ENDOMORPHISMS PRESERVING COORDINATES OF POLYNOMIAL ALGEBRAS

YUN-CHANG LI AND JIE-TAI YU

Abstract. It is proved that the Jacobian of a k-endomorphism of $k[x_1, \ldots, x_n]$ over a field k of characteristic zero taking every tame coordinate to a coordinate, must be a nonzero constant in k. It is also proved that the Jacobian of an R-endomorphism of $A := R[x_1, \ldots, x_n]$ (where R is a polynomial ring in finite number of variables over an infinite field k), taking every R-linear coordinate of A to an R-coordinate of A, is a nonzero constant in k.

1. **Introduction and the main results**

Van den Essen and Shpilrain [2] asked the following

Question 1. *Let k be a field. Is it true that every k-endomorphism of $k[x_1, ..., x_n]$ taking every coordinate to a coordinate is an automorphism?*

In [2] the question was answered by van den Essen and Shpilrain themselves in the positive for an arbitrary field k when $n = 2$. The question was solved by Jelonek [5] affirmatively for algebraically closed fields k of characteristic zero for all n by geometric method based on

Derksen's observation. (see [2]) Let k be an algebraically closed field. A k-endomorphism ϕ of $k[x_1, \ldots, x_n]$ taking every k-linear coordinate of $k[x_1, \ldots, x_n]$ to a coordinate of $k[x_1, \ldots, x_n]$ must have nonzero constant Jacobian $J(\phi)$ in k.

2010 *Mathematics Subject Classification.* Primary 13F20, 13W20, 14R10.

Key words and phrases. Automorphisms, endomorphisms, coordinates, linear coordinates, tame coordinates, polynomial algebras, Jacobian.

The research of Yun-Chang Li was partially supported by a postgraduate studentship in the University of Hong Kong.

The research of Jie-Tai Yu was partially supported by an RGC-GRF Grant.

For the related *linear coordinate preserving problem* for polynomial algebras, see Mikhalev, Yu and Zolotykh [8], Cheng and ven den Essen [1], and Gong and Yu [3]. For another related *automorphic orbit problem* for polynomial algebras, see van den Essen and Shpilrain [2], Yu [9], Gong and Yu [4], and Li and Yu [7].

The purpose of this note is to prove the following two new results.

Theorem 1.1. *Let $R := k[x_{n+1}, \dots, x_{n+m}]$ where k is an infinite field, $m > 0$. Let $\phi := (f_1, \dots, f_n)$ be an R-endomorphism of $A := R[x_1, \dots, x_n]$ taking every R-linear coordinate of A to an R-coordinate of A. Then*

$$J(\phi) = J_{x_1,\dots,x_n}(f_1, \dots, f_n) := \det[(f_i)'_{x_j}] \in k^*.$$

Note if we replace R by a field k, the statement of Theorem 1.1 is generally not true, unless k is algebraically closed (Derksen's observation). For non-algebraically closed fields k, see Mikhalev, J.-T.Yu and Zolotykh [8], and Gong and Yu [3] for counterexamples.

Theorem 1.2. *Let $\phi := (f_1, \dots, f_n)$ be a k-endomorphism of $k[x_1, \dots, x_n]$ over a field k of characteristic zero taking every tame coordinate of $k[x_1, \dots, x_n]$ to a coordinate of $k[x_1, \dots, x_n]$. Then*

$$J(\phi) = J_{x_1,\dots,x_n}(f_1, \dots, f_n) := \det[(f_i)'_{x_j}] \in k^*.$$

In the sequel k always denotes a field with a fixed algebraic closure K. Endomorphisms (automorphisms) always means k-endomorphisms (k-automorphisms) unless otherwise specified.

2. **Preliminaries**

Recall that an automorphism of $k[x_1, \cdots, x_n]$ is tame if it can be decomposed to product of linear and elementary automorphisms, and a coordinate p (i.e. a component of an automorphism) is called tame if p is a component of a tame automorphism. For an endomorphism $\phi := (f_1, \dots, f_n)$ of $k[x_1, \dots, x_n]$, we use

$$J(\phi) := J(f_1, \dots, f_n) := J_{x_1,\dots,x_n}(f_1, \dots, f_n)$$

to denote its Jacobian $\det[(f_i)'_{x_j}]$.

Let $\phi := (f_1, \dots, f_n)$ be an endomorphism of $k[x_1, \dots, x_n]$ taking a coordinate p of $k[x_1, \dots, x_n]$ to a coordinate of $k[x_1, \dots, x_n]$ and let $\sigma := (g_1, \dots, g_n)$ be any automorphism of $k[x_1, \dots, x_n]$. Obviously,

$$\sigma \circ \phi := (f(g_1, \dots, g_n), \dots, f_n(g_1, \dots, g_n))$$

is also an endomorphism of $k[x_1, \dots, x_n]$ taking the same coordinate p to a coordinate. Moreover, $J(\phi) \in k^*$ if and only if $J(\sigma \circ \phi) \in k^*$.

We need the following two lemmas.

Lemma 2.1. *Let $f_1, \dots, f_{n-1} \in k[x_1, \dots, x_n]$ over an infinite field k such that*

$$\deg_{x_n} J(f_1, \dots, f_{n-1}, x_n) > 0.$$

Then there exist $a_1, \dots, a_{n-1} \in k$, $b \in K$ and $h_1, h_2, \dots, h_{n-1} \in k[x_n]$ (without loss of generality we may assume $h_1 = 1$ after acting a transposition on $\{f_1, \dots, f_{n-1}\}$) such that the gradient (partial derivatives) of

$$g := f_1 + h_2(x_n) f_2 + \cdots + h_{n-1}(x_n) f_{n-1}$$

with respect to $(x_1, \dots, x_{n-1})$ is $(0, \dots, 0)$ at the point

$$P = (a_1, \dots, a_{n-1}, b).$$

Proof. Let

$$G(x_1, \dots, x_n) := J(f_1, \dots, f_{n-1}, x_n) = \sum_{i=0}^{m} p_i(x_1, \cdots, x_{n-1}) x_n^i$$

where $m \geq 1$ and $p_m(x_1, \cdots, x_{n-1}) \neq 0$. Since k is infinite, we may choose $a_1, \cdots, a_{n-1} \in k$ such that $p_m(a_1, \cdots, a_{n-1}) \in k^*$. Then $G(a_1, \cdots, a_{n-1}, x_n) \in k[x_n] - k$ and there exists some $b \in K$ such that

$$G(a_1, \cdots, a_{n-1}, b) = 0.$$

Hence $E = k(b) = k[b]$ is a finite algebraic extension of k. Let $P = (a_1, \cdots, a_{n-1}, b)$ be a point in E^n and for each $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$, define

$$f(x_1, \dots, x_n)|_P := f(a_1, \cdots, a_{n-1}, b)$$

We have the first $(n-1)$ rows of the determinant $J(f_1, \dots, f_{n-1}, x_n)|_P$ are $k[b]$-linearly dependent. Therefore, there exist

$$h_1(x), h_2(x), \cdots, h_{n-1}(x) \in k[x]$$

such that

$$(h_1(b)f_1 + h_2(b)f_2 + \cdots + h_{n-1}(b)f_{n-1})'_{x_i}|_P = 0$$

for all $i = 1, \cdots, n-1$ and not all $h_1(b), \cdots, h_{n-1}(b)$ are zero. Without loss of generality, we may assume $h_1(b) \neq 0$, replace $h_i(b)$ by $h_1^{-1}(b)h_i(b)$ for all i, we may assume $h_1(x) = 1$. Now define

$$g := f_1 + h_2(x_n)f_2 + \cdots + h_{n-1}(x_n)f_{n-1} \in k[x_1, \dots, x_n].$$

It is easy to see that

$$g'_{x_i}|_P = (f_1 + h_2(b)f_2 + \cdots + h_{n-1}(b)f_{n-1})'_{x_i}|_P = 0, \quad \forall i = 1, \cdots, n-1.$$

□

Lemma 2.2. *Let $f_1, \dots, f_{n-1} \in k[x_1, \dots, x_n]$ over a field k of characteristic zero such that $\deg_{x_n} J(f_1, \dots, f_n, x_n) > 0$. Then there exist $a_1, \dots, a_{n-1} \in k$, $b \in K$ and $h_1, h_2, \dots, h_{n-1}, h_n \in k[x_n]$ (without loss of generality, we may assume $h_1 = 1$ after acting a transposition on $\{f_1, \dots, f_{n-1}\}$) such that the gradient (partial derivatives) of*

$$u := f_1 + h_2(x_n)f_2 + \cdots + \cdots + h_{n-1}(x_n)f_{n-1} + h_n$$

with respect to $(x_1, \dots, x_{n-1}, x_n)$ is $(0, \dots, 0, 0)$ at the point

$$P = (a_1, \dots, a_{n-1}, b).$$

Proof. Using the same notations in the Lemma 2.1, $u = g + h_n$, where $h_n = h_n(x_n)$ is to be determined. Define $v(b) := g'_{x_n}|_P \in k[b]$ for some $v(x) = c_0 + c_1x + \cdots + c_sx^s \in k[x]$. Define (here we need k to be characteristic zero)

$$h_n(x_n) := -c_0x_n - (1/2)c_1x_n^2 - \cdots - (1/(s+1))c_sx_n^{s+1} \in k[x_n].$$

It is easy to see that

$$u'_{x_i}|_P = g'_{x_i}|_P + h_n(x_n)'_{x_i} = 0 + 0 = 0, \quad \forall i = 1, \cdots, n-1$$

and $u'_{x_n}|_P = v(b) - v(b) = 0$. □

3. **Proofs of the main results**

Proof of Theorem 1.1. For simplicity we only present the proof for $m = 1$, the general case can be proved similarly, by an enhanced version of Lemma 2.1. Suppose on the contrary, $J(\phi)$ is not constant. If x_{n+1} does not appear in $J(\phi)$. We may assume $\deg_{x_1} J(\phi)$ is the highest among all $\deg_{x_i} J(\phi)$. Replace ϕ by $\sigma \circ \phi$, where $\sigma := (x_1 + x_{n+1}, x_2, \ldots, x_n)$. So we may assume that x_{n+1} appears in $J(\phi)$. By Lemma 2.1 there exist $a_1, \ldots, a_n \in k$, $b \in K$ and $h_1, h_2, \ldots, h_n \in R[x_{n+1}]$ (Without loss of generality we may assume $h_1 = 1$) such that

$$g := f_1 + h_2(x_{n+1})f_2 + \cdots + h_n(x_{n+1})f_n$$

with gradient (partial derivatives) with respect to $(x_1, \ldots, x_n)$ is $(0, \ldots, 0)$ at the point

$$P = (a_1, \ldots, a_n, b),$$

so g cannot be an R-coordinate of A. On the other hand, as

$$p := x_1 + h_2(x_{n+1})x_2 + \cdots + h_n(x_{n+1})x_n$$

is an R-linear coordinate of A with a corresponding R-automorphism $(p, x_2, \ldots, x_n)$ of A, $g = \phi(p)$ is also an R-coordinate of A. A contradiction. □

Proof of Theorem 1.2. We may assume $f_n = x_n$, otherwise replace ϕ by $\psi \circ \phi$, where ψ is an automorphism taking the coordinate f_n to x_n. Suppose on the contrary, $J(\phi)$ is not nonzero constant. If x_n does not appear in $J(\phi)$, we may assume that $\deg_{x_1} J(\phi)$ is the highest among all $\deg_{x_i} J(\phi)$. Replace ϕ by $\sigma \circ \phi$, where $\sigma := (x_1 + x_n, x_2, \ldots, x_n)$. So we may assume that x_n appears in $J(\phi)$. By Lemma 2.2, there exist $a_1, \ldots, a_{n-1} \in k$, $b \in K$ and $h_1, h_2, \ldots, h_{n-1}, h_n \in k[x_n]$ (without loss of generality we may assume $h_1 = 1$) such that the partial derivatives of

$$u := f_1 + h_2(x_n)f_2 + \cdots + \cdots + h_{n-1}(x_n)f_{n-1} + h_n$$

with respect to $(x_1, \ldots, x_n)$ is $(0, \ldots, 0)$ at the point

$$P = (a_1, \ldots, a_{n-1}, b),$$

hence u cannot be a coordinate of $k[x_1, \ldots, x_n]$. On the other hand, as

$$q := x_1 + h_2(x_n)x_2 + \cdots + h_{n-1}(x_n)x_{n-1} + h_n(x_n) \in k[x_1, \ldots, x_n]$$

is a tame coordinate of $k[x_1, \dots, x_n]$ with a corresponding elementary automorphism

$$(q, x_2, \dots, x_{n-1}, x_n),$$

of $k[x_1, \dots, x_n]$, $u = \phi(q)$ is also a coordinate. A contradiction. $\square$

Acknowledgement. The authors would like to thank Z.Jelonek for kindly showing his very recent result [6] where Question 1 in this note is solved affirmatively for characteristic zero and arbitrary n by geometric method based on Theorem 1.2 in this note.

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Department of Mathematics, The University of Hong Kong, Hong Kong SAR, China

E-mail address: liyunch@hku.hk, liyunch@163.com

Department of Mathematics, The University of Hong Kong, Hong Kong SAR, China

E-mail address: yujt@hku.hk, yujietai@yahoo.com