THE MOTIVE OF THE CLASSIFYING STACK OF THE ORTHOGONAL GROUP

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Abstract. We compute the motive of the classifying stack of an orthogonal group in the Grothendieck ring of stacks over a field of characteristic different from two. We also discuss an application of this result to Donaldson-Thomas theory with orientifolds.

1. Introduction

The Grothendieck ring of stacks over a field k has been introduced by a number of authors [1], [5], [7], [11]. Denote this ring by $\hat{K}_0(Var_k)$. An algebraic group G defined over k is called special if any G-torsor over a k-variety is locally trivial in the Zariski topology. General linear, special linear and symplectic groups are special while special orthogonal groups are not special in dimensions greater than two. Serre proved that special groups are linear and connected [9]. Over algebraically closed fields, the special groups were classified by Grothendieck [6]. For a special group G, the motive $[G]$ is invertible in $\hat{K}_0(Var_k)$ and its inverse is equal to the motive of the classifying stack BG. This naturally raises the problem of computing the motive of BG when the group G is not special. For finite group schemes, a number of examples were computed in [4]. The case of groups of positive dimension is more difficult. In [2] it was shown that $[BPGL_n] = [PGL_n]^{-1}$ for $n = 2, 3$ with mild restrictions on the field k. In this paper, we compute the motive of the classifying stack of an orthogonal group over a field whose characteristic is not two; see Theorem 3.7. The result is that this motive is equal to the inverse of the motive of the split special orthogonal group of the same dimension. As a corollary, we are able to compute the motive of classifying stacks of special orthogonal groups in odd dimensions. We also describe an application of the result to Donaldson-Thomas theory with orientifolds.

Notation. We will work over a base field k with $\mathrm{char}(k) \neq 2$. If n is a non-negative integer we denote by $[n]_{\mathbb{L}}$ the nth Gaussian polynomial in the Lefschetz motive $\mathbb{L}$. Explicitly,

$$[n]_{\mathbb{L}} = 1 + \mathbb{L} + \cdots + \mathbb{L}^{n-1}.$$

The Gaussian polynomials $[n]_{\mathbb{L}}!$ and $\begin{bmatrix} n \\ r \end{bmatrix}_{\mathbb{L}}$ are defined in the usual way. The class of the Grassmannian $Gr(r,n)$ in the ring $\hat{K}_0(Var_k)$ is then $\begin{bmatrix} n \\ r \end{bmatrix}_{\mathbb{L}}$.

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2. Preliminaries

2.1. **The Grothendieck ring of stacks.** Fix a ground field k. Let $K_0(Var_k)$ be the Grothendieck ring of varieties over k. Its underlying abelian group is generated by symbols $[X]$, with X a variety, modulo the relations $[X] = [Y]$ if X and Y are isomorphic and

$$[X] = [X \backslash Z] + [Z]$$

if $Z \subset X$ is a closed subvariety. Cartesian product of varieties gives $K_0(Var_k)$ the structure of a commutative ring with identity. The Lefschetz motive is $\mathbb{L} = [\mathbb{A}^1_k]$.

The Grothendieck ring of stacks, $\hat{K}_0(Var_k)$, is the dimensional completion of $K_0(Var_k)$ defined as follows [1]. Let $F^m \subset K_0(Var_k)[\mathbb{L}^{-1}]$ be the additive subgroup generated by those $\mathbb{L}^{-d}[X]$ with $\dim X - d \leq -m$. This defines a descending filtration of $K_0(Var_k)[\mathbb{L}^{-1}]$ and $\hat{K}_0(Var_k)$ is the completion with respect to this filtration.

It is shown in [1] that any Artin stack that is essentially of finite type, all of whose geometric stabilizers are affine algebraic groups, has a motivic class in $\hat{K}_0(Var_k)$.

Lemma 2.1 ([1, Lemma 2.5])**.** *Let $\mathfrak{X}$ be an Artin stack (satisfying the conditions above) and let G be a special algebraic group. If $P \to \mathfrak{X}$ is G-torsor, then $[P] = [\mathfrak{X}][G]$ in $\hat{K}_0(Var_k)$.*

In particular, if G is special, applying Lemma 2.1 to the universal G-torsor $\operatorname{Spec} k \to BG$ shows that $[BG] = [G]^{-1}$. This equality is called the universal G-torsor relation.

More generally, if X is a variety acted on by an affine algebraic group G, then the quotient stack X/G has a class in $\hat{K}_0(Var_k)$. For any closed embedding $G \hookrightarrow GL_N$ there is an isomorphism of stacks $X/G \simeq (X \times_G GL_N)/GL_N$. Since GL_N is special, Lemma 2.1 implies that

$$[X/G] = \frac{[X \times_G GL_N]}{[GL_N]} \tag{1}$$

in $\hat{K}_0(Var_k)$.

2.2. **Orthogonal groups.** Recall that the ground field k is not of characteristic two. Let V be a finite dimensional vector space over k and let $Q : V \to k$ be a quadratic form. The radical of Q is the subspace of V given by

$$\mathrm{rad}_Q = \{v \in V \mid Q(v + w) = Q(v) + Q(w) \quad \forall w \in V\}.$$

The rank of Q is $\dim V - \dim \mathrm{rad}_Q$. The quadratic form Q is called nondegenerate if $\mathrm{rad}_Q = \{0\}$.

Given a nondegenerate quadratic form Q, denote by $O(Q)$ its group of isometries. If the field k is algebraically closed, then there is a unique nondegenerate quadratic form on k^n up to equivalence. The corresponding orthogonal group is unique up to isomorphism. If k is not algebraically closed, then there are will in general exist inequivalent nondegenerate quadratic forms on k^n, leading to different forms of orthogonal groups.

For each $n \geq 1$, there is a canonical nondegenerate split quadratic form on k^n. Explicitly,

$$Q_{2r} = x_1 x_2 + \cdots + x_{2r-1} x_{2r}$$

and

$$Q_{2r+1} = x_0^2 + x_1 x_2 + \cdots + x_{2r-1} x_{2r}.$$

Define $O_n = O(Q_n)$ and $SO_n = SO(Q_n)$.

3. The motive of $BO(Q)$

3.1. **Filtration of the space of quadratic forms.** Recall that $\mathrm{char}(k) \neq 2$. Denote by $Quad_n \simeq \mathbb{A}_k^{\binom{n+1}{2}}$ the affine space of quadratic forms on k^n. The group GL_n acts on $Quad_n$ by change of basis. For each $0 \leq r \leq n$, let $Quad_{n,\leq r} \subset Quad_n$ denote the closed subvariety of quadratic forms whose rank is at most r. This gives an increasing filtration of $Quad_n$ by closed subvarieties. Interpreted in $K_0(Var_k)$, this implies the identity

$$\mathbb{L}^{\binom{n+1}{2}} = \sum_{r=0}^{n} [Quad_{n,r}] \tag{2}$$

with $Quad_{n,r}$ the subvariety of quadratic forms of rank r. Denote by $Gr(m,n)$ the Grassmannian of m-planes in k^n.

Proposition 3.1. *For each $0 \leq r \leq n$, the map*

$$\pi : Quad_{n,r} \to Gr(n-r,n), \quad Q \mapsto \mathrm{rad}_Q$$

is a Zariski locally trivial fibration with fibres isomorphic to $Quad_{r,r}$.

Proof. Identify $Gr(n-r,n)$ with the quotient of the variety of $(n-r)\times n$ matrices of rank $n-r$ by the left action of GL_{n-r}. Fix coordinates $x_1,\dots,x_n$ on k^n. Consider the $(n-r)$-plane $k^{n-r} \subset k^n$ with coordinates $x_1,\dots,x_{n-r}$. A Zariski open set $U \subset Gr(n-r,n)$ containing k^{n-r} is given by the $(n-r)\times n$ matrices of the form

$$\begin{pmatrix} \mathbf{1}_{n-r} & B \end{pmatrix}$$

with $\mathbf{1}_{n-r}$ the $(n-r)\times(n-r)$ identity matrix and B an arbitrary $(n-r)\times r$ matrix. The plane k^{n-r} corresponds to the matrix $B=0$. Note that

$$\begin{pmatrix} \mathbf{1}_{n-r} & B \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{n-r} & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1}_{n-r} & B \\ 0 & \mathbf{1}_r \end{pmatrix}.$$

Let $g_B = \begin{pmatrix} \mathbf{1}_{n-r} & B \\ 0 & \mathbf{1}_r \end{pmatrix} \in GL_n$, viewed as an automorphism of k^n.

Suppose that $Q \in \pi^{-1}(U)$. Then there exists a unique matrix $B(Q)$ such that $\mathrm{rad}_Q = g_{B(Q)}(k^{n-r}) \subset k^n$. The quadratic form $g_{B(Q)} \cdot Q$ is the pullback of a nondegenerate quadratic form φ_Q in the variables $x_{n-r+1},\dots,x_n$. A trivialization of π over U is then given by

$$\pi^{-1}(U) \to U \times Quad_{r,r}, \qquad Q \mapsto (\mathrm{rad}_Q, \varphi_Q).$$

This argument can be repeated, replacing k^{n-r} with the $(n-r)$-plane with coordinates labelled by a $(n-r)$-element subset $I \subset \{1,\dots,n\}$. This gives a Zariski open cover of $Gr(n-r,n)$ over which π trivializes. □

Corollary 3.2. *The identity*

$$\mathbb{L}^{\binom{n+1}{2}} = \sum_{r=0}^{n} \begin{bmatrix} n \\ n-r \end{bmatrix}_{\mathbb{L}} [Quad_{r,r}]_{\mathbb{L}}$$

holds in the ring $K_0(Var_k)$.

Proof. It follows from Proposition 3.1 that $[Quad_{n,r}] = [Gr(n-r,n)][Quad_{r,r}]$. Since $[Gr(n-r,n)] = \begin{bmatrix} n \\ n-r \end{bmatrix}_{\mathbb{L}}$, the identity is implied by equation (2). □

3.2. **Solving the recurrence.** We form the exponential generating functions

$$P_{\text{even}}(x) = \sum_{k \geq 0} \frac{x^{2k}}{[2k]_{\mathbb{L}}!} \prod_{i=1}^{k} (\mathbb{L}^{2k+1} - \mathbb{L}^{2i})$$

and

$$P_{\text{odd}}(x) = \sum_{k \geq 0} \frac{x^{2k+1}}{[2k+1]_{\mathbb{L}}!} \prod_{i=0}^{k} (\mathbb{L}^{2k+1} - \mathbb{L}^{2i})$$

and

$$G(x) = \sum_{n \geq 0} \frac{[Quad_{n,n}] x^n}{[n]_{\mathbb{L}}!}.$$

Our goal in this section is to show that

$$G(x) = P_{\text{even}}(x) + P_{\text{odd}}(x),$$

thereby solving the recurrence relation from Corollary 3.2.

Proposition 3.3. *Denote by* $\exp_{\mathbb{L}}(x)$ *the* $\mathbb{L}$*-deformed exponential series:*

$$\exp_{\mathbb{L}}(x) = \sum_{n \geq 0} \frac{x^n}{[n]_{\mathbb{L}}!}.$$

The following equality holds:

$$G(x) = \frac{\prod_{i \geq 1} (1 + (1-\mathbb{L}) x \mathbb{L}^i)}{\exp_{\mathbb{L}}(x)}.$$

Proof. To ease notation, set $\mathcal{Q}_n = [Quad_{n,n}]$. Using Corollary 3.2 we find that

$$\begin{aligned}
G(x) &= \sum_{n \geq 0} \frac{\mathcal{Q}_n}{[n]_{\mathbb{L}}!} x^n \\
&= \sum_{n \geq 0} \left(\mathbb{L}^{\binom{n+1}{2}} - \sum_{r=0}^{n-1} \begin{bmatrix} n \\ n-r \end{bmatrix}_{\mathbb{L}} \mathcal{Q}_r \right) \frac{x^n}{[n]_{\mathbb{L}}!} \\
&= \sum_{n \geq 0} \left(\mathbb{L}^{\binom{n+1}{2}} - \sum_{r=0}^{n-1} \frac{[n]_{\mathbb{L}}!}{[n-r]_{\mathbb{L}}! [r]_{\mathbb{L}}!} \mathcal{Q}_r \right) \frac{x^n}{[n]_{\mathbb{L}}!} \\
&= \sum_{n \geq 0} \left(\mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} - \sum_{r=0}^{n-1} \frac{\mathcal{Q}_r x^r}{[r]_{\mathbb{L}}!} \frac{x^{n-r}}{[n-r]_{\mathbb{L}}!} \right) \\
&= \sum_{n \geq 0} \mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} - \sum_{n \geq 0} \sum_{r=0}^{n} \frac{\mathcal{Q}_r x^{n-r}}{[r]_{\mathbb{L}}! [n-r]_{\mathbb{L}}!} + \sum_{n \geq 0} \frac{\mathcal{Q}_n x^n}{[n]_{\mathbb{L}}!} \\
&= \sum_{n \geq 0} \mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} - \exp_{\mathbb{L}}(x) G(x) + G(x).
\end{aligned}$$

Recall that

$$[n]_{\mathbb{L}}! = \frac{(1-\mathbb{L})(1-\mathbb{L}^2) \cdots (1-\mathbb{L}^n)}{(1-\mathbb{L})^n}$$

so that we have

$$\sum_{n \geq 0} \mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} = \sum_{n \geq 0} \frac{\mathbb{L}^{\binom{n+1}{2}} (1-\mathbb{L})^n x^n}{(1-\mathbb{L})(1-\mathbb{L}^2) \cdots (1-\mathbb{L}^n)}.$$

We make use of the identity

$$\prod_{i \geq 1} (1 + (1-\mathbb{L}) x \mathbb{L}^i) = \sum_{n \geq 0} \frac{\mathbb{L}^{\binom{n+1}{2}} (1-\mathbb{L})^n x^n}{(1-\mathbb{L})(1-\mathbb{L}^2) \cdots (1-\mathbb{L}^n)},$$

see [10, Proposition 1.8.6]. Hence

$$G(x) = \frac{\prod_{i\geq 1}(1+(1-\mathbb{L})x\mathbb{L}^i)}{\exp_{\mathbb{L}}(x)},$$

which completes the proof. □

It will be convenient to make the change of variables $g(x) = G(\frac{x}{1-\mathbb{L}})$.

Proposition 3.4. *We have*

$$\begin{aligned} g(x) &= (1-x)\prod_{k\geq 1}(1-x^2\mathbb{L}^{2k}) \\ &= (1-x)\sum_{k\geq 0}\frac{(-1)^k x^{2k}\mathbb{L}^{k(k+1)}}{(1-\mathbb{L}^2)(1-\mathbb{L}^4)\cdots(1-\mathbb{L}^{2k})}. \end{aligned}$$

Proof. We compute

$$\begin{aligned} \exp_{\mathbb{L}}(x) &= \sum_{n\geq 0}\frac{x^n}{(n!)_{\mathbb{L}}} \\ &= \sum_{n\geq 0}\frac{x^n(1-\mathbb{L})^n}{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^n)} \\ &= \frac{1}{\prod_{i\geq 0}(1-(1-\mathbb{L})x\mathbb{L}^i)}, \end{aligned}$$

where the last equality is via [10, page 74]. The first assertion of the proposition now follows. The second follows from the first by [10, Proposition 1.8.6]. □

Similarly, make the change of variables $p_{\mathrm{even}}(x) = P_{\mathrm{even}}(\frac{x}{1-\mathbb{L}})$ and $p_{\mathrm{odd}}(x) = P_{\mathrm{odd}}(\frac{x}{1-\mathbb{L}})$.

Proposition 3.5. *We have*

$$p_{\mathrm{even}}(x) = \sum_{k\geq 0}\frac{(-1)^k x^{2k}\mathbb{L}^{k(k+1)}}{(1-\mathbb{L}^2)(1-\mathbb{L}^4)\cdots(1-\mathbb{L}^{2k})}$$

and

$$p_{\mathrm{odd}}(x) = \sum_{k\geq 0}\frac{(-1)^{k+1} x^{2k}\mathbb{L}^{k(k+1)}}{(1-\mathbb{L}^2)(1-\mathbb{L}^4)\cdots(1-\mathbb{L}^{2k})}.$$

Proof. The generating function P_{even} can be rewritten as

$$P_{\mathrm{even}}(x) = \sum_{k\geq 0}\frac{(1-\mathbb{L})^{2k}x^{2k}}{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^{2k})}\prod_{i=1}^{k}(\mathbb{L}^{2k+1}-\mathbb{L}^{2i}).$$

Then we have

$$\begin{aligned} p_{\mathrm{even}}(x) &= \sum_{k\geq 0}\frac{x^{2k}}{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^{2k})}\prod_{i=1}^{k}(\mathbb{L}^{2k+1}-\mathbb{L}^{2i}) \\ &= \sum_{k\geq 0}\frac{x^{2k}\mathbb{L}^{k(k+1)}}{(1-\mathbb{L})(1-\mathbb{L}^2)\cdots(1-\mathbb{L}^{2k})}\prod_{i=1}^{k}(\mathbb{L}^{2(k-i)+1}-1) \\ &= \sum_{k\geq 0}\frac{(-1)^k x^{2k}\mathbb{L}^{k(k+1)}}{(1-\mathbb{L}^2)(1-\mathbb{L}^4)\cdots(1-\mathbb{L}^{2k})}. \end{aligned}$$

The other calculation is similar. □

Corollary 3.6. *The following identity holds in* $\hat{K}_0(Var_k)$ *:*

$$G(x) = P_{\text{even}}(x) + P_{\text{odd}}(x).$$

Proof. As $(1-\mathbb{L})$ is a unit in $\hat{K}_0(Var_k)$ it suffices to show that

$$g(x) = p_{\text{even}}(x) + p_{\text{odd}}(x).$$

This follows from Propositions 3.4 and 3.5. □

3.3. **The main theorem.** We now state the main result.

Theorem 3.7. *Let k be a field whose characteristic is not 2 and let $n \geq 1$. For any nondegenerate quadratic form Q on k^n, the following equality holds in $\hat{K}_0(Var_k)$:*

$$[BO(Q)] = \begin{cases} \dfrac{\mathbb{L}^{-r}}{\prod_{i=0}^{r-1}(\mathbb{L}^{2r} - \mathbb{L}^{2i})}, & \textit{if } n = 2r+1 \\ \dfrac{\mathbb{L}^{r}}{\prod_{i=0}^{r-1}(\mathbb{L}^{2r} - \mathbb{L}^{2i})}, & \textit{if } n = 2r. \end{cases}$$

Proof. The subvariety $Quad_{n,n} \subset Quad_n$ is stable under the action of GL_n on $Quad_n$. Pick $Q \in Quad_{n,n}$. This gives rise to an orbit morphism $GL_n \to Quad_{n,n}$. Since $\pi : GL_n \to GL_n/O(Q)$ is a uniform categorical quotient [8, Theorem 1.1], the orbit morphism factors through a unique morphism $\psi : GL_n/O(Q) \to Quad_{n,n}$. We claim that ψ is an isomorphism.

Let $\overline{k}$ be an algebraic closure of k. Base change gives a morphism

$$\overline{\pi} : GL_{n,\overline{k}} \to GL_n/O(Q) \times_k \overline{k}$$

which is a categorical quotient for the action of $O(Q)_{\overline{k}}$ on $GL_{n,\overline{k}}$. Here $GL_{n,\overline{k}}$ denotes the general linear group over $\overline{k}$, and so on. The universal property of categorical quotients implies

$$GL_n/O(Q) \times_k \overline{k} \simeq GL_{n,\overline{k}}/O(Q)_{\overline{k}}.$$

Using this isomorphism and applying base change to ψ gives

$$\overline{\psi} : GL_{n,\overline{k}}/O(Q)_{\overline{k}} \to Quad_{n,n} \times_k \overline{k}.$$

Since $Quad_{n,n} \times_k \overline{k}$ is homogeneous under the action of $GL_{n,\overline{k}}$ with stabilizer $O(Q)_{\overline{k}}$, the map $\overline{\psi}$ is an isomorphism. By faithfully flat descent ψ itself is an isomorphism.

Identifying $BO(Q)$ with the quotient stack $\operatorname{Spec} k/O(Q)$, equation (1) gives

$$[BO(Q)] = \left[\frac{GL_n/O(Q)}{GL_n}\right] = \frac{[GL_n/O(Q)]}{[GL_n]} = \frac{[Quad_{n,n}]}{[GL_n]}.$$

Using Corollary 3.6, we read off from P_{even} and P_{odd} the equality

$$[Quad_{n,n}] = \begin{cases} \prod_{i=0}^{r}(\mathbb{L}^{2r+1} - \mathbb{L}^{2i}), & \text{if } n = 2r+1 \\ \prod_{i=1}^{r}(\mathbb{L}^{2r+1} - \mathbb{L}^{2i}), & \text{if } n = 2r. \end{cases}$$

A direct calculation now shows that $[BO(Q)]$ is given by the claimed formula. □

Corollary 3.8. *Suppose that $n \geq 3$ and let Q be a nondegenerate quadratic form on k^n. Then $[BO(Q)] = [SO_n]^{-1}$. Moreover, if n is odd, then $[BSO(Q)] = [SO_n]^{-1}$.*

Proof. Since $n \geq 3$, the split group SO_n is semisimple. According to [1, Lemma 2.1],

$$[SO_{2r+1}] = \mathbb{L}^r \prod_{i=0}^{r-1}(\mathbb{L}^{2r} - \mathbb{L}^{2i}), \qquad [SO_{2r}] = \mathbb{L}^{-r} \prod_{i=0}^{r-1}(\mathbb{L}^{2r} - \mathbb{L}^{2i}).$$

Comparing these expressions with Theorem 3.7 gives the first statement. Continuing, note that for nondegenerate quadratic forms in odd dimensions there is an isomorphism $O(Q) \simeq \mu_2 \times SO(Q)$. It is shown in [4, Proposition 3.2] that $[B\mu_2] = 1$. Hence

$$[BO(Q)] = [B\mu_2 \times BSO(Q)] = [B\mu_2][BSO(Q)] = [BSO(Q)].$$

The second statement now follows from the first. □

Since $PGL_2 \simeq SO_3$ over any field, Corollary 3.8 recovers the first part of [2, Theorem A] as a special case.

It follows from Corollary 3.8 that the universal torsor relations are satisfied for split special orthogonal groups in odd dimensions. In particular, the universal $SO_{2n+1}(\mathbb{C})$-torsor relation holds. In [3, Theorem 2.2] it is shown that for any non-special connected reductive complex algebraic group G there exists a G-torsor $P \to X$ over a variety such that $[P]$ is not equal to $[G][X]$. Therefore, the universal G-torsor relation does not imply the general G-torsor relation, answering a question posed in [1, Remark 3.3]. In the recent paper [2] the groups $PGL_2(\mathbb{C})$ and $PGL_3(\mathbb{C})$ were also shown to answer this question.

3.4. **Motivic Donaldson-Thomas theory with orientifolds.** Suppose that G is a linear algebraic group and let R be a finite dimensional representation of G. Lemma 2.1 and the fact that general linear groups are special implies $[R/G] = [R][BG]$. If G is a product of general linear, symplectic and orthogonal groups, then $[BG]$ can be computed explicitly using Theorem 3.7.

Exactly the above situation arises in the study of moduli stacks of self-dual representations of a quiver with involution [12]. In this case, G and R are naturally defined over finite fields of odd characteristic. As the motivic class $[R/G]$ is rational in $\mathbb{L}$, it follows that $[R/G]$ and the number of $\mathbb{F}_q$-rational points $\#[R/G](\mathbb{F}_q)$ agree under the substitution $\mathbb{L} \leftrightarrow q$. In other words, the orientifold Donaldson-Thomas series with trivial stability, defined in [12], are in fact motivic. In view of [12, Theorem 3.2], this suggests that an analogous statement holds for arbitrary σ-compatible stability. Indeed, the second author will show this in coming work.

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