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THE TWO GOLDBACH CONJECTURES

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Abstract: The article is for general readers who may have heard the name, Goldbach or Jingrun Chen (陳景潤) and would like to know some mathematical problems related to them. There are four sections in the article. The first two sections are for those who are only interested in the stories, developments, latest results of the two Goldbach conjectures, their differences in difficulty and Chen's achievement. If after having read the first two sections, one would like to know more about mathematical information and explanation on the contents in the first two sections, one may go further to the last two sections.

1 Historical Developments and Chen's Theorem

Everyone knows that all positive integers can be classified as even integers 2, 4, 6, 8, ⋯ and odd integers 1, 3, 5, 7, ⋯ On the other hand, they can also be classified as prime numbers, 2, 3, 5, 7, ⋯ (we call any prime number >2 an odd prime), composite numbers, 4, 6, 8, 9, ⋯ and the "building block" of integers, 1. These two classifications are quite different. Prime numbers are rather hard to identify while even and odd integers have a trivially simple outward appearance by their last digit. Based on human curiosity, one may ask

Are there simple relations between the above two classifications?

The following two conjectures are on these relations. On June 7, 1742 C. Goldbach (1690-1764) [Gol] in his letter to E. Euler (1707-1783) posed two conjectures on the representation of even and odd integers as sums of primes. Nowadays these two conjectures with some modifications may be stated as follows:

G(2) — Every even integer not less than 6 is a sum of two odd primes.

G(3) — Every odd integer not less than 9 is a sum of three odd primes.

We call problems concerning G(2) and G(3) Goldbach's problems. About three weeks later on June 30, 1742, Euler [Eul] replied to Goldbach that he believed the truth of G(2) (if so G(3) is also true, cf. (D-1) in Section 2 below) although he could not verify it. Note that during 1742 there were no express communications like telephone, e-mail, etc. Euler's reply was a prompt one which seemed to indicate incidentally that he agreed with Goldbach without any hesitation or reservation. Despite the prima facie statements in G(2) and G(3), mathematicians, except for numerically checking the two conjectures, did not know how to start their attack. By tedious computations during the almost two hundred years between 1742 and 1919, they could ensure nothing about the truth of G(2) and G(3) although they could not find any positive integer for which either G(2) or G(3) is false.

In the Second International Congress of Mathematicians (ICM) held at Paris 1900, the world renowned mathematician D. Hilbert (1862-1943) [Hil] made a historic speech in which he posed twenty-three unsolved problems. These problems have profound impact and influence on mathematical development and research in the twentieth century. Goldbach's problems were included in Hilbert's eighth problem. Twelve years later in 1912,

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E. Landau (1877-1938) [Lan] addressed in the fifth ICM at Cambridge, U. K. that according to the contemporary stage of human knowledge, Goldbach's problems were beyond attack.

The first significant achievement in the study of G(3) was obtained by G. H. Hardy (1877-1947) and J. E. Littlewood (1885-1977) in their celebrated series of seven joint papers, 'Partitio Numerorum' published between 1920 and 1928. In their third paper [H-L-1] of the series concerning G(3), they proved

Hardy & Littlewood's Theorem (1923). *Assume the GRH (see (A-2) in Section 3 below). Then there is a positive effectively computable constant* **V** *such that each odd integer* $\geqslant V$ *is a sum of three odd primes.*

They added two extra conditions in G(3), namely, the GRH and the constant V. Therefore, they did not settle conjecture G(3) but had progressed a big step since, in their joint work, they created a powerful method called the Circle Method (or the Hardy-Littlewood Method). The main part of the method is based on some integrals along the unit circle. Modifying the Circle Method, I. M. Vinogradov (1891-1983) [Vim] was successful in removing the essential assumption GRH and proved

Vinogradov's Theorem (1937). *There is a positive effectively computable constant V such that for every odd integer $n \geqslant V$ (i. e., every sufficiently large odd integer n) we have*

$$n = p_1 + p_2 + p_3 \quad \textit{where } p_j \textit{ are odd primes.}$$

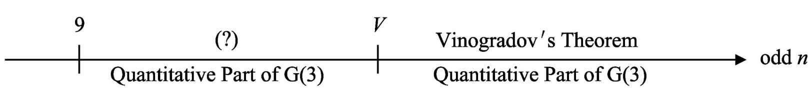
That is, conjecture G(3) is true for each odd integer not less than V. We call this result the truth of the **Qualitative Part** of G(3). However, we are still unable to claim that G(3) is true since the value of V in Vinogradov's proof is too large, e. g.,

$$\log_{10}(V) < 6,846,169.$$

We call the truth of $V = 9$ the **Quantitative Part** of G(3). Therefore, if one can prove the quantitative part, then G(3) is completely settled. Since 1937, the above bound for V has been reduced gradually by several authors. Very recently in 2002, M.-C. Liu and T. Z. Wang [L-W] obtained

$$\log_{10}(V) < 1,347.$$

This is the best known result on the Quantitative Part of G(3), i. e., the gap between 9 and the above best known bound for V forms the final unsettled part of G(3) as shown in the chart below.



The first significant breakthrough in the attack upon G(2) was made by V. Brun (1885-1978). Using a method called the Sieve Method, he obtained [Bru]

Brun's Theorem (1919/1920). *Every sufficiently large even positive integer is a sum of two integers each having at most nine prime factors, equal or distinct.*

As usual, we denote this result by (9 + 9) and so the Qualitative Part of G(2) is the truth of (1 + 1). Attacking the (1 + 1) from another point of view, about thirty years later after Brun's result, A. Renyi (1921-1970) obtained [Ren]

Renyi's Theorem (1947/1948). *There is a fixed positive integer k such that every sufficiently large even positive integer is a sum of a prime and a product of at most k primes. That is, the $(1 + k)$ result is true.*

In the past several decades both Brun's Theorem and Renyi's Theorem have been improved gradually by discoveries of new ideas and techniques in refinement of the Sieve Method, e. g., the pioneered works [Sel] by A. Selberg (1917-2007) and [Lin] by Yu. V. Linnik (1915-1972). As a culmination of these discoveries, the result (1 + 3) was obtained within a period of three years independently by E. Bombieri [Bom] in 1965, A. I. Vinogradov [Vai] in 1966, and A. A. Buchstab [Buc] in 1967. About ten years later Bombieri was awarded a Fields Medal of 1974 because of his important contribution in several branches of mathematics including the result (1 + 3). Finally, by adding an ingenious new idea to the previous methods, Jingrun Chen (陳景潤) (1933-1996) obtained [Cjr] in 1966 and 1973 the following best known result (1 + 2) on G(2).

Chen's Theorem (1966 & 1973). *There is a positive non-effectively computable constant* $\boldsymbol{N_0}$ *such that for each even integer* $n \geqslant N_0$ *we have*

$$n = p + P_2$$

where p is a prime and $\boldsymbol{P_2}$ *is either a prime or a product of two primes.*

Chen's Theorem is so deep and difficult that so far the result (1 + 2) has resisted all further improvement for almost half a century. On some occasions, Chen's Theorem has already been called by experts the Goldbach-Chen Theorem.

Note that Chen's Theorem is still not the truth of the Qualitative Part of G(2) (i. e. , not (1 + 1)) unless we can replace the P_2 there by a prime. So there still does not exist a chart on even n for G(2) like the one above for G(3). Optimists believe that Chen's Theorem is only "a stone's throw" away from (1 + 1) while pessimists trust that all present available knowledge including the Sieve Method is not powerful enough to settle (1 + 1). Anyway, Chen's Theorem constituted a climax of the Sieve Method.

The reader may feel curious that the author has mentioned two not contiguous years, 1966 and 1973 (seven years apart) for the single result (1 + 2) obtained by Chen alone. Those who are now over forties may still remember that the Cultural Revolution in China had lasted for the ten years from 1966 to 1976 (sometimes it is also called the "Ten-Year Riot"). Just before the Cultural Revolution, Chen obtained his (1 + 2) and announced it at the Chinese Academy of Sciences, Beijing and the announcement was published as a paper (of two pages only) in Kexue Tongbao (1966) [Cjr]. Then because of the Cultural Revolution, all academic activities in China had been severely interrupted for several years until 1973. Since no detailed proof was published by Chen immediately after his announcement, especially, since except Chen himself no one in the world can give a proof of what Chen announced in 1966, the result (1 + 2) was not recognized by the academic world before 1974. After having obtained Chen's paper published in Sci. Sinica, (1973) [Cjr] where a proof of (1 + 2) was given, H. Halberstam and H. -E. Richert included (1 + 2) in their book, Sieve Methods (1974) [H-R]. Then Chen's (1 + 2) was internationally recognized. That is why the author mentioned the two different years, 1966 and 1973, for Chen's Theorem. During these seven years, Chen worked in an extraordinarily chaotic environment, and suffered from great political difficulties and hardship [Wan, Chapter 9]. Unfortunately, during these seven years, in the peaceful world outside China (comparing with the riotous China in the Cultural Revolution) some even began to suspect whether Chen had made some fatal errors in his proof of (1 + 2) and could not fill the gaps.

The above story indicates that Chen's achievement on (1 + 2) was a "solo performance in the mathematical world" for seven years. The 7-year (1966 to 1973) absence of essential progress on G(2) outside China singled out Chen's ingenious ability in the attack upon G(2) which was much more well advanced than the contemporary works on Goldbach's problems in the world.

Clearly, some areas of the number theory research in China before the Cultural Revolution have once reached the very frontiers in the world. The record of Chen's (1 + 2) has already lasted for more than four decades and indeed is important enough to form a notable part of the history in mathematics.

We have seen that the historical developments of G(2) and G(3) are quite different. In the next section, we will try to discuss and explain some obvious differences between G(2) and G(3) in more details.

2 Some Differences between G(2) and G(3)

Historical developments of G(2) and G(3) indicate that G(2) is tougher than G(3) in resistance of attacks. If we go through the proofs in details of Chen's Theorem and Vinogradov's Theorem, we would not be surprised at all by these developments. For these proofs, one may read, for example, [Dav, Chapter 26] for Vinogradov's Theorem and [H-R, Chapter 11] for Chen's Theorem. In this section, instead of proofs of these two theorems, we will look into the following five quite obvious differences between G(2) and G(3). With these differences we could sense that the solution of G(2) lies much deeper than G(3).

We denote the set of all positive integers by $\mathbb{N}$ and the set of all prime numbers by $\mathbb{P}$. For $a, b \in \mathbb{N}$ if a divides b we write $\boldsymbol{a \mid b}$ or otherwise $\boldsymbol{a \nmid b}$. $\#S$ means the number of all elements in the (finite) set S.

(**D-1**). It is obvious that G(2) is a problem of two independent (prime) variables while G(3) is of three independent (prime) variables. That is, in solving G(3) we have one more freedom than G(2) in choosing independent variables from $\mathbb{P}$. Note that actually both Vinogradov's Theorem and Chen's Theorem are merely results of three independent (prime) variables since the P_2 in Chen's Theorem may be a product of two primes. On the other hand, plainly the truth of G(2) implies the truth of G(3), since the integer 3 is an odd prime and $(6 \leqslant)\ n-3$ is an even integer if $n(\geqslant 9)$ is an odd integer. In the following (D-4) we see that even from a result which is extremely far away from the truth of the Qualitative Part of G(2) we can still obtain the truth of the Qualitative Part of G(3).

(**D-2**). The constant V in Vinogradov's Theorem is effectively computable and so the attack upon the Quantitative Part of G(3) (i.e., to prove $V=9$) could be started as soon as the Qualitative Part of G(3) has been settled in 1937 (i.e., immediately after Vinogradov's Theorem was obtained). On the other hand, the constant N_0 in Chen's Theorem is non-effectively computable. That is, although the number of independent (prime) variables in the Qualitative Part of G(2) is released to possibly three but not strictly two (i.e., the (1+2) but not the (1+1)), it is still not possible to assign a numerical value to the N_0 in Chen's Theorem with the existing knowledge. If in future, for any even integer $n \geqslant N_0$ we can replace the P_2 in Chen's Theorem by a prime but the constant N_0 is still non-effectively computable then we can only claim that the Qualitative Part of G(2) (i.e., the (1+1)) is settled but do not know how to start our attack immediately upon the Quantitative Part of G(2) (i.e., to prove $N_0=6$). So it is still hopeless to finally settle G(2) unless N_0 is effectively computable. This phenomenon shows the toughness of G(2) and reflects the striking depth of Chen's work.

The reason why the N_0 in Chen's Theorem is non-effectively computable is interestingly mysterious and will be explained in Section 3, (A-3) and Section 4. It should be noted that, by the same reason causing the non-effectiveness of the N_0 in Chen's Theorem, the corresponding N_0 in the weaker result (1+3) is also non-effectively computable.

(**D-3**). Assuming the GRH and applying the Circle Method, in 1997 J. M. Deshouillers, G. Effinger, H. Te Riele and D. Zinoviev [DERZ] obtained that the constant V in Vinogradov's Theorem satisfies $V=9$. Independently, under the GRH, $V=9$ was also obtained by combining the two results by Y. Saouter [Sao] in 1998 and D. Zinoviev [Zin] in 1997. They did not settle the Quantitative Part of G(3) because of the extra assumption, GRH (cf. Hardy & Littlewood's Theorem). However, their result strongly supports the belief in the truth of G(3). On the other hand, under the GRH, the Circle Method does not work well for G(2). In 1924, G. H. Hardy and J. E. Littlewood [H-L-2] obtained that

under the GRH for any $\varepsilon > 0$ there is a positive effectively computable constant N_1 depending on ε only such that we have

$$\#E(N) \leqslant N^{1/2+\varepsilon} \text{ for any } N \geqslant N_1$$

where $E(N) = \{ n \in \mathbb{N} : 2 \mid n \leqslant N, n \neq p_1 + p_2 \text{ for any } p_1, p_2 \in \mathbb{P} \}$. (2-1)

Note that the truth of G(2) is equivalent to the much stronger result

$$\#E(N) = 1 \text{ for any } N \geqslant 2$$

since G(2) is true if and only if $E(N) = \{2\}$ for any $N \geqslant 2$. That is, unlike G(3) even under the GRH the result on $\#E(N)$ is still extremely far away from the truth of $\#E(N) = N^0$ for $N \geqslant 2$ since the difference between the two bounds for $\#E(N)$, $N^{1/2+\varepsilon}$ and $1 = N^0$ is a vast difference in order of infinity. We shall provide further evidence on essential differences between G(2) and G(3) in the next (D-4) and (D-5).

(**D-4**). One can easily verify that the following extremely weak Result (E) on G(2) is powerful enough to obtain the Qualitative Part of G(3), i.e., the Vinogradov's Theorem.

Result (E). *There is a positive effectively computable constant N_1 such that*

$$\#E(N) \leqslant N/(\log N)^2 \text{ for any integer } N \geqslant N_1.$$

Here and in what follows, log denotes the logarithm function with the number $e(=2.71828\cdots)$ as the base.

Claim: *Result (E) implies Vinogradov's Theorem.*

Proof. For any $k \in \mathbb{N} \setminus \{1,2\}$ put

$$S(2k) = S = \{2k+3-p \in [4,2k] : p \in \mathbb{P}\} \tag{2-2}$$

where $[4,2k]$ denotes the closed interval of real numbers with end points 4 and $2k$. So each p appearing in S is odd. Let $\pi(x)$ denote the number of primes $\leqslant x$, i.e., $\pi(x) = \sum_{p \leqslant x} 1$. Then

$$\#S = \pi(2k) - 1$$

since $p=2$ is not counted in S and $2k \notin \mathbb{P}$. By the Prime Number Theorem (see Section 3, (A-1)) we have

$$\lim_{k\to\infty} \#S/(2k/\log(2k)) = 1. \tag{2-3}$$

Next let

$$A(2k) = A = \{2t \in [4,2k] : t \in \mathbb{N}\} \text{ and}$$

$$F(2k) = \{2t \in [4,2k] : t \in \mathbb{N},\ 2t \neq p_1 + p_2 \text{ for any } p_1, p_2 \in \mathbb{P} \setminus \{2\}\}. \tag{2-4}$$

Note that

$$S \subset A \text{ and } F(2k) \subset A.$$

Now, since $6 \notin F(2k)$ for $k \geqslant 3$, we always have

$$A \setminus F(2k) \text{ is not an empty set.}$$

If there exists $k_1 \in \mathbb{N} \setminus \{1,2\}$ such that

$$\text{for each } s \in A \setminus F(2k_1) \text{ we always have } s \notin S \tag{2-5}$$

then

$$A \setminus F(2k_1) \subset A \setminus S.$$

This gives $\#(A \setminus F(2k_1)) \leqslant \#(A \setminus S)$

or

$$\#F(2k_1) \geqslant \#S \tag{2-6}$$

since both $F(2k_1)$ and S are subsets of A.

Next note that since by (2-1) and (2-4) the only difference between $E(2k)$ and $F(2k)$ is about the integers 2 and 4, we see that

$$\#F(2k) = \#E(2k) \text{ for every } k \geqslant 3.$$

By Result (E) with $N_1 \geqslant 6$, i.e.,

$$\#F(2k) \leqslant 2k/(\log(2k))^2 \text{ for any } 2k \geqslant N_1$$

and (2-3) we see that there is a positive effectively computable constant N_2 such that

$$\#S > \#F(2k) \text{ for any } 2k \geqslant N_2$$

since $\lim_{x\to\infty} \log(x) = \infty$. From this and (2-6) we conclude that the k_1 in (2-5) must satisfy $2k_1 < N_2$. This proves that (2-5) is false if $2k_1 \geqslant N_2$ or that

$$S \cap (A \setminus F(2k)) \text{ is not empty for any } 2k \geqslant N_2.$$

So by (2-2) and (2-4) for any $2k \geqslant N_2$ there is $p \in \mathbb{P} \setminus \{2\}$ such that $2k+3-p \in S \cap (A \setminus F(2k))$ and

$$2k+3-p = p_1 + p_2 \quad \text{for some } p_1, p_2 \in \mathbb{P} \setminus \{2\}. \tag{2-7}$$

Let $V = N_2 + 3$. For any **odd** integer $n \geqslant V$ let $2k = n - 3(\geqslant N_2)$. By (2-7) we have

$$n = 2k+3 = p + p_1 + p_2,$$

i.e., every odd integer $n \geqslant V$ is a sum of three odd primes. This is Vinogradov's Theorem and the proof of our Claim is complete.

Following the above arguments we see that actually in view of the Prime Number Theorem we can even replace the 2 in Result (E) by any fixed constant >1, i.e., by a further weaker bound than that in Result (E) above, we can still obtain Vinogradov's Theorem.

(D-5). The method applied for Vinogradov's Theorem is the Circle Method while for Chen's Theorem the

Sieve Method is applied. The two methods are entirely different although both depend heavily on results of distribution of prime numbers. However, there is a common feature of these two methods, namely, Chen (in 1966 & 1973) and Vinogradov (in 1937) were successful in showing that for even integers n ($\geqslant N_0$ for G(2)) or odd integers n ($\geqslant V$ for G(3)) the following inequalities hold.

$$\mathcal{N}(j) \geqslant \mathcal{M}(j) - \mathcal{E}(j) > 0 \text{ for } j = 2, 3$$

where $\mathcal{N}(2)$ denotes the number of solutions of the equation

$$n = p + P_2$$

and $\mathcal{N}(3)$ denotes the number of solutions of the equation

$$n = p_1 + p_2 + p_3$$

and $\mathcal{M}(j) > 0$ and $\mathcal{E}(j) > 0$ for $j = 2, 3$ denote a lower bound for the main term and an upper bound for the error term in the estimation of the $\mathcal{N}(j)$ respectively. Next, we are going to take one more step to look into these $\mathcal{M}(j)$ and $\mathcal{E}(j)$ and shall discover that the inequality for positive $\mathcal{N}(2)$

$$\mathcal{M}(2) > \mathcal{E}(2) \tag{2-8}$$

is very tight while the $\mathcal{M}(3) > \mathcal{E}(3)$ is rather slack.

In the proof of Chen's Theorem we have

$$\mathcal{M}(2) = A\Pi \text{ and } \mathcal{E}(2) = B\Pi$$

where for any even integer $n \geqslant N_0$

$$\Pi = \prod_{p \geqslant 3}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{3 \leqslant p \mid n}\left(\frac{p-1}{p-2}\right)n \Big/ (\log n)^2 > (0.66)n/(\log n)^2 \text{ where } p \in \mathbb{P} \tag{2-9}$$

and A, B are positive constants with

$$A - B > 0.67.$$

So $\mathcal{N}(2) > (0.67)\Pi$ for even integer $n \geqslant N_0$ and

$$\lim_{n \to \infty, 2 \mid n} \mathcal{M}(2)/\mathcal{E}(2) \text{ is the positive constant } A/B,$$

i. e., $\mathcal{M}(2)$ and $\mathcal{E}(2)$ are of the same order of infinity and $\mathcal{M}(2)$ dominates $\mathcal{E}(2)$ only because A is slightly larger than B. This shows the severe tightness of the inequality in (2-8). We shall come back to $\mathcal{N}(2)$ with some further explanation in Section 4, Remark 4-3.

On the other hand, in the proof of Vinogradov's Theorem, we have that for any odd integer $n \geqslant V$ $\mathcal{M}(3) = \frac{1}{2}\prod_{p \mid n}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p \nmid n}\left(1 + \frac{1}{(p-1)^3}\right)n^2 \Big/ (\log n)^3 > (0.66)n^2/(\log n)^3$ where $p \in \mathbb{P}$ and that for any fixed constant $D > 3$ there is a positive constant K depending on D only such that

$$\mathcal{E}(3) < Kn^2/(\log n)^D.$$

So $\mathcal{N}(3) > (0.65)n^2/(\log n)^3$ for $n \geqslant V$ and

$$\lim_{n \to \infty, 2 \nmid n} \mathcal{M}(3)/\mathcal{E}(3) = \infty,$$

i. e., $\mathcal{M}(3)$ is of higher order of infinity than $\mathcal{E}(3)$ and $\mathcal{M}(3)$ dominates $\mathcal{E}(3)$ overwhelmingly. Hence the inequality $\mathcal{M}(3) > \mathcal{E}(3)$ is rather slack.

Once again, this difference indicates that G(2) is much more difficult than G(3). The tightness of $\mathcal{M}(2) > \mathcal{E}(2)$ probably supports the belief by some experts in this field that Sieve Method is not powerful enough to settle G(2) even only for the Qualitative Part of G(2).

Remark 2-1. By the way it should be mentioned that the significant work in evaluation or estimation of some important constants should not be ignored although usually the estimation work may be tedious. Actually, the estimation of constants needs not only more effort and time in their research, but also much deeper understanding about the original profound ideas in those inequalities involved. Chen succeeded in proving his Theorem because of not only his ingenuity in adding new ideas in the Sieve Method, but also his ingenuity in numerical estimation of the constant B in (D-5). Another example on the importance of constant estimation is about a remarkable milestone record of the Linnik Constant. D. R. Heath-Brown in 1992 [Hea] obtained the upper bound 5.5 for the Linnik Constant by his significant numerical estimation of those constants involved in the zero-free regions and zero density of the Dirichlet L-functions.

3 The Prime Number Theorem, the GRH and Theorems S, W & B

In this section we shall give some very brief description about (A-1) The Prime Number Theorem, (A-2) The GRH and (A-3) Non-effectiveness of the N_0.

(A-1) The Prime Number Theorem.

By examining

$$\pi(x)/(x/\log(x)) \text{ for } 1\leqslant x\leqslant 10^6.$$

C. F. Gauss (1777-1855) in 1792 [Gau] and A. M. Legendre (1752-1833) in 1798 [Leg] conjectured independently that

$$\lim_{x\to\infty}\pi(x)/(x/\log(x))=1.$$

Both of them tried to prove the conjecture in vain. Since then, the conjecture has attracted the attention of eminent mathematicians for more than a century. In 1851 P. L. Chebyshev (1821-1894) [Cp1] made an important step towards the proof of the conjecture by showing that

if $\pi(x)/(x/\log(x))$ *tends to a limit then the limit is 1.*

In 1859 B. Riemann (1826-1866) [Rie] attacked the problem by using the theories and techniques in complex analysis. Although he was unable to settle completely the conjecture before his untimely death at the age of forty, his discoveries and ingenious methods for connecting the distribution of prime numbers to the properties of Riemann-zeta function had a very profound influence on this successors' work. For the first thirty years after Riemann's paper [Rie] was published, there was virtually no progress. It was as if it took the mathematical world that much time to digest Riemann's ideas. Eventually, just before the end of the 19th century, in 1896, using analytic methods due to Riemann, J. Hadamard (1865-1963) [Had] and C. de la Vallee Pous*sin*(1866-1962) [Pou] independently proved the conjecture, i. e.,

$$\lim_{x\to\infty}\pi(x)/(x/\log(x))=1.$$

The result is now called the **Prime Number Theorem.**

(A-2) The GRH or the Generalized Riemann Hypothesis.

Let $\mathbb{Z}$ denote the set of all integers. For any $m,n\in\mathbb{Z}$ we denote the **Greatest Common Divisor** of m and n by $(\boldsymbol{m},\boldsymbol{n})$. For any $q\in\mathbb{N}$ if q divides $m-n$ we write $\boldsymbol{m\equiv n(\mathrm{mod}\, q)}$.

Definition 3-1. *Let $q\in\mathbb{N}$. The complex-valued function χ defined on $\mathbb{Z}$ is called a* ***Dirichlet Character*** *(or* ***Character***) ***Modulo q*** *denoted by* $\boldsymbol{\chi(\mathrm{mod}\, q)}$ *if χ is not identically zero and satisfies*

(ⅰ) $\chi(mn)=\chi(m)\chi(n)$ *for any* $m,n\in\mathbb{Z}$,

(ⅱ) $\chi(m+q)=\chi(m)$ *for any* $m\in\mathbb{Z}$, *i. e.*, χ *is periodic with period* q,

(ⅲ) $\chi(m)=0$ *if* $(m,q)\neq 1$.

Remark 3-2. It can be proved that for each $q\in\mathbb{N}$ there are exactly $\phi(q)$ Dirichlet characters modulo q where $\boldsymbol{\phi(q)}$ is called the **Euler Function** defined by

$$\phi(q)=\#\{m\in\mathbb{N}: m\leqslant q,\ (m,q)=1\}.$$

Let $m,a\in\mathbb{Z}$ with $(a,q)=1$. It can be proved that the sum $\sum_{\chi(\mathrm{mod}\, q)}$ of all the $\phi(q)$ characters $\chi(\mathrm{mod}\, q)$ satisfies

$$\sum_{\chi(\mathrm{mod}\, q)}\chi(m)\bar{\chi}(a)=\phi(q) \text{ if } m\equiv a(\mathrm{mod}\, q) \text{ or } =0 \text{ otherwise} \tag{3-1}$$

where $\bar{\chi}(a)$ means the complex conjugate of $\chi(a)$.

Definition 3-3. *For any $q\in\mathbb{N}$, the following function f defined on $\mathbb{Z}$ by*

$$f(m)=1 \textit{ if } (m,q)=1 \textit{ or } =0 \textit{ otherwise}$$

satisfies Definition 3-1. We call this function the ***Principal*** Dirichlet Character Modulo *q and denote it by*

$\boldsymbol{\chi}_0(\bmod \boldsymbol{q})$. *Note that the only (by $\phi(1)=1$) character $\chi(\bmod 1)$ is in fact the principal character $\chi_0(\bmod 1)$ ($=1$ for any $m \in \mathbb{Z}$).*

Remark 3-4. If $\chi(\bmod q)$ has no period less than q, then we call it a **Primitive** Character Modulo q. It can be proved that for any given $\chi(\bmod q)$ there is a unique primitive character $\psi(\bmod r)$ such that $r \mid q$ and $\chi(m)=\psi(m)\chi_0(m)$ for any $m \in \mathbb{Z}$ where χ_0 is the principal character modulo q. We say the $\chi(\bmod q)$ is induced by the primitive $\psi(\bmod r)$. Actually, the converse is also true, namely, for any given primitive $\psi(\bmod r)$ if $q \in \mathbb{N}$ with $r \mid q$ then there is a unique $\chi(\bmod q)$ induced by the $\psi(\bmod r)$.

Remark 3-5. Let $\mathbb{C}$ be the set of all complex numbers. It can be proved that for any $q \in \mathbb{N}$ the absolute value of $\chi(m)$ satisfies

$$|\chi(m)|=1 \text{ if } (m,q)=1$$

and then we have $|\chi(m)| \leqslant 1$ for any $m \in \mathbb{Z}$. Therefore, for any $s \in \mathbb{C}$ we have that the series $\sum_{m=1}^{\infty}\chi(m)m^{-s}$ converges absolutely on the half-plane $\mathrm{Re}(s)>1$ where

$$m^s=\exp(s\log(m)) \text{ implies } |m^s|=m^{\mathrm{Re}(s)}.$$

It can be proved that the sum function of the series $\sum_{m=1}^{\infty}\chi(m)m^{-s}$ is analytic (i. e., the derivative of the sum function exists) for $\mathrm{Re}(s)>1$ and that the sum function can be extended analytically to $\mathbb{C}$ if $\chi(\bmod q) \neq \chi_0(\bmod q)$ and to $\mathbb{C}\setminus\{1\}$ otherwise. We also have

$$\frac{d}{ds}\sum_{m=1}^{\infty}\chi(m)m^{-s}=-\sum_{m=1}^{\infty}\log(m)\chi(m)m^{-s} \text{ for } \mathrm{Re}(s)>1. \tag{3-2}$$

Definition 3-6. *For each $\chi(\bmod q)$ the analytic extension of $\sum_{m=1}^{\infty}\chi(m)m^{-s}$ mentioned in Remark 3-5 is called the* ***Dirichlet L-function*** *denoted by $\boldsymbol{L}(\boldsymbol{s},\boldsymbol{\chi})$. In particular, if $q=1$ we call the corresponding $L(s,\chi_0)$ the* ***Riemann-zeta Function*** *denoted by $\boldsymbol{\zeta}(\boldsymbol{s})$, i. e.,*

$$\zeta(s) \textit{ is analytic on } \mathbb{C}\setminus\{1\} \textit{ and } \zeta(s)=\sum_{m=1}^{\infty}m^{-s} \textit{ for } Re(s)>1.$$

Definition 3-7. *It can be proved that there are infinitely many zeros $\rho=\beta+i\gamma$ of $L(s,\chi)$ with $0<\beta<1$ and that these zeros lie symmetrically to the vertical line $\mathrm{Re}(s)=1/2$, i. e., both $\beta+i\gamma$ and $1-\beta+i\gamma$ are zeros of $L(s,\chi)$. We call these zeros ρ* ***Non-trivial Zero*** *of $L(s,\chi)$. It was conjectured by B. Riemann [Rie] in 1859 that all non-trivial zeros ρ of $\zeta(s)$ lie on the line $\mathrm{Re}(s)=1/2$ or $\mathrm{Re}(\rho)=1/2$. The conjecture is called the* ***Riemann Hypothesis****, usually abbreviated as* ***RH****. About all non-trivial zeros ρ of the $L(s,\chi)$ in general there is a similar conjecture, namely, all $\mathrm{Re}(\rho)=1/2$. We call this conjecture the* ***Generalized Riemann Hypothesis*** *abbreviated as* ***GRH****.*

From the statements of G(2) and G(3) we see that they are closely related to the distribution of prime numbers. In Result 4-1 below we shall find a clear relation connecting some property of the distribution of prime numbers and the location of non-trivial zeros of $L(s,\chi)$. So it is natural to start attacks upon G(2) and G(3) by assuming the GRH as in Hardy-Littlewood's Theorem and (D-3).

Remark 3-8. It can be proved that if $\chi(\bmod q)$ is induced by the primitive $\psi(\bmod r)$ then

$$L(s,\chi)=L(s,\psi)\prod_{p\mid q}(1-\psi(p)p^{-s}) \text{ for } \mathrm{Re}(s)>0,$$

where $\prod_{p\mid q}$ denotes the product over all prime divisors p of q. This shows that $L(s,\chi)$ and $L(s,\psi)$ have the same set of zeros ρ with $\mathrm{Re}(\rho)>0$ since for each prime p, $|\psi(p)/p^s|<1$ if $\mathrm{Re}(s)>0$.

(A-3) Non-effectiveness of the N_0.

Here we shall give a very sketchy description on the reason why the N_0 in Chen's Theorem is non-effectively computable. However, if one would like to go into the details, one may refer to, for example [Dav, Chapters 21, 22 & 28] for Results 3-10 to 3-13 below.

Result 3-9. It can be proved that for any given $T \geqslant 2$ there is an effectively computable positive absolute constant C_1 such that there is at most one primitive character $\tilde{\chi}(\bmod \tilde{r})$ with $\tilde{r} \leqslant T$ for which the corresponding $L(s,\tilde{\chi})$ has a zero $\tilde{\rho}$ lying in the region

$$\mathrm{Re}(s) > 1 - \eta(T) \text{ and } |\mathrm{Im}(s)| \leqslant T \text{ where } \eta(T) = C_1/\log T. \quad (3\text{-}3)$$

If such $\tilde{\chi}(\bmod \tilde{r})$ exits the zero $\tilde{\rho} = \tilde{\beta} + i\tilde{\gamma}$ is real, simple and unique, i. e.,

$$\tilde{\beta} > 1 - \eta(T) \text{ and } \tilde{\gamma} = 0.$$

So by Definition 3-7, $1 - \tilde{\beta}$ is also a zero of $L(s, \tilde{\chi})$. We call the zero $\tilde{\boldsymbol{\beta}}(=\tilde{\rho})$ the **Siegel Zero** and $\tilde{\boldsymbol{\chi}}(\mathbf{mod}\,\tilde{\boldsymbol{r}})$ the **Exceptional Character.** Moreover, there is an effectively computable positive absolute constant C_2 such that the zero $1 - \tilde{\beta}$ of $L(s, \tilde{\chi})$ satisfies

$$C_2/(\tilde{r}^{1/2}(\log \tilde{r})^2) < 1 - \tilde{\beta}. \quad (3\text{-}4)$$

By Remark 3-4, Results 3-8 and 3-9, if $\chi(\bmod q)$ with $q \leqslant T$ is not induced by $\tilde{\chi}(\bmod \tilde{r})$ then (3-3) gives the zero-free region for $L(s, \chi)$. In particular,

$$\text{all } L(s, \chi) \neq 0 \text{ for } \mathrm{Re}(s) \geqslant 1. \quad (3\text{-}5)$$

Result 3-10. For a better lower bound of $1 - \tilde{\beta}$, C. L. Siegel (1896-1981) obtained [Sie]

Theorem S (1935). *For any $\varepsilon > 0$ there is a* ***non-effectively*** *computable positive number $C(\varepsilon)$ depending on ε only such that*

$$C(\varepsilon)/\tilde{r}^{\varepsilon} < 1 - \tilde{\beta} \quad (3\text{-}6)$$

where $\tilde{\beta}$ is the Siegel zero and $\tilde{r}$ is the modulus of the Exceptional Character.

Comparing the two lower bounds (3-4) and (3-6) for the $1 - \tilde{\beta}$ we see that Theorem S is considerably better but we have to sacrifice the effectiveness, i. e., to replace the effectively computable C_2 by the non-effective $C(\varepsilon)$.

Result 3-11. Applying Theorem S, A. Walfisz (1892-1962) obtained [Wa1]

Theorem W (1935). *For given fixed $k \geqslant 1$ there are non-effectively computable positive constants C_3 and C_4 depending on k only such that if for any non-principal $\chi(\bmod q)$ we have*

$$q \leqslant (\log(r))^k$$

then

$$\sum_{m \leqslant r} \Lambda(m)\chi(m) < r \exp(-C_4\sqrt{\log(r)}) \text{ for } r \geqslant C_3$$

where $\Lambda(m)$ is the ***von Mangoldt Function***, *i. e., $\Lambda(m) = \log(p)$ if $m = p^{\ell}$ (i. e., m is a power of a prime p) and $\Lambda(m) = 0$ otherwise.*

Theorem W is non-effective since in the proof we have to apply the non-effective Theorem S.

Remark 3-12. Theorem W is in fact involved in the error term of the following non-effective Theorem W-S (1935 [Wal]) on the Prime Number Theorem for Arithmetic Progressions, namely, *we have*

$$\sum_{m \leqslant r,\ m \equiv a(\bmod q)} \Lambda(\mathrm{m}) = r/\phi(q) + Error$$

if $q \leqslant (\log(r))^k$ with $k \geqslant 1$ and $r \geqslant C_3$ where $\phi(q)$ is the Euler function and the Error is dominated by the bound for $\sum_{m \leqslant r} \Lambda(m)\chi(m)$ in Theorem W.

It is not surprising that Theorem W (and Theorem W-S) is one of the key tools in the attacks of G(2) although it is non-effective since G(2) (and G(3) also) is essentially and closely related with the distribution of prime numbers. Imagine that if the word "primes" in G(2) were replaced by "integers" then trivially it becomes exercises of "finger-counting" for kids in any pre-kindergarten class while G(2) itself like a mysterious treasure trove has been tantalizing so many genius mathematicians for so many years since 1742.

Result 3-13. Concerning the number of primes lying in arithmetic progression, E. Bombieri (1940-) obtained [Bom]

Theorem B (1965). *Let $\Lambda(m)$ be the von Mangoldt function and $\phi(q)$ be the Euler function. Let $a, q \in \mathbb{N}$ with $(a, q) = 1$. For $n > r \geqslant 3$ write*

$$E(r; q, a) = \left(\sum_{m \leqslant r,\ m \equiv a(\bmod q)} \Lambda(\mathrm{m})\right) - (r/\phi(q)),$$

$$E(r,q) = \max_{(a,q)=1} |E(r;q,a)| \text{ and } E^*(n,q) = \max_{r\leqslant n} E(r,q).$$

Let $A>0$ be fixed. Then there is a non-effectively computable positive constant C_5 depending on A only such that if

$$n^{1/2}(\log n)^{-A} \leqslant Q \leqslant n^{1/2}$$

then

$$\sum_{q\leqslant Q} E^*(n,q) < C_5 n^{1/2} Q(\log n)^5. \tag{3-7}$$

Theorem B is non-effective since in the proof we need the non-effective Theorem W.

Remark 3-14. In the proof of Chen's Theorem we have to apply Theorem B and Theorem W. Therefore the constant N_0 in Chen's Theorem is non-effectively computable.

By the way, it should be remarked that for G(3), instead of the non-effective Theorem S, the effective (3-4) is powerful enough to be applied in the proof of Vinogradov's Theorem and so the constant V is effectively computable.

In the next section more mathematical details than (A-3) shall be given to further explain the relations of the non-effectiveness among Theorems S, W, B and Chen's Theorem.

4 Further Explanation

In this section we shall give some concise description and simple mathematical explanation on the non-effectiveness of Theorems W, B and Chen's Theorem as mentioned in Section 3 (A-3). All these theorems are deep and their proofs are difficult and lengthy but elegant. Here we shall only give those necessary mathematical results for our explanation and have no intention at all to touch the details of any part of the proofs of these theorems.

(A-4) Non-effectiveness of Theorem W.

For $q\in\mathbb{N}$ let $\chi(\bmod q)$ be any Dirichlet character and $\Lambda(m)$ be the von Mangoldt function. Note that the series $\sum_{m=1}^{\infty}\Lambda(m)\chi(m)m^{-s}$ is absolutely convergent for $\mathrm{Re}(s)>1$ since $|\chi(m)|\leqslant 1$ and for any (small) $\delta>0$, $(0\leqslant)\Lambda(m)$ is eventually less than m^{δ} as m increases. By Definition 3-6

$$L(s,\chi) = \sum_{m=1}^{\infty}\chi(m)m^{-s} \text{ for } \mathrm{Re}(s)>1$$

and by the multiplication of the two absolutely convergent series (i. e., the Cauchy product), we have

$$L(s,\chi)\sum_{m=1}^{\infty}\Lambda(m)\chi(m)m^{-s} = \sum_{m=1}^{\infty} m^{-s}\sum_{d|m}\chi(m/d)\Lambda(d)\chi(d) =$$
$$\sum_{m=1}^{\infty} m^{-s}\chi(m)\log(m) = -L'(s,\chi) \text{ for } \mathrm{Re}(s)>1 \text{ by (3-2)}.$$

The above second equality holds since by the definition of $\Lambda(m)$, $\Lambda(d)\neq 0$ only when d is a power of some prime p_j and so

$$\sum_{d|m}\Lambda(d) = \sum_{j=1}^{r}\sum_{h=1}^{t_j}\Lambda(p_j^h) = \sum_{j=1}^{r} t_j\log(p_j) = \log(m)$$

for $m=\prod_{j=1}^{r}p_j^{t_j}$. Note that by (3-5), there are no zeros of $L(s,\chi)$ in the half plane $\mathrm{Re}(s)\geqslant 1$. Then we obtain

$$-L'(s,\chi)/L(s,\chi) = \sum_{m=1}^{\infty}\Lambda(m)\chi(m)m^{-s} \text{ for } \mathrm{Re}(s)>1. \tag{4-1}$$

It can be proved that for non-principal $\chi(\bmod q)$ the analytic function defined by (4-1) can be extended analytically beyond the line $\mathrm{Re}(s)=1$ to $\mathrm{Re}(s)<1$ except for those zeros of $L(s,\chi)$ in the half plane $\mathrm{Re}(s)<1$. Then with some suitable closed contour on $\mathbb{C}$ by the Residue Theorem we can obtain Result 4-1 below.

Result 4-1. *Let $r\geqslant T\geqslant 2$, $q\in\mathbb{N}$ and let $\chi(\bmod q)$ be any non-principal character modulo q. We have*

$$\sum_{m\leqslant r}\Lambda(m)\chi(m) = -(\tilde{E}r^{\tilde{\beta}}/\tilde{\beta}) - \Big(\sum_{|\gamma|\leqslant T}{}' r^{\rho}/\rho\Big) + Error \tag{4-2}$$

where $\Lambda(m)$ is the von Mangoldt function, $\rho=\beta+i\gamma$ are non-trivial zeros of the Dirichlet L-function $L(s,\chi)$, $\tilde{\beta}$ is the Siegel zero (see Result 3-9) and $\tilde{E}=1$ if $\chi=\tilde{\chi}$ (the exceptional character) and $=0$ if $\chi\neq\tilde{\chi}$. Here the sum

$\sum'_{|\gamma|\leqslant T}$ *is over all* ρ *with* $|Im(\rho)|\leqslant T$ *but excludes the zeros* $\tilde{\beta}$ *and* $1-\tilde{\beta}$ (*if they exist*).

Note that Theorem W is about the estimation on upper bound for the left hand side of the equation in (4-2). Theorem W becomes non-effective since we have to apply the non-effective Theorem S to handle the possible existence of the Siegel zero $\tilde{\beta}$ in (4-2).

(**A-5**) **Non-effectiveness of Theorem B.**

For $a,q,r\in\mathbb{N}$ with $(a,q)=1$, by (3-1) we have

$$\sum_{\chi}\bar{\chi}(a)\sum_{m\leqslant r}\wedge(m)\chi(m)=\sum_{m\leqslant r}\wedge(m)\sum_{\chi}\bar{\chi}(a)\chi(m)=\phi(q)\sum_{m\leqslant r,\ m\equiv a(\mathrm{mod}\,q)}\wedge(\mathrm{m}) \tag{4-3}$$

where the sum $\sum_{\chi}$ is over all the $\phi(q)$ Dirichlet characters $\chi(\mathrm{mod}\,q)$.

Now Theorem B is on the estimate of the sum over q in (3-7) involving the term

$$\max_{r\leqslant n}\max_{(a,q)=1}\left|\left(\sum_{m\leqslant r,\ m\equiv a(\mathrm{mod}\,q)}\wedge(\mathrm{m})\right)-(r/\phi(q))\right|,$$

where the sums of $\wedge(m)$ over $m\leqslant r$ with $r\leqslant n$ and $m\equiv a(\mathrm{mod}\,q)$ form the main issue for the estimation. Then for non-principal $\chi(\mathrm{mod}\,q)$, the sums $\sum_{m\leqslant r}\wedge(m)\chi(m)$ on the left hand side of (4-3) are involved. Theorem B then becomes non-effective because we have to apply the non-effective Theorem W to handle them.

(**A-6**) **The Non-effectively Computable N_0 in Chen's Theorem.**

For clarity, we begin our explanation from the following lemma of Chen.

Let $p,p',p_j,p'_j\in\mathbb{P}$ and P_2 be either a prime or a product of two primes. For a given $n\in\mathbb{N}$ and any $t>2$ let

$$P(t)=\prod_{p'<t,p'\nmid n}p'. \tag{4-4}$$

For any given **even** $n\in\mathbb{N}$, let

$$S(n)=\{p\in\mathbb{P}:p<n,\ p\nmid n,\ (n-p,\ P(n^{1/10}))=1\}. \tag{4-5}$$

It can be shown that there is an effectively computable constant $N_1>0$ such that for any $n\in\mathbb{N}$ with $2|n$ and $n\geqslant N_1$ we have

$$\#S(n)>A\Pi$$

(and so $S(n)$ is non-empty) where $A\Pi=\mathscr{M}(2)$ was mentioned in (D-5) and (2-9). In what follows, we always assume $n\in\mathbb{N}$ with $2|n$ and $n\geqslant N_1$. We call such n a large even integer. For the given n and $p\in S(n)$ let $\sum_{(p_1)}$ be the sum over all p_1 such that

$$n^{1/10}\leqslant p_1<n^{1/3},\ p_1\nmid n \text{ and } p_1|n-p. \tag{4-6}$$

For the given $n,p\in S(n)$ and p_1 satisfying (4-6) let $\sum_{(p_2)}$ be the sum over all p_2 such that

$$n^{1/3}\leqslant p_2<(n/p_1)^{1/2},\ p_2\nmid n,\ p_2|n-p \text{ and } (n-p)/p_1p_2\in\mathbb{P}. \tag{4-7}$$

As usual, we assign zero value to an empty sum.

Lemma. *For any given large even* $n\in\mathbb{N}$ *we have*

$$\#\{n-p:p<n,\ n-p=P_2\}\geqslant\sum_{p\in S(n)}\left\{1-\frac{1}{2}\sum_{(p_1)}1-\frac{1}{2}\sum_{(p_1)}\sum_{(p_2)}1\right\}. \tag{4-8}$$

Remark 4-3. Since we shall not go into the proof of Chen's Theorem in details, for simplicity, in (4-8) we have omitted some conditions from the original version (see, for example, (2. 1) in [H-R, p. 321]) of the corresponding set $S(n)$ in Chen's proof. Anyway, the Lemma here is closed enough (for our explanation in (A-6)) to the original starting point of Chen's adventure in G(2) for the destination (1+2). Therefore, in what follows we ignore the difference between (4. 8) and the corresponding inequality in Chen's proof. Note that if we can prove that there is a positive (non-effectively computable) constant $N_0\geqslant N_1$ such that for any even integer $n\geqslant N_0$ we have a positive value of the sum $\sum_{p\in S(n)}$ in (4-8) then we obtain (1+2) or Chen's Theorem. Actually, with the $\mathscr{E}(2)$ and $B\Pi$ in (D-5) and (2-9) Chen could prove that the sum of the last two sums in (4-8) satisfies

$$\frac{1}{2}\sum_{p\in S(n)}\sum_{(P_1)}\left\{1+\sum_{(p_2)}1\right\}<B\Pi=\mathscr{E}(2) \text{ if } n\geqslant N_0 \text{ and } 2|n.$$

Therefore, by this bound together with $\#S(n)>A\Pi$ for even $n\geqslant N_0\geqslant N_1$, Chen could obtain that the right hand side of (4-8) is larger than $(A-B)\Pi=(0.67)\Pi$ which is the lower bound for the number of primes p satisfying $n-p=P_2$ for the given even integer $n\geqslant N_0$, i. e., the $\mathcal{N}(2)$ in Section 2 (D-5).

The sum $\sum\limits_{(p_1)}$ in (4-8) was introduced by pioneers who eventually obtained (1+3) while the sum $\sum\limits_{(p_1)}\sum\limits_{(p_2)}$ was introduced by Chen himself in 1966. By this additional ingenious new idea and his skillful but complicated techniques involving many deep theorems, Chen could obtain a good enough upper bound for the triple sum $\sum\limits_{p\in S(n)}\sum\limits_{(p_1)}\sum\limits_{(p_2)}$ to achieve the goal (1+2). This forms Chen's spectacular "solo performance in the mathematical world" for the seven years from 1966 to 1973.

Proof (of the Lemma). Let $S_2=\{n-p:p<n,n-p=P_2\}$. For each $p\in S(n)$ let $\{\cdots\}(p)$ denote

$$\Big\{1-\frac{1}{2}\sum_{(p_1)}1-\frac{1}{2}\sum_{(p_1)}\sum_{(p_2)}1\Big\}.$$

So (4-8) is $\#S_2\geqslant\sum\limits_{p\in S(n)}\{\cdots\}(p)$. Since $\#S_2\geqslant 0$, in order to prove the inequality (4-8) we only need to consider those $p\in S(n)$ such that $\{\cdots\}(p)>0$. Now, $\{\cdots\}(p)\leqslant 1$ and $\#S_2$ is a counting function, i. e., its value jumps up by 1 for each count. Therefore, it suffices to prove that if $\{\cdots\}(p)>0$ for some $p\in S(n)$ then this p satisfies $n-p\in S_2$. We consider two cases, (i) the sum $\sum\limits_{(p_1)}$ is empty and (ii) the sum $\sum\limits_{(p_1)}$ is non-empty. Since we are only interested in $\{\cdots\}(p)>0$, case (ii) is in fact the case $\sum\limits_{(p_1)}1=1$.

Consider case (i). In this case we have both sums $\sum\limits_{(p_1)}$ and $\sum\limits_{(p_1)}\sum\limits_{(p_2)}$ are empty and hence the values of these sums are zero. As a consequence, $\{\cdots\}(p)=1$. On the other hand, for any given $p\in S(n)$ if $\sum\limits_{(p_1)}$ is an empty sum then by (4-6) for each p_1 with

$$n^{1/10}\leqslant p_1<n^{1/3},\ p_1\nmid n \text{ we have } p_1\nmid n-p.$$

By this together with $(n-p,P(n^{1/10}))=1$ in (4-5) we have, in view of (4-4)

$$(n-p,\prod_{p'<n^{1/3},p'\nmid n}p')=1. \tag{4-9}$$

Then we have

$$p'\geqslant n^{1/3} \text{ for each } p' \text{ with } p'\mid n-p \tag{4-10}$$

since if there were $p'<n^{1/3}$ with $p'\mid n-p$ then by (4-9) we must have $p'\mid n$ and so in fact this p' equals to the p from $S(n)$. This is impossible as by (4-5) for each $p\in S(n)$ we have $p\nmid n$. Then now by (4-10) we have

$$n-p=P_2. \tag{4-11}$$

For if there were more than two prime divisors of $n-p$ let us consider $n-p=p'_1p'_2p'_3$. By (4-10) all $p'_j\geqslant n^{1/3}$. This gives $n>n-p\geqslant(n^{1/3})^3$. It is impossible and our proof for case (i) is complete, i. e., if the sum $\sum\limits_{(p_1)}$ is empty then this $p\in S(n)$ satisfies (4-11) or $n-p\in S_2$.

Next, consider case (ii) $\sum\limits_{(p_1)}1=1$. In this case, by (4-6) $n-p$ has precisely one prime divisor p_1 from the range $[n^{1/10},n^{1/3})$ with $p_1\nmid n$. This together with $(n-p,P(n^{1/10}))=1$ in (4-5) gives $n-p=p_1m$ for some $m\in\mathbb{N}$ with

$$(m,\prod_{p'<n^{1/3},p'\nmid n}p')=1. \tag{4-12}$$

If $m=1$ then obviously, $n-p\in S_2$. On the other hand, $\sum\limits_{(p_1)}\sum\limits_{(p_2)}1=0$ as there is no p_2 dividing m as in (4-7). So $\{\cdots\}(p)=1-\dfrac{1}{2}>0$ by $\sum\limits_{(p_1)}1=1$. That is, the $p\in S(n)$ causing $m=1$ is counted on both sides of the inequality (4-8).

Now, consider $m\geqslant 2$. By (4-12), if $p'\mid m$ then $p'\geqslant n^{1/3}$ and hence $m=P_2$ by the same arguments as for (4-11). For if $m=p'_1p'_2p'_3\geqslant(n^{1/3})^3$ then we have the contradiction, $n>n-p=p_1m>m\geqslant n$. Now, either (ii-a) m has exactly two prime divisors or (ii-b) m is a prime. Consider (ii-a). If m has exactly two prime divisors, by (4-12) write

$$m=p_2p_3 \text{ with } n^{1/3}\leqslant p_2\leqslant p_3.$$

Then $(n-p)/(p_1p_2)=p_3\in\mathbb{P}$ and $n-p\notin S_2$ as $n-p=p_1p_2p_3$. Now $p_2\nmid n$. For if $p_2\mid n$ then $p_2=p$ by $p_2\mid n-p$. But by (4-5), $p\in S(n)$ does not divide n. Furthermore, we must have $p_2<(n/p_1)^{1/2}$ otherwise there is the following contradiction.

$$n > n - p = p_1 p_2 p_3 \geqslant p_1 p_2^2 \geqslant p_1(n/p_1) = n.$$

Then for this $p \in S(n)$ there is at least one pair of p_1 and p_2 such that all the conditions, (4-6) and (4-7) in the sum $\sum_{(p_1)}\sum_{(p_2)}$ are satisfied and as a consequences, $\sum_{(p_1)}\sum_{(p_2)} 1 \geqslant 1$. Then $\{\cdots\}(p) \leqslant 1 - \frac{1}{2} - \frac{1}{2} = 0$. That is, the $p \in S(n)$ causing $m = p_2 p_3$ is not counted on both sides of the inequality (4-8).

It remains to consider case (ii-b). In (ii-b) m is a prime. Write $m = p_2$ or $n - p = p_1 p_2 (= P_2)$. This shows that $(n-p)/p_1 p_2 \notin \mathbb{P}$ and then the sum $\sum_{(p_1)}\sum_{(p_2)}$ is an empty sum or $= 0$, and so $\{\cdots\}(p) = 1 - \frac{1}{2} > 0$. That is, this p has positive contribution to the sum $\sum_{p \in S(n)}$ in (4-8). Meanwhile, this $p \in S(n)$ satisfies $n - p = P_2$ and is counted in the set S_2 in (4-8). This completes our proof of the Lemma.

Remark 4-4. In order to obtain a good upper bound for the term caused by the sum $\sum_{p \in S(n)}\sum_{(p_1)}$ in (4-8), in view of (4-5) and (4-6) we may follow those pioneers who obtained (1 + 3) to treat the sum

$$\sum_{n^{1/10} \leqslant p_1 < n^{1/3}, p_1 \nmid n} \#\{p \in \mathbb{P} : p < n,\ p \equiv n (\bmod p_1),\ (n - p,\ P(n^{1/10})) = 1\}. \tag{4-13}$$

Note that the main condition in (4-13) is concerned with a system of relations (usually, called congruences) $p \equiv n(\bmod p_1)$ with $(n, p_1) = 1$. It can be shown that with the help of the Chinese Remainder Theorem, this is to deal with those primes p lying in an arithmetic progression $a + q\ell$, where $\ell \in \mathbb{N}$ is a variable and $a, q \in \mathbb{N}$ are determined by n and all the p_1 from (4-13) with $a < q$ and $(a, q) = 1$. It is well known that by Dirichlet's Theorem (1837 in [Dir] by P. G. L. Dirichlet (1805-1859)) there are infinitely many primes p lying in any given arithmetic progression $a + q\ell$ with $(a, q) = 1$, $\ell \in \mathbb{N}$. Furthermore, similar to the Prime Number Theorem, the number of primes p with $p \leqslant n$ lying in $a + q\ell$ is about $n/(\phi(q)\log(n))$ where $\phi(q)$ is the Euler function. So in the treatment of (4-13), they dealt with the estimation involving

$$\left(\sum_{p \leqslant n,\ p \equiv a(\bmod q)} 1\right) - (n/(\phi(q)\log(n))) \tag{4-14}$$

by applying Theorem B where the estimation involved, in average of q, is for

$$\left(\sum_{m \leqslant n,\ m \equiv a(\bmod q)} \Lambda(\mathrm{m})\right) - (n/\phi(q)). \tag{4-15}$$

We can consider (4-15) instead of (4-14) because it is not difficult to show that estimation for (4-14) is equivalent to estimation for

$$\left(\sum_{p \leqslant n,\ p \equiv a(\bmod q)} \log(p)\right) - (n/(\phi(q))$$

which is the main term in (4-15) by the definition of the von Mangoldt function $\Lambda(m)$. The error caused by only taking the main term from (4-15) is absorbed amply by the total acceptable error of the estimation for (4-13). Next, in the treatment for a good upper bound of the term caused by the triple sum $\sum_{p \in S(n)}\sum_{(p_1)}\sum_{(p_2)}$ in (4-8), it was found that we can apply Theorem W directly together with some deep results including theorems on Large Sieve to achieve the goal. Now both Theorems B and W are non-effective and hence the N_0 in Chen Theorem is non-effective.

Remark 4-5. Plainly, the Lemma, (4.8) performs the function as a "Sieve" to select those primes p satisfying $n - p = P_2$ by shifting out all unqualified primes. This is the reason why we call the main method applied in the attacks on G(2) the Sieve Method.

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