

# Nowhere-zero 3-Flows in Signed Graphs

Yezhou Wu<sup>a\*</sup> Dong Ye<sup>b†</sup> Wenan Zang<sup>a‡</sup> Cun-Quan Zhang<sup>c§</sup>

<sup>a</sup> Department of Mathematics, University of Hong Kong  
Hong Kong, China

<sup>b</sup> Department of Mathematical Sciences, Middle Tennessee State University  
Murfreesboro, TN 37132, USA

<sup>c</sup> Department of Mathematics, West Virginia University  
Morgantown, WV 26506, USA

## Abstract

We prove that if an 8-edge-connected signed graph admits a nowhere-zero integer flow, then it has a nowhere-zero 3-flow; our result generalizes Thomassen's 3-flow theorem on 8-edge-connected graphs and strengthens Zhu's 3-flow theorem on 11-edge-connected signed graphs.

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\*Supported in part by the Research Grants Council of Hong Kong. E-mail: yezhouwu@hku.hk.

†E-mail: Dong.Ye@mtsu.edu.

‡Supported in part by the Research Grants Council of Hong Kong. E-mail: wzang@maths.hku.hk.

§Supported in part by NSF grant DMS-1264800 and NSA grant H98230-12-1-0233. E-mail: cqzhang@math.wvu.edu.

## 1 Introduction

Graphs considered in this paper may have multiple edges and loops unless otherwise stated. Let  $G = (V, E)$  be a graph and let  $k$  be a positive integer. An ordered pair  $(D, f)$  is called a  $k$ -flow of  $G$  if  $D = (V, A)$  is an orientation of  $G$  and  $f : A \mapsto \{0, \pm 1, \dots, \pm(k-1)\}$  is an assignment of flows, such that

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$$

for each  $v \in V$ , where  $E^+(v)$  is the set of all arcs leaving vertex  $v$  in  $D$  and  $E^-(v)$  is the set of all arcs entering vertex  $v$ . We say that the  $k$ -flow  $(D, f)$  is *nowhere-zero* if  $f(e) \neq 0$  for any  $e \in A$ . The concept of nowhere-zero integer flow was introduced by Tutte in 1954, and the theory of integer flows provides an interesting way to extend theorems about region-coloring planar graphs to general graphs [12, 13]. Tutte observed that every nowhere-zero  $k$ -flow on a plane graph gives rise to a  $k$ -vertex-coloring of its dual, and vice versa. Thus nowhere-zero integer flow and graph coloring can be viewed as dual concepts, and the above Tutte's observation is often referred to as the *duality theorem*. One of the major open problems in this research area is Tutte's 3-Flow Conjecture, which is exactly the dual version of Grötzsch's 3-color theorem on planar graphs [3, 4].

**Conjecture 1.1** (Tutte [12]). *Every 4-edge-connected graph has a nowhere-zero 3-flow.*

Thomassen [11] made a breakthrough in this conjecture by establishing the following weaker version.

**Theorem 1.1** (Thomassen [11]). *Every 8-edge-connected graph has a nowhere-zero 3-flow.*

This 3-flow theorem has recently been strengthened by Lovász, Thomassen, Wu, and Zhang [8] as follows.

**Theorem 1.2** (Lovász et al. [8]). *Every 6-edge-connected graph has a nowhere-zero 3-flow.*

As proved by Kochol [7], a minimum counterexample to the 3-Flow Conjecture is 5-edge-connected. Therefore the above theorem is actually just one step away from the resolution.

The aforementioned duality theorem cannot be extended directly to embedded graphs (see DeVos et al. [2] for an asymptotic version). Nevertheless, Jaeger [5] showed that if a graph  $G$  has a face- $k$ -colorable 2-cell embedding in some orientable surface, then it has a nowhere-zero  $k$ -flow. Interestingly, if the surface is non-orientable, then this coloring corresponds to a nowhere-zero  $k$ -flow in a signed graph arising from  $G$ . It is due to their great theoretical interest that integer flows in sign graphs have also been subjects of extensive research.

Let us define a few terms before proceeding. A *signed graph* is a pair  $(G, \sigma)$ , where  $G$  is a graph and  $\sigma : E(G) \rightarrow \{1, -1\}$  is a *signature* of  $G$ . An edge  $e$  is called *positive* if  $\sigma(e) = 1$  and *negative* otherwise. Each edge  $e = xy$  of a signed graph  $(G, \sigma)$  is comprised of two half-edges  $h_x$  and  $h_y$ , where  $h_x$  is incident with  $x$  and  $h_y$  is incident with  $y$ . An *orientation*  $D$  of  $(G, \sigma)$  assigns every half-edge a direction in the following way: if  $e = xy$  is positive, then  $h_x$  and  $h_y$  are directed both from  $x$  to  $y$ , or both from  $y$  to  $x$  (see Figure 1); if  $e = xy$  is negative, then the

directions of  $h_x$  and  $h_y$  are opposite (there are two possibilities: (1)  $h_x$  is directed to  $x$  and  $h_y$  is directed to  $y$ ; and (2)  $h_x$  is directed from  $x$  and  $h_y$  is directed from  $y$ . See Figure 1).

A negative edge  $e = xy$  is called a *source edge* if  $e$  is directed towards both  $x$  and  $y$  and called a *sink edge* otherwise. In the literature an oriented signed graph is also called a *bidirected graph*. If all edges of  $(G, \sigma)$  are positive, then a signed graph is equivalent to a graph. So we can view signed graphs as generalizations of graphs.

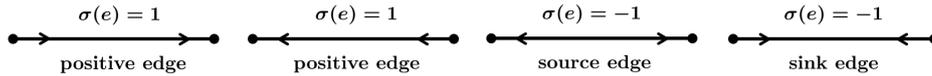


Figure 1: Orientations of positive and negative edges.

The concept of nowhere-zero integer flow in graphs carries over naturally to signed graphs, and the following is a well-known conjecture on integer flows in signed graphs.

**Conjecture 1.2** (Bouchet [1]). *Every signed graph admitting a nowhere-zero integer flow has a nowhere-zero 6-flow.*

Despite tremendous research effort, this conjecture remains open; Xu and Zhang [14] confirmed it for 6-edge-connected signed graphs. In [10], Raspaud and Zhu established that every 4-edge-connected signed graph has a nowhere-zero 4-flow provided it admits a nowhere-zero integer flow. Based on Theorem 1.2, Zhu [16] obtained the following result recently.

**Theorem 1.3** (Zhu [16]). *Every 11-edge-connected signed graph admitting a nowhere-zero integer flow has a nowhere-zero 3-flow.*

What is the least edge-connectivity that can guarantee the existence of nowhere-zero 3-flows in signed graphs? Zhu posed this as an open question in [16]. With the motivation to improve the bound in Theorem 1.3 and extend the setting of Theorem 1.1, we establish the following main result in this paper.

**Theorem 1.4.** *Every 8-edge-connected signed graph admitting a nowhere-zero integer flow has a nowhere-zero 3-flow.*

It is worthwhile pointing out that the assertion no longer holds if 8 is replaced by 4: Let  $(G, \sigma)$  be the signed graph with three vertices in which each pair of vertices is connected by precisely one positive edge and precisely one negative edge. Clearly,  $G$  is 4-edge-connected and has a nowhere-zero 4-flow. Nevertheless, it is routine to check that  $G$  admits no nowhere-zero 3-flow.

In response to Zhu's open question [16], we offer the following conjecture whose validity would imply Tutte's 3-flow Conjecture (see Kochol [7]).

**Conjecture 1.3.** *Every 5-edge-connected signed graph admitting a nowhere-zero integer flow has a nowhere-zero 3-flow.*

## 2 Operations

In this section we introduce some operations on signed graphs which will be employed in subsequent proofs.

**Flipping.** Let  $(G, \sigma)$  be a signed graph and let  $A$  be a subset of  $V(G)$ . Define  $\sigma' : E(G) \rightarrow \{1, -1\}$  as

$$\sigma'(e) = \begin{cases} -\sigma(e) & \text{if } e \in [A, \bar{A}], \\ \sigma(e) & \text{otherwise,} \end{cases}$$

where  $\bar{A} = V(G) \setminus A$  and  $[A, \bar{A}]$  is the cut in  $G$  consisting of all edges between  $A$  and  $\bar{A}$ . We say that the signed graph  $(G, \sigma')$  is obtained from  $(G, \sigma)$  by *flipping* all edges in  $[A, \bar{A}]$ .

Two signed graphs  $(G, \sigma)$  and  $(G, \sigma')$  are called *equivalent* if one can be obtained from the other by flipping all edges in a cut. Let us show that this flipping operation does not affect the existence of a nowhere-zero integer flow in a signed graph.

**Lemma 2.1.** *Let  $(G, \sigma)$  and  $(G, \sigma')$  be two equivalent signed graph and let  $k$  be a positive integer. Then  $(G, \sigma)$  has a nowhere-zero  $k$ -flow if and only if so does  $(G, \sigma')$ .*

**Proof.** Suppose  $(G, \sigma')$  is obtained from  $(G, \sigma)$  by flipping all edges in a cut  $[A, \bar{A}]$ , and suppose  $(D, f)$  is a nowhere-zero  $k$ -flow of  $(G, \sigma)$ . Let  $D'$  be obtained from  $D$  by changing the direction of each half-arc incident with a vertex in  $A$ . Then  $(D', f)$  is a nowhere-zero  $k$ -flow of  $(G, \sigma')$ . The reverse direction can be established similarly. ■

Throughout we use  $n(G, \sigma)$  to denote the minimum number of negative edges contained in a signed graph equivalent to  $(G, \sigma)$ .

**Lemma 2.2.** *If a signed graph  $(G, \sigma)$  admits a nowhere-zero integer flow, then  $n(G, \sigma) \neq 1$ .*

**Proof.** Assume the contrary:  $n(G, \sigma) = 1$ . Then some equivalent signed graph of  $(G, \sigma)$ , denoted by  $(G, \sigma')$ , has precisely one negative edge  $e$ . By Lemma 2.1 and the hypothesis,  $(G, \sigma')$  also admits a nowhere-zero integer flow  $(D, f)$ . Let  $D'$  be the digraph obtained from  $D$  by inserting a dummy vertex  $v$  in the middle of  $e$ , and let  $e_1$  and  $e_2$  denote the two arcs arising from  $e$ . Set  $f(e_i) = f(e)$  for  $i = 1, 2$ . Then  $f$  is conserved at all vertices in  $D'$  except  $v$ , which is impossible. ■

**Contraction.** Let  $(G, \sigma)$  be a signed graph and let  $A$  be a subset of  $V(G)$ . The signed graph obtained from  $(G, \sigma)$  by *contracting*  $A$ , denoted by  $(G/A, \sigma)$ , is the graph arising from  $(G, \sigma)$  by identifying all vertices in  $A$  to a single vertex, in which each edge of  $G$  with both ends in  $A$  becomes a loop, and each edge has the same sign as in  $(G, \sigma)$ .

Since the sign of a loop is not effected by a flipping operation, the following statement holds.

**Lemma 2.3.** *Let  $(G, \sigma)$  be a signed graph with precisely  $n(G, \sigma)$  negative edges. Then  $n(G/A, \sigma) = n(G, \sigma)$  for any proper subset  $A$  of  $V(G)$ .* ■

**Lifting.** Let  $(G, \sigma)$  be a signed graph, let  $xy, xz$  be two edges of  $G$ , and let  $G'$  be obtained from  $G$  by deleting  $xy, xz$  and adding a new edge  $e_0$  between  $y$  and  $z$ . Define  $\sigma' : E(G') \rightarrow \{1, -1\}$

as

$$\sigma'(e) = \begin{cases} \sigma(xy)\sigma(xz) & \text{if } e = e_0, \\ \sigma(e) & \text{otherwise.} \end{cases}$$

We say that the signed graph  $(G', \sigma')$  is obtained from  $(G, \sigma)$  by *lifting*  $xy$  and  $xz$ ; see Figure 2 for an illustration. Note that  $x, y, z$  are not necessary distinct in this definition.

An orientation of  $(G', \sigma')$  can be extended naturally to an orientation of  $(G, \sigma)$  by orienting the two half-edges incident with  $x$  as follows: one enters  $x$  and the other leaves  $x$ ; see Figure 2 for the case when  $\sigma(xy) = \sigma(xz) = -1$ .

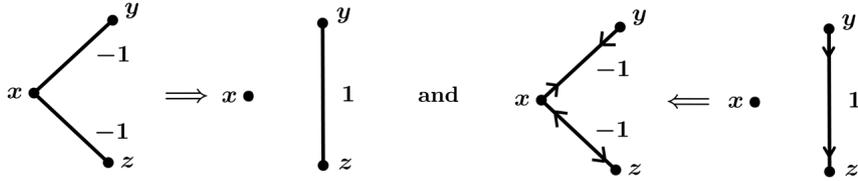


Figure 2: A lifting of  $xy$  and  $xz$  and an orientation extension.

**Lemma 2.4.** *Let  $(G, \sigma)$  be a signed graph and let  $xy, xz$  be two edges of  $G$ . If  $(G', \sigma')$  is the signed graph obtained from  $(G, \sigma)$  by lifting  $xy$  and  $xz$ , then*

$$n(G', \sigma') \geq n(G, \sigma) - 2.$$

**Proof.** For each  $U \subseteq V(G)$ , let  $[U, \bar{U}]_{G'}$  (resp.  $[U, \bar{U}]_G$ ) denote the cut consisting of all edges between  $U$  and  $\bar{U}$  in  $G'$  (resp. in  $G$ ). Suppose the signed graph  $(G', \sigma'')$  obtained from  $(G', \sigma')$  by flipping all edges in a cut  $[A, \bar{A}]_{G'}$  has precisely  $n(G', \sigma'')$  negative edges. Consider the signed graph  $(G, \bar{\sigma})$  obtained from  $(G, \sigma)$  by flipping all edges in  $[A, \bar{A}]_G$ . It is easy to see that the number of negative edges in  $(G, \bar{\sigma})$  is at most two plus the number of negative edges in  $(G', \sigma'')$ . Hence  $n(G, \sigma) \leq n(G', \sigma'') + 2$ , as desired. ■

Let  $G$  be a graph and let  $x, y$  be two distinct vertices of  $G$ . The *local edge-connectivity* of  $G$  between  $x$  and  $y$ , denoted by  $\lambda_G(x, y)$ , is the maximum number of edge-disjoint paths connecting  $x$  and  $y$  in  $G$ . The following Mader's theorem [9] asserts that the local edge-connectivity is preserved under some lifting operation.

**Theorem 2.5** (Mader [9]). *Let  $G$  be a connected loopless graph and let  $v_0$  be a vertex of degree at least 4 such that no edge incident with  $v_0$  is a cut-edge of  $G$ . Then  $G$  contains two edges  $v_0v_1$  and  $v_0v_2$  such that  $\lambda_H(x, y) = \lambda_G(x, y)$  for any two vertices  $x, y$  different from  $v_0$ , where  $H$  is the graph obtained from  $G$  by lifting  $v_0v_1$  and  $v_0v_2$ .*

### 3 Orientations: Modulo and Beyond

Let  $(G, \sigma)$  be a signed graph. For each  $A \subseteq V(G)$ , the *degree* of  $A$ , denoted by  $d(A)$ , is the number of edges between  $A$  and  $\bar{A}$ ; we write  $d(A) = d(a)$  if  $A = \{a\}$  (notice that the contribution

to  $d(a)$  made by any loop incident with  $a$ , if any, is zero). For each orientation  $D$  of  $(G, \sigma)$ , let  $d_D^+(v)$  (resp.  $d_D^-(v)$ ) denote the number of half-arcs leaving (resp. entering) a vertex  $v$ ; we may drop the subscript  $D$  if there is no danger of confusion. Note that, by definition, each loop incident with  $v$  (if any) contributes two to  $d_D^+(v) + d_D^-(v)$ , so  $d(v) < d_D^+(v) + d_D^-(v)$  if such a loop exists.

An orientation  $D$  of  $(G, \sigma)$  is called a *modulo 3-orientation* if  $d_D^+(v) \equiv d_D^-(v) \pmod{3}$  for all  $v \in V(G)$ . As shown by Tutte [12], a graph  $G$  admits a modulo 3-orientation if and only if it has a nowhere-zero 3-flow; this equivalence relation can be further extended to signed graphs.

**Lemma 3.1** (Xu and Zhang [14]). *Let  $(G, \sigma)$  be a signed graph. Then  $(G, \sigma)$  admits a modulo 3-orientation if and only if it has a nowhere-zero 3-flow.*

To prove Theorem 1.4, we shall actually establish the following assertion.

**Theorem 3.2.** *Let  $(G, \sigma)$  be a signed graph with  $n(G, \sigma) \geq 2$ , and let  $V_0 = \emptyset$  or  $V_0 = \{v_0\}$ , where  $v_0$  is a vertex of  $G$  such that no loop is incident with  $v_0$  and that  $d(v_0) \leq 6$  and is even. If  $|V(G) \setminus V_0| \geq 2$  and  $\lambda_G(x, y) \geq 8$  for any distinct vertices  $x, y$  in  $V(G) \setminus V_0$ , then  $(G, \sigma)$  admits a modulo 3-orientation.*

To see the implication, let  $(G, \sigma)$  be an 8-edge-connected signed graph with a nowhere-zero integer flow. By Lemma 2.2, we have  $n(G, \sigma) \neq 1$ . From Theorem 1.1 and Lemma 2.1 (if  $n(G, \sigma) = 0$ ) and from Theorem 3.2 with  $V_0 = \emptyset$  and Lemma 3.1 (if  $n(G, \sigma) \geq 2$ ), we can thus deduce that  $(G, \sigma)$  admits a nowhere-zero 3-flow.

The remainder of this paper is devoted to a proof of Theorem 3.2. The proof proceeds by induction on  $|V(G)| + |E(G)|$ ; to make the induction work, we need a generalized concept of graph orientation and a set function from [8], which is a variant of the one introduced by Thomassen in [11].

Let  $G$  be a loopless graph. A mapping  $\beta : V(G) \mapsto \mathbb{Z}_3 = \{0, 1, 2\}$  is called a  $\mathbb{Z}_3$ -boundary of  $G$  if  $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{3}$ . Given a  $\mathbb{Z}_3$ -boundary  $\beta$  of  $G$ , an orientation  $D$  of  $G$  is called a  $\beta$ -orientation if  $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod{3}$  for all  $v \in V(G)$ . The set function is a mapping  $\tau : V(G) \mapsto \{0, \pm 1, \pm 2, \pm 3\}$  such that

$$\tau(v) \equiv \begin{cases} \beta(v) \pmod{3} \\ d(v) \pmod{2} \end{cases}$$

for all  $v \in V(G)$ . This mapping  $\tau$  can be further extended to any nonempty  $A \subseteq V(G)$  as follows:

$$\tau(A) \equiv \begin{cases} \beta(A) \pmod{3} \\ d(A) \pmod{2}, \end{cases}$$

where  $\beta(A) \equiv \sum_{v \in A} \beta(v) \pmod{3}$ . Since  $d(A)$  and  $\tau(A)$  have the same parity, the following inequality holds.

**Lemma 3.3** (Lovász et al. [8]). *If  $d(A) \geq 6$ , then  $d(A) \geq 4 + |\tau(A)|$ .* ■

Theorem 1.2 is an immediate corollary of the following result, which was derived by refining Thomassen's technique [11] and will be used in our proof.

**Theorem 3.4** (Lovász et al. [8]). *Let  $G$  be a loopless graph, let  $\beta$  be a  $\mathbb{Z}_3$ -boundary of  $G$ , let  $z_0 \in V(G)$ , and let  $D(z_0)$  be a pre-orientation of the set  $E(z_0)$  of all edges incident with  $z_0$ . Assume that*

- (i)  $|V(G)| \geq 3$ ;
- (ii)  $d(z_0) \leq 4 + |\tau(z_0)|$  and  $d^+(z_0) - d^-(z_0) \equiv \beta(z_0) \pmod{3}$ ;
- (iii)  $d(A) \geq 4 + |\tau(A)|$  for each nonempty  $A \subseteq V(G) \setminus \{z_0\}$  with  $|V(G) \setminus A| \geq 2$ .

*Then  $D(z_0)$  can be extended to a  $\beta$ -orientation  $D$  of the entire graph  $G$ .*

When restricted to the disjoint union of an isolated vertex  $z_0$  and a 6-edge-connected loopless graph, the preceding theorem yields the following statement.

**Theorem 3.5** (Lovász et al. [8]). *Let  $G$  be a loopless graph and let  $\beta$  be a  $\mathbb{Z}_3$ -boundary of  $G$ . If  $G$  is 6-edge-connected, then  $G$  has a  $\beta$ -orientation.*

We now proceed to prove two technical lemmas, which will play important roles in our proof of Theorem 3.2.

**Lemma 3.6.** *Let  $(G, \sigma)$  be a 6-edge-connected signed graph with only 2 or 3 negative edges. Then  $(G, \sigma)$  admits a modulo 3-orientation.*

**Proof.** Let  $m$  be the number of negative edges of  $(G, \sigma)$ . Set  $r = 1$  if  $m = 2$  and  $r = 0$  if  $m = 3$ . Let  $H$  be the graph obtained from  $G$  by first orienting  $r$  negative edges as sink edges and the remaining  $m - r$  negative edges as source edges, then inserting a new vertex to each negative edge, and finally identifying all these newly inserted vertices to a single vertex  $z_0$ . Let  $G' = H$  if  $m = 2$  and let  $G'$  be obtained from  $H$  by replacing one arc leaving  $z_0$  with two parallel arcs entering  $z_0$  if  $m = 3$ . For each  $A \subseteq V(G')$ , we use  $d'(A)$  and  $\tau'(A)$  to denote the degree of  $A$  in  $G'$  and the value of the set function at  $A$ , respectively. If  $m = 2$ , then  $d'(z_0) = 4 \leq 4 + |\tau'(z_0)|$ . If  $m = 3$ , then  $d'(z_0) = 7$ . So  $|\tau'(z_0)| = 3$  by definition and thus  $d'(z_0) = 4 + |\tau'(z_0)|$ . Hence the inequality  $d'(z_0) \leq 4 + |\tau'(z_0)|$  holds in either case. By Lemma 3.3, we have  $d'(A) \geq 6 \geq 4 + |\tau'(A)|$  for each nonempty  $A \subseteq V(G') \setminus \{z_0\}$  with  $|V(G') \setminus A| \geq 2$ . Therefore, by Theorem 3.4, the pre-orientation of the arcs incident with  $z_0$  can be extended to a modulo 3-orientation of the entire graph  $G'$ , which clearly yields a modulo 3-orientation of  $(G, \sigma)$ . ■

**Lemma 3.7.** *Let  $G$  be a loopless graph, let  $\beta$  be a  $\mathbb{Z}_3$ -boundary of  $G$ , let  $z_0 \in V(G)$ , let  $D(z_0)$  be a pre-orientation of the set  $E(z_0)$  of all edges incident with  $z_0$ , and let  $S = \{v \in V(G) \setminus \{z_0\} \mid d(v) = 5 \text{ and } \beta(v) = 0\}$ . Assume that*

- (i)  $|V(G)| \geq 3$ ;
- (ii)  $d(z_0) \leq 5$  and  $d^+(z_0) - d^-(z_0) \equiv \beta(z_0) \pmod{3}$ ;
- (iii)  $d(v) \geq 4 + |\tau(v)|$  for each  $v \in V(G) \setminus (S \cup \{z_0\})$ ; and
- (iv)  $d(A) \geq 6$  for each  $A \subseteq V(G) \setminus \{z_0\}$  with  $\min\{|A|, |V(G) \setminus A|\} \geq 2$ .

*If  $|S| \leq 2$ , then  $D(z_0)$  can be extended to a  $\beta$ -orientation  $D$  of the entire graph  $G$ .*

**Proof.** By definition,  $d(z_0)$  and  $\tau(z_0)$  have the same parity, so  $|\tau(z_0)| \geq 1$  if  $d(z_0) = 5$ . Hence  $d(z_0) \leq 4 + |\tau(z_0)|$ . If  $S = \emptyset$ , then the statement follows instantly from Theorem 3.4. Thus we may assume  $S \neq \emptyset$ .

Let  $p$  be the integer in  $\mathbb{Z}_3$  with  $\beta(z_0) - d(z_0) + 1 \equiv 2p \pmod{3}$  and let  $q = 7 - d(z_0) - p$ . Then  $q \geq 0$  and  $p + q \geq 2$  as  $d(z_0) \leq 5$ . Let  $G'$  be obtained from  $G$  by adding a set  $P$  of  $p$  arcs from  $S$  to  $z_0$  and adding a set  $Q$  of  $q$  arcs from  $z_0$  to  $S$  such that each vertex in  $S$  has degree at least six in  $G'$  (this  $G'$  is available because  $|S| \leq 2$ ). Let  $\beta'(z_0)$  be the integer in  $\mathbb{Z}_3$  with  $\beta'(z_0) \equiv \beta(z_0) + q - p \pmod{3}$ . By the definitions of  $p$  and  $q$ , we obtain  $\beta'(z_0) \equiv (d(z_0) - 1 + 2p) + (7 - d(z_0) - p) - p \equiv 0 \pmod{3}$ . So  $\beta'(z_0) = 0$ . For each vertex  $v \neq z_0$ , let  $P(v)$  (resp.  $Q(v)$ ) be the set of all arcs in  $P$  (resp.  $Q$ ) incident with  $v$ , and let  $\beta'(v)$  be the integer in  $\mathbb{Z}_3$  with  $\beta'(v) \equiv \beta(v) + |P(v)| - |Q(v)| \pmod{3}$ . Then  $\sum_{v \in V(G')} \beta'(v) = \sum_{v \in V(G)} \beta(v) + (q - p) + \sum_{v \neq z_0} (|P(v)| - |Q(v)|) = \sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{3}$ . Hence  $\beta'$  is a  $\mathbb{Z}_3$ -boundary of  $G'$ .

Let  $d'(A)$  and  $\tau'(A)$  denote the degree of  $A$  in  $G'$  and the value of the set function at  $A$ , respectively. Since  $d'(z_0) = 7$  and  $\beta'(z_0) = 0$ , we have  $|\tau'(z_0)| = 3$ . So  $d'(z_0) = 4 + |\tau'(z_0)|$ . Since  $d'(v) \geq 6$  for each  $v \in S$ , from Lemma 3.3 it follows that  $d'(v) \geq 4 + |\tau'(v)|$ . Therefore, by Theorem 3.4, the pre-orientation of the arcs incident with  $z_0$  can be extended to a  $\beta'$ -orientation of the entire graph  $G'$ , which clearly yields a  $\beta$ -orientation of  $(G, \sigma)$ . ■

#### 4 Proof of Theorem 3.2

The proof proceeds by induction on  $|V(G)| + |E(G)|$ . Assume on the contrary that  $(G, \sigma)$  is a smallest counterexample and, subject to this, the number of negative edges in  $(G, \sigma)$  is minimum.

For each nonempty proper  $A \subseteq V(G)$ , we use  $g(A, \sigma)$  (resp.  $h(A, \sigma)$ ) to denote the number of positive (resp. negative) edges of  $(G, \sigma)$  contained in the cut  $[A, \bar{A}]$  of  $G$ , and set  $g(A, \sigma) = g(a, \sigma)$  (resp.  $h(A, \sigma) = h(a, \sigma)$ ) if  $A = \{a\}$ .

**Claim 1.** For each nonempty proper  $A \subseteq G$ , we have  $g(A, \sigma) \geq h(A, \sigma)$ . Hence  $(G, \sigma)$  contains exactly  $n(G, \sigma)$  negative edges.

Otherwise,  $g(A, \sigma) < h(A, \sigma)$ . Let  $(G, \sigma')$  be the signed graph obtained from  $(G, \sigma)$  by flipping all edges in the cut  $[A, \bar{A}]$ . Then the number of negative edges in  $(G, \sigma')$  is less than that in  $(G, \sigma)$ . By Lemma 2.1 and Lemma 3.1,  $(G, \sigma')$  admits no modulo 3-orientation. Thus the existence of  $(G, \sigma')$  contradicts the minimality assumption on  $(G, \sigma)$ .

From the definition, it follows instantly that  $(G, \sigma)$  contains exactly  $n(G, \sigma)$  negative edges. Thus Claim 1 is justified.

**Claim 2.**  $n(G, \sigma) \geq 4$ .

Assume the contrary:  $n(G, \sigma) = 2$  or  $3$ . By Lemma 3.6, we have  $V_0 = \{v_0\}$  and  $d(v_0) \leq 4$ . In view of Claim 1,  $g(v_0, \sigma) \geq h(v_0, \sigma)$ . Thus we can partition all the edges incident with  $v_0$  into pairs so that each pair contains at most one negative edge. Let  $(G', \sigma')$  be the signed graph obtained from  $(G, \sigma)$  by lifting each of these edge pairs and deleting the resulting isolated vertex  $v_0$ . Then  $(G', \sigma')$  has the same number of negative edges as  $(G, \sigma)$ . For each nonempty proper  $A \subseteq V(G')$ , let  $d'(A)$  be the degree of  $A$  in  $G'$  and let  $\bar{A} = V(G') \setminus A$ . Then  $d'(A) + d'(\bar{A}) \geq d(A) + d(\bar{A}) - d(v_0) \geq 8 + 8 - 4 = 12$ . Since  $d'(A) = d'(\bar{A})$ , we have  $d'(A) \geq 6$ . Thus  $G'$  is

6-edge-connected. By Lemma 3.6,  $(G', \sigma')$  admits a modulo 3-orientation, which clearly yields a modulo 3-orientation of  $(G, \sigma)$ ; this contradiction proves Claim 2.

**Claim 3.**  $(G, \sigma)$  contains no loops.

Suppose on the contrary that  $e_1$  is a loop incident with a vertex  $x$ . Let  $e_2$  be an edge connecting  $x$  and one of its neighbors  $y$ , and let  $(G', \sigma')$  be the signed graph obtained from  $(G, \sigma)$  by lifting  $e_1$  and  $e_2$ . By Claim 2 and Lemma 2.4, we have  $n(G', \sigma') \geq n(G, \sigma) - 2 \geq 4 - 2 = 2$ . Hence, by induction hypothesis,  $(G', \sigma')$  admits a modulo 3-orientation, which clearly yields a modulo 3-orientation of  $(G, \sigma)$ ; this contradiction establishes Claim 3.

**Claim 4.**  $|V(G)| \neq 2$ .

Otherwise,  $|V(G)| = 2$ ; let  $V(G) = \{x, y\}$ . By hypothesis, we have  $V_0 = \emptyset$ . By Claim 3, the edges of  $(G, \sigma)$  are all between  $x$  and  $y$ . Recall Claim 1, the number of negative edges between  $x$  and  $y$  is  $n(G, \sigma)$ , so the number of positive edges between  $x$  and  $y$  is  $|E(G)| - n(G, \sigma) \geq n(G, \sigma) \geq 4$  by Claim 2.

Let  $p$  be the integer in  $\mathbb{Z}_3$  such that  $p \equiv n(G, \sigma) - p \pmod{3}$ . Orient  $p$  negative edges as source edges and the remaining  $n(G, \sigma) - p$  negative edges as sink edges.

Let  $q$  be the integer in  $\mathbb{Z}_3$  such that  $q \equiv (E(G) - n(G, \sigma)) - q \pmod{3}$ . Orient  $q$  positive edges from  $x$  to  $y$  and the remaining  $(E(G) - n(G, \sigma)) - q$  positive edges from  $y$  to  $x$ .

Clearly, the resulting orientation is a modulo 3-orientation of  $(G, \sigma)$ ; this contradiction implies Claim 4.

**Claim 5.**  $d(v)$  is odd for each  $v \in V(G)$ . So  $V_0 = \emptyset$  and hence  $|V(G)| \geq 4$  by Claim 4.

Suppose on the contrary that some vertex of  $G$  has even degree; let  $u$  be such vertex with the smallest  $d(u)$ . By Theorem 2.5,  $G$  contains two edges  $uv_1$  and  $uv_2$  such that  $\lambda_{G'}(x, y) = \lambda_G(x, y)$  for any two distinct vertices  $x$  and  $y$  different from  $u$ , where  $(G', \sigma')$  is the signed graph obtained from  $(G, \sigma)$  by lifting  $uv_1$  and  $uv_2$ . Let  $d'(v)$  stand for the degree of a vertex  $v$  in  $G'$ . Then  $d'(u) = d(u) - 2$ . Depending on the value of  $d(u)$ , we define  $V'_0$  as follows.

CASE 1.  $d(u) \leq 6$ .

In this case  $V_0 = \{u\}$  because, by hypothesis and Menger's theorem, all vertices except  $v_0$  in  $V_0$  have degree at least eight. If  $d'(u) = 0$ , with a slight abuse of notation, we still use  $G'$  to denote the graph obtained from  $G'$  by deleting  $u$ , and set  $V'_0 = \emptyset$ . If  $d'(u) > 0$ , set  $V'_0 = \{u\}$ . Since  $V(G') \setminus V'_0 = V(G) \setminus V_0$ , by hypothesis  $|V(G') \setminus V'_0| \geq 2$ .

CASE 2.  $d(u) \geq 8$ .

In this case  $V_0 = \emptyset$  by the choice of  $u$ . If  $d(u) \geq 10$ , then  $d'(u) \geq 8$ ; set  $V'_0 = \emptyset$ . If  $d(u) = 8$ , then  $d'(u) = 6$ ; set  $V'_0 = \{u\}$ . By Claim 4,  $|V(G') \setminus V'_0| \geq |V(G) \setminus \{u\}| \geq 2$ .

In either case, by Claim 2 and Lemma 2.4, we obtain  $n(G', \sigma') \geq n(G, \sigma) - 2 \geq 4 - 2 = 2$ . Thus, by induction hypothesis,  $(G', \sigma')$  admits a modulo 3-orientation, which clearly yields a modulo 3-orientation of  $(G, \sigma)$ , a contradiction. So Claim 5 is established.

**Claim 6.** For each  $v \in V(G)$ , either  $g(v, \sigma) \geq 6$  or  $g(v, \sigma) = 5$  and  $h(v, \sigma) = 4$ .

By Claim 1,  $g(v, \sigma) \geq h(v, \sigma)$ . By Claim 5,  $g(v, \sigma) + h(v, \sigma)$  is odd. By hypothesis,  $g(v, \sigma) + h(v, \sigma) \geq 8$  and hence is at least 9. So the statement follows.

**Claim 7.** For some nonempty proper  $A \subseteq V(G)$ , we have  $g(A, \sigma) \leq 5$ .

Suppose on the contrary that  $g(A, \sigma) \geq 6$  for each nonempty proper  $A \subseteq V(G)$ . Let  $G'$  be the graph obtained from  $G$  by deleting all negative edges. Then  $G'$  is 6-edge-connected. By Claim 3,  $G'$  is also loopless.

Let  $p$  be the integer in  $\mathbb{Z}_3$  such that  $p \equiv n(G, \sigma) - p \pmod{3}$ . We partition the set of all negative edges into two subsets  $P$  and  $Q$  with  $|P| = p$ . Then  $|Q| = n(G, \sigma) - p$  by Claim 1. Let us orient all negative edges in  $P$  (resp. in  $Q$ ) as source (resp. sink) edges. For each  $v \in V(G')$ , let  $P(v)$  (resp.  $Q(v)$ ) be the set of all arcs in  $P$  (resp.  $Q$ ) incident with  $v$ , and let  $\beta'(v)$  be the integer in  $\mathbb{Z}_3$  with  $\beta'(v) \equiv |P(v)| - |Q(v)| \pmod{3}$ . Clearly,  $\sum_{v \in V(G')} \beta'(v) \equiv 0 \pmod{3}$ . So  $\beta'$  is a  $\mathbb{Z}_3$ -boundary of  $G'$ .

By Theorem 3.5,  $(G', \sigma')$  admits a  $\beta$ -orientation, which clearly yields a modulo 3-orientation of  $(G, \sigma)$ ; this contradiction justifies Claim 7.

In the remainder of our proof, we reserve the symbol  $A$  for a nonempty proper subset of  $V(G)$  such that

- (1)  $g(A, \sigma) \leq 5$ ;
- (2)  $|A| \geq 2$ ; and
- (3)  $g(B, \sigma) \geq 6$  for any  $B \subseteq A$  with  $2 \leq |B| < |A|$ .

Such  $A$  is available because  $|A| + |\bar{A}| \geq 4$  by Claim 5; we may interchange  $A$  and  $\bar{A}$  if  $|A| = 1$ . By hypothesis,  $d(A) \geq 8$ . So  $h(A, \sigma) = d(A) - g(A, \sigma) \geq 8 - g(A, \sigma)$ . By (1), we thus have

- (4)  $h(A, \sigma) \geq 3$ .

Let  $k(A, \sigma)$  be the number of negative edges with both ends in  $A$ . By Lemma 2.3 and Lemma 2.4, we obtain  $n(G/A, \sigma) = n(G, \sigma) \geq k(A, \sigma) + h(A, \sigma)$ . It follows from (4) that

- (5)  $n(G/A, \sigma) - k(A, \sigma) \geq 3$ .

Let  $v_A$  be the vertex of  $(G/A, \sigma)$  resulting from contracting  $A$ . By Claim 3, all loops of  $(G/A, \sigma)$  are incident with  $v_A$ , and precisely  $k(A, \sigma)$  of them are negative. By (1) and Claim 1, we have  $d(A) \leq 10$ . By Claim 5,  $V_0 = \emptyset$ , so the minimum degree of  $G$  is at least eight by Menger's theorem, and hence some edge of  $G$  has two ends in  $A$  (see (2)). Let  $(G', \sigma')$  be the signed graph obtained from  $(G/A, \sigma)$  by replacing all loops incident with  $v_A$  by a new loop  $e$ , such that

- (6)  $\sigma'(e) = 1$  if  $k(A, \sigma) \equiv 0 \pmod{3}$  and  $\sigma'(e) = -1$  otherwise.

Notice that  $e$  does not necessarily correspond to an edge of  $G$ . Let  $d'(U)$  stand for the degree of  $U$  in  $G'$  for each  $U \subseteq V(G')$ . Since  $d(A) \geq 8$ , we have  $d'(v_A) \geq 8$ . Set  $V'_0 = \emptyset$ . It is clear that

- $|V(G') \setminus V'_0| = |V(G) \setminus A| + 1 \geq 2$ ;
- $n(G', \sigma') \geq n(G/A, \sigma) - k(A, \sigma) \geq 3$  by (5); and
- $\lambda_{G'}(x, y) \geq 8$  for any two vertices  $x$  and  $y$  of  $G'$  by Menger's theorem.

Thus, by (2) and induction hypothesis,  $(G', \sigma')$  has a modulo 3-orientation  $D'$ , which yields a partial orientation of  $(G, \sigma)$ . Reversing the directions of all half-arcs in  $D'$  if necessary, we may assume that

- (7)  $e$  is a source edge in  $D'$  when  $\sigma'(e) = -1$ .

Let  $G''$  be the loopless graph (with no signature) obtained from the signed graph  $(G/\bar{A}, \sigma)$  by first deleting all negative edges and then deleting all loops incident with  $z_0$ , the vertex arising from contracting  $\bar{A}$ . We orient all edges between  $A$  and  $z_0$  in  $G''$  as follows: Suppose edge  $xz_0$  in  $G''$ , with  $x \in A$ , corresponds to edge  $v_A y$  in  $G'$ , with  $y \in \bar{A}$ . Then the direction of  $xz_0$  in  $G''$  is exactly the same as the direction of  $v_A y$  in  $D'$ . For convenience, we denote this pre-orientation of edges incident with  $z_0$  by  $D(z_0)$ . Let  $p(z_0)$  (resp.  $q(z_0)$ ) be the number of all resulting arcs

entering (resp. leaving)  $z_0$ , we define  $\beta''(z_0)$  to be the integer in  $\mathbb{Z}_3$  with  $\beta''(z_0) \equiv q(z_0) - p(z_0) \pmod{3}$ .

Let  $F_1$  be the set of all negative edges of  $G$  with both ends in  $A$ . Recall that

$$(8) |F_1| = k(A, \sigma).$$

We orient all edges in  $F_1$  as sink edges if  $k(A, \sigma) \equiv 2 \pmod{3}$ , and orient all edges in  $F_1$  as source edges otherwise. Let  $F_2$  be the set of all negative edges between  $A$  and  $\bar{A}$  in  $G$ ; for each edge  $f \in F_2$ , we orient it as in  $D'$ . Set  $F = F_1 \cup F_2$ . For each  $v \in A$ , let  $p(v)$  (resp.  $q(v)$ ) be the number of all half-arcs entering (resp. leaving)  $v$  in  $F$ , we define  $\beta''(v)$  to be the integer in  $\mathbb{Z}_3$  with  $\beta''(v) \equiv p(v) - q(v) \pmod{3}$ . We propose to show that

$$(9) \beta'' \text{ is a } \mathbb{Z}_3\text{-boundary of } G''.$$

To justify this, let  $p_1$  (resp.  $q_1$ ) be the number of positive edges directed from  $A$  to  $\bar{A}$  (resp. from  $\bar{A}$  to  $A$ ) in  $D'$ , and let  $p_2$  (resp.  $q_2$ ) be the number of source (resp. sink) edges between  $A$  and  $\bar{A}$  in  $D'$ . Note that

$$(10) p_1 = p(z_0) \text{ and } q_1 = q(z_0).$$

Since  $d_{D'}^+(v_A) \equiv d_{D'}^-(v_A) \pmod{3}$ , the following equality holds.

$$(11) p_1 + q_2 \equiv q_1 + p_2 \pmod{3} \text{ if } \sigma'(e) = 1 \text{ and } p_1 + q_2 \equiv q_1 + p_2 + 2 \pmod{3} \text{ if } \sigma'(e) = -1.$$

Observe that in  $F$  there are precisely  $p_2$  half-arcs entering  $A$  and precisely  $q_2$  half-arcs leaving  $A$ . By direct computation, we obtain

$$(12) \sum_{v \in A} \beta''(v) = p_2 - q_2 - 2|F_1| \text{ if } k(A, \sigma) \equiv 2 \pmod{3} \text{ and } \sum_{v \in A} \beta''(v) = p_2 - q_2 + 2|F_1| \text{ otherwise.}$$

If  $k(A, \sigma) \equiv 0 \pmod{3}$  then, by (6) and (8), we have  $\sigma'(e) = 1$  and  $|F_1| \equiv 0 \pmod{3}$ . It follows from (12), (10), and (11) that  $\sum_{v \in A} \beta''(v) + \beta''(z_0) \equiv p_2 - q_2 + q_1 - p_1 \equiv 0 \pmod{3}$ .

If  $k(A, \sigma) \equiv 1 \pmod{3}$  then, by (6) and (8), we have  $\sigma'(e) = -1$  and  $|F_1| \equiv 1 \pmod{3}$ . It follows from (12), (10), and (11) that  $\sum_{v \in A} \beta''(v) + \beta''(z_0) \equiv p_2 - q_2 + 2 + q_1 - p_1 \equiv 0 \pmod{3}$ .

If  $k(A, \sigma) \equiv 2 \pmod{3}$  then, by (6) and (8), we have  $\sigma'(e) = -1$  and  $|F_1| \equiv 2 \pmod{3}$ . It follows from (12), (10), and (11) that  $\sum_{v \in A} \beta''(v) + \beta''(z_0) \equiv p_2 - q_2 - 4 + q_1 - p_1 \equiv 0 \pmod{3}$ .

Combining the above three cases, we arrive at (9).

Let us now verify that  $G''$  satisfies all the hypotheses of Lemma 3.7. By (2), we have  $|V(G'')| \geq |A| + 1 \geq 3$ . From (1) and the construction of  $G''$ , we see that  $d_{G''}(z_0) = g(A, \sigma) \leq 5$ ; with respect to  $D(z_0)$ , the equality  $d^+(z_0) - d^-(z_0) \equiv \beta(z_0) \pmod{3}$  clearly holds. For each  $v \in V(G'') \setminus (S \cup \{z_0\})$ , we have  $d''(v) \geq 5$  by Claim 6. If  $d''(v) \geq 6$ , then  $d''(v) \geq 4 + |\tau''(v)|$  by Lemma 3.3. If  $d''(v) = 5$ , then  $|\tau''(v)| = 1$  by definition as  $\beta''(v) \neq 0$ . Hence the inequality  $d''(v) \geq 4 + |\tau''(v)|$  also holds. Each  $B \subseteq V(G'') \setminus \{z_0\}$  with  $\min\{|B|, |V(G'') \setminus B|\} \geq 2$  is a proper subset of  $A$ , so  $d''(B) \geq 6$  by (3). Moreover, for each  $v \in S$ , Claim 6 implies  $g(v, \sigma) = 5$  and  $h(v, \sigma) = 4$ . Since  $\beta''(v) = 0$  and since negative edges with both ends in  $A$  are either all source edges or all sink edges, there are at least two negative edges between  $v$  and  $z_0$ . Since  $h(v, \sigma) \leq 5$  by (1) and Claim 1, we obtain  $|S| \leq 2$ . Thus, by Lemma 3.7,  $D(z_0)$  can be extended to a  $\beta''$ -orientation  $D''$  of the entire graph  $G$ . Combining  $D''$  with  $D' \setminus \{e\}$ , we obtain a modulo 3-orientation of  $(G, \sigma)$ ; this contradiction completes the proof of our theorem.  $\blacksquare$

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