THE HALL MODULE OF AN EXACT CATEGORY WITH DUALITY

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Abstract. We construct from a finitary exact category with duality $\mathcal{A}$ a module over its Hall algebra, called the Hall module, encoding the first order self-dual extension structure of $\mathcal{A}$. We study in detail Hall modules arising from the representation theory of a quiver with involution. In this case we show that the Hall module is naturally a module over the specialized reduced σ-analogue of the quantum Kac-Moody algebra attached to the quiver. For finite type quivers, we explicitly determine the decomposition of the Hall module into irreducible highest weight modules.

Introduction

Let $\mathcal{A}$ be an abelian category with finite Hom and Ext^1 sets, called finitary below. In [20] Ringel constructed from $\mathcal{A}$ an associative algebra $\mathcal{H}_{\mathcal{A}}$, the Hall algebra, whose multiplication encodes the first order extension structure of $\mathcal{A}$. There is also a coalgebra structure on $\mathcal{H}_{\mathcal{A}}$ which, if $\mathcal{A}$ is hereditary, makes $\mathcal{H}_{\mathcal{A}}$ into a (twisted) bialgebra [10]. The category $Rep_{\mathbb{F}_q}(Q)$ of representations of a quiver over a finite field is an example of a finitary hereditary category. The corresponding Hall algebra $\mathcal{H}_Q$ contains a subalgebra isomorphic to the positive part of the quantum Kac-Moody algebra associated to Q, specialized at $\sqrt{q}$ [21], [10]. A second example of a finitary hereditary category is Coh_X, the category of coherent sheaves over a smooth projective curve X defined over $\mathbb{F}_q$. The simplest case is $X = \mathbb{P}^1$, where the Hall algebra contains a subalgebra isomorphic to a positive part of the quantum affine algebra $U_{\sqrt{q}}(\hat{\mathfrak{sl}}_2)$ [13]. Hall algebras can be defined more generally for exact categories [12] and often give algebras behaving like quantum nilpotent groups [3].

The first goal of this paper is to introduce an analogue of the Hall algebra when objects of $\mathcal{A}$ are allowed to carry non-degenerate forms. We work in the framework of exact categories with duality, where a self-dual object is an object of $\mathcal{A}$ together with a symmetric isomorphism with its dual. In Theorem 2.4, we associate to an exact category with duality a $\mathcal{H}_{\mathcal{A}}$-module, called the Hall module and denoted $\mathcal{M}_{\mathcal{A}}$, encoding the self-dual extension structure of $\mathcal{A}$. Similarly, $\mathcal{M}_{\mathcal{A}}$ is a $\mathcal{H}_{\mathcal{A}}$-comodule. In Theorem 2.6, we modify this construction to obtain modules over the Ringel-Hall algebra of $\mathcal{A}$, whose (co)multiplication differs from that of the standard Hall algebra by a twist by the Euler form. The module twist is defined using an integer valued function $\mathcal{E}$ on the Grothendieck group of $\mathcal{A}$ that can be seen as a self-dual version of the Euler form. In Theorem 2.9, we prove an identity relating $\mathcal{E}$, the Euler form and the stacky number of self-dual extensions in $\mathcal{A}$. This identity is used in Section 3 but is also of independent interest. Its proof uses the combinatorics of self-dual analogues of Grothendieck's *extensions panachées* [11], [4].

In Section 3 we study Hall modules arising from the representation theory of a quiver with involution (Q, σ). From the involution and a choice of signs we define a duality structure on $Rep_{\mathbb{F}_q}(Q)$, with q odd. For particular signs, self-dual objects

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coincide with the orthogonal and symplectic representations of [6]. The module and comodule structures are incompatible in that $\mathcal{M}_Q$ is not a Hopf module. In Theorem 3.5, we instead show that the action and coaction of the simple representations $[S_i] \in \mathcal{H}_Q$ make $\mathcal{M}_Q$, with its $\mathcal{E}$-twisted module structure, a module over $B_\sigma(\mathfrak{g}_Q)$, the reduced σ-analogue of $U_{\sqrt{q}}(\mathfrak{g}_Q)$. The proof is combinatorial in nature and involves counting configurations of pairs of self-dual exact sequences. Furthermore, in Theorem 3.10 we describe the decomposition of $\mathcal{M}_Q$ into irreducible highest weight $B_\sigma(\mathfrak{g}_Q)$-modules, with generators being elements of $\mathcal{M}_Q$ annihilated by the coaction of each $[S_i]$. The proof uses a canonically defined non-degenerate bilinear form on $\mathcal{M}_Q$ and a characterization of irreducible highest weight modules due to Enomoto-Kashiwara [8].

In Section 4 we restrict attention to finite type quivers. Unlike ordinary quiver representations, self-dual representations may have non-trivial $\overline{\mathbb{F}}_q/\mathbb{F}_q$-forms. In Proposition 4.2 we extend results of [6] to explicitly describe all such forms and classify indecomposable self-dual $\mathbb{F}_q$-representations. We use this result in Theorems 4.4 and 4.6 to explicitly describe the decomposition of finite type Hall modules into irreducible highest weight $B_\sigma(\mathfrak{g}_Q)$-modules. The generators are written as alternating sums of $\overline{\mathbb{F}}_q/\mathbb{F}_q$-forms of self-dual indecomposables.

Enomoto [7] proved a result related to Theorems 3.5 and 3.10, showing that induction and restriction along $[S_i]$ endow the Grothendieck group of a category of perverse sheaves on the moduli stack of complex orthogonal representations with the structure of a highest weight $B_\sigma(\mathfrak{g}_Q)$-module. In the terminology of the present paper, the weight module in [7] is generated by the trivial orthogonal representation, whereas Theorems 3.5 and 3.10 hold for arbitrary dualities and describe the decomposition of the entire Hall module. The methods of [7] follow Lusztig's geometric framework [15] and are completely different from those used in this paper. The existence of both approaches suggests a self-dual analogue of Lusztig's purity result [16] for multiplicity complexes of perverse sheaves. This would provide a direct link between [7] and the present paper.

Notations and assumptions: In this paper, all fields are assumed to have characteristic different from two. In particular, if $\mathbb{F}_q$ is a finite field with q elements, then q is odd. All categories are assumed to be essentially small and we write $Iso(\mathcal{A})$ for the set of isomorphism classes of objects.

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1. The Hall algebra of an exact category

Let $\mathcal{A}$ be an exact category in the sense of Quillen [18]. In particular, $\mathcal{A}$ is additive and is equipped with a collection $\mathcal{F}$ of kernel-cokernel pairs (i, π) called short exact sequences and denoted

$$U \overset{i}{\rightarrowtail} X \overset{\pi}{\twoheadrightarrow} V, \tag{1}$$

satisfying a collection of axioms [18]. A morphism i is an admissible monic if it occurs in a pair $(i, \pi) \in \mathcal{F}$. Abelian categories are an important class of exact categories. More generally, an extension-closed full subcategory of an abelian category inherits a canonical exact structure.

Denote by $\underline{\mathcal{F}}_{U,V}^X$ the set of short exact sequences of the form (1). Assume that $\mathcal{A}$ is finitary, that is, for all $U, V \in \mathcal{A}$ the set $\operatorname{Hom}(U, V)$ is finite and $\underline{\mathcal{F}}_{U,V}^X$ is non-empty for only finitely many $X \in Iso(\mathcal{A})$. The Hall numbers are then the cardinalities

$$F_{U,V}^{X} = |\{\tilde{U} \subset X \mid \tilde{U} \simeq U,\ X/\tilde{U} \simeq V\}|,$$

where the subobjects $\tilde{U}$ are required to be admissible. Setting $a(U) = |Aut(U)|$ we have $|\underline{\mathcal{F}}_{U,V}^{X}| = a(U)a(V)F_{U,V}^{X}$.

Fix an integral domain R containing $\mathbb{Q}$, a unit $\nu \in R$ and a bilinear form $c : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$. The Hall algebra of $\mathcal{A}$ is the free R-module with basis $Iso(\mathcal{A})$,

$$\mathcal{H}_{\mathcal{A}} = \bigoplus_{U \in Iso(\mathcal{A})} R[U],$$

with associative multiplication given by [20], [12]

$$[U][V] = \nu^{c(V,U)} \sum_{X \in Iso(\mathcal{A})} F_{U,V}^{X}[X]. \tag{2}$$

Similarly, $\mathcal{H}_{\mathcal{A}}$ is a topological coassociative coalgebra with coproduct [10]

$$\Delta[X] = \sum_{U,V \in Iso(\mathcal{A})} \nu^{c(V,U)} \frac{a(U)a(V)}{a(X)} F_{U,V}^{X}[U] \otimes [V].$$

In general, Δ takes values in the completion $\mathcal{H}_{\mathcal{A}} \hat{\otimes}_R \mathcal{H}_{\mathcal{A}}$ consisting of all formal linear combinations $\sum_{U,V} c_{U,V}[U] \otimes [V]$; see [22] for details. Both the product and coproduct respect the natural (Grothendieck group) $K(\mathcal{A})$-grading of $\mathcal{H}_{\mathcal{A}}$.

Suppose that $\mathcal{A}$ is $\mathbb{F}_q$-linear and of finite homological dimension with finite Ext^i sets, $i \geq 0$. In this case, its Euler form is the bilinear form on $K(\mathcal{A})$ defined by

$$\langle U, V \rangle = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{F}_q} \operatorname{Ext}^i(U, V).$$

Its symmetrization is denoted $(\cdot, \cdot)$. With the choices $R = \mathbb{Q}[\nu, \nu^{-1}]$, $\nu = \sqrt{q}^{-1} \in \mathbb{R}$ and $c = -\langle \cdot, \cdot \rangle$, $\mathcal{H}_{\mathcal{A}}$ is called the Ringel-Hall algebra of $\mathcal{A}$.

The following fundamental result asserts the compatibility of the product and coproduct when $\mathcal{A}$ is hereditary, i.e. of homological dimension at most one.

Theorem 1.1 ([10]). *Let $\mathcal{H}_{\mathcal{A}}$ be the Ringel-Hall algebra of a hereditary abelian category $\mathcal{A}$. Equip $\mathcal{H}_{\mathcal{A}} \hat{\otimes} \mathcal{H}_{\mathcal{A}}$ with the algebra structure given on homogeneous elements $x, y, z, w \in \mathcal{H}_{\mathcal{A}}$ by*

$$(x \otimes y)(z \otimes w) = \nu^{-(y,z)} xz \otimes yw.$$

Then $\Delta : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{A}} \hat{\otimes} \mathcal{H}_{\mathcal{A}}$ is an algebra homomorphism.

Finally, in [10] an R-valued non-degenerate symmetric bilinear form on $\mathcal{H}_{\mathcal{A}}$ is defined by $([U], [V])_{\mathcal{H}} = \dfrac{\delta_{U,V}}{a(U)}$. This form satisfies

$$(x \otimes y, \Delta z)_{\mathcal{H} \otimes \mathcal{H}} = (xy, z)_{\mathcal{H}}, \quad x, y, z \in \mathcal{H}_{\mathcal{A}}$$

where $(x \otimes y, x' \otimes y')_{\mathcal{H} \otimes \mathcal{H}} = (x, x')_{\mathcal{H}} (y, y')_{\mathcal{H}}$.

The category $Rep_{\mathbb{F}_q}(Q)$ satisfies the assumptions of Theorem 1.1. Its Hall algebra will be discussed in Section 3. We describe below a second example only briefly. The reader is referred to [22] for detailed examples of Hall algebras.

Example. Let X be a smooth projective curve defined over $\mathbb{F}_q$. Theorem 1.1 applies to the category of coherent sheaves over X. The full subcategory of vector bundles defines a subalgebra $\mathcal{H}_{Vect_X} \subset \mathcal{H}_{Coh_X}$. The adèlic description of the stack of vector bundles over X shows that $\mathcal{H}_{Vect_X}$ consists of the unramified automorphic forms for GL defined over $\mathbb{F}_q(X)$, with multiplication given by the parabolic Eisenstein series map. Incorporating torsion sheaves gives a Hall algebraic realization of Hecke operators. The quadratic identities satisfied by cusp eigenforms become quadratic relations in $\mathcal{H}_{Coh_X}$ [13]. These relations imply that $\mathcal{H}_{Coh_{\mathbb{P}^1}}$ is isomorphic to the semidirect product of the Hall algebra of torsion sheaves (a tensor product of classical Hall algebras) with a negative part of the quantum affine algebra $U_{\sqrt{q}^{-1}}(\hat{\mathfrak{sl}}_2)$ [13]. In higher genus the quadratic relations no longer determine $\mathcal{H}_{Coh_X}$. See [23], [14] for higher genus results. ◁

2. The Hall module of an exact category with duality

2.1. Exact categories with duality.

A basic reference for exact categories with duality is [2].

Definition. *(1) An exact category with duality is a triple $(\mathcal{A}, S, \Theta)$ consisting of an exact category $\mathcal{A}$, an exact contravariant functor $S : \mathcal{A} \to \mathcal{A}$ and an isomorphism $\Theta : 1_{\mathcal{A}} \xrightarrow{\sim} S^2$ satisfying $S(\Theta_U)\Theta_{S(U)} = 1_{S(U)}$ for all $U \in \mathcal{A}$.*
(2) A self-dual object of $(\mathcal{A}, S, \Theta)$ is an object $N \in \mathcal{A}$ together with an isomorphism $\psi_N : N \xrightarrow{\sim} S(N)$ satisfying $S(\psi_N)\Theta_N = \psi_N$.

The notation $(N, \psi_N) \in \mathcal{A}_S$, or sometimes just $N \in \mathcal{A}_S$ if ψ_N is understood, indicates that (N, ψ_N) is a self-dual object. We also sometimes refer to $\mathcal{A}$, instead of $(\mathcal{A}, S, \Theta)$, as an exact category with duality. We say that $N, N' \in \mathcal{A}_S$ are isometric, notation $N \simeq_S N'$, if there exists an isomorphism $\phi : N \xrightarrow{\sim} N'$ satisfying $S(\phi)\psi_{N'}\phi = \psi_N$. The set of isometry classes of self-dual objects $Iso(\mathcal{A}_S)$ is an abelian monoid under orthogonal direct sum.

Example. Let X be a smooth variety defined over a field k and write $Vect_X$ for the exact category of vector bundles over X. Given $s \in \{\pm 1\}$ and a line bundle $\mathcal{L} \to X$, define a duality functor on $Vect_X$ by $S(\mathcal{V}) = \mathcal{V}^\vee \otimes_{\mathcal{O}_X} \mathcal{L}$, where $\mathcal{V}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{V}, \mathcal{O}_X)$. Let $\Theta_{\mathcal{V}} : \mathcal{V} \xrightarrow{\sim} \mathcal{V}^{\vee\vee}$ be the signed evaluation map, given at the level of sections by $\Theta(f)(x) = s \cdot f(x)$, $x \in X$. Then $(Vect_X, S, \Theta)$ is an exact category with duality, the self-dual objects being $\mathcal{L}$-valued orthogonal or symplectic vector bundles over X. ◁

Definition. *An isotropic subobject of $N \in \mathcal{A}_S$ is an admissible monic $U \overset{i}{\rightarrowtail} N$ such that $S(i)\psi_N i = 0$ and the induced monic $U \hookrightarrow U^\perp := \ker S(i)\psi_N$ is admissible.*

The notation $U \overset{\perp}{\subset} N$ indicates that U is isotropic in N. We will use the following categorical version of orthogonal and symplectic reduction by isotropic subobjects.

Proposition 2.1 ([17, Proposition 5.2]). *If $i : U \rightarrowtail N$ is isotropic, then there exists a self-dual structure $\tilde{\psi}$ on $N//U := U^\perp/U$, unique up to isometry, making the following exact diagram commute:*

$$\begin{array}{ccccc}
U & \overset{j}{\rightarrowtail} & E & \overset{\pi}{\twoheadrightarrow} & N//U \\
\| & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle S(\pi)\tilde{\psi}} \\
U & \overset{i}{\rightarrowtail} & N & \overset{S(k)\psi_N}{\twoheadrightarrow} & S(E) \\
 & & \downarrow{\scriptstyle S(i)\psi_N} & & \downarrow{\scriptstyle S(j)} \\
 & & S(U) & = & S(U)
\end{array} \tag{3}$$

Here $E \overset{k}{\rightarrowtail} N$ is a kernel of $S(i)\psi_N$ and π is a cokernel for the induced monic $U \overset{j}{\rightarrowtail} E$.

Motivated by Proposition 2.1, we make the following definition.

Definition. *Given $U \in \mathcal{A}$, $M, N \in \mathcal{A}_S$, let $\underline{\mathcal{G}}_{U,M}^N$ be the set of equivalence classes of exact commutative diagrams $(E; i, j, k, \pi)$ of the form (3), with $(N//U, \tilde{\psi})$ replaced with (M, ψ_M). Two such diagrams E, E', are equivalent if there exists an isomorphism $E \xrightarrow{\sim} E'$ making all appropriate diagrams commute.*

Elements of $\underline{\mathcal{G}}_{U,M}^N$ are called self-dual exact sequences and written $U \overset{i}{\rightarrowtail} N \overset{\pi}{\dashrightarrow\!\!\!\!\rightarrow} M$.

2.2. **Hall modules.** Let $\mathcal{A}$ be a finitary exact category with duality. For $U \in \mathcal{A}$ and $M, N \in \mathcal{A}_S$ define the self-dual Hall number by

$$G_{U,M}^N = |\{\tilde{U} \overset{\perp}{\subset} N \mid \tilde{U} \simeq U, \;\; N//\tilde{U} \simeq_S M\}|. \tag{4}$$

Let $\mathcal{G}_{U,M}^N = |\underline{\mathcal{G}}_{U,M}^N|$ and $a_S(M) = |Aut_S(M)|$, the number of isometries of M.

Lemma 2.2. *The equality $G_{U,M}^N = \dfrac{\mathcal{G}_{U,M}^N}{a(U)a_S(M)}$ holds.*

Proof. The group $Aut(U) \times Aut_S(M)$ acts freely on $\underline{\mathcal{G}}_{U,M}^N$ by

$$(g, h) \cdot (E; i, j, k, \pi) = (E; ig^{-1}, jg^{-1}, k, h\pi), \quad (g, h) \in Aut(U) \times Aut_S(M).$$

The map $(E; i, j, k, \pi) \mapsto \mathrm{im}(i)$ induces a bijection from $\underline{\mathcal{G}}_{U,M}^N / Aut(U) \times Aut_S(M)$ to the right-hand side of equation (4). $\square$

Lemma 2.3. *For fixed $U \in \mathcal{A}$ and $M \in \mathcal{A}_S$, the set $\underline{\mathcal{G}}_{U,M}^N$ is non-empty for at most finitely many $N \in Iso(\mathcal{A}_S)$.*

Proof. As $\mathcal{A}$ is finitary, only finitely many isomorphism types of E, and hence N, can appear in the diagram (3). By Hom-finiteness, any such N admits at most finitely many self-dual structures and the statement follows. $\square$

Let $\mathcal{M}_{\mathcal{A}}$ be the free R-module with basis $Iso(\mathcal{A}_S)$,

$$\mathcal{M}_{\mathcal{A}} = \bigoplus_{M \in Iso(\mathcal{A}_S)} R[M].$$

The next result defines the Hall module of $\mathcal{A}$. For now, take $c = 0$ in equation (2).

Theorem 2.4. *The formulae*

$$[U] \star [M] = \sum_{N \in Iso(\mathcal{A}_S)} G_{U,M}^N [N]$$

and

$$\rho[N] = \sum_{U \in Iso(\mathcal{A})} \sum_{M \in Iso(\mathcal{A}_S)} \frac{a(U)a_S(M)}{a_S(N)} G_{U,M}^N [U] \otimes [M]$$

make $\mathcal{M}_{\mathcal{A}}$ a left $\mathcal{H}_{\mathcal{A}}$-module and topological left $\mathcal{H}_{\mathcal{A}}$-comodule, respectively.

Proof. Lemmas 2.2 and 2.3 imply that the above formulae are well-defined. A direct calculation shows that the $\mathcal{H}_{\mathcal{A}}$-action is associative if and only if

$$\sum_{W \in Iso(\mathcal{A})} F_{U,V}^W G_{W,M}^N = \sum_{P \in Iso(\mathcal{A}_S)} G_{U,P}^N G_{V,M}^P, \quad U, V \in \mathcal{A}, \quad M, N \in \mathcal{A}_S. \tag{5}$$

Interpreting this equation in terms of isotropic filtrations shows that it is equivalent to a self-dual version of the Second Isomorphism theorem. Precisely, this says that for fixed $U \overset{\perp}{\subset} N$, the map $V \mapsto V/U$ gives a bijection

$$\{V \overset{\perp}{\subset} N \mid U \subset V\} \longleftrightarrow \{\tilde{V} \overset{\perp}{\subset} N//U\}$$

satisfying $(N//U)//(V/U) \simeq_S N//V$. When $\mathcal{A}$ is abelian, this is proved in [17, Proposition 6.5]. The same argument applies to exact categories once admissibility of all subobjects involved is verified, which is straightforward.

Turning to coassociativity, we first show that the composition

$$(1 \otimes \rho) \circ \rho : \mathcal{M}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{A}} \hat{\otimes} \mathcal{H}_{\mathcal{A}} \hat{\otimes} \mathcal{M}_{\mathcal{A}}$$

is well-defined, the completion consisting of all formal linear combinations; the map $(\Delta \otimes 1) \circ \rho$ is dealt with similarly. For any $\xi \in \mathcal{M}_{\mathcal{A}}$, the terms of $\rho(\xi)$ contributing to the coefficient of $[U_1] \otimes [U_2] \otimes [M]$ in $(1 \otimes \rho) \circ \rho(\xi)$ are proportional to $[U_1] \otimes [N]$ where $N//U_2 \simeq_S M$. The number of such terms is finite by Lemma 2.3. A direct calculation now shows that coassociativity is equivalent to equation (5). □

We now introduce a generalized grading on the Hall module. Recall that an object $N \in \mathcal{A}_S$ is called metabolic if it contains a Lagrangian, an isotropic subobject U with $U^{\perp} = U$. For example, the hyperbolic object $H(U)$ on any $U \in \mathcal{A}$,

$$H(U) = \Big(U \oplus S(U), \left(\begin{smallmatrix} 0 & 1_{S(U)} \\ \Theta_U & 0 \end{smallmatrix}\right)\Big) \in \mathcal{A}_S,$$

is metabolic.

Definition (See [2])**.** *(1) The Grothendieck-Witt group $GW(\mathcal{A})$ is the Grothendieck group of $Iso(\mathcal{A}_S)$ modulo the relation $|N| = |H(U)|$ whenever U is a Lagrangian in N.*
(2) The Witt group $W(\mathcal{A})$ is the abelian monoid $Iso(\mathcal{A}_S)$ modulo the submonoid of metabolic objects.

The map $U \mapsto H(U)$ extends to a group homomorphism $H : K(\mathcal{A}) \to GW(\mathcal{A})$. The groups $GW(\mathcal{A})$ and $W(\mathcal{A})$ give two R-module decompositions of $\mathcal{M}_{\mathcal{A}}$,

$$\mathcal{M}_{\mathcal{A}} = \bigoplus_{\gamma \in GW(\mathcal{A})} \mathcal{M}_{\mathcal{A}}(\gamma), \qquad \mathcal{M}_{\mathcal{A}} = \bigoplus_{w \in W(\mathcal{A})} \mathcal{M}_{\mathcal{A}}(w),$$

where $\mathcal{M}_{\mathcal{A}}(\gamma)$ is spanned by self-dual objects of class $\gamma \in GW(\mathcal{A})$, and similarly for $\mathcal{M}_{\mathcal{A}}(w)$.

Proposition 2.5. *The homomorphism H intertwines the $K(\mathcal{A})$ and $GW(\mathcal{A})$-gradings of $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}$: for all $\alpha \in K(\mathcal{A})$ and $\gamma \in GW(\mathcal{A})$*

$$\mathcal{H}_{\mathcal{A}}(\alpha) \star \mathcal{M}_{\mathcal{A}}(\gamma) \subset \mathcal{M}_{\mathcal{A}}(H(\alpha) + \gamma).$$

Moreover, for each $w \in W(\mathcal{A})$, $\mathcal{M}_{\mathcal{A}}(w) \subset \mathcal{M}_{\mathcal{A}}$ is an $\mathcal{H}_{\mathcal{A}}$-submodule. Analogous statements hold for the comodule structure.

Proof. The first statement follows from that fact that if $U \overset{\perp}{\subset} N$, then in $GW(\mathcal{A})$ the identity

$$|N| = |N//U| + |H(U)|$$

holds [17]. The second part now follows from the exact sequence of abelian groups

$$K(\mathcal{A}) \xrightarrow{H} GW(\mathcal{A}) \to W(\mathcal{A}) \to 0. \tag{6}$$

□

To extend Theorem 2.4 to c-twisted Hall algebras, suppose we are given a function $\tilde{c} : GW(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ satisfying, for all $\alpha, \beta \in K(\mathcal{A})$ and $\gamma \in GW(\mathcal{A})$,

$$c(\alpha, \beta) + \tilde{c}(\gamma, \alpha + \beta) = \tilde{c}(\gamma, \alpha) + \tilde{c}(\gamma + H(\alpha), \beta). \tag{7}$$

This guarantees that the twisted action

$$[U] \star [M] = \nu^{\tilde{c}(M,U)} \sum_{N \in Iso(\mathcal{A}_S)} G^N_{U,M} [N]$$

makes $\mathcal{M}_{\mathcal{A}}$ a module over the c-twisted Hall algebra. The comodule structure is similarly modified.

Since we are primarily interested in the Ringel-Hall algebra, we seek a $\tilde{c}$ compatible with $c = -\langle \cdot, \cdot \rangle$. For each $U \in \mathcal{A}$, the pair (S, Θ) generates a linear $\mathbb{Z}_2$-action on $\mathrm{Ext}^i(S(U), U)$ and we denote by $\mathrm{Ext}^i(S(U), U)^{pS}$ the subspace of symmetric $(p = 1)$ or skew-symmetric $(p = -1)$ extensions with respect to this action.

Theorem 2.6. *Let $\mathcal{A}$ be a $\mathbb{F}_q$-linear abelian category of finite homological dimension with finite Ext^i-sets. Then the function $\mathcal{E} : Iso(\mathcal{A}) \to \mathbb{Z}$ given by*

$$\mathcal{E}(U) = \sum_{i \geq 0} (-1)^i \dim_k \mathrm{Ext}^i(S(U), U)^{(-1)^{i+1} S}$$

descends to $K(\mathcal{A})$. Moreover, the function

$$\tilde{c}(M, U) = -\langle M, U \rangle - \mathcal{E}(U)$$

satisfies equation (7) *with $c = -\langle \cdot, \cdot \rangle$.*

Proof. Applying the bifunctor $\mathrm{Hom}(-,-)$ to the short exact sequence

$$0 \to U \to W \to V \to 0$$

and its dual gives six long exact sequences fitting into the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(S(U),U) & \longrightarrow & \mathrm{Hom}(S(U),W) & \longrightarrow & \mathrm{Hom}(S(U),V) \xrightarrow{-} \\
 & & \downarrow & & -\downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(S(W),U) & \longrightarrow & \mathrm{Hom}(S(W),W) & \longrightarrow & \mathrm{Hom}(S(W),V) \longrightarrow \\
 & & \downarrow & & -\downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(S(V),U) & \longrightarrow & \mathrm{Hom}(S(V),W) & \longrightarrow & \mathrm{Hom}(S(V),V) \longrightarrow \\
 & & \downarrow & & -\downarrow & & \downarrow \\
 & & \mathrm{Ext}^1(S(U),U) & \longrightarrow & \mathrm{Ext}^1(S(U),W) & \longrightarrow & \mathrm{Ext}^1(S(U),V) \longrightarrow \\
 & & \downarrow & & -\downarrow & & \downarrow \\
 & & \mathrm{Ext}^1(S(W),U) & \longrightarrow & \mathrm{Ext}^1(S(W),W) & \longrightarrow & \mathrm{Ext}^1(S(W),V) \xrightarrow{-} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

The minus signs, indicating that negatives of the canonical maps are taken, ensure that each square of the diagram anti-commutes. Consider the total complex, obtained by summing over the diagonal. Its first few terms are

$$0 \to \mathrm{Hom}(S(U),U) \to \begin{array}{c} \mathrm{Hom}(S(U),W) \\ \oplus \\ \mathrm{Hom}(S(W),U) \end{array} \to \begin{array}{c} \mathrm{Hom}(S(U),V) \\ \oplus \\ \mathrm{Hom}(S(W),W) \\ \oplus \\ \mathrm{Hom}(S(V),U) \end{array} \to \cdots$$

There is a $\mathbb{Z}_2$-action on the total complex, commuting with all differentials, defined by letting the generator act by $(-1)^i S$ on $\mathrm{Ext}^i(S(W), W)$ and by $(-1)^{i+1} S$ on the remaining ith extension groups. Viewed as a virtual $\mathbb{Z}_2$-representation, the character of the total complex is zero, implying the following relation between virtual dimensions of $\mathbb{Z}_2$-invariants:

$$0 = \mathcal{E}(U) - \langle S(U), W \rangle + \langle S(U), V \rangle + (\langle S(W), W \rangle - \mathcal{E}(W)) - \langle S(W), V \rangle + \mathcal{E}(V).$$

This can be rewritten as

$$\mathcal{E}(W) = \mathcal{E}(U) + \mathcal{E}(V) + \langle S(U), V \rangle. \tag{8}$$

The right-hand side of this equation is equal to $\mathcal{E}(U \oplus V)$, proving that $\mathcal{E}$ descends to $K(\mathcal{A})$. Finally, equation (7) is verified using equation (8). $\square$

Definition. *With $\tilde{c}$ as in Theorem 2.6, $\mathcal{M}_{\mathcal{A}}$ is called the Ringel-Hall module.*

The following analogue of Green's bilinear form will be used in Section 3.

Lemma 2.7. *The R-valued symmetric bilinear form on $\mathcal{M}_{\mathcal{A}}$ defined by $([M],[N])_{\mathcal{M}} = \frac{\delta_{M,N}}{a_S(M)}$ is non-degenerate and satisfies*

$$(x \otimes \xi, \rho(\zeta))_{\mathcal{H}\otimes\mathcal{M}} = (x \star \xi, \zeta)_{\mathcal{M}}, \quad x \in \mathcal{H}_{\mathcal{A}},\ \xi, \zeta \in \mathcal{M}_{\mathcal{A}}.$$

We now give two examples.

Given an exact category $\mathcal{A}$, the triple $(H\mathcal{A}, S_H, 1_{H\mathcal{A}})$, with $H\mathcal{A} = \mathcal{A} \times \mathcal{A}^{op}$ and $S_H(A,B) = (B,A)$, is called the hyperbolic exact category with duality; all its self-dual objects are hyperbolic. Let $\mathcal{H}_{\mathcal{A}}^{op-cop}$ be the (co)algebra obtained from $\mathcal{H}_{\mathcal{A}}$ by taking the opposite (co)multiplication. Then $\mathcal{H}_{H\mathcal{A}} \simeq \mathcal{H}_{\mathcal{A}} \otimes_R \mathcal{H}_{\mathcal{A}}^{op-cop}$.

Proposition 2.8. *The map $[X] \mapsto [H(X)]$ extends to an isomorphism $\mathcal{H}_{\mathcal{A}} \xrightarrow{\sim} \mathcal{M}_{H\mathcal{A}}$ of left $\mathcal{H}_{\mathcal{A}} \otimes_R \mathcal{H}_{\mathcal{A}}^{op-cop}$-(co)modules preserving Grothendieck-Witt gradings and Green forms.*

Proof. For simplicity set $c = \tilde{c} = 0$. The above map clearly defines an R-module isomorphism. A subobject of $H(X)$, $X \in \mathcal{A}$, is necessarily of the form $U_1 \oplus S_H(U_2)$ for some $U_1, U_2 \in \mathcal{A}$, and is isotropic if and only if $S_H(U_2) \subset S_H(X/U_1)$. Summing over isomorphism types of X/U_1 shows

$$G_{U_1 \oplus S_H(U_2), H(Y)}^{H(X)} = \sum_{W \in Iso(\mathcal{A})} F_{U_1,W}^{X} F_{Y,U_2}^{W},$$

where we have used $F_{S_H(U_2),S_H(Y)}^{S_H(W)} = F_{Y,U_2}^{W}$. This implies that $G_{U_1 \oplus S_H(U_2), H(Y)}^{H(X)}$ is the coefficient of $[X]$ in $[U_1][Y][U_2]$, all multiplication in $\mathcal{H}_{\mathcal{A}}$, and gives the $\mathcal{H}_{\mathcal{A}} \otimes_R \mathcal{H}_{\mathcal{A}}^{op}$-module isomorphism $\mathcal{H}_{\mathcal{A}} \simeq \mathcal{M}_{H\mathcal{A}}$. Using

$$Aut(U_1 \oplus S_H(U_2)) \simeq Aut(U_1) \times Aut(U_2), \quad Aut_S(H(X)) \simeq Aut(X),$$

a similar argument gives the comodule isomorphism and shows that Green forms are preserved. That the gradings are respected follows from the fact that the restriction of the hyperbolic functor to $\mathcal{A} \subset H\mathcal{A}$ induces an isomorphism $K(\mathcal{A}) \xrightarrow{\sim} GW(H\mathcal{A})$. $\square$

Example. Let $\mathcal{H}_{Vect_X}$ and $\mathcal{M}_{Vect_X}$ be the Hall algebra and module associated to a smooth projective curve X over $\mathbb{F}_q$, with duality determined by a line bundle $\mathcal{L}$. Following the automorphic interpretation of $\mathcal{H}_{Vect_X}$, $\mathcal{M}_{Vect_X}$ is identified with the space of $\mathcal{L}$-twisted unramified automorphic forms for symplectic or orthogonal groups over $\mathbb{F}_q(X)$. The Witt group of $(Vect_X, \mathcal{L}, \Theta)$ is finite [1] and therefore provides a finite $\mathcal{H}_{Vect_X}$-module decomposition of $\mathcal{M}_{Vect_X}$. As the duality does not extend to Coh_X there is no obvious Hall module interpretation of Hecke operators on $\mathcal{M}_{Vect_X}$. $\triangleleft$

2.3. **An identity for self-dual Hall numbers.** In this section we prove the following theorem.

Theorem 2.9. *Let $\mathcal{A}$ be a $\mathbb{F}_q$-linear hereditary finitary abelian category. For all $U \in \mathcal{A}$ and $M \in \mathcal{A}_S$, the following identity holds:*

$$\sum_{N \in Iso(\mathcal{A}_S)} \frac{\mathcal{G}_{U,M}^{N}}{a_S(N)} = q^{-\langle M,U \rangle - \mathcal{E}(U)}.$$

We proceed in steps. Fix $\mathfrak{s} = (E; i, j, k, \pi) \in \underline{\mathcal{G}}^N_{U,M}$. Applying $\mathrm{Hom}(S(U), -)$ to the exact sequence $\mathfrak{s}_- = (j, \pi)$ gives the long exact sequence

$$\cdots \to \mathrm{Hom}(S(U), M) \xrightarrow{\delta_-} \mathrm{Ext}^1(S(U), U) \xrightarrow{j_*} \mathrm{Ext}^1(S(U), E) \to \cdots$$

Similarly, applying $\mathrm{Hom}(-, U)$ to $\mathfrak{s}_| = (S(\pi)\psi_M, S(j))$ gives

$$\cdots \to \mathrm{Hom}(M, U) \xrightarrow{\delta_|} \mathrm{Ext}^1(S(U), U) \xrightarrow{S(j)^*} \mathrm{Ext}^1(S(E), U) \to \cdots$$

Define $\delta^S_{\mathfrak{s}} : \mathrm{Hom}(M, U) \to \mathrm{Ext}^1(S(U), U)^S$ by

$$\beta \mapsto \delta_| \beta + \delta_-(\psi_M^{-1} S(\beta)).$$

Since $\delta^S_{\mathfrak{s}}$ depends only on the class $[\mathfrak{s}_-] = \xi \in \mathrm{Ext}^1(M, U)$, we denote it by δ^S_ξ.

The set of self-dual extensions of M by U is defined to be

$${}^S\mathrm{Ext}^1(M, U) = \bigsqcup_{N \in Iso(\mathcal{A}_S)} \underline{\mathcal{G}}^N_{U,M} / Aut_S(N),$$

where $\phi \in Aut_S(N)$ acts by $\phi \cdot (E; i, j, k, \pi) = (E; \phi i, j, \phi k, \pi)$. The assignment $\mathfrak{s} \mapsto [\mathfrak{s}_-]$ defines maps $T : \underline{\mathcal{G}}^N_{U,M} \to \mathrm{Ext}^1(M, U)$ and $\tilde{T} : {}^S\mathrm{Ext}^1(M, U) \to \mathrm{Ext}^1(M, U)$.

Lemma 2.10. *If $\mathfrak{s} \in \underline{\mathcal{G}}^N_{U,M}$ satisfies $T(\mathfrak{s}) = \xi$, then*

$$|\mathrm{Stab}_{Aut_S(N)}(\mathfrak{s})| = |\ker \delta^S_\xi||\mathrm{Hom}(S(U), U)^{-S}|.$$

Proof. An element $\phi \in Aut_S(N)$ fixes $\mathfrak{s}$ if and only if there exists $r \in Aut(E)$ with

$$j = rj, \quad \pi = \pi r^{-1}, \quad \phi k = kr^{-1}.$$

The first two equations imply $r = r_\beta = 1_E - j\beta\pi$ for a unique $\beta \in \mathrm{Hom}(M, U)$, and the last equation implies

$$\phi = \phi_\tau = 1_N + k\tau S(k)\psi_N$$

for a unique $\tau \in \mathrm{Hom}(S(E), E)$. The map ϕ_τ is an isometry if and only if

$$\Theta_E^{-1} S(\tau) + \tau + \Theta_E^{-1} S(\pi\tau)\psi_M \pi\tau = 0. \tag{9}$$

Restricting ϕ_τ to E shows $j\beta = \tau S(\pi)\psi_M$. Hence τ fits into the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{S(\pi)\psi_M} & S(E) & \xrightarrow{S(j)} & S(U) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle -\psi_M^{-1}S(\beta)} & & \\
0 & \longrightarrow & U & \xrightarrow[j]{} & E & \xrightarrow[\pi]{} & M & \longrightarrow & 0
\end{array} \tag{10}$$

where we have used equation (9) to determine the map $S(U) \to M$.

Conversely, for fixed β, the existence of a lift τ is equivalent to the condition $\beta \in \ker \delta^S_\xi$, in which case the set of lifts of β forms a $\mathrm{Hom}(S(U), U)$-torsor. For such a lift τ_0, we can write $\Theta_E^{-1} S(\tau_0) + \tau_0 = j\mu S(j)$ for a unique $\mu \in \mathrm{Hom}(S(U), U)^S$. Putting

$$\tau_1 = \tau_0 - \frac{1}{2} j \left(\mu + \beta\psi_M^{-1} S(\beta)\right) S(j)$$

gives $\phi_{\tau_1} \in \mathrm{Stab}_{Aut_S(N)}(\mathfrak{s})$ lifting β. Finally, the action

$$\tilde{\mu} \cdot \phi = \phi + j\tilde{\mu} S(j), \quad \tilde{\mu} \in \mathrm{Hom}(S(U), U)^{-S}.$$

makes the set of lifts of $\beta \in \ker \delta^S_\xi$ to $\mathrm{Stab}_{Aut_S(N)}(\mathfrak{s})$ a $\mathrm{Hom}(S(U), U)^{-S}$-torsor. $\square$

Remark. The ordinary version of Theorem 2.9 is a corollary of the formula [19]

$$\frac{\mathcal{F}_{U,V}^{X}}{a(X)} = \frac{|\mathrm{Ext}^1(V,U)_X|}{|\mathrm{Hom}(V,U)|},$$

where $\mathrm{Ext}^1(V,U)_X \subset \mathrm{Ext}^1(V,U)$ are the extensions with middle term isomorphic to X. This formula follows from the fact that all elements of $\mathcal{F}_{U,V}^{X}$ have $Aut(X)$-stabilizer isomorphic to $\mathrm{Hom}(V,U)$. That $\mathrm{Stab}_{Aut_S(N)}(\mathfrak{s})$ depends on more data than just U and M complicates the proof of Theorem 2.9.

Turning to the problem of describing the fibres of $\tilde{T}$, suppose that

$$\mathfrak{t}_- : \qquad U \overset{j}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} M$$

represents a class $\xi \in \mathrm{Ext}^1(M,U)$. Since $\mathcal{A}$ is hereditary, there exists an exact commutative diagram $\mathfrak{t}$ extending $\mathfrak{t}_-$:

$$t: \qquad \begin{array}{ccccc} U & \overset{j}{\rightarrowtail} & E & \overset{\pi}{\twoheadrightarrow} & M \\ \| & & \downarrow k & & \downarrow S(\pi)\psi_M \\ U & \overset{i}{\rightarrowtail} & N & \overset{\rho}{\twoheadrightarrow} & S(E) \\ & & \downarrow \tau & & \downarrow S(j) \\ & & S(U) & = & S(U) \end{array}$$

In [11] such a diagram is called an *extension panachée* of $S(E)$ by E. The dual diagram $S(\mathfrak{t})$ (after using Θ) is another such extension panachée. By [11, §9.3.8.b] there exists a unique $\gamma_{\mathfrak{t}} \in \mathrm{Ext}^1(S(U),U)$ such that $S(\mathfrak{t})$ and $\mathfrak{t}\bullet\gamma_{\mathfrak{t}}$ are isomorphic extensions panachées. Here $\bullet$ is the simply transitive action of $\mathrm{Ext}^1(S(U),U)$ on isomorphism classes of extensions panachées of $S(E)$ by E. Precisely, $\mathfrak{t}\bullet\gamma_{\mathfrak{t}}$ is the canonical lift of the Baer sum $N + j_*\gamma_{\mathfrak{t}} \in \mathrm{Ext}^1(S(U),E)$ to an extension panachée of $S(E)$ by E.

Lemma 2.11. *There exists a self-dual structure ψ_N on N satisfying $\rho = S(k)\psi_N$ (i.e. $\mathfrak{t} \in \underline{\mathcal{G}}_{U,M}^{N}$) if and only if $\gamma_{\mathfrak{t}} = 0$. Moreover, such a self-dual structure is unique up to isometry.*

Proof. The implication is clear. Conversely, if $\gamma_{\mathfrak{t}} = 0$ then $\mathfrak{t} \simeq S(\mathfrak{t})$, i.e. there exists an isomorphism $\psi : N \to S(N)$ satisfying $\psi k = S(\rho)\Theta_E$ and $S(k)\psi = \rho$. These equations imply there exists a unique $\mu \in \mathrm{Hom}(S(U),U)^{-S}$ such that

$$S(\psi)\Theta_N - \psi = S(\tau)\Theta_U \mu\tau.$$

Then $\psi(1_N + \frac{1}{2}i\mu\tau)$ is the desired self-dual structure. □

Lemma 2.12 (See also [4, Lemme 3])**.** *The class $\gamma_{\mathfrak{t}}$ satisfies $\gamma_{\mathfrak{t}} + \Theta_{U*}^{-1}S(\gamma_{\mathfrak{t}}) = 0$.*

Proof. We show that $S(\mathfrak{t})\bullet\Theta_{U*}^{-1}S(\gamma_{\mathfrak{t}}) = \mathfrak{t}$. The definition of $\gamma_{\mathfrak{t}}$ and the freeness of the $\mathrm{Ext}^1(S(U),U)$-action then imply the lemma. According to [4, Lemme A.1], $S(\mathfrak{t})\bullet\Theta_{U*}^{-1}S(\gamma_{\mathfrak{t}})$ can also be described as the lift of $S(N) + S(j)^*\Theta_{U*}^{-1}S(\gamma_{\mathfrak{t}}) \in \mathrm{Ext}^1(S(E),U)$ to an extension panachée. Since $S(N) + S(j)^*\Theta_{U*}^{-1}S(\gamma_{\mathfrak{t}})$ is also the middle horizontal exact sequence of $S(\mathfrak{t}\bullet\gamma_{\mathfrak{t}})$ the claim follows. □

The proof of [4, Théorème 1] shows that for each $\lambda \in \mathrm{Ext}^1(S(U),U)$ we have

$$\gamma_{\mathfrak{t}\bullet\lambda} = \gamma_{\mathfrak{t}} + \lambda - \Theta_{U*}^{-1}S(\lambda).$$

Together with Lemma 2.12, this implies $\gamma_{\mathfrak{t}\bullet(-\frac{1}{2}\gamma_{\mathfrak{t}})} = 0$, and so by Lemma 2.11 the diagram $\mathfrak{t}\bullet(-\frac{1}{2}\gamma_{\mathfrak{t}})$ extends to an element of $\underline{\mathcal{G}}_{U,M}^N$. In particular, $\tilde{T}^{-1}(\xi)$ is non-empty.

Lemma 2.13. *The action of* $\mathrm{Ext}^1(S(U),U)^S$ *on* $\tilde{T}^{-1}(\xi)$ *is transitive with stabilizer* $\mathrm{im}\,\delta_\xi^S$. *In particular,*

$$|\tilde{T}^{-1}(\xi)| = \frac{|\mathrm{Ext}^1(S(U),U)^S|}{|\,\mathrm{im}\,\delta_\xi^S|}.$$

Proof. Transitivity can be verified as in [4, Théorèm 1]. Consider the diagrams $\mathfrak{t}$ and $\mathfrak{t}\bullet\delta_\xi^S\beta = (\mathfrak{t}\bullet\delta_-\beta)\bullet\delta_|\psi_M^{-1}S(\beta)$. By [4, Lemme A.2], $\mathfrak{t}\bullet\delta_-\beta$ is obtained from $\mathfrak{t}$ by replacing k with kr_β^{-1}. Similarly, $\mathfrak{t}\bullet\delta_\xi^S\beta$ is obtained from $\mathfrak{t}\bullet\delta_-\beta$ by replacing $S(\rho)$ with $S(r_\beta^{-1})S(\rho)$. In all, $\mathfrak{t}$ and $\mathfrak{t}\bullet\delta_\xi^S\beta$ differ by an automorphism of $\mathfrak{t}_-$ and its induced action on $S(\mathfrak{t}_-)$. Therefore, $\mathfrak{t}$ and $\mathfrak{t}\bullet\delta_\xi^S\beta$ are equal as self-dual exact sequences and $\mathrm{im}\,\delta_\xi^S$ acts trivially.

On the other hand, suppose that $\mathfrak{t},\mathfrak{t}'\in\mathcal{G}_{U,M}^N$ are equal in ${}^S\mathrm{Ext}^1(M,U)$. Without loss of generality, we can assume $\mathfrak{t}_- = \mathfrak{t}'_-$. Then there exists $\phi\in Aut_S(N)$ satisfying $\phi k = k' r_\beta^{-1}$ for some $\beta\in Hom(M,U)$. Replacing k' with $k'r_\beta^{-1}$ gives the same self-dual exact sequence but a different extension panachée, namely $\mathfrak{t}'\bullet\delta_\xi^S\beta$. The map ϕ is now an isomorphism of extensions panachées $\mathfrak{t}\simeq\mathfrak{t}'\bullet\delta_\xi^S\beta$. In particular, if $\mathfrak{t}=\mathfrak{t}\bullet\gamma$ in ${}^S\mathrm{Ext}^1(M,U)$, then $\gamma\in\mathrm{im}\,\delta_\xi^S$. □

Proof of Theorem 2.9. We compute using Burnside's lemma

$$\begin{aligned}
\sum_{N\in Iso(\mathcal{A}_S)}\frac{\mathcal{G}_{U,M}^N}{a_S(N)} &= \sum_{[\mathfrak{s}]\in{}^S\mathrm{Ext}^1(M,U)}\frac{1}{|\mathrm{Stab}_{Aut_S(N)}(\mathfrak{s})|}\\
&= \sum_{\xi\in\mathrm{Ext}^1(M,U)}\frac{|\tilde{T}^{-1}(\xi)|}{|\ker\delta_\xi^S||\mathrm{Hom}(S(U),U)^{-S}|} && \text{(Lemma 2.10)}\\
&= \sum_{\xi\in\mathrm{Ext}^1(M,U)}\frac{|\mathrm{Ext}^1(S(U),U)^S|}{|\,\mathrm{im}\,\delta_\xi^S||\ker\delta_\xi^S||\mathrm{Hom}(S(U),U)^{-S}|} && \text{(Lemma 2.13)}\\
&= \sum_{\xi\in\mathrm{Ext}^1(M,U)}\frac{|\mathrm{Ext}^1(S(U),U)^S|}{|\mathrm{Hom}(M,U)||\mathrm{Hom}(S(U),U)^{-S}|}\\
&= q^{-\langle M,U\rangle-\mathcal{E}(U)}.
\end{aligned}$$

□

3. Hall modules from quivers with involution

For the remainder of the paper we assume that Hall algebras and modules are given the Ringel twist.

3.1. Quantum groups and the Hall algebra of a quiver.

Let $A=(a_{ij})_{i,j=1}^n$ be a symmetric generalized Cartan matrix with associated derived Kac-Moody algebra $\mathfrak{g}$. The root lattice $\Phi=\bigoplus_{i=1}^n\mathbb{Z}\epsilon_i$ is generated by the simple roots $\epsilon_1,\dots,\epsilon_n$. The Cartan form $(-,-):\Phi\times\Phi\to\mathbb{Z}$ satisfies $(\epsilon_i,\epsilon_j)=a_{ij}$.

Let $\mathbb{Q}(v)$ be the field of rational functions in an indeterminate v. Define

$$[n]_v=\frac{v^n-v^{-n}}{v-v^{-1}},\qquad [n]_v! = \prod_{i=1}^n [i]_v,\qquad \left[\begin{array}{c} n\\ k\end{array}\right]_v = \frac{[n]_v!}{[k]_v![n-k]_v!},\qquad n,k\in\mathbb{Z}_{\geq 0}.$$

Definition. *The quantum Kac-Moody algebra $U_v(\mathfrak{g})$ is the $\mathbb{Q}(v)$-algebra generated by symbols E_i, F_i, T_i, T_i^{-1}, for $i = 1, \ldots, n$, subject to the relations*

1. $[T_i, T_j] = 0$ *and* $T_i T_i^{-1} = 1$ *for* $i = 1, \ldots, n$.
2. $T_i E_j T_i^{-1} = v^{(\epsilon_i, \epsilon_j)} E_j$ *and* $T_i F_j T_i^{-1} = v^{-(\epsilon_i, \epsilon_j)} F_j$ *for* $i, j = 1, \ldots, n$.
3. $[E_i, F_j] = \delta_{ij} \frac{T_i - T_i^{-1}}{v - v^{-1}}$ *for* $i, j = 1, \ldots, n$.
4. *(quantum Serre relations) For any* $i, j = 1, \ldots, n$, *with* $i \neq j$,

$$\sum_{p=0}^{a} (-1)^p \left[\begin{array}{c} a \\ p \end{array} \right]_v F_i^p F_j F_i^{a-p} = 0, \qquad \sum_{p=0}^{a} (-1)^p \left[\begin{array}{c} a \\ p \end{array} \right]_v E_i^p E_j E_i^{a-p} = 0$$

where $a = 1 - (\epsilon_i, \epsilon_j)$.

Let $U_v^-(\mathfrak{g})$ be the subalgebra of $U_v(\mathfrak{g})$ generated by F_i, $i = 1, \ldots, n$. For $\nu \in \mathbb{C}^\times$ not a root of unity, the specialized quantum groups $U_\nu(\mathfrak{g}), U_\nu^-(\mathfrak{g})$ are the $\mathbb{Q}[\nu, \nu^{-1}]$-algebras with generators and relations as above but with v replaced with ν.

We now recall the connection between quantum groups and Hall algebras of quivers. Consider a quiver Q with finite sets of nodes Q_0 and arrows Q_1 together with head and tail maps $h, t : Q_1 \to Q_0$. A k-representation of Q is a pair $(V, v = \{v_\alpha\}_{\alpha \in Q_1})$ where $V = \bigoplus_{i \in Q_0} V_i$ is a finite dimensional Q_0-graded k-vector space and $V_{t(\alpha)} \xrightarrow{v_\alpha} V_{h(\alpha)}$ is a linear map. The category $Rep_k(Q)$ of k-representations is abelian and hereditary. The abelian group $\mathbb{Z}^{Q_0}$ of virtual dimension vectors has a natural basis $\{\epsilon_i\}_{i \in Q_0}$ consisting of unit vectors supported at $i \in Q_0$. The simple representation with dimension vector ϵ_i and all structure maps zero is denoted by S_i.

If Q has no loops, then its symmetrized Euler form in the basis $\{\epsilon_i\}_{i \in Q_0}$ is a generalized Cartan matrix. Denote by $\mathfrak{g}_Q$ the corresponding derived Kac-Moody algebra and let $\mathcal{H}_Q$ be the Hall algebra of $Rep_{\mathbb{F}_q}(Q)$.

Theorem 3.1 ([21], [10])**.** *If Q has no loops, then the subalgebra of $\mathcal{H}_Q$ generated by $[S_i]$, for all $i \in Q_0$, is isomorphic to $U_\nu^-(\mathfrak{g}_Q)$.*

3.2. Representations of a quiver with involution.

Definition. *An involution of Q is a pair of involutions $Q_i \xrightarrow{\sigma} Q_i$, $i = 0, 1$, such that for all $\alpha \in Q_1$, $h(\sigma(\alpha)) = \sigma(t(\alpha))$ and if $\sigma(t(\alpha)) = h(\alpha)$ then $\sigma(\alpha) = \alpha$.*

Not every quiver admits an involution. For example, the only simply laced Dynkin quivers that admit involutions are of type A. On the other hand, the double of a quiver always admits an involution.

Let (Q, σ) be a quiver with involution. To construct a duality on $Rep_k(Q)$, let ι be an involutive field automorphism of k with fixed subfield k_0. Fix also functions $s : Q_0 \to \{\pm 1\}$ and $\tau : Q_1 \to \{\pm 1\}$ satisfying $s_i = s_{\sigma(i)}$ and $\tau_\alpha \tau_{\sigma(\alpha)} = s_i s_j$ for all $i \xrightarrow{\alpha} j$. The functor $S : Rep_k(Q) \to Rep_k(Q)$ is defined by setting $S(U, u)$ to be

$$S(U)_i = \overline{U}_{\sigma(i)}, \qquad S(u)_\alpha = \tau_\alpha u_{\sigma(\alpha)}^\vee,$$

where

$$\overline{U}_{\sigma(i)} = \{ f \in \mathrm{Hom}_{k_0}(U_{\sigma(i)}, k) \mid f(cv) = \iota(c) f(v), \;\; v \in U_{\sigma(i)}, \;\; c \in k \}.$$

Given a morphism $U \xrightarrow{\phi} U'$, the components of $S(U') \xrightarrow{S(\phi)} S(U)$ are $S(\phi)_i = \phi_{\sigma(i)}^\vee$. Put $\Theta = \bigoplus_{i \in Q_0} s_i \cdot \overline{\mathrm{ev}}_i$, where $\overline{\mathrm{ev}}$ is the composition of the evaluation map with ι. Then $(Rep_k(Q), S, \Theta)$ is a k_0-linear abelian category with duality.

Geometrically, a self-dual structure ψ_M defines a non-degenerate form on M by $\langle v, w \rangle = \psi_M(v)(w)$ that is linear in the first variable and ι-linear in the second variable. Moreover, M_i and M_j are orthogonal unless $i = \sigma(j)$., in which case the restriction of the form to $M_i + M_{\sigma(i)}$ is s_i-symmetric (resp. s_i-hermitian) if ι is trivial (resp. non-trivial). Finally, the structure maps satisfy

$$\langle m_\alpha v, w\rangle - \tau_\alpha \langle v, m_{\sigma(\alpha)} w\rangle = 0, \qquad v \in M_{t(\alpha)},\ w \in M_{\sigma(h(\alpha))}.$$

When ι is the identity, $\tau = -1$ and s is constant, self-dual objects are called orthogonal and symplectic representations (referred to as the $s = 1$ and -1 cases, respectively) and were originally introduced by Derksen-Weyman [6]. When ι is non-trivial and $(s, \tau) = (1, -1)$ self-dual objects are called unitary representations (referred to as the $s = 0$ case). In particular, if $k = \mathbb{F}_q$ then q must be a perfect square. We then regard $Rep_{\mathbb{F}_q}(Q)$ as a $\mathbb{F}_{\sqrt{q}}$-linear category, so that $R = \mathbb{Q}[\nu_0, \nu_0^{-1}]$ with $\nu_0 = \sqrt[4]{q}^{-1}$ in the definition of $\mathcal{H}_Q$. We also rescale $\mathcal{E}$ by a factor of $\frac{1}{2}$. While $\mathcal{E}$ is then only half-integral, the quantity $\nu^{-\mathcal{E}(U)} = \sqrt{q}^{\mathcal{E}(U)}$ is integral.

Example. The quiver $\bullet\!\longrightarrow\!\bullet$ has a unique involution, swapping nodes and fixing the arrow. An orthogonal representation is a skew-symmetric map $V \to V^\vee$. Isometry classes of orthogonal representations are parameterized by $\Lambda^2 k^n / GL_n$. $\triangleleft$

Example. Consider the Jordan quiver  with the trivial involution. A symplectic representation consists of a symplectic vector space M and $m \in \mathfrak{sp}(M)$. Isometry classes of symplectic representations are parametrized by $\mathfrak{sp}_{2n}/Sp_{2n}$. $\triangleleft$

Example. For any quiver Q let Q^{op} be the quiver obtained by reversing the orientations of all arrows of Q. Then $Q^{\sqcup} = Q \sqcup Q^{op}$ has an involution that sends a node (arrow) of Q to the corresponding node (arrow) of Q^{op}. For any duality (S, Θ) on $Rep_{\mathbb{F}_q}(Q^{\sqcup})$, there are $\mathcal{H}_Q \otimes_R \mathcal{H}_Q^{op-cop}$-(co)module isomorphisms

$$\mathcal{M}_{Q^{\sqcup}} \simeq \mathcal{M}_{HRep(Q)} \simeq \mathcal{H}_Q.$$

Indeed, the functor $F : HRep_k(Q) \to (Rep_k(Q^{\sqcup}), S, \Theta)$ given by $F(A, B) = A \oplus S(B)$ and isomorphism $\left(\begin{smallmatrix} 0 & 1 \\ \Theta & 0 \end{smallmatrix}\right) : F \circ S_H \to S \circ F$ define an equivalence of categories with duality. This gives the first isomorphism. The second follows from Proposition 2.8. $\triangleleft$

For any quiver, $K(Rep_k(Q))$ is the free abelian group with basis the set of isomorphism classes $\mathfrak{S}$ of simple representations. Write $\mathfrak{S} = \mathfrak{S}^+ \sqcup \mathfrak{S}^S \sqcup \mathfrak{S}^-$ where $\mathfrak{S}^S$ consists of simples fixed by S and $S(\mathfrak{S}^+) = \mathfrak{S}^-$.

Proposition 3.2. *There are canonical group isomorphisms*

$$GW(Rep_k(Q)) \simeq \mathbb{Z}\mathfrak{S}^+ \oplus \bigoplus_{U \in \mathfrak{S}^S} GW(\mathcal{A}_U), \qquad W(Rep_k(Q)) \simeq \bigoplus_{U \in \mathfrak{S}^S} W(\mathcal{A}_U)$$

where $\mathcal{A}_U$ is the semisimple abelian category with duality generated by U.

Proof. Let $U \overset{i}{\rightarrowtail} N$ be a simple subrepresentation. By Schur's lemma, $S(i)\psi_N i$ is zero or an isomorphism. In the former case U is isotropic and $|N| = |H(U)| + |N/\!/U|$ in $GW(Rep_k(Q))$, while in the latter case $S(i)\psi_N i$ is a self-dual structure on U, so that $N \simeq_S U \oplus \tilde{N}$, implying $|N| = |U| + |\tilde{N}|$. As every representation has a finite composition series we can repeatedly apply the above procedure, giving the description of $GW(Rep_k(Q))$. The description of $W(Rep_k(Q))$ now follows from the exact sequence (6). $\square$

If Q is acyclic then $\mathfrak{S} = \{S_i\}_{i \in Q_0}$ and $\mathcal{A}_U \simeq Vect_k$ with duality determined by s. When $k = \mathbb{F}_q$, $GW(Vect_{\mathbb{F}_q})$ is isomorphic to $\mathbb{Z}$ (resp. $\mathbb{Z}^2$) if $s = -1, 0$ (resp.

$s = 1$) and

$$W(Vect_{\mathbb{F}_q}) \simeq \begin{cases} \{1\}, & \text{if } s = -1 \\ \mathbb{Z}_2, & \text{if } s = 0 \\ \mathbb{Z}_4, & \text{if } s = 1 \text{ and } \operatorname{char}(\mathbb{F}_q) \equiv 3 \pmod 4 \\ \mathbb{Z}_2 \times \mathbb{Z}_2, & \text{if } s = 1 \text{ and } \operatorname{char}(\mathbb{F}_q) \equiv 1 \pmod 4 \end{cases}$$

In particular, for $s = 1$, this is the classical Witt group $W(\mathbb{F}_q)$ of orthogonal forms. Note that the Grothendieck-Witt class of a representation is essentially its dimension vector, with additional decorations at σ-fixed vertices with $s_i = 1$.

We now determine explicitly the function $\mathcal{E}$. Given representations V, W, define

$$A^0(V, W) = \bigoplus_{i \in Q_0} \operatorname{Hom}_k(V_i, W_i), \qquad A^1(V, W) = \bigoplus_{i \xrightarrow{\alpha} j \in Q_1} \operatorname{Hom}_k(V_i, W_j).$$

There is a differential $A^0(V, W) \xrightarrow{\delta} A^1(V, W)$ given by $\delta\{f_i\}_i = \{w_\alpha f_i - f_j v_\alpha\}_\alpha$. The resulting complex $A^\bullet(V, W)$ fits into the exact sequence

$$0 \to \operatorname{Hom}(V, W) \to A^0(V, W) \xrightarrow{\delta} A^1(V, W) \to \operatorname{Ext}^1(V, W) \to 0. \tag{11}$$

It follows that the Euler form depends only on the dimension vectors of its arguments:

$$\langle d, d' \rangle = \sum_{i \in Q_0} d_i d'_i - \sum_{i \xrightarrow{\alpha} j} d_i d'_j, \qquad d, d' \in \mathbb{Z}^{Q_0}.$$

Proposition 3.3. *When $\mathcal{A} = Rep_k(Q)$, $\mathcal{E}(U)$ depends only on $u = \mathbf{dim}\, U$ and is given by*

$$\mathcal{E}(U) = \sum_{i \in Q_0^\sigma} \frac{u_i(u_i - s_i)}{2} + \sum_{i \in Q_0^+} u_{\sigma(i)} u_i - \sum_{(\sigma(i) \xrightarrow{\alpha} i) \in Q_1^\sigma} \frac{u_i(u_i + \tau_\alpha s_i)}{2} - \sum_{(i \xrightarrow{\alpha} j) \in Q_1^+} u_{\sigma(i)} u_j.$$

Here $Q_0 = Q_0^+ \sqcup Q_0^\sigma \sqcup Q_0^+$, where Q_0^σ consists of the σ-fixed vertices and $\sigma(Q_0^+) = Q_0^-$ and Q_1 is decomposed analogously.

Proof. Define an involution of $A^\bullet(S(U), U)$ by the composition

$$A^i(S(U), U) \xrightarrow{S} A^i(S(U), S^2(U)) \xrightarrow{\Theta_{U*}^{-1}} A^i(S(U), U).$$

This involution anticommutes with δ so that the subcomplex $B^\bullet(U)$ of (anti-)fixed points $A^0(S(U), U)^{-S} \xrightarrow{\delta} A^1(S(U), U)^S$ fits into the exact sequence

$$0 \to \operatorname{Hom}(S(U), U)^{-S} \to B^0(U) \xrightarrow{\delta} B^1(U) \to \operatorname{Ext}^1(S(U), U)^S \to 0.$$

Taking the Euler characteristic gives the claimed formula for $\mathcal{E}$. □

3.3. **$B_\sigma(\mathfrak{g}_Q)$-module structure of $\mathcal{M}_Q$.** In this section we assume that Q has no loops. Theorems 2.4 and 3.1 imply that $\mathcal{M}_Q$ is a representation of $U_\nu^-(\mathfrak{g}_Q)$. Since $\mathcal{H}_Q$ itself is a quantum Borcherds algebra [24, Theorem 1.1], $\mathcal{M}_Q$ is also a representation of a much larger quantum group. However, without a better understanding of the full structure of $\mathcal{H}_Q$ it is difficult to use this to say much about $\mathcal{M}_Q$. Instead, we focus on incorporating the comodule structure. The naïve guess that $\mathcal{M}_Q$ is a Hopf module, possibly after a twist as in Theorem 1.1, is already false for the quiver consisting of a single node and no arrows. In this section we seek a replacement of the Hopf module condition.

To begin, we recall a modification of Kashiwara's q-boson algebra. Keeping the notation of Section 3.1, suppose that σ is an involution of the set of simple roots of $\mathfrak{g}$ that preserves the Cartan form.

Definition ([8]). *The reduced σ-analogue $B_\sigma(\mathfrak{g})$ is the $\mathbb{Q}(v)$-algebra generated by symbols E_i, F_i, T_i, T_i^{-1}, for $i = 1, \dots, n$, subject to the relations*

(1) $[T_i, T_j] = 0$, $T_i T_i^{-1} = 1$ *and* $T_i = T_{\sigma(i)}$ *for* $i = 1, \dots, n$.
(2) $T_i E_j = v^{(\epsilon_j + \epsilon_{\sigma(j)}, \epsilon_i)} E_j T_i$ *and* $T_i F_j = v^{-(\epsilon_j + \epsilon_{\sigma(j)}, \epsilon_i)} F_j T_i$ *for* $i, j = 1, \dots, n$.
(3) $E_i F_j = v^{-(\epsilon_i, \epsilon_j)} F_j E_i + \delta_{i,j} + \delta_{i,\sigma(j)} T_i$ *for* $i, j = 1, \dots, n$.
(4) *quantum Serre relations for the* E_i *and* F_i.

Later we will use the following characterization of highest-weight $B_\sigma(\mathfrak{g})$-modules.

Proposition 3.4 ([8, Proposition 2.11]). *Let $\lambda \in \mathrm{Hom}(\Phi, \mathbb{Z})$ be a σ-invariant integral weight of $\mathfrak{g}$. Then there exists a $B_\sigma(\mathfrak{g})$-module $V_\sigma(\lambda)$ generated by a non-zero vector ϕ_λ such that $T_i \phi_\lambda = v^{\lambda(\epsilon_i)} \phi_\lambda$ for all $i = 1, \dots, n$ and*

$$\{x \in V_\sigma(\lambda) \mid E_i x = 0, \;\; i = 1, \dots, n\} = \mathbb{Q}(v)\phi_\lambda.$$

Moreover, $V_\sigma(\lambda)$ is irreducible and is unique up to isomorphism.

We require two straightforward variations of Proposition 3.4. The first is the extension to σ-invariant half-integral weights $\lambda \in \mathrm{Hom}(\Phi, \frac{1}{2}\mathbb{Z})$, in which case $V_\sigma(\lambda)$ is a $B_\sigma(\mathfrak{g}) \otimes_{\mathbb{Q}(v)} \mathbb{Q}(v^{\frac{1}{2}})$-module. The second is an extension to representations of generic specializations $B_\sigma(\mathfrak{g})_\nu$, the resulting modules written $V_\sigma(\lambda)_\nu$. The proof in [8] carries over directly in both cases.

Returning to Hall modules, define operators $E_i, F_i, T_i \in \mathrm{End}_R(\mathcal{M}_Q)$ as follows. See also [7]. Put

$$F_i[M] = [S_i] \star [M] = \nu^{-\langle M, S_i \rangle - \mathcal{E}(S_i)} \sum_N G^N_{S_i, M}[N].$$

and let E_i be the projection of ρ onto $[S_i] \otimes \mathcal{M}_Q \subset \mathcal{H}_Q \otimes \mathcal{M}_Q$:

$$E_i[N] = \sum_M \nu^{-\langle M, S_i \rangle - \mathcal{E}(S_i)} \frac{a(S_i) a_S(M)}{a_S(N)} G^N_{S_i, M}[M].$$

Finally,

$$T_i[M] = \nu^{-(\mathbf{dim}\, M, \epsilon_i) - \mathcal{E}(\epsilon_i) - \mathcal{E}(\epsilon_{\sigma(i)})}[M].$$

Abusing notation slightly, let

$$B_\sigma(\mathfrak{g}_Q)_{\nu_0} = B_\sigma(\mathfrak{g}_Q)_\nu \otimes_{\mathbb{Q}[\nu, \nu^{-1}]} \mathbb{Q}[\nu_0, \nu_0^{-1}].$$

If $\nu_0 = \nu$ there is no conflict of notation, but if $\nu_0 = \sqrt{\nu}$, $B_\sigma(\mathfrak{g}_Q)_{\nu_0}$ is not the specialization of $B_\sigma(\mathfrak{g}_Q)$ to ν_0. We now state the first main result of this section.

Theorem 3.5. *The operators E_i, F_i, T_i, for $i \in Q_0$, give $\mathcal{M}_Q$ the structure of a $B_\sigma(\mathfrak{g}_Q)_{\nu_0}$-module.*

Beginning of proof. The first two parts of the first relation satisfied by $B_\sigma(\mathfrak{g}_Q)$ are clear while $T_i = T_{\sigma(i)}$ because $(d, \epsilon_i) = (d, \epsilon_{\sigma(i)})$ for all σ-symmetric $d \in \mathbb{Z}^{Q_0}$. The second relation follows from the fact that F_i (resp. E_i) increases (resp. decreases) the dimension vector by $\epsilon_i + \epsilon_{\sigma(i)}$. The quantum Serre relations for F_i follow from Theorems 2.4 and 3.1. Lemma 2.7 gives

$$(F_i \xi, \zeta)_{\mathcal{M}} = \frac{1}{\nu^{-2} - 1} (\xi, E_i \zeta)_{\mathcal{M}}, \quad \xi, \zeta \in \mathcal{M}_Q.$$

The quantum Serre relations for E_i now follow from those of F_i and the non-degeneracy of $(\cdot, \cdot)_{\mathcal{M}}$. To complete the proof it remains to verify the third relation, whose proof we break into a number of parts. □

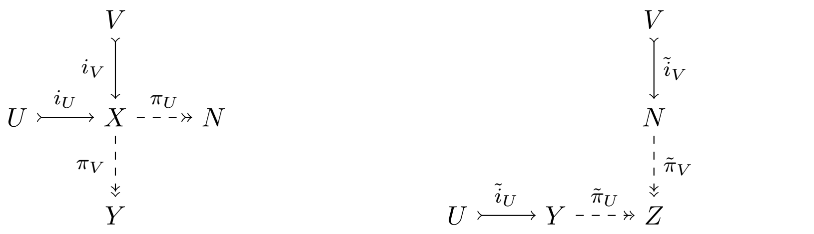

FIGURE 1. (Left) a cross diagram; (right) a corner diagram.

Using Lemma 2.2, the third relation is seen to be equivalent to the following identity, for all $i, j \in Q_0$ and self-dual representations N, Y:

$$\sum_X \frac{\mathcal{G}^X_{S_i,N}\mathcal{G}^X_{S_j,Y}}{a_S(X)} = \frac{|\mathrm{Ext}^1(S_{\sigma(j)}, S_i)|}{|\mathrm{Hom}(S_{\sigma(j)}, S_i)|}\sum_Z \frac{\mathcal{G}^Y_{S_i,Z}\mathcal{G}^N_{S_j,Z}}{a_S(Z)} + \delta_{i,\sigma(j)}\delta_{N,Y}a(S_i)a_S(N)$$

$$+ \delta_{i,j}\delta_{N,Y}a(S_i)a_S(N)\frac{|\mathrm{Ext}^1(N,S_i)||\mathrm{Ext}^1(S_{\sigma(i)},S_i)^S|}{|\mathrm{Hom}(N,S_i)||\mathrm{Hom}(S_{\sigma(i)},S_i)^{-S}|}. \tag{12}$$

We will complete the proof of Theorem 3.5 by proving this identity.

Given representations U, V and self-dual representations X, Y, N, let $C_X(U, V; N, Y)$ be the set of crosses of self-dual exact sequences, as in Figure 1. The group $Aut_S(X)$ acts on $C_X(U, V; N, Y)$ with orbit space $\tilde{C}_X(U, V; N, Y)$. Then

$$\sum_X \frac{\mathcal{G}^X_{S_i,N}\mathcal{G}^X_{S_j,Y}}{a_S(X)} = \sum_X \frac{|C_X(i,j;N,Y)|}{a_S(X)}$$

where $C_X(i, j; N, Y) = C_X(S_i, S_j; N, Y)$. Similarly, for a self-dual representation Z let $D_Z(U, V; N, Y)$ be the set of all corners of self-dual exact sequences, as in Figure 1. The group $Aut_S(Z)$ acts freely on $D_Z(U, V; N, Y)$ with orbit space $\tilde{D}_Z(U, V; N, Y)$ and the sum on the right-hand side of (12) becomes

$$\sum_Z \frac{\mathcal{G}^Y_{S_i,Z}\mathcal{G}^N_{S_j,Z}}{a_S(Z)} = \sum_Z \frac{|D_Z(i,j;N,Y)|}{a_S(Z)} = \sum_Z |\tilde{D}_Z(i,j;N,Y)|.$$

In the notation of Figure 1, if i_U and i_V present $U \oplus V$ as an isotropic subrepresentation of X, by reducing X in stages along U and V we obtain a corner on $Z = X//U \oplus V$. In this case, we say that the cross descends to this corner.

Lemma 3.6. *If $\mathcal{C} \in C_X(i, j; N, Y)$ does not descend to a corner, then $N \simeq_S Y$.*

Proof. The cross fails to descend if and only if $\mathrm{im}\, i_{S_i} + \mathrm{im}\, i_{S_j}$ is not a two dimensional isotropic subrepresentation. This occurs if $\mathrm{im}\, i_{S_i} = \mathrm{im}\, i_{S_j}$, in which case clearly $N \simeq_S Y$, or if $\mathrm{im}\, i_{S_i} + \mathrm{im}\, i_{S_j}$ is non-degenerate, in which case it is isometric to $H(S_i)$. In the latter case $X \simeq_S H(S_i) \oplus N \simeq_S H(S_i) \oplus Y$ and again $N \simeq_S Y$. □

To prove equation (12) we will show that the sum on the right-hand side counts (with weights) crosses that descend to corners while the other two terms count crosses that fail to descend for the two reasons indicated in the proof of Lemma 3.6. Since the left-hand side of (12) counts all crosses, the equation will follow.

Lemma 3.7. *There are exactly $a(U)a_S(N)$ crosses in $\tilde{C}_X(U, V; N, Y)$ such that $\mathrm{im}\, i_U$ and $\mathrm{im}\, i_V$ intersect trivially and $\mathrm{im}\, i_U \oplus \mathrm{im}\, i_V$ is non-degenerate.*

Proof. The assumptions imply $\mathrm{im}\, i_U \oplus \mathrm{im}\, i_V \simeq_S H(U)$ and $X \simeq_S H(U) \oplus N$. Acting by $Aut_S(H(U))$ and $Aut_S(N)$ (both are subgroups of $Aut_S(X)$) we may take i_U to be the standard inclusion $U \rightarrowtail H(U)$, i_V to factor through the standard inclusion

$S(U) \rightarrowtail H(U)$ and π_U to be the projection $N \oplus \operatorname{im} i_V \twoheadrightarrow N$. The set of pairs (i_V, π_V) completing the cross is a $Aut(U) \times Aut_S(N)$-torsor, with different pairs giving different classes in $\tilde{C}_X(U, V; N, Y)$. ☐

Lemma 3.8. *Let $\mathcal{C} \in C_X(U, V; N, Y)$.*
(1) If $\mathcal{C}$ descends to a corner, then $\operatorname{Stab}_{Aut_S(X)}(\mathcal{C}) \simeq \operatorname{Hom}(S(U), V)$.
(2) If $U = S_i$, $V = S_j$ and $\operatorname{im} i_U + \operatorname{im} i_V$ is non-degenerate, then $\operatorname{Stab}_{Aut_S(X)}(\mathcal{C}) = \{1\}$.

Proof. Suppose that $\mathcal{C}$ descends to a corner and let $\phi \in \operatorname{Stab}_{Aut_S(X)}(\mathcal{C})$. From the proof of Lemma 2.10, the restrictions $\phi_{|E_U}$ and $\phi_{|E_V}$ factor through maps $N \to U$ and $Y \to V$, respectively. As ϕ also stabilizes the induced self-dual exact sequence

$$0 \to U \oplus V \xrightarrow{i_U \oplus i_V} X \dashrightarrow Z \to 0,$$

the restriction of ϕ to $E_{U \oplus V} = E_U \cap E_V$ factors through a map $Z \to U \oplus V$. Compatibility with $\phi_{|E_U}$ and $\phi_{|E_V}$ requires that this map vanish. Then ϕ is uniquely determined by an element of $\operatorname{Hom}(S(U \oplus V), U \oplus V)^{-S}$. Again, compatibility with $\phi_{|E_U}$ and $\phi_{|E_V}$ imply that only the summand

$$(\operatorname{Hom}(S(U), V) \oplus \operatorname{Hom}(S(U), V))^{-S} \simeq \operatorname{Hom}(S(U), V)$$

contributes to ϕ. Reversing this argument shows that each element of $\operatorname{Hom}(S(U), V)$ gives rise to an element of $\operatorname{Stab}_{Aut_S(X)}(\mathcal{C})$. The proof of the second statement is similar. ☐

Proposition 3.9. *Exactly $|\operatorname{Ext}^1(S(U), V)|$ elements of $\bigsqcup_X \tilde{C}_X(U, V; N, Y)$ descend to each element of $\bigsqcup_Z \tilde{D}_Z(U, V; N, Y)$.*

Proof. Fix a corner as in Figure 1 and consider the pullback of $\tilde{\pi}_U$ along $\tilde{\pi}_V$:

$$\begin{array}{ccccc}
 & & V & = & V \\
 & & {\scriptstyle j'_V}\downarrow & & \downarrow{\scriptstyle \tilde{j}_V} \\
U & \xrightarrow{j'_U} & \mathbb{E} & \xrightarrow{\pi'_U} & \tilde{E}_V \\
\| & & {\scriptstyle \pi'_V}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}_V} \\
U & \xrightarrow[\tilde{j}_U]{} & \tilde{E}_U & \xrightarrow[\tilde{\pi}_U]{} & Z
\end{array}$$

From Lemma 2.13, $\operatorname{Ext}^1(S(U \oplus V), U \oplus V)^S$ acts transitively on the set of lifts of the exact sequence $U \oplus V \rightarrowtail \mathbb{E} \twoheadrightarrow Z$ to self-dual extensions $U \oplus V \rightarrowtail X \dashrightarrow Z$. Fix such a lift. After acting by $\operatorname{Ext}^1(S(U), U)^S$ and $\operatorname{Ext}^1(S(V), V)^S$, we can assume that $X//V \simeq_S Y$ and $X//U \simeq_S N$. The definition of $\mathbb{E}$ ensures that X, viewed as a cross, descends to the original corner. Consider the induced commutative diagrams

$$\begin{array}{ccccc}
U & \xrightarrow{j'_U} & \mathbb{E} & \xrightarrow{\pi'_U} & \tilde{E}_V \\
\| & & {\scriptstyle l_U}\downarrow & & \downarrow \\
U & \rightarrowtail & E_U & \xrightarrow{\pi_U} & N \\
 & & \downarrow & & \downarrow \\
 & & S(V) & = & S(V)
\end{array}
\qquad
\begin{array}{ccccc}
V & \xrightarrow{j'_V} & \mathbb{E} & \xrightarrow{\pi'_V} & \tilde{E}_U \\
\| & & {\scriptstyle l_V}\downarrow & & \downarrow \\
V & \rightarrowtail & E_V & \xrightarrow{\pi_V} & Y \\
 & & \downarrow & & \downarrow \\
 & & S(U) & = & S(U)
\end{array}$$

First, note that the only data in the left (say) diagram not determined by the corner is $(E_U; l_U, \pi_U)$ and that this data is determined only up to automorphisms of E_U. Second, since the pushout of l_U along l_V gives $\mathbb{E} \rightarrowtail X \twoheadrightarrow S(U \oplus V)$, the diagrams recover X up to isomorphism. Therefore, to count crosses that lift the corner it suffices to count the pairs $(E_U; l_U, \pi_U)$ and $(E_V; l_V, \pi_V)$ that make the above diagrams commute and that are compatible in the sense that the corresponding central term of the cross admits a self-dual structure. From [11, §9.3.8.b] (see also [10]) the set of $(E_U; l_U, \pi_U)$ making the left diagram commute, up to automorphisms of E_U, is an $\mathrm{Ext}^1(S(V), U)$-torsor. Similarly, the data for the diagram on the right is an $\mathrm{Ext}^1(S(U), V)$-torsor. Compatibility requires these group actions be dependent. Namely, only the subgroup

$$\mathrm{Ext}^1(S(U), V) \simeq (\mathrm{Ext}^1(S(V), U) \oplus \mathrm{Ext}^1(S(U), V)^S \subset \mathrm{Ext}^1(S(U \oplus V), U \oplus V)^S$$

preserves the condition that the central term of the cross be self-dual. This completes the proof. □

Completion of the proof of Theorem 3.5. Write

$$C_X(i,j;N,Y) = C_X^{(1)}(i,j;N,Y) \bigsqcup C_X^{(2)}(i,j;N,Y)$$

with $C_X^{(1)}(i,j;N,Y)$ the set of crosses that descend to corners. Burnside's lemma and the first part of Lemma 3.8 give

$$\sum_X \frac{|C_X(i,j;N,Y)|}{a_S(X)} = \sum_X \frac{|\tilde{C}_X^{(1)}(i,j;N,Y)|}{|\mathrm{Hom}(S_{\sigma(i)}, S_j)|} + \sum_X \frac{|C_X^{(2)}(i,j;N,Y)|}{a_S(X)}.$$

By Proposition 3.9 the first sum is

$$\sum_X \frac{|\tilde{C}_X^{(1)}(i,j;N,Y)|}{|\mathrm{Hom}(S_{\sigma(i)}, S_j)|} = \frac{|\mathrm{Ext}^1(S_{\sigma(j)}, S_i)|}{|\mathrm{Hom}(S_{\sigma(j)}, S_i)|} \sum_Z |\tilde{D}_Z(i,j;N,Y)|$$

while Lemma 3.7 and the second part of Lemma 3.8 give for the second sum

$$\sum_X \frac{|C_X^{(2)}(i,j;N,Y)|}{a_S(X)} = \delta_{N,Y}\delta_{i,j} a(S_i) a_S(N) \sum_X \frac{\mathcal{G}_{S_i,N}^X}{a_S(X)} + \delta_{N,Y}\delta_{i,\sigma(j)} a(S_i) a_S(N).$$

Here the crosses counted by the first (resp. second) term on the right-hand side fail to descend because $\mathrm{im}\, i_{S_i} = \mathrm{im}\, i_{S_j}$ (resp. $\mathrm{im}\, i_{S_i} + \mathrm{im}\, i_{S_j}$ is non-degenerate). Finally, using Theorem 2.9 to evaluate $\sum_X \frac{\mathcal{G}_{S_i,N}^X}{a_S(X)}$ establishes equation (12). □

We now discuss the decomposition of $\mathcal{M}_Q$ into irreducible $B_\sigma(\mathfrak{g}_Q)_{\nu_0}$-modules.

Definition. *A non-zero element $\xi \in \mathcal{M}_Q$ is called cuspidal if $E_i \xi = 0$ for all $i \in Q_0$.*

Fix a homogeneous orthogonal basis $\mathcal{C}_Q$ for the R-module of cuspidals. Given $\xi \in \mathcal{C}_Q$, define a σ-invariant weight λ_ξ by

$$\lambda_\xi(\epsilon_i) = -(\mathbf{dim}\, \xi, \epsilon_i) - \mathcal{E}(\epsilon_i) - \mathcal{E}(\epsilon_{\sigma(i)}).$$

Theorem 3.10. *The Hall module $\mathcal{M}_Q$ admits an orthogonal direct sum decomposition into irreducible highest weight $B_\sigma(\mathfrak{g}_Q)_{\nu_0}$-modules generated by elements of $\mathcal{C}_Q$:*

$$\mathcal{M}_Q = \bigoplus_{\xi \in \mathcal{C}_Q} V_\sigma(\lambda_\xi)_{\nu_0}.$$

Proof. We first show that the submodule $\langle \xi \rangle \subset \mathcal{M}_Q$ generated by $\xi \in \mathcal{C}_Q$ is isomorphic to $V_\sigma(\lambda_\xi)_{\nu_0}$. Indeed, suppose that $x \in \langle \xi \rangle$ is non-zero with $E_i x = 0$ for all $i \in Q_0$. If $x = \sum_{i \in Q_0} F_i y_i$ for some $y_i \in \langle \xi \rangle$, then

$$(x,x)_{\mathcal{M}} = \sum_{i \in Q_0} (x, F_i y_i)_{\mathcal{M}} = \frac{1}{\nu^{-2}-1} \sum_{i \in Q_0} (E_i x, y_i)_{\mathcal{M}} = 0.$$

However, writing x in the natural basis of $\mathcal{M}_Q$ as $x = \sum_M c_M[M]$ shows

$$(x,x)_{\mathcal{M}} = \sum_M \frac{c_M^2}{a_S(M)} > 0,$$

a contradiction. So, x is a scalar multiple of ξ and Proposition 3.4 implies $\langle \xi \rangle \simeq V_\sigma(\lambda_\xi)_{\nu_0}$.

Suppose now that $\xi_1, \xi_2 \in \mathcal{C}_Q$ are distinct, and hence orthogonal. It follows that $\langle \xi_1 \rangle$ and $\langle \xi_2 \rangle$ are also orthogonal, giving an inclusion

$$\bigoplus_{\xi \in \mathcal{C}_Q} V_\sigma(\lambda_\xi)_{\nu_0} \hookrightarrow \mathcal{M}_Q.$$

To prove that this is an isomorphism, note that the restriction of $(\cdot,\cdot)_{\mathcal{M}}$ to $\langle \xi \rangle$, and hence to $\bigoplus_{\xi \in \mathcal{C}_Q} V_\sigma(\lambda_\xi)_{\nu_0}$, is non-degenerate. Let $0 \neq x \in \mathcal{M}_Q$ be orthogonal to $\bigoplus_{\xi \in \mathcal{C}_Q} V_\sigma(\lambda_\xi)_{\nu_0}$ and of minimal dimension with this property. As x is not cuspidal, $E_i x \neq 0$ for some $i \in Q_0$. By the minimality assumption, $E_i x \in \bigoplus_{\xi \in \mathcal{C}_Q} V_\sigma(\lambda_\xi)_{\nu_0}$. Since $F_i E_i x \in \bigoplus_{\xi \in \mathcal{C}_Q} V_\sigma(\lambda_\xi)_{\nu}$,

$$(E_i x, E_i x)_{\mathcal{M}} = (\nu^{-2} - 1)(x, F_i E_i x)_{\mathcal{M}} = 0,$$

contradicting $E_i x \neq 0$, completing the proof. □

As a special case of Theorem 3.10, note that for any quiver with duality we have $\langle [0] \rangle \simeq V_\sigma(\lambda_{[0]})_{\nu_0}$. A geometric version of this isomorphism was obtained by Enomoto [7, Theorem 5.12] by studying perverse sheaves on the moduli stack of orthogonal representations. Moreover, a lower global basis of $V_\sigma(0)$ was obtained, giving an orthogonal analogue of Lusztig's construction of the lower global basis of $U_v^-(\mathfrak{g}_Q)$ [15]. In [27] Enomoto's approach was generalized to construct lower global bases of $V_\sigma(\lambda)$ for general λ.

A result stronger than Theorem 3.5, but valid only for the Jordan quiver, was obtained in [26]. Following Zelevinsky [28] and interpreting $\mathcal{M}_Q$ in terms of unipotent characters of classical groups, van Leeuwen constructed a ring homomorphism $\Phi : \mathcal{H}_Q \to \mathcal{H}_Q \otimes_R \mathcal{H}_Q$, third order in the Hall numbers, satisfying $\rho([U] \star [M]) = \Phi([U]) \star \rho([M])$. See also [25] for a p-adic analogue. Theorem 3.5 recovers a particular component of this Φ-twisted Hopf module structure.[1] It would be very interesting to extend this result to arbitrary (Q, σ).

4. Finite type Hall modules

4.1. Classification of self-dual representations over finite fields.

A quiver Q is called finite type if it has only finitely many isomorphism classes of indecomposable representations over any field. By [9], a connected finite type quiver is an orientation of an ADE Dynkin diagram and its indecomposables are in bijection with the positive roots of $\mathfrak{g}_Q$.

Example. Let Q be an orientation of A_{2n} or A_{2n+1}. Label the nodes $-n, \dots, n$ (omitting 0 for A_{2n}) with i and $i+1$ adjacent. The indecomposables are $\{I_{i,j}\}_{-n \leq i \leq j \leq n}$, where $I_{i,j}$ has dimension vector $\epsilon_i + \cdots + \epsilon_j$ and all intermediate structure maps the identity. ◁

Similarly, (Q, σ) is called finite type if it has only finitely many isometry classes of indecomposable self-dual representations over any field whose characteristic is not two. By [6, Theorem 3.1], (Q, σ) is finite type if and only if Q is finite type. In *loc. cit.* the authors work with orthogonal and symplectic representations but

[1] While Theorem 3.5 is stated for loopless quivers, the verification of (12) above holds without this assumption.

their proof applies to the more general dualities considered here. It follows that if (Q,σ) is finite type and not a disjoint union of quivers with involution, then Q is of Dynkin type A or $Q = Q'^{\sqcup}$ with Q' of Dynkin type ADE.

Lemma 4.1. *(1) The representation underlying a self-dual indecomposable is either indecomposable or of the form $I \oplus S(I)$ for some indecomposable I.*
(2) Let Q be finite type. If an indecomposable I does not admit a self-dual structure, then, up to isometry, $H(I)$ is the unique self-dual structure on $I \oplus S(I)$.

Proof. The first statement is given in [6, Proposition 2.7] for algebraically closed fields but the proof works without this assumption.

If Q is finite type, then there is a total order $\preceq$ on the set of indecomposables such that $\mathrm{Hom}(I,J) = \mathrm{Ext}^1(J,I) = 0$ if $J \prec I$; see [5]. Writing a self-dual structure ψ on $I \oplus S(I)$ as

$$I \oplus S(I) \xrightarrow{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} S(I) \oplus S^2(I)$$

we see that $S(a)\Theta_I = a$. If $I \simeq S(I)$, then $\mathrm{Hom}(I,S(I)) \simeq k$ and $a = 0$; otherwise a is a self-dual structure on I. Similarly $d = 0$. It is then straightforward to verify that ψ is isometric to $H(I)$. If instead $I \not\simeq S(I)$, we may assume $S(I) \prec I$. Again $a = 0$ and acting by $Aut(I)$ we may take $b = 1_{S(I)}$ and $c = \Theta_I$. Then $\begin{pmatrix} 1 & -\frac{1}{2}d \\ 0 & 1 \end{pmatrix}$ is an isometry from ψ to $H(I)$. □

For the purpose of studying Hall modules of finite type quivers it suffices to restrict attention to orthogonal, symplectic and unitary representations. Indeed, any other choice of duality is seen to be equivalent to one of these choices.

We can use Lemma 4.1 to describe the self-dual indecomposables of finite type quivers over finite fields. For $Q^{\sqcup}$ the self-dual indecomposables are in bijection with the indecomposables of Q. Indecomposable representations in type type A_{2n+1} (resp. A_{2n}) do not admit symplectic (resp. orthogonal) structures. Therefore, in these cases the self-dual indecomposables are exactly the hyperbolics $\{H(I_{i,j})\}$. For orthogonal (resp. symplectic) representations in type A_{2n+1} (resp. A_{2n}) the indecomposables $I_{-i,i}$ admit two self-dual structures, denoted by R_i^c according to the following rule. Composing the structure maps of R_i^c gives an isomorphism from the vector space attached to the $-i$th node to that of the ith node. Together with the self-dual structure of R_i^c, this gives a one dimensional orthogonal form over $\mathbb{F}_q$ whose Witt class is $c \in W(\mathbb{F}_q)$. We must replace $H(I_{-i,i})$ in the above set with the two R_i^c. Finally, $I_{-i,i}$ admits a unique unitary structure R_i and replaces $H(I_{-i,i})$ in the above set.

Example. There are six indecomposable orthogonal representations of $\bullet \to \bullet \to \bullet$:

$$H(S_1) : k \to 0 \to k, \quad H(I_{0,1}) : k \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} k^2 \xrightarrow{(0\ -1)} k,$$

$$R_0^c : 0 \to k \to 0, \quad R_1^c : k \xrightarrow{1} k \xrightarrow{-c} k.$$

The orthogonal forms on the middle nodes are hyperbolic for $H(I_{0,1})$ and have Witt index c for R_i^c. ◁

Over algebraically closed fields self-dual indecomposables of finite type quivers admit a partial interpretation in terms of root systems [6]. The next result extends this to finite fields. Denote by $\mathcal{I}_Q^{\mathfrak{g}}$ the set of dimension vectors (with multiplicity) of self-dual $\mathbb{F}_q$-indecomposables of Q.

Theorem 4.2. *Let (Q,σ) be finite type. Then $\mathcal{I}_Q^{\mathfrak{g}}$ is independent of the orientation of Q and the finite field $\mathbb{F}_q$. Precisely,*

(1) if Q is of type A_{2n}, then $\mathcal{I}_Q^{\mathfrak{o}}$ is in bijection with BC_n^+ while $\mathcal{I}_Q^{\mathfrak{sp}}$ (resp. $\mathcal{I}_Q^{\mathfrak{u}}$) surjects onto B_n^+, the short roots having fibre of cardinality three (resp. two);
(2) if Q is of type A_{2n+1}, then $\mathcal{I}_Q^{\mathfrak{sp}}$ (resp. $\mathcal{I}_Q^{\mathfrak{u}}$) is in bijection with C_{n+1}^+ (resp. B_{n+1}^+), while $\mathcal{I}_Q^{\mathfrak{o}}$ surjects onto B_{n+1}^+, the short roots having fibre of cardinality two;
(3) $\mathcal{I}_{Q^{\sqcup}}^{\mathfrak{g}}$ is in bijection with $\Delta_{\mathfrak{g}_Q}^+$.

Proof. Suppose Q is of type A_{2n}. Recall that $B_n^+ = \{\varepsilon_i \pm \varepsilon_j, \varepsilon_i \mid 0 \leq i \leq j \leq n-1\}$ and $BC_n^+ = B_n^+ \bigsqcup \{2\varepsilon_i\}_{i=0}^{n-1}$. For orthogonal representations, the bijection is given by

$$H(I_{i,j}) \mapsto \begin{cases} \varepsilon_{n-j} - \varepsilon_{n-i+1}, & \text{for } 1 \leq i \leq j \leq n \\ \varepsilon_{n-i} + \varepsilon_{n-j}, & \text{for } 1 \leq i < n \text{ and } 1 \leq j \leq n. \end{cases}$$

For symplectic (resp. unitary) representations the bijection is as above, but now R_i^c (resp. R_i) also maps to $2\varepsilon_{n-i}$.

The case of type A_{2n+1} is similar. The last part follows from Gabriel's theorem [9] and the bijection between self-dual indecomposables of $Q^{\sqcup}$ and indecomposables of Q. □

4.2. **Application to Hall modules.** A weak version of the Krull-Schmidt theorem holds for self-dual representations: a self-dual representation decomposes into an orthogonal direct sum of self-dual indecomposables. However, while the type c of a summand $R_i^{\oplus r_{i,c}}$ is well-defined, the type of its indecomposable summands may not be.

Proposition 4.3. *Let (Q,σ) and (Q',σ) be finite type quivers with involution with the same underlying graph and duality. Then the decompositions of $\mathcal{M}_Q$ and $\mathcal{M}_{Q'}$ into irreducible $B_\sigma(\mathfrak{g}_Q)_{\nu_0}$-modules coincide.*

Proof. Let $\mathrm{ch}(\mathcal{M}_Q)$ be the generating function of the ranks of the T_i-weight spaces of $\mathcal{M}_Q$. Observe that the T_i-weight of a self-dual representation depends only on its dimension vector and not on the orientation of Q. Theorem 4.2 and the weak Krull-Schmidt theorem therefore imply $\mathrm{ch}(\mathcal{M}_Q) = \mathrm{ch}(\mathcal{M}_{Q'})$. Since Q is finite type, the symmetrized Euler form (Cartan form) is non-degenerate. It follows that the weight λ subspace of $V_\sigma(\lambda)$ is rank one. From this we conclude that the characters $\{\mathrm{ch}(V_\sigma(\lambda))\}_{\lambda \in \mathrm{Hom}(\Phi, \frac{1}{2}\mathbb{Z})}$ are linearly independent. The proposition now follows. □

We first deal with those (Q,σ) admitting only hyperbolics.

Theorem 4.4. *If a duality structure on (Q,σ) admits only hyperbolic representations, then $\mathcal{M}_Q = \langle [0] \rangle \simeq V_\sigma(\lambda_{[0]})_{\nu_0}$.*

Proof. By assumption, an arbitrary self-dual representation is of the form

$$H(U) \simeq_S \bigoplus_{i=1}^{l} H(I_i)^{\oplus m_i}, \quad m_i \geq 0$$

for indecomposables I_i satisfying $I_i \not\simeq I_j$ and $I_i \not\simeq S(I_j)$ for $i \neq j$. Without loss of generality we may assume $S(I_i) \preceq I_i \prec I_{i+1} \prec \cdots \prec I_l$ for $i = 1, \dots, l$. This implies $\mathrm{Ext}^1(S(I_i), I_j) = 0$ for all $i \leq j$, and by duality, also for $i \geq j$. Hence $\mathrm{Ext}^1(S(U), U) = 0$ and we have

$$[U] \star [0] = \nu^{-\mathcal{E}(U)} G_{U,0}^{H(U)} [H(U)].$$

The equality $\mathcal{M}_Q = \langle [0] \rangle$ now follows from the fact that the Hall algebra of a finite type quiver is generated by simple representations. □

Hall modules of unitary, symplectic and orthogonal representations of A_n, A_{2n} and A_{2n+1}, respectively, are not covered by Theorem 4.4. To deal with these cases we will use the following simple fact.

Lemma 4.5. *An element $\xi \in \mathcal{M}_Q$ is cuspidal if and only if $\rho(\xi) = [0] \otimes \xi$.*

Proof. Since Q is finite type, any $[0] \neq [U] \in \mathcal{H}_Q$ is a non-trivial sum of products of simple representations. If ξ is cuspidal, Lemma 2.7 implies $(\xi, [U] \star [M])_{\mathcal{M}} = 0$ for any $[M] \in \mathcal{M}_Q$. As the coefficient of $[U] \otimes [M]$ in $\rho(\xi)$ is proportional to $(\xi, [U] \star [M])_{\mathcal{M}}$, it follows that $\rho(\xi) = [0] \otimes \xi$. The converse is trivial. □

The classical Witt group $W(\mathbb{F}_q)$ has a ring structure given by tensor product. The subset $W_{(1)} \subset W(\mathbb{F}_q)$ spanned by one dimensional orthogonal spaces is stable under this product and is isomorphic to $\mathbb{Z}_2$, which we identify with $\{1, -1\}$. Given $\underline{c} = (c_i)_{j \in J} \in W_{(1)}^J$ we write $R^{\underline{c}} = \oplus_{j \in J} R_j^{c_j}$.

Denote by $\vec{A}_{2n}$ and $\vec{A}_{2n+1}$ the Dynkin diagrams with orientation $-n \to \cdots \to n$. Together with Theorem 3.10 and Proposition 4.3, the following result completes the decomposition of finite type Hall modules into irreducible representations.

Theorem 4.6. *Homogeneous bases for the submodules of cuspidals are given as follows:*

(1) $\mathcal{C}^{\mathtt{u}}_{\vec{A}_{2n}} = \{[0]\}$ *and* $\mathcal{C}^{\mathtt{u}}_{\vec{A}_{2n+1}} = \{[0], [R_0]\}$,

(2) $\mathcal{C}^{\mathtt{sp}}_{\vec{A}_{2n}} = \{[0], \xi_1, \dots, \xi_n\}$, *where* $\xi_j = \sum_{\underline{c} \in W_{(1)}^{[1,j]}} a_{\underline{c}} [R^{\underline{c}}]$ *and* $a_{\underline{c}} = \prod_{i \text{ odd}} c_i$,

(3) $\mathcal{C}^{\mathtt{o}}_{\vec{A}_{2n+1}} = \{[0], \xi_0^b, \dots, \xi_n^b\}$, *where* $\xi_j^b = \sum_{\underline{c}} a_{\underline{c}} [R^{\underline{c}}]$, $a_{\underline{c}}$ *is as above and the sum is over all* $\underline{c} \in W_{(1)}^{[0,j]}$ *satisfying* $\sum_{i=0}^j c_i = b \in W(\mathbb{F}_q)$.

Proof. Fix the following explicit choice of total order $\prec$:

$$I_{i,j} \prec I_{k,l} \text{ if and only if } i > k \text{ or } i = k \text{ and } j \geq l.$$

Then $S(I_{i,j}) \preceq I_{i,j}$ if and only if $i + j \leq 0$.

Suppose that $N = H(U) \oplus R$ where R has no hyperbolic summands and the indecomposable summands of U are ordered as in the proof of Theorem 4.4. When $i + j \leq 0$, we can verify directly that $\text{Ext}^1(I_{-k,k}, I_{i,j}) = 0$ for all k. Since the representation underlying R is a direct sum of indecomposables of the form $I_{-k,k}$, $\text{Ext}^1(R, U) = 0$ and dually $\text{Ext}^1(S(U), R) = 0$. It follows that N is the only self-dual extension of R by U. Therefore, if $[N]$ appears with non-zero coefficient in a cuspidal, by Lemma 4.5 we must have $U = 0$.

In the unitary case homogeneous cuspidals then of the form $\xi = [\oplus_{j \in J} R_j]$. Note that $E_j \xi \neq 0$ whenever $0 \neq j \in J$. Hence, either $J = \emptyset$ or $Q = \vec{A}_{2n+1}$ and $J = \{0\}$, giving the claimed cuspidals.

Consider now a homogeneous cuspidal $\xi \in \mathcal{M}^{\mathtt{sp}}_{\vec{A}_{2n}}$. Then ξ does not contain the term $[R \oplus R_i^{\oplus 2, c}]$, with $c \neq 0$ (so that $R_i^{\oplus 2, c}$ is not hyperbolic) and R containing no R_i summand; otherwise $[S_i^{\oplus 2}] \otimes [R \oplus R_{i-1}^{\oplus 2, c}]$ would appear with non-zero coefficient in $\rho(\xi)$, contradicting Lemma 4.5. Therefore,

$$\xi = \sum_{\underline{c} \in W_{(1)}^J} a_{\underline{c}} [R^{\underline{c}}]$$

for some $J \subset [1, n]$ and $a_{\underline{c}} \in \mathbb{Q}$. Denoting by $\eta : \mathbb{F}_q^\times \to \{1, -1\}$ the quadratic character, for each $i > 0$ we have

$$R_i^c // S_i \simeq_S R_{i-1}^{\eta(-1)c}.$$

This implies that if $2 \leq i \in J$, then $i - 1 \in J$, as otherwise $E_i \xi \neq 0$. Hence $J = [1, j]$ for some $1 \leq j \leq n$. The condition $E_1 \xi = 0$ is equivalent to $a_{\underline{c}} = -a_{\underline{c}'}$ whenever $\underline{c}$ and $\underline{c}'$ agree except in their first slot. For $2 \leq i \leq j$, the condition $E_i \xi = 0$ is equivalent to $a_{\underline{c}} = -a_{\underline{c}'}$ if $\underline{c}$ and $\underline{c}'$ agree except in their $(i-1)$th and ith slots and satisfy

$$c_{i-1} + \eta(-1) c_i = \tilde{c}_{i-1} + \eta(-1) \tilde{c}_i \in W(\mathbb{F}_q).$$

It is straightforward to verify that, up to a non-zero scalar multiple, $a_{\underline{c}}$ must be as claimed.

The argument for the final case is similar. The index $b \in W(\mathbb{F}_q)$ labels the Witt summand of $\mathcal{M}^{\circ}_{\vec{A}_{2n+1}}$ in which $\langle \xi_j^b \rangle$ lies; see Proposition 2.5. □

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