [J, x, y]$ and $Y = [K, u, v]^{-,-}$ with $J, K \subset \Gamma$, $(x, y) \in W^J \times W$, and $(u, v) \in W^K \times W$, or if $X = Z_{J,\delta,w}$ and $Y = [K, x, y]^{-,+}$ with $J, K \subset \Gamma, w \in W^J$ and $(x, y) \in W^K \times W$, and if $\overline{X} \cap \overline{Y} \neq \emptyset$, we show that $\overline{X}$ and $\overline{Y}$ intersect properly in $\overline{Z_{J \cup K}}$ (Corollary 3.5).

Our discussions in this paper, and especially that in §2.9 and §3.5, also apply to intersections of R-stable pieces and $B \times B$-orbits, where R is a certain class of connected subgroups of $G \times G$ as in [17], as long as $R \cap (B \times B)$ is connected and that $\mathrm{Lie}(R) + \mathrm{Lie}(B \times B) = \mathrm{Lie}(G \times G)$.

1.4. We set up more notation for the rest of the paper.

For $\alpha \in \Gamma$, let U^α be the one dimensional unipotent subgroup of G defined by α. For a subset J of Γ, let P_J and P_J^- be respectively the standard parabolic subgroups of G determined by J that contain B and B^-, and let U_J and U_J^- be respectively the uniradicals of P_J and P_J^-. Let $M_J = P_J \cap P_J^-$ be the common Levi factor of P_J and P_J^-, and let $\mathrm{Cen}(M_J)$ be the center of M_J.

The longest element in W is denoted by w_0. If $J \subset \Gamma$, denote by w_0^J the longest element in W_J, and let ${}^J W = \{w^{-1} : w \in W^J\}$. If J' is a another subset of Γ, let ${}^{J'} W^J = {}^{J'} W \cap W^J$.

Throughout the paper, $\bigsqcup$ always means disjoint union.

2. Intersections in $Z_{\mathcal{C}} = (G \times G)/R_{\mathcal{C}}$

2.1. Following [17], an *admissible quadruple* for G is a quadruple $\mathcal{C} = (J, J', c, L)$, where J and J' are subsets of Γ, $c : J \to J'$ is a bijective map preserving the inner products between the simple roots, and L is a connected closed subgroup of $M_J \times M_{J'}$ of the form

$$L = \{(m, m') \in M_J \times M_{J'} : \theta_c(mC) = m'C'\},$$

with $C \subset \mathrm{Cen}(M_J)$ and $C' \subset \mathrm{Cen}(M_{J'})$ being closed subgroups and $\theta_c : M_J/C \to M_{J'}/C'$ a group isomorphism mapping $H/\mathrm{Cen}(M_J)$ to $H/\mathrm{Cen}(M_{J'})$ and U^α to $U^{c(\alpha)}$ for every $\alpha \in J$. For an admissible quadruple $\mathcal{C} = (J, J', c, L)$, let

$$R_{\mathcal{C}} = L(U_J \times U_{J'}) \subset P_J \times P_{J'}. \tag{3}$$

For example, $R_{\mathcal{C}} = B \times B$ for $\mathcal{C} = (\emptyset, \emptyset, \mathrm{Id}, H \times H)$ and $R_{\mathcal{C}} = G_{\mathrm{diag}}$ for $\mathcal{C} = (\Gamma, \Gamma, \mathrm{Id}, G_{\mathrm{diag}})$. For an admissible quadruple $\mathcal{C} = (J, J', c, L)$, let

$$Z_{\mathcal{C}} = (G \times G)/R_{\mathcal{C}}.$$

When G is of adjoint type, the $G\times G$-orbits in the De Concini-Procesi compactification $\overline{G}$ of G are all of the form $Z_{\mathcal{C}}$ for some admissible quadruples $\mathcal{C}$ (see §3.1).

2.2. For $(x, y) \in W^J \times W$, let

$$\begin{aligned}
[\mathcal{C}, x, y] &= (B \times B)(x, y).R_{\mathcal{C}} \subset Z_{\mathcal{C}}, \\
[\mathcal{C}, x, y]^{-,+} &= (B^- \times B)(x, y).R_{\mathcal{C}} \subset Z_{\mathcal{C}}, \\
[\mathcal{C}, x, y]^{-,-} &= (B^- \times B^-)(x, y).R_{\mathcal{C}} \subset Z_{\mathcal{C}}.
\end{aligned}$$

It follows from [25, Lemma 1.3] that

$$Z_{\mathcal{C}} = \bigsqcup_{(x,y)\in W^J\times W} [\mathcal{C}, x, y] = \bigsqcup_{(x,y)\in W^J\times W} [\mathcal{C}, x, y]^{-,+} = \bigsqcup_{(x,y)\in W^J\times W} [\mathcal{C}, x, y]^{-,-}$$

are the partitions of $Z_{\mathcal{C}}$ by the $B \times B$, $B^- \times B$, and $B^- \times B^-$-orbits, respectively.

2.3. Let δ be an automorphism of G preserving both H and B, and let

$$G_\delta = \{(g,\, \delta(g)) :\, g \in G\} \subset G \times G$$

be the graph of δ. For $w \in W^J$, let

$$Z_{\mathcal{C},\delta,w} = G_\delta(B \times B)(w, 1).R_{\mathcal{C}} \subset Z_{\mathcal{C}}.$$

The sets $Z_{\mathcal{C},\delta,w}$ for $w \in W^J$ will be called the G_δ-*stable pieces* in $Z_{\mathcal{C}}$. By [10, 17, 26], each $Z_{\mathcal{C},\delta,w}$ is a locally closed smooth irreducible subset of $Z_{\mathcal{C}}$, and

$$Z_{\mathcal{C}} = \bigsqcup_{w\in W^J} Z_{\mathcal{C},\delta,w}$$

is the partition of $Z_{\mathcal{C}}$ by the G_δ-stable pieces.

2.4. We now recall the closure relations of the $B \times B$-orbits and G_δ-stable pieces in $Z_{\mathcal{C}}$. For $X \subset Z_{\mathcal{C}}$, let $\overline{X}$ be the closure of X in $Z_{\mathcal{C}}$.

1) For $(x, y) \in W^J \times W$, $\overline{[\mathcal{C}, x, y]} = \bigsqcup [\mathcal{C}, x', y']$, where (x', y') runs over elements in $W^J \times W$ such that $x'u \leqslant x$ and $y'c(u) \leqslant y$ for some $u \in W_J$. See [17, Corollary 4.1].

2) For $w \in W^J$, $\overline{Z_{\mathcal{C},\delta,w}} = \bigsqcup Z_{\mathcal{C},\delta,w'}$, where w' runs over elements in W^J such that $\delta^{-1}(c(u))w'u^{-1} \leqslant w$ for some $u \in W_J$. See [11, Corollary 5.9].

Using that facts that

$$\begin{aligned}
[\mathcal{C}, x, y]^{-,+} &= (w_0, 1)[\mathcal{C}, w_0 x w_0^J, y w_0^{J'}], \\
[\mathcal{C}, x, y]^{-,-} &= (w_0, w_0)[\mathcal{C}, w_0 x w_0^J, w_0 y w_0^{J'}],
\end{aligned}$$

one has the following variations of 1).

3) For $(x, y) \in W^J \times W$, $\overline{[\mathcal{C}, x, y]^{-,+}} = \bigsqcup [\mathcal{C}, x', y']^{-,+}$, where (x', y') runs over elements in $W^J \times W$ such that $x'u \geqslant x w_0^J$ and $y'c(u) \leqslant y w_0^{J'}$ for some $u \in W_J$.

4) For $(x, y) \in W^J \times W$, $\overline{[\mathcal{C}, x, y]^{-,-}} = \bigsqcup [\mathcal{C}, x', y']^{-,-}$, where (x', y') runs over elements in $W^J \times W$ such that $x'u \geqslant xw_0^J$ and $y'c(u) \geqslant yw_0^{J'}$ for some $u \in W_J$.

2.5. Recall that the monoid operation $*$ on W is defined by $\overline{B(x*y)B} = \overline{BxByB}$ for $x, y \in W$. Similarly, for $x, y \in W$, define $x \triangleright y \in W$ and $x \triangleleft y \in W$ by

$$\overline{BxByB^-} = \overline{B(x \triangleright y)B^-} \quad \text{and} \quad \overline{B^-xByB} = \overline{B^-(x \triangleleft y)B}.$$

Then

$$(W, *) \times W \longrightarrow W : \ (x, y) \longmapsto x \triangleright y, \quad x, y \in W$$

is a left monoid action of $(W, *)$ on W, and

$$W \times (W, *) \longrightarrow W : \ (x, y) \longmapsto x \triangleleft y, \quad x, y \in W$$

is a right monoidal action of $(W, *)$ on W. More properties of $*, \triangleright$ and $\triangleleft$ are reviewed in the Appendix.

2.6. We now determine when the intersection of a $B \times B$-orbit and a $B^- \times B^-$-orbit in $Z_{\mathcal{C}}$ is non-empty.

Proposition 2.1. *For any $(x, y), (u, v) \in W^J \times W$, the following conditions are equivalent:*
1) $[\mathcal{C}, x, y] \cap [\mathcal{C}, u, v]^{-,-} \neq \emptyset$,
2) $u \leqslant x$ and $\min(vW_{J'}) \leqslant y$,
3) $u \leqslant x$ and $v \leqslant \max(yW_{J'})$.

Proof. Since $x \in W^J$, we have that

$$\begin{aligned}(B \times B)(x, y)(B \times B).R_{\mathcal{C}} &= (B \times B)(x, y)((B \cap M_J) \times (B \cap M_{J'})).R_{\mathcal{C}} \\ &= (B \times B)(x, y)((B \cap M_J) \times \{1\}).R_{\mathcal{C}} \\ &= (B \times B)(x, y).R_{\mathcal{C}} = [\mathcal{C}, x, y].\end{aligned}$$

Similarly, $[\mathcal{C}, u, v]^{-,-} = (B^- \times B^-)(uw_0^J, vw_0^{J'})(B \times B).R_{\mathcal{C}}$. Therefore, $[\mathcal{C}, x, y] \cap [\mathcal{C}, u, v]^{-,-} \neq \emptyset$ if and only if

$$(BxB, ByB) \cap \big((B^- \times B^-)(uw_0^J, vw_0^{J'})(B \times B)R_{\mathcal{C}}(B \times B)\big) \neq \emptyset.$$

Since

$$(B \times B)R_{\mathcal{C}}(B \times B) = \bigcup_{z \in W_J} (B \times B)(z, c(z))(B \times B), \tag{4}$$

$[\mathcal{C}, x, y] \cap [\mathcal{C}, u, v]^{-,-} \neq \emptyset$ if and only if

$$(BxB, \ ByB) \cap (B^-uw_0^JBzB, \ B^-vw_0^{J'}Bc(z)B) \neq \emptyset \tag{5}$$

for some $z \in W_J$. By 1) and 6) of Lemma 4.7 in the Appendix, (5) is the same as

$$uw_0^J \leqslant x * z^{-1} \quad \text{and} \quad vw_0^{J'} \leqslant y * c(z)^{-1} \quad \text{for some } z \in W_J.$$

By Lemma 4.1 in the Appendix, for any $z \in W_J$, $x * z \leqslant \max(xW_J)$ and $y * c(z) \leqslant \max(yW_{J'})$ and both inequalities become equalities when

$z = w_0^J$, (5) is equivalent to $uw_0^J \leqslant \max(xW_J)$ and $vw_0^{J'} \leqslant \max(yW_{J'})$ which, by Lemma 4.5 in the Appendix, are in turn equivalent to $u \leqslant x$ and $\min(vW_{J'}) \leqslant y$, or $u \leqslant x$ and $v \leqslant \max(yW_{J'})$. $\square$

Example 2.1. When $R_{\mathcal{C}} = G_{\mathrm{diag}}$ and $Z_{\mathcal{C}}$ is identified with G, the intersections in Proposition 2.1 are of the form $ByB \cap B^-wB^-$ for $y, w \in W$, and are called *double Bruhat cells* [7]. It is well-known (see, for example, [7]) that the intersection $(ByB) \cap (B^-wB^-)$ is non-empty for all $y, w \in W$, which can also be seen from Proposition 2.1.

2.7. We now determine when the intersection of a G_δ-stable piece and a $B^- \times B$-orbit in $Z_{\mathcal{C}}$ is nonempty.

Proposition 2.2. *For $w \in W^J$ and $(x, y) \in W^J \times W$, the following conditions are equivalent:*
1) $Z_{\mathcal{C},\delta,w} \cap [\mathcal{C}, x, y]^{-,+} \neq \emptyset$,
2) $y^{-1} \triangleright \delta(x) \leqslant \max(W_{J'}\, \delta(w))$,
3) $\min(W_{J'}(y^{-1} \triangleright \delta(x))) \leqslant \delta(w)$.

Proof. Using the facts that $w, x \in W^J$, it is easy to see that

$$Z_{\mathcal{C},\delta,w} = G_\delta(B \times B)(w, 1)(B \times B).R_{\mathcal{C}},$$
$$[\mathcal{C}, x, y]^{-,+} = (B^- \times B)(xw_0^J, yw_0^{J'})(B \times B).R_{\mathcal{C}}.$$

Thus $Z_{\mathcal{C},\delta,w} \cap [\mathcal{C}, x, y]^{-,+} \neq \emptyset$ if and only if

$$G_\delta \cap \Big((B^- \times B)(xw_0^J, yw_0^{J'})(B \times B)R_{\mathcal{C}}(B \times B)(w^{-1}, 1)(B \times B)\Big) \neq \emptyset.$$

By (4), $Z_{\mathcal{C},\delta,w} \cap [\mathcal{C}, x, y]^{-,+} \neq \emptyset$ if and only if

$$G_\delta \cap \Big((B^- \times B)(xw_0^J, yw_0^{J'})(B \times B)(z, c(z))(B \times B)(w^{-1}, 1)(B \times B)\Big) \neq \emptyset$$

for some $z \in W_J$, which is equivalent to

$$(B^-\delta(xw_0^J)B\delta(z)B\delta(w^{-1})B) \cap (Byw_0^{J'}Bc(z)B) \neq \emptyset. \tag{6}$$

Since $l(zw^{-1}) = l(z) + l(w^{-1})$ for every $z \in W_J$, $wz^{-1} = w * z^{-1}$ and $B\delta(z)B\delta(w^{-1})B = B\delta(zw^{-1})B$. Thus (6) is equivalent to

$$(B^-\delta(xw_0^J)B\delta(zw^{-1})B) \cap (Byw_0^{J'}Bc(z)B) \neq \emptyset$$

which, by 1) and 6) of Lemma 4.7 in the Appendix, is equivalent to

$$\delta(xw_0^J) \leqslant (yw_0^{J'}) * c(z) * \delta(w) * \delta(z^{-1}) \tag{7}$$

for some $z \in W_J$. By Lemma 4.2 and Lemma 4.4 in the Appendix,

$$\begin{aligned}
&(yw_0^{J'}) * c(z) * \delta(w) * \delta(z^{-1}) \leqslant (yw_0^{J'}) * c(w_0^J) * \delta(w) * \delta(w_0^J) \\
&= ((yw_0^{J'}) * w_0^{J'}) * \delta(w) * \delta(w_0^J) = \max(yW_{J'}) * \delta(w) * \delta(w_0^J) \\
&= y * w_0^{J'} * \delta(w) * \delta(w_0^J) = y * \max(W_{J'}\, \delta(w)\, W_{\delta(J)})
\end{aligned}$$

for any $z \in W_J$, and the inequality becomes an equality when $z = w_0^J$. Thus (7) is equivalent to

$$\delta(xw_0^J) \leqslant y * \max(W_{J'}\, \delta(w)\, W_{\delta(J)}). \tag{8}$$

Since $\delta(x) \leqslant \delta(xw_0^J)$, (8) leads to

$$\delta(x) \leqslant y * \max(W_{J'}\, \delta(w)\, W_{\delta(J)}). \tag{9}$$

Conversely, if (9) holds, then

$$\begin{aligned} \delta(xw_0^J) = \delta(x) * w_0^{\delta(J)} &\leqslant y * \max(W_{J'}\, \delta(w)\, W_{\delta(J)}) * w_0^{\delta(J)} \\ &= y * \max(W_{J'}\, \delta(w)\, W_{\delta(J)}). \end{aligned}$$

Thus (8) is equivalent to (9), which, by 3) of Lemma 4.3 in the Appendix, is equivalent to

$$y^{-1} \triangleright \delta(x) \leqslant \max(W_{J'}\, \delta(w)\, W_{\delta(J)}). \tag{10}$$

Since $y^{-1} \triangleright \delta(x) \in W^{\delta(J)}$ by Lemma 4.4 in the Appendix, and since

$$\max(W_{J'}\, \delta(w)\, W_{\delta(J)}) = \max(W_{J'}\, \delta(w)) * w_0^{\delta(J)},$$

(10) is equivalent to $y^{-1} \triangleright \delta(x) \leqslant \max(W_{J'}\, \delta(w))$ by Lemma 4.5 in the Appendix. The equivalence of 2) and 3) also follows from Lemma 4.5 in the Appendix. $\square$

2.8. We now discuss some consequences of the results in §2.6 and §2.7.

Corollary 2.1. *Let $J \subset \Gamma$. For $(x, y) \in W^J \times W$. Set*

$$w_{x,y} = \min(W_{\delta^{-1}(J')}(\delta^{-1}(y^{-1}) \triangleright x)).$$

Then $w_{x,y} \in {}^{\delta^{-1}(J')}W^J$ and for $w \in W^J$,

$$Z_{\mathcal{C},\delta,w} \cap [\mathcal{C}, x, y]^{-,+} \neq \emptyset \quad \textit{iff} \quad Z_{\mathcal{C},\delta,w_{x,y}} \subset \overline{Z_{\mathcal{C},\delta,w}}.$$

Proof. By definition, $w_{x,y} \in {}^{\delta^{-1}(J')}W$. By Lemma 4.4 in the Appendix, $\delta^{-1}(y^{-1}) \triangleright x \in W^J$ and $w_{x,y} = w_0^{\delta^{-1}(J')} \triangleright (\delta^{-1}(y^{-1}) \triangleright x) \in W^J$. If $Z_{\mathcal{C},\delta,w} \cap [\mathcal{C}, x, y]^{-,+} \neq \emptyset$, by Proposition 2.2, $w_{x,y} \leqslant w$. By §2.4, 2), $Z_{\mathcal{C},\delta,w_{x,y}} \subset \overline{Z_{\mathcal{C},\delta,w}}$. On the other hand, if $Z_{\mathcal{C},\delta,w_{x,y}} \subset \overline{Z_{\mathcal{C},\delta,w}}$, then there exists $u \in W_J$ such that $\delta^{-1}(c(u))w_{x,y}u^{-1} \leqslant w$. Since $w_{x,y} \in {}^{\delta^{-1}(J')}W^J$, $w_{x,y} \leqslant \delta^{-1}(c(u))w_{x,y}u^{-1} \leqslant w$. By Proposition 2.2, $Z_{\mathcal{C},\delta,w} \cap [\mathcal{C}, x, y]^{-,+} \neq \emptyset$. $\square$

Proposition 2.3. *Let $\pi : Z_{\mathcal{C}} \to (G \times G)/(P_J \times P_{J'})$ be the natural projection induced by the inclusion $R_{\mathcal{C}} \subset P_J \times P_{J'}$. Then for any $w \in W^J$ and $(x, y) \in W^J \times W$,*

$$Z_{\mathcal{C},\delta,w} \cap [\mathcal{C}, x, y]^{-,+} \neq \emptyset \quad \textit{iff} \quad \pi(Z_{\mathcal{C},\delta,w}) \cap \pi([\mathcal{C}, x, y]^{-,+}) \neq \emptyset.$$

Proof. Clearly $Z_{\mathcal{C},\delta,w} \cap [\mathcal{C}, x, y]^{-,+} \neq \emptyset$ implies that $\pi(Z_{\mathcal{C},\delta,w}) \cap \pi([\mathcal{C}, x, y]^{-,+}) \neq \emptyset$. Assume now that $\pi(Z_{\mathcal{C},\delta,w}) \cap \pi([\mathcal{C}, x, y]^{-,+}) \neq \emptyset$. Let $y' = \min(yW_{J'}) \in W^{J'}$ and $w' = \min(W_{\delta^{-1}(J')}w)$. By Lemma 4.4 in the Appendix, $w' \in {}^{\delta^{-1}(J')}W^J$. Then

$$\pi(Z_{\mathcal{C},\delta,w}) = G_\delta(w', 1)(P_J \times P_{J'}),$$
$$\pi([\mathcal{C}, x, y]^{-,+}) = (B^- \times B)(x, y')(P_J \times P_{J'}).$$

By definition, $\max(W_{J'}\delta(w)) = \max(W_{J'}\delta(w'))$. By Lemma 4.2 in the Appendix, $y^{-1} \triangleright \delta(x) \leqslant (y')^{-1} \triangleright \delta(x)$. Now $Z_{\mathcal{C},\delta,w} \cap [\mathcal{C}, x, y]^{-,+} \neq \emptyset$ follows from Proposition 2.2 and the following Lemma 2.1. □

Lemma 2.1. *For $w \in {}^{\delta^{-1}(J')}W^J$ and $(x, y) \in W^J \times W^{J'}$,*

$$G_\delta(w, 1)(P_J \times P_{J'}) \cap (B^- \times B)(x, y)(P_J \times P_{J'}) \neq \emptyset$$

if and only if $y^{-1} \triangleright \delta(x) \leqslant \max(W_{J'}\, \delta(w))$.

Proof. First note that

$$G_\delta(w, 1)(P_J \times P_{J'}) = G_\delta(B \times B)(w, 1)(P_J \times P_{J'}).$$

Thus $G_\delta(w, 1)(P_J \times P_{J'}) \cap (B^- \times B)(x, y)(P_J \times P_{J'}) \neq \emptyset$ if and only if

$$G_\delta \cap (B^- \times B)(x, y)(P_J \times P_{J'})(w^{-1}, 1)(B \times B) \neq \emptyset. \tag{11}$$

Using $P_J \times P_{J'} = \cup_{z \in W_J, z' \in W_{J'}} (B \times B)(z, z')(B \times B)$ and the fact that $BzBw^{-1}B = Bzw^{-1}B$ for any $z \in W_J$, one sees that (11) is equivalent to

$$\bigcup_{z \in W_J, z' \in W_{J'}} G_\delta \cap (B^- x B z w^{-1} B \times ByBz'B) \neq \emptyset,$$

or $(B^- \delta(x) B \delta(zw^{-1})B) \cap (ByBz'B) \neq \emptyset$ for some $z \in W_J$ and $z' \in W_{J'}$, which, by 1) and 6) of Lemma 4.7 and 3) of Lemma 4.3 in the Appendix, is in turn equivalent to

$$y^{-1} \triangleright \delta(x) \leqslant z' * \delta(wz^{-1}) = \delta(x) \leqslant z' * \delta(w) * \delta(z^{-1})$$

for some $z \in W_J$ and $z' \in W_{J'}$. Since for any $z \in W_J$ and $z' \in W_{J'}$,

$$z' * \delta(w) * \delta(z^{-1}) \leqslant w_0^{J'} * \delta(w) * w_0^{\delta(J)} = \max(W_{J'}\, \delta(w)\, W_{\delta(J)}),$$

and the inequality becomes an equality when $z = w_0^J$ and $z' = w_0^{J'}$, (11) is equivalent to

$$y^{-1} \triangleright \delta(x) \leqslant \max(W_{J'}\, \delta(w)\, W_{\delta(J)}). \tag{12}$$

Since $\delta(x) \in W^{\delta(J)}$, it follows from Lemma 4.4 in the Appendix that $y^{-1} \triangleright \delta(x) \in W^{\delta(J)}$ and (12) is equivalent to $y^{-1} \triangleright \delta(x) \leqslant \max(W_{J'}\, \delta(w))$. □

2.9. To study the geometry and closures of the non-empty intersections in §2.6 and §2.7, we first recall some elementary facts on intersections of subvarieties in an algebraic variety.

The following Lemma 2.2 is a generalization of [23, Corollary 1.5] of Richardson. Our proof of Lemma 2.2 is essentially the same as that of [23, Theorem 1.4].

Lemma 2.2. *Let A be a connected algebraic group and let H, K and L be closed connected subgroups of A. Assume that $H \cap K$ is connected and that* $\mathrm{Lie}(H) + \mathrm{Lie}(K) = \mathrm{Lie}(A)$. *Let Y be an irreducible subvariety of A/L such that $HY \subset A/L$ is smooth. Then for any K-orbit O in A/L such that $(HY) \cap O \neq \emptyset$, HY and O intersect transversally in A/L and $HY \cap O$ is a smooth irreducible subvariety of A/L with*

$$\dim((HY) \cap O) = \dim HY + \dim O - \dim A/L.$$

Proof. Since HY is a union of H-orbits in A/L, it follows from [23, Corollary 1.5] and [23, Proposition 1.2] that HY and O intersect transversally and that the intersection $(HY) \cap O$ is smooth. Moreover, each irreducible component of $(HY) \cap O$ has dimension equal to $\dim HY + \dim O - \dim A/L$.

It remains to show that $(HY) \cap O$ is irreducible. Fix an $x \in O$ and consider the diagram

$$O \xleftarrow{p} H \times K \xrightarrow{m} A \xrightarrow{q} A/L,$$

where $p(h, k) = kx$, $m(h, k) = h^{-1}k$, and $q(a) = ax$ for $h \in H, k \in K$, and $a \in A$. Let

$$E = \{(h, k) \in H \times K : \ h^{-1}kx \in Y\} \subset H \times K.$$

Then $(HY) \cap O = p(E)$, so it is enough to show that E is irreducible.

Since L is connected and $Y \subset A/L$ is irreducible, $q^{-1}(Y) \subset A$ is irreducible by [23, Lemma 1.3]. As in the proof of [23, Theorem 1.4], HK is open in A, so $HK \cap q^{-1}(Y)$ is an irreducible subvariety of HK. The map m induces an isomorphism $m : (H \times K)/D \to HK$, where $D = \{(g, g) : g \in H \cap K\}$. Let $\nu : H \times K \to (H \times K)/D$ be the natural projection. Since D is connected, by [23, Lemma 1.3], $E = \nu^{-1}(m^{-1}(HK \cap q^{-1}(Y)))$ is also irreducible. □

The following Lemma 2.3 is useful in determining the irreducible components of intersections of algebraic varieties and will be used several times in the paper.

Lemma 2.3. *Let Y be an algebraic variety over an algebraically closed field* $\mathbf{k}$. *Suppose that $l \geq 0$ is an integer such that every irreducible component of Y has dimension at least l, and suppose that $Y = \bigsqcup_{k \in \mathcal{K}} Y_k$ is a finite disjoint union, where each Y_k is an irreducible subvariety of Y with $\dim Y_k \leqslant l$. Then the irreducible components of Y are precisely the closures $\overline{Y_k}$ of those Y_k's, where $k \in \mathcal{K}$ for which $\dim Y_k = l$.*

Proof. Let S be any irreducible component of Y. Then

$$S = \bigcup_{k\in\mathcal{K}: S\cap Y_k \neq \emptyset} \overline{S \cap Y_k}.$$

Since S is irreducible, $S = \overline{S \cap Y_k} \subset \overline{Y_k}$ for some $k \in \mathcal{K}$. Since $\overline{Y_k}$ is irreducible, $S = \overline{Y_k}$, and it follows from the dimension assumptions that $\dim Y_k = l$. Since the Y_k's are pair-wise disjoint, the closures $\overline{Y_k}$ with $\dim Y_k = l$ are pair-wise distinct irreducible components. $\square$

Lemma 2.4. *Let Z be a smooth irreducible algebraic variety and let*

$$Z = \bigsqcup_{i\in\mathcal{I}} X_i = \bigsqcup_{j\in\mathcal{J}} Y_j$$

be two partitions of Z such that each non-empty intersection $X_i \cap Y_j$ is transversal and irreducible. Then for any $(i,j) \in \mathcal{I}\times\mathcal{J}$, $X_i \cap Y_j \neq \emptyset$ if and only if $\overline{X_i} \cap \overline{Y_j} \neq \emptyset$, and in this case,

$$\overline{X_i \cap Y_j} = \overline{X_i} \cap \overline{Y_j}.$$

In particular, $Z = \bigsqcup_{(i,j)\in\mathcal{K}} (X_i \cap Y_j)$ is again a partition of Z. Here $\mathcal{K} = \{(i,j) \in \mathcal{I}\times\mathcal{J} : X_i \cap Y_j \neq \emptyset\}$.

Proof. Let $(i,j) \in \mathcal{I}\times\mathcal{J}$ be such that $\overline{X_i} \cap \overline{Y_j} \neq \emptyset$, and let

$$\mathcal{K}_{ij} = \{(i',j') \in \mathcal{I}\times\mathcal{J} :\ X_{i'} \subset \overline{X_i},\ Y_{j'} \subset \overline{Y_j},\ X_{i'} \cap Y_{j'} \neq \emptyset\}.$$

Then

$$\overline{X_i} \cap \overline{Y_j} = \bigsqcup_{(i',j')\in\mathcal{K}_{ij}} X_{i'} \cap Y_{j'}$$

is a disjoint union. By [8, Page 222], every irreducible component of $\overline{X_i} \cap \overline{Y_j}$ has dimension at least $\dim X_i + \dim Y_j - \dim Z$. On the other hand, for any $(i',j') \in \mathcal{K}_{ij}$,

$$\begin{aligned} \dim X_{i'} \cap Y_{j'} &= \dim X_{i'} + \dim Y_{j'} - \dim Z \\ &\leqslant \dim X_i + \dim Y_j - \dim Z. \end{aligned} \tag{13}$$

Since $\overline{X_i}$ is irreducible, $\dim X_{i'} < \dim X_i$ for any $i' \in \mathcal{I}$ such that $X_{i'} \subset \overline{X_i}$ and $i' \neq i$. Similarly, $\dim Y_{j'} < \dim Y_j$ for any $j' \in \mathcal{J}$ such that $Y_{j'} \subset \overline{Y_j}$ and $j' \neq j$. Thus the inequality in (13) is an equality if and only if $(i',j') = (i,j)$. By Lemma 2.3, $X_i \cap Y_j \neq \emptyset$ and $\overline{X_i \cap Y_j} = \overline{X_i} \cap \overline{Y_j}$. $\square$

Theorem 2.1. *Let $w \in W^J$ and $(x,y),(u,v) \in W^J \times W$. Then*

1) $Z_{\mathcal{C},\delta,w} \cap [\mathcal{C},x,y]^{-,+} \neq \emptyset$ if and only if $\overline{Z_{\mathcal{C},\delta,w}} \cap \overline{[\mathcal{C},x,y]^{-,+}} \neq \emptyset$. In this case, $Z_{\mathcal{C},\delta,w}$ and $[\mathcal{C},x,y]^{-,+}$ intersect transversally in $Z_{\mathcal{C}}$, the intersection is smooth and irreducible, and

$$\overline{Z_{\mathcal{C},\delta,w} \cap [\mathcal{C},x,y]^{-,+}} = \overline{Z_{\mathcal{C},\delta,w}} \cap \overline{[\mathcal{C},x,y]^{-,+}}.$$

2) $[\mathcal{C},x,y]\cap[\mathcal{C},u,v]^{-,-}\neq\emptyset$ *if and only if* $\overline{[\mathcal{C},x,y]}\cap\overline{[\mathcal{C},u,v]^{-,-}}\neq\emptyset$. *In this case,* $[\mathcal{C},x,y]$ *and* $[\mathcal{C},u,v]^{-,-}$ *intersect transversally in* $Z_{\mathcal{C}}$*, the intersection is smooth and irreducible, and*

$$\overline{[\mathcal{C},x,y]\cap[\mathcal{C},u,v]^{-,-}}=\overline{[\mathcal{C},x,y]}\cap\overline{[\mathcal{C},u,v]^{-,-}}.$$

Proof. Since $R_{\mathcal{C}}$ is connected, $G_\delta\cap(B^-\times B)$ is connected, and

$$\mathrm{Lie}(G_\delta)+\mathrm{Lie}(B^-\times B)=\mathrm{Lie}(G\times G),$$

Lemma 2.2 applies. By taking $A=G\times G$,

$$H=G_\delta,\ \ K=B^-\times B,\ \ L=R_{\mathcal{C}},\ \ \text{and}\ \ Y=(B\times B)(w,1).R_{\mathcal{C}}$$

in Lemma 2.2, one sees that when $Z_{\mathcal{C},\delta,w}\cap[\mathcal{C},x,y]^{-,+}\neq\emptyset$, $Z_{\mathcal{C},\delta,w}$ and $[\mathcal{C},x,y]^{-,+}$ intersect transversally in $Z_{\mathcal{C}}$, and that the intersection $Z_{\mathcal{C},\delta,w}\cap[\mathcal{C},x,y]^{-,+}$ is smooth and irreducible. By applying Lemma 2.4 to the two partitions

$$Z_{\mathcal{C}}=\bigsqcup_{w\in W^J}Z_{\mathcal{C},\delta,w}=\bigsqcup_{(x,y)\in W^J\times W}(B^-\times B)(x,y).R_{\mathcal{C}}$$

of $Z_{\mathcal{C}}$, one proves part 1). Part 2) can be proved in the same way. □

Remark 2.1. In both 1) and 2) in Theorem 2.1, the fact that the intersection is non-empty if and only if the intersection of the closures is non-empty can also been obtained using §2.4 and Proposition 2.1 and Proposition 2.2. However, the proof we give is more conceptual.

3. Intersections in $\overline{G}$

3.1. Let G be a connected semi-simple adjoint group and $\overline{G}$ be the De Concini-Procesi compactification. It is known [2] that the $G\times G$-orbits in $\overline{G}$ are in one to one correspondence with subsets of Γ. For $J\subset\Gamma$, let Z_J be the corresponding $G\times G$-orbit in $\overline{G}$. A distinguished point $h_J\in Z_J$ can be chosen such that the stabilizer subgroup of $G\times G$ at h_J is

$$R_J^-\overset{\text{def}}{=}(U_J^-\times U_J)\{(m_1,m_2)\in M_J\times M_J:\ \pi_J(m_1)=\pi_J(m_2)\},$$

where $\pi_J:M_J\to M_J/\mathrm{Cen}(M_J)$ is the natural projection and $\mathrm{Cen}(M_J)$ is the center of M_J.

For $J\subset\Gamma$, let $\mathcal{C}_J=(J^*,J,c,L)$, where $J^*=-w_0(J)$, $c=(w_0w_0^J)^{-1}$, and

$$L=\{(\dot{w}_0\dot{w}_0^Jm_1(\dot{w}_0\dot{w}_0^J)^{-1},m_2):m_1,m_2\in M_J,\pi_J(m_1)=\pi_J(m_2)\}$$

with $\dot{w}_0$ and $\dot{w}_0^J$ being any representatives of w_0 and w_0^J in $N_G(H)$. By [17, Section 5], $\mathcal{C}_J$ is an admissible quadruple for G, and

$$R_{\mathcal{C}_J}=(\dot{w}_0\dot{w}_0^J,1)R_J^-(\dot{w}_0\dot{w}_0^J,1)^{-1}.$$

One thus has the isomorphism

$$Z_J\longrightarrow Z_{\mathcal{C}_J}:\quad (g,g').h_J\longmapsto(gw_0^Jw_0,g').R_{\mathcal{C}_J},\qquad g,g'\in G. \tag{14}$$

3.2. For $J \subset \Gamma$ and $(x, y) \in W^J \times W$, let

(15) $[J, x, y] = (B \times B)(x, y).h_J,$

(16) $[J, x, y]^{-,+} = (B^- \times B)(x, y).h_J = (w_0, 1)[J,\ w_0 x w_0^J,\ y w_0^J],$

(17) $[J, x, y]^{-,-} = (B^- \times B^-)(x, y).h_J = (w_0, w_0)[J,\ w_0 x w_0^J,\ w_0 y w_0^J].$

For $J \subset \Gamma$ and $w \in W^J$, let

$$Z_{J,\delta,w} = G_\delta (B \times B)(w, 1).h_J.$$

The $Z_{J,\delta,w}$'s will be called the G_δ-stable pieces in $\overline{G}$. By [25, Lemma 1.3] and [20, 12.3], one has the following partitions of $\overline{G}$:

$$\begin{aligned}\overline{G} &= \bigsqcup_{J \subset \Gamma, (x,y) \in W^J \times W} [J, x, y] = \bigsqcup_{J \subset \Gamma, (x,y) \in W^J \times W} [J, x, y]^{-,-} \\ &= \bigsqcup_{J \subset \Gamma, (x,y) \in W^J \times W} [J, x, y]^{-,+} = \bigsqcup_{J \subset \Gamma, w \in W^J} Z_{J,\delta,w}.\end{aligned}$$

For a an irreducible subvariety $X \subset Z_J$, let $\mathrm{Codim}_{Z_J}(X)$ be the codimension of X in Z_J. Let l be the length function of W. One has, for any $J \subset \Gamma$ and $w \in W^J$,

1) $\dim Z_J = \dim G - \dim \mathrm{Cen}(M_J) = \dim G - |\Gamma| + |J|$. See [2, Theorem 3.1].

2) $\mathrm{Codim}_{Z_J}[J, x, y] = l(w_0) + l(x) - l(y)$. See [25, Lemma 1.3].

3) $\mathrm{Codim}_{Z_J} Z_{J,\delta,w} = l(w)$. See [20, Section 8].

By (16) and (17), one also has

4) $\mathrm{Codim}_{Z_J}[J, x, y]^{-,+} = 2l(w_0) - l(x w_0^J) - l(y w_0^J)$.

5) $\mathrm{Codim}_{Z_J}[J, x, y]^{-,-} = l(w_0) - l(x w_0^J) + l(y w_0^J)$.

3.3. The closure of a $G \times G$-orbit is described in [2, Theorem 3.1 & Theorem 5.2] as follows.

1) For $J \subset \Gamma$, $\overline{Z_J} = \bigsqcup_{K \subset J} Z_K$ is a smooth subvariety of $\overline{G}$.

The closure of a $B \times B$-orbit is described in [25, Proposition 2.4], and the following simplified version in 2) is found in [13, Proposition 6.3] and [17, Example 1.3]. The following 3) and 4) are obtained using (16) and (17).

2) For $J \subset \Gamma$ and $(x, y) \in W^J \times W$, $\overline{[J, x, y]} = \bigsqcup [K, x', y']$, where $K \subset J$, $(x', y') \in W^K \times W$ and there exists $u \in W_J$ such that $xu \leqslant x', y' \leqslant yu$.

3) For $J \subset \Gamma$ and $(x, y) \in W^J \times W$, $\overline{[J, x, y]^{-,+}} = \bigsqcup [K, x', y']^{-,+}$, where $K \subset J$, $(x', y') \in W^K \times W$ and there exists $u \in W_J$ such that $x' w_0^K \leqslant xu, y' w_0^K \leqslant yu$.

4) For $J \subset \Gamma$ and $(x, y) \in W^J \times W$, $\overline{[J, x, y]^{-,-}} = \bigsqcup [K, x', y']^{-,-}$, where $K \subset J$, $(x', y') \in W^K \times W$ and there exists $u \in W_J$ such that $x' w_0^K \leqslant xu, y' w_0^K \geqslant yu$.

For $J \subset \Gamma$ and $w \in W$, let

$$C_J(w) = \{\delta^{-1}(u) w u^{-1} : u \in W_J\},$$

and denote by $\mathrm{Min}(C_J(w))$ the set of minimal length elements in $C_J(w)$. It is easy to see that $w \in \mathrm{Min}(C_J(w))$ for any $w \in W^J$. The closure of a G_δ-stable piece is described in [10, Sections 3 and 4] as follows:

5) For $J \subset \Gamma$ and $w \in W^J$, $\overline{Z_{J,\delta,w}} = \bigsqcup Z_{K,\delta,w'}$, where $K \subset J, w' \in W^K$, and $w' \geqslant w_1$ for some $w_1 \in \mathrm{Min}(C_J(w))$.

3.4. We can now prove our first main result in this paper.

Proposition 3.1. *For $J \subset \Gamma$, $w \in W^J$, and $(x,y),(u,v) \in W^J \times W$,*

$$[J,x,y] \cap [J,u,v]^{-,-} \neq \emptyset \ \text{ iff } \ x \leqslant u,\ v \leqslant \max(yW_J), \tag{18}$$

$$Z_{J,\delta,w} \cap [J,x,y]^{-,+} \neq \emptyset \ \text{ iff } \ \min(W_J\,\delta(w)) \leqslant y^{-1} * \delta(x). \tag{19}$$

Proof. Let $\mathcal{C}_J$ be as in §3.1 and recall the isomorphism $Z_J \to Z_{\mathcal{C}_J}$ in (14). Since $W^{J^*} = W^J w_0^J w_0$, one has

$$[J,x,y] \cap [J,u,v]^{-,-} \neq \emptyset \ \text{ iff } \ [\mathcal{C}_J, xw_0^J w_0, y] \cap [\mathcal{C}_J, uw_0^J w_0, v]^{-,-} \neq \emptyset,$$

which, by Proposition 2.1, is equivalent to $uw_0^J w_0 \leqslant xw_0^J w_0$ and $v \leqslant \max(yW_J)$. Note that $uw_0^J w_0 \leqslant xw_0^J w_0$ if and only if $uw_0^J \geqslant xw_0^J$, which is equivalent to $u \geqslant x$ since $x, u \in W^J$. Thus (18) is proved.

Similarly, $Z_{J,\delta,w} \cap [J,x,y]^{-,+} \neq \emptyset$ iff $Z_{\mathcal{C},\delta,ww_0^J w_0} \cap [\mathcal{C}, xw_0^J w_0, y]^{-,+} \neq \emptyset$, which, by Proposition 2.2, is equivalent to

$$y^{-1} \triangleright (\delta(x) w_0^{\delta(J)} w_0) \leqslant \max(W_J \delta(w) w_0^{\delta(J)} w_0). \tag{20}$$

By Lemma 4.5 and Lemma 4.3 in the Appendix, (20) is equivalent to

$$\min(W_J \delta(w) w_0^{\delta(J)}) \leqslant y^{-1} * (\delta(x) w_0^{\delta(J)}),$$

which is in turn equivalent to $\min(W_J \delta(w)) \leqslant y^{-1} * \delta(x)$. □

3.5. Admissible partitions of $\overline{G}$. In order to generalize Theorem 2.1 to $\overline{G}$, we will introduce the notion "admissible partitions" and discuss some of their properties.

Definition 3.1. A partition of $\overline{G}$ is said to be *admissible* if it is of the form

$$\overline{G} = \bigsqcup_{J \subset \Gamma} \bigsqcup_{\alpha \in \mathcal{A}_J} X_{J,\alpha}, \tag{21}$$

where for each $J \subset \Gamma$, $\mathcal{A}_J$ is a finite index set, and for each $\alpha \in \mathcal{A}_J$, $X_{J,\alpha} \subset Z_J$ and

$$\mathrm{Codim}_{Z_K} X_{K,\alpha'} \geqslant \mathrm{Codim}_{Z_J} X_{J,\alpha}$$

for every $K \subset J$ and $\alpha' \in \mathcal{A}_K$ such that $X_{K,\alpha'} \subset \overline{X_\alpha} \cap Z_K$. An admissible partition is said to be *strongly admissible* if $\overline{X_{J,\alpha}} \cap Z_K \neq \emptyset$ for every $K \subset J$ and $\alpha \in \mathcal{A}_J$.

Note that the partition $\overline{G} = \bigsqcup_{J \subset \Gamma} Z_J$ is strongly admissible.

Proposition 3.2. *The partitions*

$$\overline{G} = \bigsqcup_{J\subset\Gamma,(x,y)\in W^J\times W} [J,x,y] = \bigsqcup_{J\subset\Gamma,(x,y)\in W^J\times W} [J,x,y]^{-,-} \tag{22}$$
$$= \bigsqcup_{J\subset\Gamma,(x,y)\in W^J\times W} [J,x,y]^{-,+} = \bigsqcup_{J\subset\Gamma,w\in W^J} Z_{J,\delta,w}$$

are strongly admissible.

Proof. Let $K \subset J \subset \Gamma$ and $(x,y) \in W^J \times W$. If $(x',y') \in W^K \times W$ is such that $[K,x',y'] \subset \overline{[J,x,y]}$, one knows from §3.3 that there exists $u \in W_J$ such that $x' \geqslant xu$ and $y' \leqslant yu$. By 2) of §3.2,

$$\begin{aligned}\mathrm{Codim}_{Z_K}[K,x',y'] &= l(w_0)+l(x')-l(y') \geqslant l(w_0)+l(xu)-l(yu)\\ &= l(w_0)+l(x)+l(u)-l(yu) \geqslant l(w_0)+l(x)-l(y)\\ &= \mathrm{Codim}_{Z_J}[J,x,y].\end{aligned}$$

Regard x as in W^K. By §3.3, $[K,x,y] \subset \overline{[J,x,y]} \cap Z_K$. Thus the first partition in (22) is strongly admissible. The second and third partitions of $\overline{G}$ in (22), being the translations by $(w_0,1)$ and by (w_0,w_0) of the first one, are thus also strongly admissible.

Consider now the partition of $\overline{G}$ into the G_δ-stable pieces. Let $K \subset J \subset \Gamma$ and $w \in W^J$. If $w' \in W^K$ is such that $Z_{K,\delta,w'} \subset \overline{Z_{J,\delta,w}}$, one knows from §3.3 that there exists $w_1 \in \mathrm{Min}(C_J(w))$ such that $w' \geqslant w_1$. By 3) of §3.2,

$$\mathrm{Codim}_{Z_K}(Z_{K,\delta,w'}) = l(w') \geqslant l(w_1) = l(w) = \mathrm{Codim}_{Z_J}(Z_{J,\delta,w}).$$

Regard w as in W^K. By §3.3, $Z_{K,\delta,w} \subset \overline{Z_{J,\delta,w}} \cap Z_K$. Thus the partition $\overline{G} = \bigsqcup_{J\subset\Gamma,w\in W^J} Z_{J,\delta,w}$ is strongly admissible. $\square$

Definition-Notation 3.1. Two admissible partitions

$$\overline{G} = \bigsqcup_{J\subset\Gamma}\bigsqcup_{\alpha\in\mathcal{A}_J} X_{J,\alpha} = \bigsqcup_{J\subset\Gamma}\bigsqcup_{\beta\in\mathcal{B}_J} Y_{J,\beta} \tag{23}$$

of $\overline{G}$ are said to be *compatible* if for any $J \subset \Gamma$, $\alpha \in \mathcal{A}_J$, and $\beta \in \mathcal{B}_J$ with $X_{J,\alpha} \cap Y_{J,\beta} \neq \emptyset$, $X_{J,\alpha}$ and $Y_{J,\beta}$ intersect transversally in Z_J and $X_{J,\alpha} \cap Y_{J,\beta}$ is irreducible. For two such partitions of $\overline{G}$, and for $K \subset J \subset \Gamma$, $\alpha \in \mathcal{A}_J$, and $\beta \in \mathcal{B}_J$, let

$$\mathcal{A}_K^\alpha = \{\alpha' \in \mathcal{A}_K : \ X_{K,\alpha'} \subset \overline{X_{J,\alpha}}, \ \ \mathrm{Codim}_{Z_K} X_{K,\alpha'} = \mathrm{Codim}_{Z_J} X_{J,\alpha}\},$$
$$\mathcal{B}_K^\beta = \{\beta' \in \mathcal{B}_K : \ Y_{K,\beta'} \subset \overline{Y_{J,\beta}}, \ \ \mathrm{Codim}_{Z_K} Y_{K,\beta'} = \mathrm{Codim}_{Z_J} Y_{J,\beta}\}.$$

Proposition 3.3. *1) Any admissible partition of $\overline{G}$ is compatible with the partition of $\overline{G}$ into $G \times G$-orbits;*

2) The partitions of $\overline{G}$ into G_δ-stable pieces and into $B^- \times B$-orbits are compatible;

3) The partitions of $\overline{G}$ into $B \times B$-orbits and into $B^- \times B^-$-orbits are compatible.

Proof. Part 1) follows directly from the definition.

2) Notice that a G_δ-stable piece $Z_{J,\delta,w}$ and a $B^- \times B$-orbit $[K,x,y]^{-,+}$ intersect only if $J = K$. Applying Lemma 2.2 to the two subvarieties $Z_{J,\delta,w}$ and $[J,x,y]^{-,+}$ of Z_J, we have that the intersection is transversal and irreducible.

Part 3) can be proved in the same way. □

Recall [9, Page 427] that two irreducible subvarieties X and Y of a smooth irreducible variety Z with $X \cap Y \neq \emptyset$ are said to *intersect properly* in Z if every irreducible component of $X \cap Y$ has dimension equal to $\dim X + \dim Y - \dim Z$.

Theorem 3.1. *Let two compatible admissible partitions of $\overline{G}$ be given as in (23). Then for any $J, K \subset \Gamma$ and $\alpha \in \mathcal{A}_J$ and $\beta \in \mathcal{B}_K$, if $\overline{X_{J,\alpha}} \cap \overline{Y_{K,\beta}} \neq \emptyset$, then $\overline{X_{J,\alpha}}$ and $\overline{Y_{K,\beta}}$ intersect properly in $\overline{Z_{J\cup K}}$, and*

$$\overline{X_{J,\alpha}} \cap \overline{Y_{K,\beta}} = \bigcup_{(\alpha',\beta') \in \mathcal{I}_{J\cap K}^{\alpha,\beta}} \overline{X_{J\cap K,\alpha'} \cap Y_{J\cap K,\beta'}}$$

is the decomposition of $\overline{X_{J,\alpha}} \cap \overline{Y_{K,\beta}}$ into (distinct) irreducible components, where

$$\mathcal{I}_{J\cap K}^{\alpha,\beta} = \{(\alpha',\beta') \in \mathcal{A}_{J\cap K}^{\alpha} \times \mathcal{B}_{J\cap K}^{\beta} : \ X_{J\cap K,\alpha'} \cap Y_{J\cap K,\beta'} \neq \emptyset\}.$$

In particular, $\mathcal{I}_{J\cap K}^{\alpha,\beta} \neq \emptyset$.

Proof. Let $J, K \subset \Gamma$, $\alpha \in \mathcal{A}_J$, and $\beta \in \mathcal{B}_K$ be such that $\overline{X_{J,\alpha}} \cap \overline{Y_{K,\beta}} \neq \emptyset$. Regard both $\overline{X_{J,\alpha}}$ and $\overline{Y_{K,\beta}}$ as subvarieties of $\overline{Z_{J\cup K}}$. Since $\overline{Z_{J\cup K}}$ is smooth and irreducible with

$$\dim Z_{J\cup K} = \dim Z_J + \dim Z_K - \dim Z_{J\cap K},$$

every irreducible component of $\overline{X_{J,\alpha}} \cap \overline{Y_{K,\beta}}$ has dimension at least

$$\begin{aligned} l &= \dim X_{J,\alpha} + \dim Y_{K,\beta} - \dim Z_{J\cup K} \\ &= \dim Z_{J\cap K} - \mathrm{Codim}_{Z_J} X_{J,\alpha} - \mathrm{Codim}_{Z_K} Y_{K,\beta}. \end{aligned}$$

On the other hand,

$$\overline{X_{J,\alpha}} \cap \overline{Y_{K,\beta}} = \bigsqcup_{\substack{I \subset J\cap K,\, \alpha' \in \mathcal{A}_I,\, \beta' \in \mathcal{B}_I \\ X_{I,\alpha'} \subset \overline{X_{J,\alpha}},\ Y_{I,\beta'} \subset \overline{Y_{K,\beta}}}} X_{I,\alpha'} \cap Y_{I,\beta'}. \tag{24}$$

For each non-empty intersection on the right hand side of (24),

$$\begin{aligned} \dim X_{I,\alpha'} \cap Y_{I,\beta'} &= \dim Z_I - \mathrm{Codim}_{Z_I} X_{I,\alpha'} - \mathrm{Codim}_{Z_I} Y_{I,\beta'} \\ &\leqslant \dim Z_I - \mathrm{Codim}_{Z_J} X_{J,\alpha} - \mathrm{Codim}_{Z_K} Y_{K,\beta} \\ &\leqslant \dim Z_{J\cap K} - \mathrm{Codim}_{Z_J} X_{J,\alpha} - \mathrm{Codim}_{Z_K} Y_{K,\beta} \\ &= l. \end{aligned} \tag{25}$$

By Lemma 2.3 and 1) of Proposition 3.3, every irreducible component of $\overline{X_{J,\alpha}} \cap \overline{Y_{K,\beta}}$ has dimension l, and the irreducible components are exactly as described in Theorem 3.1. $\square$

By taking the second admissible partition in Theorem 3.1 to be the one by $G \times G$-orbits, we have the following Corollary 3.1.

Corollary 3.1. *Let a strongly admissible partition of $\overline{G}$ be given as in (21), and let $J \subset \Gamma$ and $\alpha \in \mathcal{A}_J$. Then for any $K \subset \Gamma$, $\overline{X_{J,\alpha}} \cap \overline{Z_K} \neq \emptyset$ and $\overline{X_{J,\alpha}}$ and $\overline{Z_K}$ intersect properly in $\overline{Z_{J\cup K}}$. Moreover,*

$$\overline{X_{J,\alpha}} \cap \overline{Z_K} = \bigcup_{\alpha' \in \mathcal{A}^{\alpha}_{J\cap K}} \overline{X_{J\cap K,\, \alpha'}}$$

is the decomposition of $\overline{X_{J,\alpha}} \cap \overline{Z_K}$ into (distinct) irreducible components.

Corollary 3.2. *Let two compatible partitions of $\overline{G}$ be given as in (23). Then for any $J \subset \Gamma$ and $\alpha, \beta \in \mathcal{A}_J$, $X_{J,\alpha} \cap Y_{J,\beta} \neq \emptyset$ if and only if $\overline{X_{J,\alpha}} \cap \overline{Y_{J,\beta}} \neq \emptyset$, and in this case,*

$$\overline{X_{J,\alpha}} \cap \overline{Y_{J,\beta}} = \overline{X_{J,\alpha} \cap Y_{J,\beta}}. \tag{26}$$

In particular,

$$\overline{G} = \bigsqcup_{J\subset\Gamma,\ X_{J,\alpha}\cap Y_{J,\beta}\neq\emptyset} X_{J,\alpha} \cap Y_{J,\beta}$$

is an admissible partition of $\overline{G}$.

Proof. Take $K = J$ in Theorem 3.1. If $\overline{X_{J,\alpha}} \cap \overline{Y_{J,\beta}} \neq \emptyset$, then $\mathcal{I}_J^{\alpha,\beta}$ consists of one element, namely, (α, β). Thus $X_{J,\alpha} \cap Y_{J,\beta} \neq \emptyset$ and (26) holds. The condition on codimensions in Definition 3.1 follows from (25) in the proof of Theorem 3.1. $\square$

3.6. We now prove our second main result in this paper. Let

$$\begin{aligned}
\mathcal{J} = \{(J,x,y,u,v) :\ & J \subset \Gamma,\ (x,y),(u,v) \in W^J \times W, \\
& [J,x,y] \cap [J,u,v]^{-,-} \neq \emptyset\} \\
= \{(J,x,y,u,v) :\ & J \subset \Gamma,\ x,u \in W^J, y,v \in W, \\
& x \leqslant u,\ v \leqslant \max(yW_J)\}, \\
\mathcal{K} = \{(J,w,x,y) :\ & J \subset \Gamma,\ (w,x,y) \in W^J \times W^J \times W, \\
& Z_{J,\delta,w} \cap [J,x,y]^{-,+} \neq \emptyset\} \\
= \{(J,w,x,y) :\ & J \subset \Gamma,\ (w,x,y) \in W^J \times W^J \times W, \\
& \min(W_J\,\delta(w)) \leqslant y^{-1} * \delta(x)\}.
\end{aligned}$$

Theorem 3.2. *Let $J \subset \Gamma$ and $(x,y),(u,v) \in W^J \times W$. Then*

$$[J,x,y] \cap [J,u,v]^{-,-} \neq \emptyset \quad \textit{iff} \quad \overline{[J,x,y]} \cap \overline{[J,u,v]^{-,-}} \neq \emptyset,$$

and in this case,

$$\overline{[J,x,y]\cap[J,u,v]^{-,-}}=\overline{[J,x,y]}\cap\overline{[J,u,v]^{-,-}}. \tag{27}$$

In particular,

$$\overline{G}=\bigsqcup_{(J,x,y,u,v)\in\mathcal{J}}[J,x,y]\cap[J,u,v]^{-,-} \tag{28}$$

is a strongly admissible partition of $\overline{G}$.

Proof. Assume that $\overline{[J,x,y]}\cap\overline{[J,u,v]^{-,-}}\neq\emptyset$. It follows from 3) of Proposition 3.3 and Corollary 3.2 that $[J,x,y]\cap[J,u,v]^{-,-}\neq\emptyset$ and (27) holds.

By Corollary 3.2, the partition (28) of $\overline{G}$ is admissible. To show that it is also strongly admissible, let $(J,x,y,u,v)\in\mathcal{J}$ and let $K\subset J$. By definition, there exists $z\in W_J$ such that $yz=\max(yW_J)$. Set $z'=\min(zW_K)\in W^K$. Then $xz',uz'\in W^K$ and

$$vz'\leqslant\max(vW_J)\leqslant\max(yW_J)=yz\leqslant\max(yz'W_K).$$

By Proposition 3.1, $[K,xz',yz']\cap[K,uz',vz']^{-,-}\neq\emptyset$. By 2) and 4) of §3.3, $[K,xz',yz']\subset\overline{[J,x,y]}$ and $[K,uz',vz']^{-,-}\subset\overline{[J,u,v]^{-,-}}$. Therefore

$$\overline{[J,x,y]\cap[J,u,v]^{-,-}}\cap Z_K\supset[K,xz',yz']\cap[K,uz',vz']^{-,-}\neq\emptyset.$$

This shows that the partition (28) is strongly admissible. $\square$

Theorem 3.3. *Let* $J\subset\Gamma$, $w\in W^J$, *and* $(x,y)\in W^J\times W$. *Then*

$$Z_{J,\delta,w}\cap[J,x,y]^{-,+}\neq\emptyset \quad \textit{iff} \quad \overline{Z_{J,\delta,w}}\cap\overline{[J,x,y]^{-,+}}\neq\emptyset,$$

and in this case,

$$\overline{Z_{J,\delta,w}}\cap\overline{[J,x,y]^{-,+}}=\overline{Z_{J,\delta,w}\cap[J,x,y]^{-,+}}. \tag{29}$$

In particular,

$$\overline{G}=\bigsqcup_{(J,w,x,y)\in\mathcal{K}}Z_{J,\delta,w}\cap[J,x,y]^{-,+} \tag{30}$$

is a strongly admissible partition of $\overline{G}$.

Proof. Assume that $\overline{Z_{J,\delta,w}}\cap\overline{[J,x,y]^{-,+}}\neq\emptyset$. It follows from 2) of Proposition 3.3 and Corollary 3.2 that $Z_{J,\delta,w}\cap[J,x,y]^{-,+}\neq\emptyset$ and (29) holds.

By Corollary 3.2, the partition (30) of $\overline{G}$ is admissible. To show that it is also strongly admissible, let $(J,w,x,y)\in\mathcal{K}$ and let $K\subset J$. By definition, there exists $z\in W_J$ such that $yz=\max(yW_J)$. Set $z'=\min(zW_K)\in W^K\cap W_J$. Then $xz'\in W^K$ and $xz'\geqslant x$. Let $z=z'z''$ with $z''\in W_K$. By Lemma 4.4 in the Appendix,

$$\begin{aligned} w_0^K*(yz')^{-1}&=\max(W_K(yz')^{-1})=\max(W_K(yz)^{-1})\\ &=\max(W_K\max(W_Jy^{-1}))=\max(W_Jy^{-1})=w_0^J*y^{-1}.\end{aligned}$$

By Lemma 4.2 and Lemma 4.4 in the Appendix,

$$\begin{aligned} w_0^K * (yz')^{-1} * \delta(xz') = w_0^J * y^{-1} * \delta(xz') &\geqslant w_0^J * (y^{-1} * \delta(x)) \\ &= \max(W_J(y^{-1} * \delta(x)). \end{aligned}$$

Since $\min(W_J\delta(w)) \leqslant y^{-1} * \delta(x)$, one has

$$\begin{aligned} \delta(w) &\leqslant \max(W_J(y^{-1} * \delta(x)) \leqslant w_0^K * ((yz')^{-1} * \delta(xz')) \\ &= \max(W_K(yz')^{-1} * \delta(xz')). \end{aligned}$$

By Lemma 4.5 in the Appendix, $\min(W_K\delta(w)) \leqslant (yz')^{-1} * \delta(xz')$, and

$$\overline{Z_{J,\delta,w} \cap [J,x,y]^{-,+}} \cap Z_K \supset Z_{K,\delta,w} \cap [K, xz', yz']^{-,+} \neq \emptyset.$$

This shows that the partition (30) of $\overline{G}$ is strongly admissible. □

Remark 3.1. Assume that $\overline{[J,x,y]} \cap \overline{[J,u,v]^{-,-}} \neq \emptyset$, we can also use Proposition 3.1 to prove directly that $[J,x,y] \cap [J,u,v]^{-,-} \neq \emptyset$. Similarly, assume that $\overline{Z_{J,\delta,w}} \cap \overline{[J,x,y]^{-,+}} \neq \emptyset$. One can use Proposition 3.1 to prove directly that $Z_{J,\delta,w} \cap [J,x,y]^{-,+} \neq \emptyset$. We omit the details.

Consider now the following four strongly admissible partitions of $\overline{G}$:

(31)

$$\begin{aligned} \overline{G} &= \bigsqcup_{J\subset\Gamma,(x,y)\in W^J\times W} [J,x,y]^{-,+} = \bigsqcup_{J\subset\Gamma, w\in W^J} Z_{J,\delta,w} \\ &= \bigsqcup_{(J,x,y,u,v)\in\mathcal{J}} [J,x,y] \cap [J,u,v]^{-,-} = \bigsqcup_{(J,x,y,u,v)\in\mathcal{K}} Z_{J,\delta,w} \cap [J,x,y]^{-,+}. \end{aligned}$$

As a direct consequence of Corollary 3.1, we have

Corollary 3.3. *Let $J \subset \Gamma$ and let X be any of the subvarieties of Z_J appearing in either one of the four partitions in (31). Then for any $K \subset \Gamma$, $\overline{X} \cap \overline{Z_K} \neq \emptyset$, and $\overline{X}$ and $\overline{Z_K}$ intersect properly in $\overline{Z_{J\cup K}}$.*

Corollary 3.1 also allows us to describe the irreducible components of the non-empty intersections in Corollary 3.3.

Corollary 3.4. *1) For any $J \subset \Gamma$, $(x,y) \in W^J \times W$, and $K \subset \Gamma$, the irreducible components of $\overline{[J,x,y]^{-,+}} \cap \overline{Z_K}$ are precisely of the form $\overline{[J\cap K, xu, yu]^{-,+}}$, where $u \in W_J \cap W^{J\cap K}$ and $l(yu) = l(y) + l(u)$.*

2) For any $J \subset \Gamma$, $w \in W^J$, and $K \subset \Gamma$, the irreducible components of $\overline{Z_{J,\delta,w}} \cap \overline{Z_K}$ are precisely of the form $\overline{Z_{J\cap K,\delta,w'}}$ with $w' \in W^{J\cap K} \cap$ $\mathrm{Min}(C_J(w))$.

3) For any $(J,x,y,u,v) \in \mathcal{J}$ and $K \subset \Gamma$, the irreducible components of the intersection $\overline{[J,x,y] \cap [J,u,v]^{-,-}} \cap \overline{Z_K}$ are the non-empty intersections of irreducible components of $\overline{[J,x,y]} \cap \overline{Z_K}$ and the irreducible components of $\overline{[J,u,v]^{-,-}} \cap \overline{Z_K}$.

4) For any $(J, w, x, y) \in \mathcal{K}$ and $K \subset \Gamma$, the irreducible components of $\overline{Z_{J,\delta,w} \cap [J, x, y]^{-,+} \cap Z_K}$ are the non-empty intersections of irreducible components of $\overline{Z_{J,\delta,w}} \cap \overline{Z_K}$ and the irreducible components of $\overline{[J, x, y]^{-,+}} \cap \overline{Z_K}$.

Remark 3.2. Corollary 3.3 and Corollary 3.4 in the case of the intersections $\overline{[J, x, y]^{-,+}} \cap \overline{Z_K}$ have also been obtained by M. Brion in [1] (using $B \times B^-$-orbits instead of $B^- \times B$-orbits).

Applying Theorem 3.1, we have

Corollary 3.5. *1) If $J, K \subset \Gamma$ and $(x, y) \in W^J \times W, (u, v) \in W^K \times W$ are such that $\overline{[J, x, y]} \cap \overline{[K, u, v]^{-,-}} \neq \emptyset$, then $\overline{[J, x, y]}$ and $\overline{[K, u, v]^{-,-}}$ intersect properly in $\overline{Z_{J \cup K}}$, and the irreducible components of $\overline{[J, x, y]} \cap \overline{[K, u, v]^{-,-}}$ are the non-empty intersections of irreducible components of $\overline{[J, x, y]} \cap \overline{Z_{J \cap K}}$ and the irreducible components of $\overline{[K, u, v]^{-,-}} \cap \overline{Z_{J \cap K}}$.*

2) If $J, K \subset \Gamma$ and $(w, x, y) \in W^J \times W^K \times W$ are such that $\overline{Z_{J,\delta,w}} \cap \overline{[K, x, y]^{-,+}} \neq \emptyset$, then $\overline{Z_{J,\delta,w}}$ and $\overline{[K, x, y]^{-,+}}$ intersect properly in $\overline{Z_{J \cup K}}$, and the irreducible components of $\overline{Z_{J,\delta,w}} \cap \overline{[K, x, y]^{-,+}}$ are the non-empty intersections of irreducible components of $\overline{Z_{J,\delta,w}} \cap \overline{Z_{J \cap K}}$ and the irreducible components of $\overline{[K, x, y]^{-,+}} \cap \overline{Z_{J \cap K}}$.

4. Appendix

Recall that W is the Weyl group of G. We now prove some properties of the operations $*, \triangleleft, \triangleright$ on W as defined in §2.5. In fact, many properties also hold for arbitrary Coxeter groups. See [12].

Lemma 4.1. [10, Lemma 3.3] *For any $x, y \in W$,*

1) $x * y \in W$ is the unique maximal element in the set $\{uy : u \leqslant x\}$ as well as in the set $\{xv; v \leqslant y\}$. Moreover, $x * y = x_1 y = x y_1$ for some $x_1 \leqslant x, y_1 \leqslant y$ with $l(x * y) = l(x_1) + l(y) = l(x) + l(y_1)$;

2) $x \triangleright y \in W$ is the unique minimal element in the set $\{uy : u \leqslant x\}$, and $x \triangleright y = x_1 y$ for some $x_1 \leqslant x$ with $l(x \triangleright y) = l(y) - l(x_1)$;

3) $x \triangleleft y \in W$ is the unique minimal element in the set $\{xv; v \leqslant y\}$, and $x \triangleleft y = x y_1$ for some $y_1 \leqslant y$ with $l(x \triangleleft y) = l(x) - l(y_1)$.

Lemma 4.2. *Let $x, x', y, y' \in W$. If $x \leqslant x'$ and $y \leqslant y'$, then*

$$x * y \leqslant x' * y', \quad x' \triangleright y \leqslant x \triangleright y', \quad \text{and} \quad x \triangleleft y' \leqslant x' \triangleleft y.$$

Proof. We have that

$$\begin{aligned} \overline{B(x * y)B} &= \overline{BxByB} = \overline{BxB}\ \overline{ByB} \subset \overline{Bx'B}\ \overline{By'B} = \overline{Bx'By'B} \\ &= \overline{B(x' * y')B}. \end{aligned}$$

Here the second equality follows from the properness of the multiplication map $\overline{BxB} \times_B \overline{ByB} \to G$. So $x * y \leqslant x' * y'$. Similarly, since

$$\begin{aligned}\overline{B(x \triangleright y')B^-} &= \overline{BxBy'B^-} = \overline{BxB}\ \overline{By'B^-} \subset \overline{Bx'B}\ \overline{ByB^-} = \overline{Bx'ByB^-} \\ &= \overline{B(x' \triangleright y)B^-},\end{aligned}$$

one has $x' \triangleright y \leqslant x \triangleright y'$. Similarly, $x \triangleleft y' \leqslant x' \triangleleft y$. $\square$

Lemma 4.3. *For any* $x, y, z \in W$,

1) $x \triangleright y = (x * (yw_0))w_0$ *and* $x \triangleleft y = w_0((w_0 x) * y)$;

2) $(x \triangleleft y)^{-1} = y^{-1} \triangleright x^{-1}$ *and* $(x * y)^{-1} = y^{-1} * x^{-1}$;

3) $x \triangleright y \leqslant z$ *if and only if* $y \leqslant x^{-1} * z$;

4) $y \triangleleft x \leqslant z$ *if and only if* $y \leqslant z * x^{-1}$;

5) $(x \triangleright y) \triangleleft z = x \triangleright (y \triangleleft z)$.

Proof. 1) Since

$$\begin{aligned}\overline{B(x \triangleright y)B^-} &= \overline{BxByB^-} = \overline{BxByw_0Bw_0} = \overline{B(x * (yw_0))Bw_0} \\ &= \overline{B\big(x * (yw_0)\big)w_0B^-},\end{aligned}$$

one has $x \triangleright y = x \triangleright (yw_0)w_0$. Similarly, $x \triangleleft y = w_0((w_0 x) * y)$.

2) Let τ be the inverse map of G. Then $(x \triangleleft y)^{-1} = y^{-1} \triangleright x^{-1}$ follows by applying τ to $\overline{B^-(x \triangleleft y)B} = \overline{B^- xByB}$. Similarly, $(x * y)^{-1} = y^{-1} * x^{-1}$.

3) Since $y \in \{u(x \triangleright y) : u \leqslant x^{-1}\}$, $y \leqslant x^{-1} * (x \triangleright y)$. If $x \triangleright y \leqslant z$, then $y \leqslant x^{-1} * z$ by Lemma 4.2. Similarly, $z \in \{u(x^{-1} * z) : u \leqslant x\}$, so $x \triangleright (x^{-1} * z) \leqslant z$. If $y \leqslant x^{-1} * z$, then by Lemma 4.2, $x \triangleright y \leqslant x \triangleright (x^{-1} * z) \leqslant z$.

Part 4) can be proved in the same way as part 3).

5) By §2.5,

$$\begin{aligned}\overline{B((x \triangleright y) \triangleleft z)B^-} &= \overline{B(x \triangleright y)B^- zB^-} = \overline{B(x \triangleright y)B^-}\ \overline{B^- zB^-} \\ &= \overline{BxByB^-}\ \overline{B^- zB^-} = \overline{BxB}\ \overline{ByB^-}\ \overline{B^- zB^-} \\ &= \overline{BxB}\ \overline{ByB^- zB^-} = \overline{BxB}\ \overline{B(y \triangleleft z)B^-} \\ &= \overline{BxB(y \triangleleft z)B^-} = \overline{B(x \triangleright (y \triangleleft z))B^-}.\end{aligned}$$

Thus $(x \triangleright y) \triangleleft z = x \triangleright (y \triangleleft z)$. $\square$

Lemma 4.4. *For* $J, J' \subset \Gamma$, $x \in W$, $y \in W^J$ *and* $z \in {}^J W$, *one has* $x \triangleright y \in W^J$, $z \triangleleft x \in {}^J W$, *and*

$$w_0^{J'} \triangleright x \triangleleft w_0^J = \min(W_{J'} x W_J), \quad w_0^{J'} * x * w_0^J = \max(W_{J'} x W_J).$$

Proof. By Lemma 4.1, $x \triangleright y = x_1 y$ for some $x_1 \leqslant x$ and $l(x \triangleright y) = l(y) - l(x_1)$. For any $u \in W_J$, $l(x_1^{-1}(x \triangleright y)u) = l(yu) = l(y) + l(u) = l(x_1^{-1}) + l(x \triangleright y) + l(u)$. Hence $l((x \triangleright y)u) = l(x \triangleright y) + l(u)$ for $u \in W_J$ and $x_1 y \in W^J$. Similarly one has $z \triangleleft x \in {}^J W$. By 2) of Lemma 4.1, $w_0^{J'} \triangleright x = \min\{ux ; u \leqslant w_0^{J'}\} = \min\{ux ; u \in W_{J'}\} = \min(W_{J'} x)$. Similarly, $x \triangleleft w_0^J = \min(xW_J) \in W^J$. Thus $w_0^{J'} \triangleright x \triangleleft w_0^J \in {}^{J'}W^J$. By

Lemma 4.1 and 5) of Lemma 4.3, $w_0^{J'} \triangleright x \triangleleft w_0^J \in W_{J'} x W_J$. Thus $w_0^{J'} \triangleright x \triangleleft w_0^J = \min(W_{J'} x W_J)$. Similarly, $w_0^{J'} * x * w_0^J = \max(W_{J'} x W_J)$. $\square$

Combining Lemma 4.4 with 4) of Lemma 4.3, we have the following consequence.

Lemma 4.5. *For any $J \subset \Gamma$ and $x, y \in W$, $x \leqslant \max(yW_J)$ if and only if $\min(xW_J) \leqslant y$ and $x \leqslant \max(W_J y)$ if and only if $\min(W_J x) \leqslant y$.*

The following Lemma 4.6 can be found in [4, Corollary 1.2] and [15, 1.2].

Lemma 4.6. *For $x, y \in W$, the following conditions are equivalent:*
1) $BxB \subset \overline{ByB}$;
2) $B^- yB \subset \overline{B^- xB}$;
3) $(B^- xB) \cap (ByB) \neq \emptyset$;
4) $\overline{B^- xB} \cap \overline{ByB} \neq \emptyset$;
5) $x \leqslant y$.

The following result is used several times in our paper.

Lemma 4.7. *For $x, y, u, v \in W$, the following conditions are equivalent:*
1) $(BxByB) \cap (B^- uBvB) \neq \emptyset$;
2) $\overline{BxByB} \cap (B^- uBvB) \neq \emptyset$;
3) $(BxByB) \cap \overline{B^- uBvB} \neq \emptyset$;
4) $\overline{BxByB} \cap \overline{B^- uBvB} \neq \emptyset$;
5) $u \triangleleft v \leqslant x * y$.
6) $u \leqslant x * y * v^{-1}$.

Proof. Clearly 1) implies 2) and 3), 2) or 3) implies 4), 4) implies 5) by Lemma 4.6, and 5) is equivalent to 6) by 4) of Lemma 4.3. It suffices to show that 5) implies 1).

Suppose that $u \triangleleft v \leqslant x * y$. Then $(B(x * y)B) \cap (B^-(u \triangleleft v)B) \neq \emptyset$ by Lemma 4.6. Since $B(x * y)B \subset BxByB$ and $B^-(u \triangleleft v)B \subset B^- uBvB$, we have $(BxByB) \cap (B^- uBvB) \neq \emptyset$. Hence 5) implies 1). $\square$

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