On proper holomorphic mappings among irreducible bounded symmetric domains of rank at least 2

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1 Introduction

Proper holomorphic mappings among domains on Euclidean spaces is a classical topic in Several Complex Variables. The literature can date back to the earliest results like the theorem of H. Alexander [1] which says that any proper holomorphic self-map of the complex unit n-ball is a biholomorphism if $n \geq 2$. Since then, the study of the proper holomorphic mappings between complex unit balls of different dimensions has become a very popular topic in the field. Many important inputs from various perspectives have been made, like Algebraic Geometry, Chern-Moser Theory, Segre variety and Bergman kernel, etc. It is apparent by now that the complexity of the problem grows with the codimension and one in general must impose certain regularity assumptions on the proper maps in order to give any satisfactory classification.

Comparing with those of the complex unit balls, the proper holomorphic mapping problems for irreducible bounded symmetric domains of higher rank are of a very different nature. On the one hand, the methods in the rank-1 case find limited applicability on the higher-rank cases due to the vast difference in their boundary structures. While the boundary of a complex unit ball is a smooth strictly pseudoconvex hypersurface defined by a simple real analytic equation in a Euclidean space, the boundary of a higher-rank irreducible bounded symmetric domain is non-smooth and contains complex analytic submanifolds. On the other hand, in the higher-rank cases, it appears that it is the rank difference which defines the difficulty of the problem rather than the codimension. This can be illustrated by, for example, the following statement which was originally conjectured by Mok and later proven by Tsai [2]

Theorem 1.1 (Tsai). *Let Ω_1, Ω_2 be two irreducible bounded symmetric domains and $F : \Omega_1 \to \Omega_2$ be a proper holomorphic map. If rank$(\Omega_1) \geq$ rank$(\Omega_2) \geq 2$, then F is a totally geodesic isometric embedding (up to a normalization constant) with respect to the Bergman metrics.*

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The above theorem generalizes the classical result of Henkin-Tumanov [3] on the special case where $\Omega_1 = \Omega_2$. The proof given in [2] is sophisticated and involves a very careful analysis on the infinitesimal behavior of the map F and a complete classification of the invariantly geodesic subspaces of irreducible bounded symmetric domains. While many of these intermediate results like the classification of invariantly geodesic subspaces are of great significance to the theory of bounded symmetric domains, it is still worth the effort to look for a simpler and more direct proof of Theorem 1.1. To this end, we first remark that the major goal behind Tsai's proof is to show that F maps minimal disks of Ω_1 properly into minimal disks of Ω_2 and after this a simple argument will lead to the desired result. This approach has also been adopted in the subsequent works of Tu ([4],[5]) on the rigidity of proper holomorphic maps among higher-rank irreducible bounded symmetric domains. The purpose of the current article is to demonstrate that the total geodesy of a map between two higher-rank irreducible bounded symmetric domains can also be characterized by a weaker condition as follows.

Proposition 1.2. *Let Ω_1, Ω_2 be irreducible bounded symmetric domains of rank at least 2 and let $F : \Omega_1 \to \Omega_2$ be a holomorphic map. If F maps minimal disks of Ω_1 properly into rank-1 invariantly geodesic subspaces of Ω_2, then F is a totally geodesic isometric embedding (up to a normalization constant) with respect to the Bergman metrics.*

When F is a proper and $\mathrm{rank}(\Omega_1) \geq \mathrm{rank}(\Omega_2) \geq 2$, it is a lot easier to verify the hypotheses of the above proposition than to show that F maps minimal disks into minimal disks. Thus, by proving the above proposition we will obtain a much simpler proof for Theorem 1.1. The rest of the article will be devoted to this task.

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2 Preliminaries

In what follows, for every $a \in \mathbb{C}^n$ and $r > 0$, we write

$$B(a, r) := \{z \in \mathbb{C}^n : \|z - a\| < r\}.$$

We also denote the set of bounded holomorphic functions on a complex manifold X by $H^\infty(X)$.

Let $D \subset \mathbb{C}^n$ be a domain. A point $b \in \partial D$ is said to be a *local peak point* (Rudin [6]) of D if there is an $r > 0$ and a function h, holomorphic in $D \cap B(b, r)$, continuous on $\overline{D \cap B(b, r)}$ such that $h(b) = 1$ but $|h(z)| < 1$ for every $z \in \overline{D \cap B(b, r)} - \{b\}$.

We have the following lemma by the maximal principle [6].

Lemma 2.1. *Let $D \Subset \mathbb{C}^n$ be bounded domain and $b \in \partial D$ be a local peak point of D. Suppose $\{F_i\}$ is a sequence of holomorphic maps from a domain $U \subset \mathbb{C}^k$ into D and there is a point $z_0 \in U$ such that $F_i(z_0) \to b$ as $i \to \infty$. Then $F_i(z) \to b$ uniformly on every compact subset of U.*

We need another lemma which is proven in [7] using Fatou's theorem on the existence of radial limits for bounded holomorphic functions.

Lemma 2.2. *Let Δ be the unit disk and $W \Subset \mathbb{C}^n$ be a bounded domain. Write the coordinates for a point $p \in \Delta \times W$ as $p = (z, w)$. Suppose $f \in H^\infty(\Delta \times W)$, then for almost every $b \in \partial\Delta$, we have that*

$$f_b(w) = \lim_{r \to 1^-} f(rb, w)$$

exists for every $w \in W$. Moreover, $f_b \in H^\infty(W)$. If f_b is constant for $b \in E \subset \partial\Delta$, where E is of positive measure with respect to the measure of $\partial\Delta$, then f is independent on w.

It is well known that a polydisk cannot be properly mapped into any complex unit ball. We follow the idea of the proof of this fact and obtain the following:

Proposition 2.3. *Let $W \Subset \mathbb{C}^n$ be a bounded domain and $F : \Delta \times W \longrightarrow \mathbb{B}^k$ be a holomorphic map such that $F(z, 0) : \Delta \longrightarrow \mathbb{B}^k$ is proper. Then $F(z, w) = F(z, 0)$.*

Proof. As all the component functions of F are in $H^\infty(\Delta \times W)$, by Lemma 2.2, for almost every $b \in \partial\Delta$,

$$F_b(w) = \lim_{r \to 1^-} F(rb, w)$$

exists for every $w \in W$ and $F_b(w)$ is a holomorphic map from W into $\mathbb{C}^k$. Fix one $b \in \partial\Delta$ such that F_b exists. As $F(z, 0)$ is proper, we have $F(rb, 0) \to \zeta(b)$ for some $\zeta(b) \in \partial\mathbb{B}^n$ as $r \to 1^-$. Since every boundary point of $\mathbb{B}^k$ is a local peak point, by Lemma 2.1, $F_b(w) \equiv \zeta(b)$. Hence we have $F(z, w) = F(z, 0)$ by Lemma 2.2. □

3 Proof of Proposition 1.2

Let Ω_1, Ω_2 be irreducible bounded symmetric domains of rank at least 2. In what follows, we refer the reader to [7] for the details of the notions like characteristic vector, minimal disk and invariantly geodesic subspace.

Lemma 3.1. *Let $F : \Omega_1 \to \Omega_2$ be a holomorphic map such that F maps minimal disks of Ω_1 properly into rank-1 invariantly geodesic subspaces of Ω_2. Let $\mu, \nu \in T_p^{(1,0)}(\Omega_1)$ be two characteristic vectors at $p \in \Omega_1$ such that $R^{(1)}_{\mu\bar{\mu}\nu\bar{\nu}} = 0$, where $R^{(1)}_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is the curvature tensor of Ω_1. Then $g_2(dF(\mu), \overline{dF(\nu)}) = 0$, where g_2 is the Bergman metric of Ω_2.*

Proof. Without loss of generality, we may take p to be the origin and assume that $F(0) = 0$. Since μ, ν are characteristic and $R^{(1)}_{\mu\bar{\mu}\nu\bar{\nu}} = 0$, we can find a totally geodesic two-disk $\Delta^2 \subset \Omega_1$ and choose a coordinate system (z, w) of Δ^2 which is the restriction of some choice of Harish-Chandra coordinates of Ω_1 such that, $\Delta_\mu := \{(z, 0) \in \Delta^2\}$, $\Delta_\nu := \{(0, w) \in \Delta^2\}$ are minimal disks of Ω_1 and μ, ν are tangent to Δ_μ and Δ_ν respectively. We may simply take $\mu = \frac{\partial}{\partial z}(0)$ and $\nu = \frac{\partial}{\partial w}(0)$.

By hypotheses we have $F(\Delta_\mu) \subset B \subset \Omega_2$, where $B \cong \mathbb{B}^k$, $k \in \mathbb{N}^+$, is a rank-1 invariantly geodesic subspace of Ω_2. By applying an automorphism in Ω_2, we may assume that $B = \{(z_1, \ldots, z_k, 0, \ldots, 0) \in \Omega_2 : |z_1|^2 + \cdots + |z_k|^2 < 1\}$, where $(z_1, \ldots, z_k, \ldots, z_m)$ are Harish-Chandra coordinates of $\Omega_2 \Subset \mathbb{C}^m$. It follows from Hermann Convexity Theorem [8] that $\pi(\Omega_2) = \mathbb{B}^k$, where $\pi : \mathbb{C}^m \to \mathbb{C}^k$ is the canonical projection to the first k direct factors. Therefore if we write $F = (f_1, \ldots, f_k, \ldots, f_m)$, and $F_k := (f_1, \ldots, f_k)$, then the restriction $F_k|_{\Delta_\mu} : \Delta_\mu \to \mathbb{B}^k$ is a proper holomorphic map.

Now by Proposition 2.3, we have $F_k(z, w) = F_k(z, 0)$, i.e. $f_j(z, w) = f_j(z, 0)$ for $1 \leq j \leq k$ and this implies that $f_j(0, w) \equiv 0$ for $1 \leq j \leq k$. Hence, we have $F(z, 0) = (f_1(z, 0), \ldots, f_k(z, 0), 0, \ldots, 0)$ and $F(0, w) = (0, \ldots, 0, f_{k+1}(0, w), \ldots, f_m(0, w))$. It then follows that $g_{\mathbb{C}^m}(dF(\mu), \overline{dF(\nu)}) = 0$, where $g_{\mathbb{C}^m}$ is the Euclidean metric. But the Bergman metric g_2 of Ω_2 agrees with $g_{\mathbb{C}^m}$ at the origin and thus the lemma follows. $\square$

We are now ready to prove our main proposition.

Proof of Proposition 1.2. Let g_1 and g_2 be the Bergman metrics of Ω_1 and Ω_2 respectively. We are going to show that $F^* g_2 = c g_1$ for some $c > 0$.

Take two distinct unit-length characteristic vectors $\mu, \nu \in T_0^{(1,0)}(\Omega_1)$ at the origin such that $R^{(1)}_{\mu\bar{\mu}\nu\bar{\nu}} = 0$. Choose Harish-Chandra coordinates $(z_1, \ldots, z_n)$ on $\Omega_1 \Subset \mathbb{C}^n$ such that $\mu = \frac{\partial}{\partial z_1}(0)$, $\nu = \frac{\partial}{\partial z_2}(0)$ and $(z_1, z_2, 0, \ldots, 0)$ is a totally geodesic two-disk $\Delta^2 \subset \Omega_1$ as in Lemma 3.1.

Let $h = F^* g_2$ and write $h_{i\bar{j}} = h(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$. Note that as for μ and ν, the vector fields $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_2}$ give a zero bisectional curvature everywhere in Δ^2 and thus by Lemma 3.1, we have $h_{1\bar{2}} = h_{2\bar{1}} \equiv 0$ on Δ^2. Since h is Kähler, it follows that $\frac{\partial h_{1\bar{1}}}{\partial z_2} = \frac{\partial h_{2\bar{1}}}{\partial z_1} \equiv 0$ on Δ^2. Let R^h be the curvature tensor of h, by direct computation, we have

$$R^h_{1\bar{1}2\bar{2}} = -\frac{\partial^2 h_{1\bar{1}}}{\partial z_2 \partial \bar{z}_2} + \sum_{1 \leq i,j \leq n} h^{i\bar{j}} \frac{\partial h_{1\bar{j}}}{\partial z_2} \frac{\partial h_{i\bar{1}}}{\partial \bar{z}_2}.$$

Since h has non-positive bisectional curvature, it follows that

$$0 \geq R^h_{1\bar{1}2\bar{2}}(0) = -\frac{\partial^2 h_{1\bar{1}}}{\partial z_2 \partial \bar{z}_2}(0) + \sum_{1\leq i,j\leq n} h^{i\bar{j}} \frac{\partial h_{1\bar{j}}}{\partial z_2} \frac{\partial h_{i\bar{1}}}{\partial \bar{z}_2}\Bigg|_{z=0} = \sum_{1\leq i,j\leq n} h^{i\bar{j}} \frac{\partial h_{1\bar{j}}}{\partial z_2} \frac{\partial h_{i\bar{1}}}{\partial \bar{z}_2}\Bigg|_{z=0} \geq 0.$$

Hence, $\frac{\partial h_{1\bar{j}}}{\partial z_2}(0) = 0$ for every j. As Harish-Chandra coordinates are complex geodesic coordinates at the origin, we have for every $\eta \in T_0^{(1,0)}(\Omega_1)$,

$$\nabla_\mu h_{\nu\bar{\eta}}(0) = 0.$$

By a polarization argument as given by Mok ([9], Chapter 6), the complex vector space $T_0^{(1,0)}(\Omega_1) \otimes T_0^{(1,0)}(\Omega_1)$ is spanned by its elements of the form $\xi \otimes \zeta$, where ξ, ζ are characteristic and $R_{\xi\bar{\xi}\zeta\bar{\zeta}} = 0$. We therefore can conclude that $\nabla h = 0$ at the origin. By exploiting the homogeneity, we deduce furthermore that $\nabla h \equiv 0$, i.e. h is parallel and hence $h = cg_1$ for some $c \geq 0$ as Ω_1 is irreducible. Since F is non-constant, we have $c \neq 0$ and hence F is an isometric embedding up to a normalizing constant. By the preservation of the zeros of the bisectional curvature by a holomorphic isometric embedding between bounded symmetric domains, it follows easily that the associated second fundamental form is identically zero and hence the embedding is totally geodesic. The proof is complete. □

Theorem 1.1 is now a simple corollary.

Proof of Theorem 1.1. If $F : \Omega_1 \to \Omega_2$ is a proper holomorphic map, then F maps maximal characteristic symmetric subspaces of Ω_1 into maximal characteristic symmetric subspaces of Ω_2. (This fact has been essentially proven in [7] and later explicitly stated in [2]). Since every maximal characteristic symmetric subspace is of rank one less than that of the ambient space, it follows by induction that F maps rank-1 characteristic symmetric subspaces of Ω_1 into rank-1 characteristic symmetric subspaces of Ω_2. But every minimal disk of Ω_1 is contained in some rank-1 characteristic symmetric subspace and every characteristic symmetric subspace is also an invariantly geodesic subspace, therefore F satisfies the hypotheses of Proposition 1.2 and hence is a totally geodesic isometric embedding up to a normalization constant. □

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